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On the minimum $s-t$ cut problem with budget constraints

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Abstract

We consider in this paper the budgeted minimum $s-t$ cut problem. Suppose that we are given a directed graph $G = (V, A)$ with two distinguished nodes s and t, k non-negative cost functions $c^1, \dots, c^k : A \rightarrow \mathbb{Z}_+$, and $k-1$ budget bounds $b_1, \dots, b_{k-1}$. The goal is to find a $s-t$ cut C satisfying the budget constraints $c^h(C) \leq b_h$, for $h = 1, \dots, k-1$, and whose cost $c^k(C)$ is minimum. In this paper we discuss the linear relaxation of the problem and introduce a strict partial ordering on its solutions. We give a necessary and sufficient condition for which it has an integral optimal minimal (with respect to this ordering) basic solution. We also show that recognizing whether this is the case is NP-hard.

Keywords Minimum s−t cut problem · Budget constraints · Linear programming

Mathematics Subject Classification 90C27 · 90C10 · 90C05

1 Introduction

We consider in this paper the budgeted version of the well known minimum $s-t$ cut problem. Consider a directed graph $G = (V, A)$ with two distinguished nodes s and t, k cost functions or *criteria* $c^1, \dots, c^k : A \rightarrow \mathbb{Z}_+$ defined on its arcs, and bounds b_h associated with criteria c^h, for $h = 1, \dots, k-1$. A *cut* C of G is a subset $C \subseteq V$ of nodes such that $\emptyset \neq C \neq V$. For a given cut C, $\delta^+(C)$ is the set of arcs such

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that the tails are in C and the heads in $V\backslash C$. The *cost* of cut C w.r.t. criterion h is $c^h(C) \equiv c^h(\delta^+(C))$. A $s-t$ *cut* is a cut C such that $s \in C$ and $t \in V\backslash C$. The *budgeted minimum* $s-t$ *cut problem* (BMCP for short) is to find a $s-t$ cut C satisfying the budget constraints $c^h(C) \leq b_h$, for $h = 1, \ldots, k-1$ such that $c^k(C)$ is minimum. The BMCP problem has many practical applications.

In the context of algorithmic game theory, the minimum cut game is a special case of the BMCP. Given a graph $G = (V, A)$ with two distinguished nodes s and t and an arc cost function $c : A \to \mathbb{Z}_+$, the degree of a node $v \in V$ is $d(v) = \sum_{(v,u)\in A} c_{vu}$. Each vertex $v \in V\backslash\{s, t\}$ is an agent. A $s-t$ cut C is said to be *stable* for agent $v \in V\backslash\{s, t\}$ if the total cost of the arcs leaving v and crossing the cut C is at most half the degree of v, that is $\sum_{(v,u)\in\delta^+(C)} c_{vu} \leq \frac{d(v)}{2}$. Observe that all the vertices $v \in V\backslash\{C, t\}$ are stable since $\sum_{(v,u)\in\delta^+(C)} c_{vu} = 0$. In this game, a stable cut corresponds to a Nash equilibrium, that is, a state where no agent has an incentive to change his choice in being in the cut C or in $V\backslash C$. The goal here is to find a cut C, stable for all the nodes, that minimizes the total cost $c(C)$. An extensive literature is available on the related maximum cut game [8, 13, 14, 21].

The BMCP also arises in the contexts of disaster, military, and crime containment. In all these applications, a limited amount of resources is allocated to block the arcs of the graph by which the disaster could spread, or people could escape. At the same time, the area to which the disaster is confined should be as small as possible. Hayrapetyan et al. [10] show that the containment of damage problem can be reduced to the minimum size bounded cut problem. Given a graph $G = (V, E)$ with edge capacities c_e, $e \in E$, source and sink nodes s and t, as well as a bound b, the goal here is to find a $s-t$ cut C such that $c(C) \leq b$ and $|C|$ is minimum. Clearly, this is a special case of the BMCP.

The BMCP can also be used to find small dense subgraphs and communities [10]. Detecting communities in networks has been extensively studied in the context of social networks analysis and the World Wide Web [5, 12]. Given a graph $G = (V, E)$ with weights c_{uv}, $uv \in E$, on the edges, and a scalar α, a *community* (see [10]) is a node set S, such that $\frac{c(S)}{d(S)} \geq \alpha$, where $c(S) = \sum_{uv\in E: u,v\in S} c_{uv}$, and $d(S)$ is the sum of the degrees of the nodes in S. In [10], the authors reduce the problem consisting in finding a community to the minimum size bounded cut problem.

While the minimum $s-t$ cut problem can be solved in polynomial time, Papadimitriou and Yannakakis [18] proved that the BMCP is strongly NP-hard even for $k = 2$. This implies that even in the case $k = 3$, it is strongly NP-hard to test if the problem has a feasible solution. Therefore, the BMCP is not approximable when $k \geq 3$. Chestnut and Zenkluzen [2] give an $O(n)$-approximation algorithm for BMCP if $k = 2$ and show that it is hard to approximate.

1.1 Related work

There has been a substantial interest in the study of multicriteria and budgeted versions of several combinatorial optimization problems [1, 9, 11, 15, 16, 20]. In general, these versions are harder than the original ones. One exception is given in [1] where Armon and Zwick show that the budgeted minimum cut problem is polynomial-time solvable if graph G is undirected. Ravi and Goemans [20] consider the minimum spanning tree problem with a single budget constraint, and give a polynomial-time approximation scheme. Grandoni et al. [9] give polynomial-time approximation schemes for the spanning tree, matroid basis, and bipartite matching problems with a fixed number of budget constraints. Levin and Woeginger [15] consider the problem of minimizing the weighted sum of job completion times on a single machine with a budget constraint on the weighted sum of job completion times. The authors provide a polynomial time approximation scheme for this problem. All these algorithms exploit structural properties of the basic optimal solutions of the linear relaxations of the problems.

Hayrapetyan et al. [10] consider the minimum size bounded cut problem. They give a $(\frac{1}{\varepsilon}, 1)$ or a $(1, \frac{1}{1-\varepsilon})$ pseudo-approximation algorithm for any $0 < \varepsilon < 1$, that is to say that the algorithm returns either a super optimal solution violating the budget by a factor $\frac{1}{\varepsilon}$ or a feasible solution with $\frac{1}{1-\varepsilon}$ times more vertices on the s side than the optimal one. However, it is not known *a priori* which case occurs. Zhang [23] considers the related problem of the minimum b-size $s - t$ cut problem. The goal is to find a $s - t$ cut with minimum cost such that the s-side has a size at most b, for some integer $b \geq 0$. The author gives a $\frac{b+1}{b+1-b^*}$-approximation algorithm for the problem, where b^* is the size of the s-side optimal solution.

1.2 Our contributions

The minimum $s-t$ cut problem is one of the few polynomial-time solvable combinatorial optimization problems which become strongly NP-hard by adding a single budget constraint. The difficulty stems from the topology of the graph, the arbitrary nonnegative values of the coefficients of the knapsack constraints, and the budget bounds. In contrast, several graph optimization problems are hard because of the structure of the graph only. For instance, the linear relaxation of the vertex packing problem is integral if and only if the graph is bipartite [4, 17]. Therefore, one can exhibit fractional solutions associated with odd cycles in the graph. For the BMCP problem, the forbidden structure must depend on the graph structure, the costs values, and the budget bounds.

In this paper we investigate structural properties of the linear relaxation of the BMCP problem. We show that the support graph of any fractional basic solution contains few nodes but may have many arcs with fractional values. We introduce a partial ordering on the solutions, and show that the linear relaxation has an integral optimal minimal (with respect to this ordering) basic solution if and only if the underlying graph does not contain a certain configuration. We also show that recognizing such a configuration is NP-hard. Furthermore, we describe a class of instances of the BMCP problem where there is no such a configuration.

1.3 Organization

The paper is organized as follows. In the next section, we discuss the structural properties of the linear relaxation and state our main result. In Sect. 3 we examine the complexity of the recognition of the interdicted configuration.

2 Structural properties of the linear relaxation

A natural formulation of the minimum $s - t$ cut problem can be obtained from the dual of the maximum flow problem [7].

$$\min \sum_{(u,v)\in A} c_{uv} d_{uv} \tag{1a}$$

$$d_{uv} \geq p_u - p_v, \text{ for all arcs } (u, v) \in A, \tag{1b}$$

$$p_s = 1,\ p_t = 0, \tag{1c}$$

$$p_u \gtrless 0 \text{ for all } u \in V\backslash\{s, t\} \tag{1d}$$

$$d_{uv} \geq 0 \text{ for all arcs } (u, v) \in A. \tag{1e}$$

Variable p_v (refered to as the potential of v) is set to 1 if $v \in C$ and $p_v = 0$ otherwise, and variable $d_{uv} = 1$ if $u \in C$ and $v \notin C$, and $d_{uv} = 0$ otherwise. Here C is a node subset representing the desired cut. Constraints (1b) imply that for each $s - t$ path $\mathcal{P}$ we have $\sum_{(u,v)\in\mathcal{P}} d_{uv} \geq 1$. Intuitively, this states that along each $s - t$ path at least one arc must be in the $s - t$ cut. By adding the budget constraints to the above formulation, we get the linear relaxation of the BMCP.

$$\min \sum_{(u,v)\in A} c^k_{uv} d_{uv} \tag{2a}$$

$$d_{uv} \geq p_u - p_v, \text{ for all arcs } (u, v) \in A, \tag{2b}$$

$$p_s = 1,\ p_t = 0, \tag{2c}$$

$$\sum_{(u,v)\in A} c^h_{uv} d_{uv} \leq b_h, \text{ for } h = 1, \ldots, k - 1, \tag{2d}$$

$$p_u \gtrless 0 \text{ for all } u \in V\backslash\{s, t\}, \tag{2e}$$

$$d_{uv} \geq 0 \text{ for all arcs } (u, v) \in A. \tag{2f}$$

Let $P(G, c, b)$ denote the polyhedron given by inequalities (2b)-(2f), where $c = (c^1, \ldots, c^k)$ denotes the cost vectors and $b = (b_1, \ldots, b_{k-1})$ the budget bounds. We consider the BMCP under the following assumption.

Assumption 1 All nodes in $V\backslash\{s, t\}$ are part of an $s - t$ path.

If there exists a node u connected to s but not to t, then there exists an optimal solution of BMCP where u is merged with s. The symmetric case can be handled similarly.

In this section we study structural properties of the linear relaxation of BMCP and give a necessary and sufficient condition for which it has integral optimal minimal basic solution for nonegative cost vectors $c^1, \ldots, c^k$.

Example 1 Fig. 1 depicts an instance of the BMCP problem defined by a graph $G_1 = (V_1, A_1)$, $k = 2$ and a cost bound is $b_1 = 2M - 1$, where M is an arbitrary non-negative integer value.

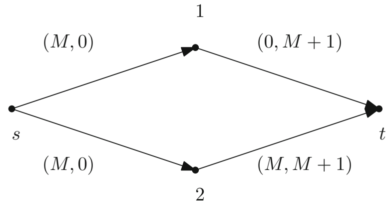


Fig. 1 Instance of the BMCP with an arbitrary high integrality gap

An optimal integer solution is $\bar{p}_1 = 1, \bar{p}_2 = 0, \bar{d}_{s1} = 0, \bar{d}_{s2} = 1, \bar{d}_{1t} = 1, \bar{d}_{2t} = 0$, with cost values $(M, M+1)$. However, a fractional basic optimal solution of the linear relaxation is $p_1 = \frac{1}{M}, p_2 = 0, d_{s1} = 1 - \frac{1}{M}, d_{s2} = 1, d_{1t} = \frac{1}{M}, d_{2t} = 0$, with cost values $(2M-1, 1+\frac{1}{M})$. The integrality gap is M which may be large.

Before stating our main result, we first give some definitions.

Definition 1 A solution (d, p) of $P(G, c, b)$ is said to be relevant if the following hold:

(i) $d_{uv} = \max\{p_u - p_v, 0\}$ for all arc $(u, v) \in A$, and
(ii) $0 \le p_u \le 1$ for all node $u \in V$.

Definition 2 Given two solutions (d^1, p^1) and (d^2, p^2) in $P(G, c, b)$, we say that (d^1, p^1) *dominates* (d^2, p^2) if $d^1_{uv} \le d^2_{uv}$ for all arcs $(u, v) \in A$ and $d^1_{u_0v_0} < d^2_{u_0v_0}$ for at least one arc $(u_0, v_0) \in A$.

Note that the dominance relation is a strict partial order. It is irreflexive (not (d, p) dominates (d, p)), asymmetric (if (d_1, p_1) dominates (d_2, p_2), then (d_2, p_2) does not dominate (d_1, p_1)), and transitive (if (d_1, p_1) dominates (d_2, p_2) and (d_2, p_2) dominates (d_3, p_3), then (d_1, p_1) dominates (d_3, p_3)). A solution (d, p) in $P(G, c, b)$ will be called *minimal* if it is not dominated by any other solution in $P(G, c, b)$.

In what follows we will give some structural properties of the linear relaxation of BMCP based on excluded cuts satisfying some specific properties. If C is a cut of G, we denote by (d^C, p^C) the incidence vector of C.

Definition 3 A set $\mathcal{S} = \{C^1, \ldots, C^r\}$ formed by $1 < r \le k$ nested $s-t$ cuts $C^1 \subset C^2 \subset \cdots \subset C^r$ is called FRACTIONAL if the following conditions are met:

1. $\mathcal{S}$ contains at least one infeasible cut C^q for some $q \in \{1, \ldots, r\}$, i.e., C^q violates at least one budget constraint, and
2. there exist scalars $\alpha_1 \ge 0, \ldots, \alpha_r \ge 0$ such that $\sum_{i=1}^r \alpha_i = 1$ and $(d, p) = \sum_{C^i \in \mathcal{S}} \alpha_i (d^{C^i}, p^{C^i})$ is a minimal basic solution of $P(G, c, b)$.

Since solution (d, p) in the above definition is a basic solution, there exists at least an infeasible cut $C^q \in \mathcal{S}$ whose associated multiplier α_q in the convex combination verifies $0 < \alpha_q < 1$ (otherwise, (d, p) is a convex combination of feasible cuts). The concept of FRACTIONAL cuts stems from the fact that solution (d, p) may be fractional.

Example 1 continued: Fig. 2 depicts a set $\mathcal{S} = \{C^1, C^2\}$ formed by two nested cuts C^1 and C^2. Observe that C^1 is infeasible. Define $\alpha_1 = 1 - \frac{1}{M}$ and $\alpha_2 = \frac{1}{M}$. The solution $(d, p) = \alpha_1(d^{C^1}, p^{C^1}) + \alpha_2(d^{C^2}, p^{C^2})$ is a convex combination of the incidence

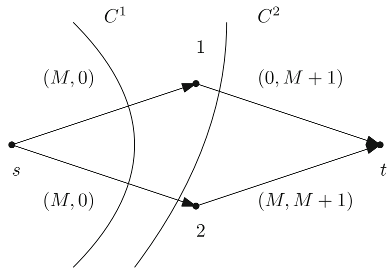


Fig. 2 The set $\mathcal{S} = \{C^1, C^2\}$ is FRACTIONAL

vectors of the cuts C^1 and C^2 and corresponds to the fractional basic optimal solution of the linear relaxation $p_1 = \frac{1}{M}$, $p_2 = 0$, $d_{s1} = 1 - \frac{1}{M}$, $d_{s2} = 1$, $d_{1t} = \frac{1}{M}$, $d_{2t} = 0$. Observe that this solution satisfies with equality all the constraints (2b) and the two $s - t$ paths $s, 1, t$ and $s, 2, t$ satisfy $d_{s1} + d_{1t} = d_{s2} + d_{1t2} = 1$. Suppose that solution (d, p) is dominated by a solution $(\bar{d}, \bar{p}) \in P(G, c, b)$. It then follows that, $\bar{d}_{uv} \leq d_{uv}$ for all arcs $(u, v) \in A$ with at least one strict inequality, say $\bar{d}_{s1} < d_{s1}$. This implies that for the path $s, 1, t$ we have $\bar{d}_{s1} + \bar{d}_{1t} < d_{s1} + d_{1t} = 1$. Consequently, solution $(\bar{d}, \bar{p})$ is infeasible which is a contradiction. Therefore, solution (d, p) is a minimal basic solution and the set $\mathcal{S}$ is FRACTIONAL.

Remark that the pathological example depicted in Fig. 1 can be expanded by applying the following operations:

- O_1 : Replace a node by an arbitrary strongly connected component where the c^1 and c^2 costs of the arcs in these components are 0 and M, respectively.
- O_2 : Subdivide an arc (i, j) and replace it by a path i, i_0, j such that the c^1 and c^2 costs of the arcs (i, i_0) and (i_0, j) coincide with those of (i, j).

It is not hard to show the following result.

Lemma 1 *Let $G = (V, A)$ be a directed graph with two cost functions c^1, c^2 associated with its arcs. Let b_1 be a cost bound on c^1. Let $G' = (V', A')$ be a graph obtained by repeated application of Operations O_1, O_2. If G contains a* FRACTIONAL, *then G' contains a FRACTIONAL set.*

Figure 3 shows a graph G_2 obtained by the application of the above operations. Observe that the graph G_1, depicted in Fig. 1, is a minor graph of G_2 obtained by contracting all the newly created arcs and components. As a consequence, the cut set given in Fig. 2 is associated to a FRACTIONAL set in G_2.

Now we can state the main result of this section.

Theorem 1 *Given nonnegative cost vectors $c^1, \ldots, c^{k-1}$, the linear relaxation of the BMCP has no minimal fractional optimal basic solution if and only if there exists no* FRACTIONAL *set of nested $s - t$ cuts.*

Note that Theorem 1 does not characterize all basic optimal solutions of the BMCP but only the minimal ones. Also note that by Theorem 1, there may exist a fractional optimal basic solution even though all minimal optimal basic solutions are integral. The following example, depicted in Fig. 4, illustrates this situation. Assume that $k = 2$

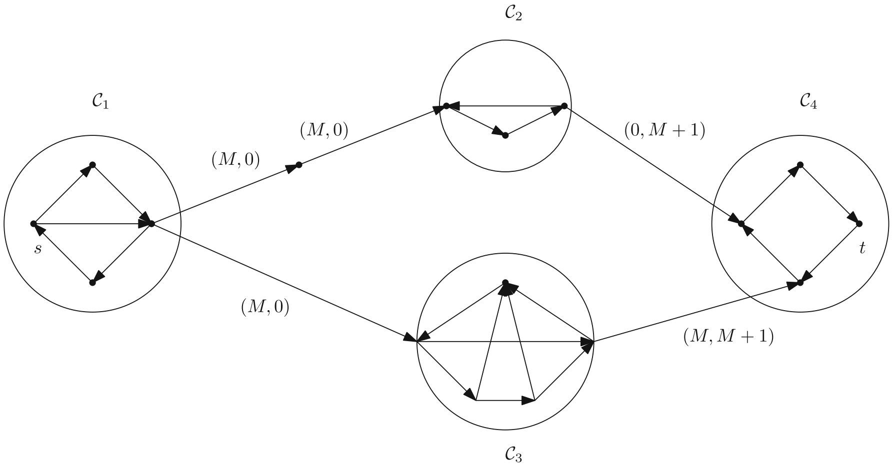
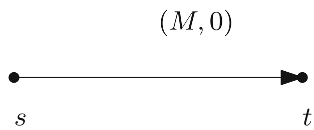


Fig. 3 An example of applications of the graph transformation operations O_1, O_2

Fig. 4 Instance of BMCP with a fractional optimal basic solution and an integral minimal optimal basic solution



and the cost bound is $b_1 = M + 1$, where M is an arbitrary non-negative integer value. This trivial graph has only one $s - t$ cut. This cut is feasible and thus, there exists no FRACTIONAL set of cuts (a set containing at least one infeasible cut). In this case, the condition of Theorem 1 holds and, consequently, the BMCP has no minimal fractional optimal basic solution. The only minimal basic optimal solution is $p_s = 1, p_t = 0, d_{st} = 1$ with cost values $(M, 0)$. However, the solution $p'_s = 1, p'_t = 0, d'_{st} = 1 + \frac{1}{M}$ is a fractional basic optimal solution with cost values $(M + 1, 0)$. This solution is dominated by (d, p).

The proof of Theorem 1 will be obtained as a consequence of the following series of lemmas.

Lemma 2 *Any minimal solution of $P(G, c, b)$ is relevant.*

Proof Consider a minimal solution (d, p) of $P(G, c, b)$ and suppose that it is not relevant. First, assume that Condition i) of Definition 1 does not hold. Hence, there exists an arc $(u_0, v_0) \in A$ such that $d_{u_0 v_0} > \max\{p_{u_0} - p_{v_0}, 0\}$. Let $(\bar{d}, \bar{p})$ be the solution given by

$$\begin{cases} \bar{p}_u = p_u \text{ for all } u \in V, \\ \bar{d}_{uv} = \max\{\bar{p}_u - \bar{p}_v, 0\} \text{ for all } (u, v) \in A. \end{cases}$$

Since the cost vector c is nonnegative and $\bar{d} \leq d$ (with at least a strict inequality), $(\bar{d}, \bar{p})$ satisfies the budget constraints (2d). Moreover, $(\bar{d}, \bar{p})$ fulfills constraints (2b). Therefore, $(\bar{d}, \bar{p})$ belongs to $P(G, c, b)$ and $(\bar{d}, \bar{p})$ dominates (d, p). This contradicts the minimality of (d, p).

Now suppose that Condition (ii) of Definition 1 does not hold. Then, there exists at least one node $u_0 \in V$ such that, say, $p_{u_0} > 1$. The case where $p_{u_0} < 0$ can be handled similarly. Consider the solution $(\bar{d}, \bar{p})$ given by

$$\bar{p}_u = \begin{cases} 1 \text{ if } p_u > 1, \\ 0 \text{ if } p_u < 0, \\ p_u \text{ otherwise,} \end{cases} \tag{3}$$

and

$$\bar{d}_{uv} = \max\{\bar{p}_u - \bar{p}_v, 0\} \text{ for all } (u, v) \in A. \tag{4}$$

Clearly $\bar{d} \leq d$. Let V_1 denote the set of nodes $u \in V$ such that $p_u > 1$. By Assumption 1, any node $u \in V_1$ is part of an $s - t$ path. Since $s, t \notin V_1$, it follows that there exists at least an arc $(u_1, v_1) \in A$ such that $u_1 \in V_1$ and $v_1 \notin V_1$. Hence, $p_{u_1} > 1$ and $p_{v_1} \leq 1$. As $d_{u_1 v_1} \geq p_{u_1} - p_{v_1}$ and $\bar{p}_{u_1} < p_{u_1}$, it follows that $\bar{d}_{u_1 v_1} < d_{u_1 v_1}$. This implies that $(\bar{d}, \bar{p})$ dominates (d, p), contradicting again the minimality of (d, p). □

Lemma 3 *For any non minimal solution (d, p) of $P(G, c, b)$, there exists a minimal solution of $P(G, c, b)$ that dominates it.*

Proof As (d, p) is not minimal, there exists a solution (d^1, p^1) in $P(G, c, b)$ that dominates it. By Definition 2, we have

$$\sum_{(u,v)\in A} d^1_{uv} < \sum_{(u,v)\in A} d_{uv}.$$

If (d^1, p^1) is minimal, then we are done. If not, then (d^1, p^1) would be dominated by a solution (d^2, p^2) of $P(G, c, b)$. By transitivity of the dominance relation, we have that (d^2, p^2) dominates (d, p), yielding

$$\sum_{(u,v)\in A} d^2_{uv} < \sum_{(u,v)\in A} d^1_{uv}.$$

Since the optimal value of

$$\min_{(d,p)\in P(G,c,b)} \sum_{(u,v)\in A} d_{uv}$$

is finite, this process must stop at a minimal solution which by transitivity dominates (d, p). □

The following result gives a constructive proof of the existence of a minimal optimal basic solution of the linear relaxation of BMCP.

Lemma 4 *The linear relaxation of the BMCP has a minimal optimal basic solution.*

Proof Consider the linear program defined as

$$\min_{(d,p)\in\mathcal{S}(G,c,b)} \sum_{(u,v)\in A} d_{uv}, \tag{5}$$

where $\mathcal{S}(G, c, b)$ is the set of all the optimal solutions of the linear relaxation of BMCP. Let (d^*, p^*) denote a basic optimal solution of (5). This solution minimizes the sum of the d_{uv}'s among all the basic optimal solutions of the linear relaxation of the BMCP. Observe that (d^*, p^*) is also a basic optimal solution of the linear relaxation of the BMCP. We claim that (d^*, p^*) is minimal. In fact, if this is not the case, then by Lemma 3, there exists a minimal solution $(\bar{d}, \bar{p})$ in $P(G, c, b)$ that dominates it. Since the cost function c^k is nonnegative, it follows that $(\bar{d}, \bar{p})$ is an optimal solution of BMCP and we have $\sum_{(u,v)\in A} \bar{d}_{uv} < \sum_{(u,v)\in A} d^*_{uv}$. This contradicts the optimality of (d^*, p^*). □

The following result gives a structural property that is satisfied by all minimal solution.

Lemma 5 *For every minimal solution (d, p) of $P(G, c, b)$, there exists a $s - t$ path $P = u_0(= s), u_1, \ldots, u_q(= t)$ such that $p_{u_0} \geq p_{u_1} \geq \cdots \geq p_{u_q}$ and $\sum_{(u_j,u_{j+1})\in P} d_{u_j u_{j+1}} = 1$.*

Proof By Lemma 2, as (d, p) is minimal, it follows that it is relevant and thus $0 \leq p_u \leq 1$ for all $u \in V$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ denote the partition of the nodes of V into classes according to the decreasing potential values p_i, i.e., for any nodes $u, u' \in \mathcal{C}_i$, we have $p_u = p_{u'}$ and for any $u \in \mathcal{C}_j$ and $v \in \mathcal{C}_l$ such that $j < l$, we have $p_u > p_v$. The sets $\mathcal{C}_1$ and $\mathcal{C}_r$ correspond to the sets of nodes $u \in V$ such that $p_u = 1$ and $p_u = 0$, respectively.
Starting from node $u_q = t$, we will iteratively construct the desired path by computing some shortest paths between pairs of nodes lying in different sets $\mathcal{C}_i$. The length $l(P_{uv})$ of a path P_{uv} between nodes u and v corresponds to the number of its arcs. Let P^*_{uv} denote a shortest path between u and v. By Assumption 1, all the nodes in $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{r-1}$ are part of $s - t$ paths in G. Therefore, there exists a node

$$u_i \in \arg\min_{u \notin \mathcal{C}_r} l(P^*_{ut}).$$

Since $P^*_{u_i t} = u_i, u_{i+1}, \ldots, u_q = t$ is a shortest $u - u_q$ path among the nodes $u \in \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{r-1}$, the nodes $u_{i+1}, \ldots, u_q$ are in $\mathcal{C}_r$. For otherwise, if $u_{i+1} \notin \mathcal{C}_r$, then $P_{u_{i+1}t}$ is a shorter $u - t$ path, a contradiction. This shows that

$$p_{u_i} \geq \cdots \geq p_{u_q}. \tag{6}$$

As (d, p) is relevant, we have therefore $d_{u_i u_{i+1}} = p_{u_i} - p_{u_{i+1}} > 0$ and $d_{u_j u_{j+1}} = p_{u_j} - p_{u_{j+1}} = 0$ for any $j \in \{i+1, \dots, q-1\}$. This yields

$$\begin{aligned} d_{u_i u_{i+1}} + d_{u_{i+1} u_{i+2}} + \cdots + d_{u_{q-1} u_q} &= \sum_{j=i}^{q-1} p_{u_j} - p_{u_{j+1}}, \\ &= p_{u_i} - p_{u_q}, \\ &= p_{u_i}. \end{aligned} \tag{7}$$

Let $h \in \{1, \dots, r\}$ such that $u_i \in \mathcal{C}_h$, $G_h = (V_h, A_h)$ denote the graph obtained from G such that $V_h = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_h$, and A_h consists of the set of arcs in A induced by V_h. For any node $u \in V_h$, let $P^*_{uu_i}$ denote a shortest path in G_h between u and u_i. The following result shows that there exists at least one node in $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{h-1}$ connected to u_i in G_h.

Claim 1 *There exists a $u - u_i$ path in G_h from some node $u \in \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{h-1}$ to u_i.*

Proof of Claim 1: Suppose by contradiction that for any node $u \in \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{h-1}$ there exists no $u - u_i$ path in G_h. Let $S \subseteq \mathcal{C}_h$ denote the largest set of nodes u in $\mathcal{C}_h$ connected to u_i by a $u - u_i$ path. Observe that no node u in $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{h-1}$ or in $\mathcal{C}_h \backslash S$ is connected in G_h to any node $v \in S$ by a $u - v$ path. Let $r^+(S)$ denote the smallest index $l > h$ such that $\mathcal{C}_l$ contains a node u incident to an arc $(u, v) \in A$ with $v \in S$. Consider the solution (d', p') obtained from (d, p) by moving all the nodes in S to $\mathcal{C}_{r^+(S)}$ given by

$$p'_u = \begin{cases} p_u \text{ if } u \notin S, \\ p_v \text{ for some } v \in \mathcal{C}_{r^+(S)} \text{ if } u \in S, \end{cases}$$

and

$$d'_{uv} = \max\{p'_u - p'_v, 0\} \text{ for any arc } (u, v) \in A.$$

See Fig. 5 for an illustration. By construction, the values of the arc variables are not increased ($d' \leq d$). However, since $u_{i+1} \in \mathcal{C}_{r^+(S)}$, it follows that $d'_{u_i u_{i+1}} < d_{u_i u_{i+1}}$ and hence (d', p') dominates (d, p). This is a contradiction since (d, p) is minimal. □

By Claim 1, there exists a node

$$u_l \in \arg \min_{u \in \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{h-1}} l(P^*_{uu_i}). \tag{8}$$

Since $P^*_{u_l u_i}$ is a shortest $u - u_i$ path in G_h among the nodes $u \in \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{h-1}$ to u_i, $u_l \in \mathcal{C}_f$ for some $f < h$ and all the remaining nodes of $P^*_{u_l u_i}$ are in $\mathcal{C}_h$. Therefore by (6), the potentials of the nodes in the $u_l - u_q$ path obtained from $P^*_{u_l u_i} \cup P^*_{u_i u_q}$ are nonincreasing

$$p_{u_l} \geq \cdots \geq p_{u_i} \geq \cdots \geq p_{u_q}. \tag{9}$$

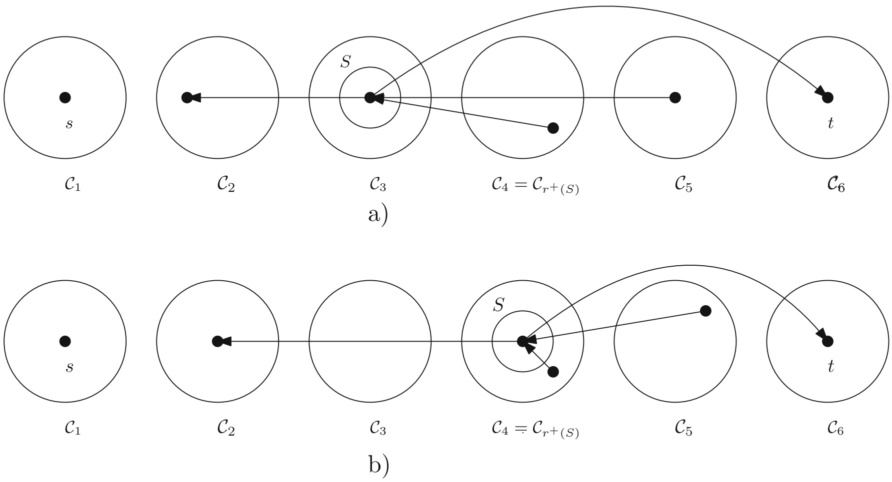


Fig. 5 The origins of all the arcs entering S are in $\mathcal{C}_4 \cup \mathcal{C}_5$. Solution (d', p') is obtained from (d, p) by moving S to $\mathcal{C}_4 = \mathcal{C}_{r^+(S)}$

Furthermore, by using the same argument as in (7), one can show that the arcs of path $u_l - u_q$ satisfy

$$\sum_{r=l}^{q-1} d_{u_r u_{r+1}} = p_{u_l} - p_{u_q} = p_{u_l}. \tag{10}$$

By applying inductively the arguments in Claim 1, (9), and (10) to node u_l, one can show that there exists a $u_0 - u_q$ path $P_{sq} = u_0(= s), u_1, \ldots, u_q$ such that $p_{u_0} \geq \cdots \geq p_{u_q}$ and

$$d_{u_0 u_1} + d_{u_1 u_2} + \cdots + d_{u_{q-1} u_q} = p_{u_0} = 1,$$

and the result follows. □

Next we introduce a reduction operation. Given a solution (d, p) of $P(G, c, b)$, consider the following operation:

O: contract an arc $(u, v) \in A$ such that $d_{uv} = p_u - p_v$ and $d_{uv} = 0$.

If an arc (u, v) is contracted and w is the node that arises from the contraction (which we call supernode), then we set $p_w = p_u = p_v$. Operation O preserves the d_{uv}'s, that is to say, for any arc $(i, j) \in A$ such that $i \in \{u, v\}$ and $j \notin \{u, v\}$, we have $d_{wj} = d_{ij}$. Let $G' = (V', A')$ and (d', p') denote the minor digraph and the solution obtained from (d, p) by repeated applications of Operation O. For sake of notation, the supernodes in V' containing s and t are denoted also by s and t, respectively. Let c' denote the restriction of the cost vector c on A'. The following results show that Operation O also preserves the extremality of the relevant solutions and yields a reduced graph with at most $k + 1$ nodes.

Lemma 6 *If (d, p) is a relevant basic solution of $P(G, c, b)$, then (d', p') is a relevant basic solution of $P(G', c', b)$.*

Proof If (d, p) is relevant, clearly, $0 \leq p'_{u'} \leq 1$ for all the nodes $u' \in V'$. Moreover, for any arc $(u', v') \in A'$, corresponding to an arc $(u, v) \in A$, we have

$$d'_{u'v'} = d_{uv} = \max\{p_u - p_v, 0\} = \max\{p'_{u'} - p'_{v'}, 0\}.$$

This shows that (d', p') is also relevant. It remains to show that this solution is also a basic solution of $P(G', c', b)$.
Suppose that $(d', p') = \frac{1}{2}(d^{1'}, p^{1'}) + \frac{1}{2}(d^{2'}, p^{2'})$ where $(d^{1'}, p^{1'})$ and $(d^{2'}, p^{2'})$ are two different solutions of $P(G', c', b)$. Let (d^1, p^1) and (d^2, p^2) denote the solutions obtained from $(d^{1'}, p^{1'})$ and $(d^{2'}, p^{2'})$ as follows:

- for all arc $(u, v) \in A$ contracted by O, set $d^1_{uv} = d^2_{uv} = 0$,
- for all arc $(u, v) \in A$ not contracted by O and corresponding to an arc $(w_i, w_j) \in A'$, set $d^1_{uv} = d^{1'}_{w_i w_j}$ and $d^2_{uv} = d^{2'}_{w_i w_j}$, and
- for all node $u \in V$ contained in a supernode w_i, set $p^1_u = p^{1'}_{w_i}$ and $p^2_u = p^{2'}_{w_i}$.

If $(u, v) \in A$ is not contracted by O, then it corresponds to an arc $(u', v') \in A'$ and we have

$$d_{uv} = d'_{u'v'} = \frac{1}{2}d^{1'}_{u'v'} + \frac{1}{2}d^{2'}_{u'v'} = \frac{1}{2}d^1_{uv} + \frac{1}{2}d^2_{uv}.$$

If arc $(u, v) \in A$ is contracted by O, we have $d_{uv} = d^1_{uv} = d^2_{uv} = 0$. Since Operation O preserves the p_u's, for any node $u \in V$ contained in a supernode $u' \in V'$ we have

$$p_u = p'_{u'} = \frac{1}{2}p^{1'}_{u'} + \frac{1}{2}p^{2'}_{u'} = \frac{1}{2}p^1_u + \frac{1}{2}p^2_u.$$

This shows that $(d, p) = \frac{1}{2}(d^1, p^1) + \frac{1}{2}(d^2, p^2)$ and thus contradicts that (d, p) is a basic solution of $P(G, c, b)$. □

Lemma 7 *Let (d, p) be a relevant basic solution of $P(G, c, b)$. Then, the graph $G' = (V', A')$ obtained from $G = (V, A)$ by all possible applications of Operation O has at most $k + 1$ nodes.*

Proof Let (d', p') denote the solution obtained from (d, p) by all possible applications of Operation O. By Lemma 6, (d', p') is a relevant basic solution of $P(G', c', b)$. Note that no arc $(w_i, w_j) \in A'$ satisfies $d'_{w_i w_j} = p'_{w_i} - p'_{w_j} = 0$. Otherwise, (w_i, w_j) would have been contracted. Since (d', p') is a basic solution of $P(G', c', b)$, (d', p') is the unique solution of a system S of $|A'| + |V'|$ linearly independent equations among the inequalities of $P(G', c', b)$. These tight constraints must be chosen among $d_{w_i, w_j} \geq p_{w_i} - p_{w_j}$, $d_{w_i, w_j} \geq 0$ for all arcs $(w_i, w_j) \in A'$, the budget constraints $\sum_{(u,v)\in A} c^h_{uv} d_{uv} \leq b_h$, for $h = 1, \ldots, k - 1$, and $p_s = 1$, $p_t = 0$. This implies that $|V'| \leq k + 1$ and the result follows. □

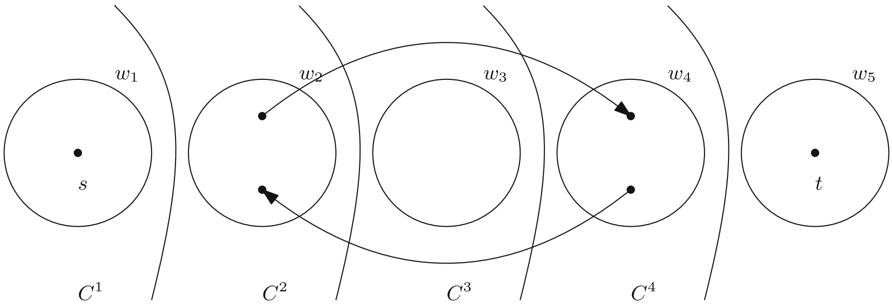


Fig. 6 In the digraph $G' = (V', A')$, the arc joining nodes w_2 and w_4 belongs to the cuts C^2 and C^3. In contrast, the arc joining nodes w_4 and w_2 does not belong to any cut C^i. The supernode w_3 is contained in the cuts C^3 and C^4

In what follows, we will discuss the relationship between the minimal solutions and the FRACTIONAL sets of nested $s-t$ cuts.

Lemma 8 *For any non integral minimal basic solution (d, p) of $P(G, c, b)$, there exists a* FRACTIONAL *set $\mathcal{S}$ formed by $|\mathcal{S}| \leq k$ nested $s-t$ cuts such that (d, p) is a solution satisfying Condition 2 of Definition 3.*

Proof We will show that (d, p) is a convex combination of the incidence vectors of a set of $r \leq k$ cuts $C^1, \ldots, C^r$ containing at least one infeasible cut C^q for some $q \in \{1, \ldots, r\}$. Let $G' = (V', A')$ and (d', p') denote the graph and the solution obtained from G and (d, p), respectively, after all possible applications of Operation O. Suppose that $V' = \{w_1 = s, \ldots, w_{r-1}, w_r = t\}$ and $p'_s = p'_{w_1} \geq p'_{w_2} \geq \cdots \geq p'_{w_{r-1}} \geq p'_{w_r} = p'_t$. Let $\mathcal{S}$ be the set of nested cuts $C^1 = \{s\}$, $C^2 = \{s, w_2\}, \ldots, C^{r-1} = \{s, w_1, \ldots, w_{r-1}\}$ in G'. By Lemma 7, we have $r \leq k+1$ and thus $|\mathcal{S}| \leq k$.
As (d, p) is minimal, by Lemma 2, it is relevant and as it is basic, by Lemma 6, (d', p') is a relevant basic solution of $P(G', c', b)$. Hence $d'_{w_i w_j} = \max\{p'_{w_i} - p'_{w_j}, 0\}$ for any arc $(w_i, w_j) \in A'$. Observe that if $p'_{w_i} \geq p'_{w_j}$, arc $(w_i, w_j) \in A'$ crosses cuts $C^i, \ldots, C^{j-1}$, that is to say, $d^{C^h}_{w_i w_j} = 1$ for $i \leq h \leq j-1$ and $d^{C^h}_{w_i w_j} = 0$ if $h \geq j$ or $h < i$, see Fig. 6. Therefore, $d'_{w_i w_j}$ can be written as

$$
\begin{aligned}
d'_{w_i w_j} &= p'_{w_i} - p'_{w_j} \\
&= (p'_{w_i} - p'_{w_{i+1}}) d^{C^i}_{w_i w_j} + (p'_{w_{i+1}} - p'_{w_{i+2}}) d^{C^{i+1}}_{w_i w_j} + \cdots + (p'_{w_{j-1}} - p'_{w_j}) d^{C^{j-1}}_{w_i w_j} \\
&= \sum_{h=i}^{j-1} (p'_{w_h} - p'_{w_{h+1}}) d^{C^h}_{w_i w_j} \\
&= \sum_{h=1}^{r-1} (p'_{w_h} - p'_{w_{h+1}}) d^{C^h}_{w_i w_j} (d^{C^h}_{w_i w_j} = 0 \text{ if } h \geq j \text{ or } h < i).
\end{aligned} \tag{11}
$$

Now if $p'_{w_i} \leq p'_{w_j}$, then $d'_{w_i w_j} = 0$ and arc (w_i, w_j) does not cross any cut C^i for $i = 1, \ldots, r-1$, i.e., $d^{C^h}_{w_i w_j} = 0$ for $h = 1, \ldots, r-1$ (see Fig. 6). Therefore,

$$d'_{w_i w_j} = \sum_{h=1}^{r-1} (p'_{w_h} - p'_{w_{h+1}}) d^{C^h}_{w_i w_j}. \tag{12}$$

Each node $w_i \in V'$ belongs to cuts $C^i, \ldots, C^{r-1}$, i.e., $p^{C^h}_{w_i} = 1$ for $h = i, \ldots, r-1$ and $p^{C^h}_{w_i} = 0$ for $h < i$ (see Fig. 6). Hence, we have

$$\begin{aligned} p'_{w_i} &= (p'_{w_i} - p'_{w_{i+1}}) p^{C^i}_{w_i} + (p'_{w_{i+1}} - p'_{w_{i+2}}) p^{C^{i+1}}_{w_i} + \cdots + (p'_{w_{r-1}} - p'_{w_r}) p^{C^{r-1}}_{w_i} \\ &= \sum_{h=i}^{r-1} (p'_{w_h} - p'_{w_{h+1}}) p^{C^h}_{w_i} \\ &= \sum_{h=1}^{r-1} (p'_{w_h} - p'_{w_{h+1}}) p^{C^h}_{w_i} \, (p^{C^h}_{w_i} = 0 \text{ for } h < i). \end{aligned} \tag{13}$$

Set $\mu_i = p'_{w_i} - p'_{w_{i+1}} \geq 0$ for $i = 1, \ldots, r-1$. Thus

$$\sum_{i=1}^{r-1} \mu_i = p'_{w_1} - p'_{w_r} = 1.$$

By (11)–(13), (d', p') is a convex combination of the cuts of $\mathcal{S}$ with coefficients μ_i, i.e., $(d', p') = \sum_{i=1}^{r-1} \mu_i (d^{C^i}, p^{C^i})$. As by Lemma 6, (d', p') is a basic solution of $P(G', c', b)$, it follows that $\mathcal{S}$ contains at least one infeasible cut C^q.
Consider now in G the set $\bar{\mathcal{S}}$ of nested $s - t$ cuts $\bar{C}^h$, $h = 1, \ldots, r$, obtained by expanding the cuts C^h, $h = 1, \ldots, r$. Let $(d^{\bar{C}^h}, p^{\bar{C}^h})$ denote the incidence vector of cut $\bar{C}^h$ defined as follows: $d^{\bar{C}^h}_{uv} = d^{C^h}_{w_i w_j}$ for any arc $(u, v) \in A$ not contracted by Operation O and corresponding to some arc $(w_i, w_j) \in A'$, and $d^{\bar{C}^i}_{uv} = 0$ if not. Furthermore, $p^{\bar{C}^h}_u = p^{C^h}_{w_i}$ for any node $u \in V$ contained in a supernode $w_i \in V'$.
We exploit in the following the fact that Operation O preserves the d_{uv}'s and the potential values p_u. This property is denoted by $\mathcal{P}$. For any arc $(u, v) \in A$ corresponding to some arc $(w_i, w_j) \in A'$ we have by (11) and $\mathcal{P}$ that

$$d_{uv} = d'_{w_i w_j} = \sum_{i=1}^{r-1} \mu_i d^{C^i}_{w_i w_j} = \sum_{i=1}^{r-1} \mu_i d^{\bar{C}^i}_{uv}. \tag{14}$$

For an arc $(u,v)\in A$, contracted by Operation O, we have $d_{uv}=0$ and $d_{uv}^{\bar{C}^h}=0$ for $h=1,\dots,r$. This implies that

$$d_{uv}=\sum_{i=1}^{r-1}\mu_i d_{uv}^{\bar{C}^i}. \tag{15}$$

Now, for a node $u\in V$ contained in a supernode $w_i\in V'$, we have by (13) and $\mathcal{P}$ that

$$p_u=p'_{w_i}=\sum_{i=1}^{r-1}\mu_i p_{w_i}^{C^i}=\sum_{i=1}^{r-1}\mu_i p_u^{\bar{C}^i}. \tag{16}$$

By (14)–(16), it follows that (d,p) is a convex combination of the cuts in $\bar{\mathcal{S}}$. Moreover, as proved before, one of these cuts is infeasible, and the proof is complete. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1 ($\Leftarrow$) Suppose that there exists no FRACTIONAL set of nested $s-t$ cuts. By Lemma 4, the linear relaxation of the BMCP has an optimal minimal basic solution (d^*,p^*). If (d^*,p^*) is non integral, then by Lemma 8 it can be decomposed into a FRACTIONAL set of nested $s-t$ cuts. This contradicts the fact that there is no FRACTIONAL set of nested $s-t$ cuts. Therefore, (d^*,p^*) is integral.

($\Rightarrow$) Suppose that there exists a FRACTIONAL set $\mathcal{S}=\{C^1,\dots,C^r\}$ of nested $s-t$ cuts in G containing an infeasible cut C^q for some $1\le q\le r$. Let (d^*,p^*) denote the fractional minimal basic solution given in Definition 3. We will construct a cost function $c^k:A\to\{0,1\}$ such that (d^*,p^*) is an optimal basic solution of $P(G,c,b)$ with respect to function c^k.

Let $\mathcal{C}_1,\dots,\mathcal{C}_r$ denote the partition of the nodes into sets according to the decreasing potential values p_i^*. By Lemma 5, there exists a $s-t$ path $P=u_0(=s),u_1,\dots,u_q=t$ such that $p_{u_0}^*\ge\cdots\ge p_{u_q}^*$ and $\sum_{(u_j,u_{j+1})\in P}d_{u_ju_{j+1}}^*=1$. Define the cost vector c^k such that $c_{uv}^k=1$ for each arc $(u,v)\in P$ and $c_{uv}^k=0$ otherwise. Any solution (d,p) in $P(G,c,b)$ satisfies

$$\begin{aligned}\sum_{(u,v)\in A}c_{uv}^k d_{uv} &= \sum_{(u_j,u_{j+1})\in P} d_{u_ju_{j+1}}\\ &\ge \sum_{(u_j,u_{j+1})\in P} p_{u_j}-p_{u_{j+1}} \text{ (by constraint (2b))}\\ &= p_{u_0}-p_{u_q}=1.\end{aligned}$$

This shows that (d^*,p^*) is an optimal basic solution of $P(G,c,b)$ with respect to function c^k. □

In what follows, we give some consequences of Theorem 1.

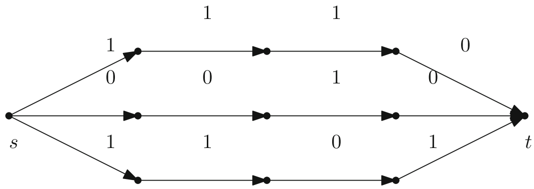


Fig. 7 A graph $G = (V, A)$ formed by three node disjoint $s - t$ paths. The c^1 costs of all the arcs are in $\{0, 1\}$

Corollary 1 *Suppose that* $c^k_{uv} > 0$ *for all arc* $(u, v) \in A$*. Then the linear relaxation of the BMCP has no fractional optimal basic solution if and only if there exists no* FRACTIONAL *set of nested* $s - t$ *cuts.*

Proof It is sufficient to show that any basic optimal solution (d, p) is minimal. Suppose that this is not the case and (d, p) is dominated by a solution (d', p') in $P(G, c, b)$, i.e., $d' \leq d$ and $d'_{u_0 v_0} < d_{u_0 v_0}$ for an arc $(u_0, v_0) \in A$. Since the cost function c^k is positive, it follows that $c^k d' < c^k d$. This contradicts the optimality of (d, p). □

Corollary 2 *Consider a graph* $G = (V, A)$ *that consists of a number of node disjoint* $s - t$ *paths together with* $k = 2$ *cost functions* c^1 *and* c^2 *such that* $c^1_{ij} \in \{0, 1\}$ *for all arcs* $(i, j) \in A$*, and a budget bound* $b \in \mathbb{N}$*. See Fig. 7. Then, there exists no minimal fractional basic solution.*

Proof By contradiction, suppose that there exists a minimal fractional basic solution (d, p). By Lemma 8, there exists a FRACTIONAL set $\mathcal{S} = \{C_1, C_2\}$ formed by $k = 2$ nested $s - t$ cuts such that one of them is infeasible, say C_2, and (d, p) is a convex combination of their incidence vectors (Condition 2 of Definition 3). Since (d, p) is fractional, $(d^{C_1}, p^{C_1}) \neq (d, p) \neq (d^{C_2}, p^{C_2})$, where (d^{C_1}, p^{C_1}) and (d^{C_2}, p^{C_2}) denote the incidence vectors of C_1 and C_2. The feasibility of (d, p) and $c^1(C_2) > b$ imply that $c^1(C_1) < b$.

Let $G' = (V', A')$ and (d', p') denote the graph and the solution obtained from G and (d, p) by all possible applications of Operation O. Since G is formed by node disjoint $s - t$ paths, no nodes u and v belonging to different $s - t$ paths will be merged during the contraction process. See Fig. 8. As (d, p) is minimal, by Lemma 2, it is relevant and as it is basic, by Lemma 6, (d', p') is a relevant basic solution of $P(G', c', b)$. Hence for any arc $(w_i, w_j) \in A'$, corresponding to an arc $(u, v) \in A$, we have $d_{uv} = d'_{w_i w_j} = \max\{p_u - p_v, 0\} = \max\{p'_{w_i} - p'_{w_j}, 0\}$. Since (d, p) is fractional, it follows that (d', p') is also fractional. Furthermore, by Lemma 7, G' has at most $k + 1 = 3$ supernodes, where one of them contains s and the other contains t. Let w_1 and w_2 denote these nodes. If $|V'| = 2$, then all the arcs in A' joining w_1 and w_2 satisfy $d'_{w_1 w_2} = p'_{w_1} - p'_{w_2} = 1$, a contradiction with the fact that (d', p') is fractional. Therefore, G' has exactly three supernodes. See Fig. 8. The third one, denoted by w_3, has an indegree and an outdgree one. We then get $b < c^1(C_2) \leq c^1(C_1) + 1 \leq b$, where the last two inequalities follow as $c^1_{ij} \in \{0, 1\}$ and $c^1(C_1) < b$, which is impossible. □

Figure 9 depicts an instance of the BMCP problem where the graph $G = (V, A)$ consists of three disjoint $s - t$ paths and the c^1 costs of all the arcs are in $\{0, 1\}$. If $k = 2$, then any FRACTIONAL set $\mathcal{S} = \{C, C'\}$ must contain an infeasible cut,

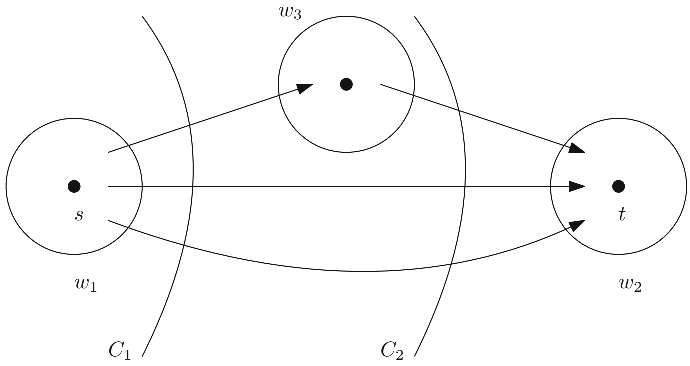


Fig. 8 The minimal fractional basic solution (d, p) is a convex combination of the incidence vectors of cuts C_1 and C_2 forming a FRACTIONAL set. Since G consists of a number of node disjoint $s - t$ paths, all the nodes in $C_2 \backslash C_1$ belong to one $s - t$ path of G

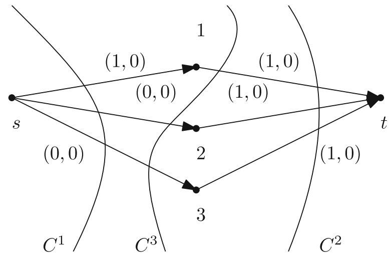


Fig. 9 Instance of the BMCP problem where the budget bound $b_1 = 2$

say C, and a feasible cut C' such that $c^1(C') < b_1$. Sets $\{C^2, C^1\}$ and $\{C^2, C^3\}$ are the only ones satisfying the later condition. Consider any feasible solution $(d, p) = \alpha_1(d^{C^1}, p^{C^1}) + \alpha_2(d^{C^2}, p^{C^2})$, where scalars $\alpha_1 > 0, \alpha_2 > 0$ satisfy $\alpha_1 + \alpha_2 = 1$. Observe that all the variables except p_s and p_t are fractional. Therefore, Operation O can not be applied to reduce graph G. By Lemma 7, (d, p) is not a basic solution (the graph G has five nodes and not $k + 1 = 3$ as expected). This implies that the set $\{C^2, C^1\}$ is not FRACTIONAL. By using the same arguments, one can show that the set $\{C^2, C^3\}$ is not FRACTIONAL. Consequently, there exists no FRACTIONAL set of nested cuts.

3 Complexity of detecting FRACTIONAL sets

In this section, we investigate the complexity of checking the existence of FRACTIONAL sets of nested $s - t$ cuts.

Theorem 2 *It is NP-hard to determine whether there exists a* FRACTIONAL *set of nested $s - t$ cuts even if $k = 2$.*

Proof The reduction is from the directed maximum $s - t$ cut problem. Given a directed unweighted graph $G = (V, A)$ and two specific nodes $s, t \in V$, the directed maximum $s - t$ cut problem aims to find a cut $C^>$ whose size $|\delta^+(C^>)|$ is maximum, where

$\delta^+(C^>)$ denotes the set of arcs leaving the node set $C^>$. It is known that this problem is NP-hard [6]. W.l.o.g., we may assume that the size of a minimum $s-t$ cut $C^<$ satisfies $|\delta^+(C^<)| < |\delta^+(C^>)|-2$. If $|\delta^+(C^<)| \geq |\delta^+(C^>)|-2$, then the maximum $s-t$ cut problem can be solved in strongly polynomial time using Vazirani and Yannakakis's [22] algorithm for enumerating the $s-t$ cuts by increasing size. Indeed, call this algorithm in order to compute in strongly polynomial time the $s-t$ cuts C' and C'' of second and third minimum size. We have

$$|\delta^+(C')| \geq |\delta^+(C^<)| + 1 \geq |\delta^+(C^>)| - 1,$$

and

$$|\delta^+(C'')| \geq |\delta^+(C')| + 1 \geq |\delta^+(C^>)|.$$

This implies that C'' is a maximum $s-t$ cut.

We construct an instance of the minimum $s-t$ cut problem with a single budget constraint on a complete and symmetric graph $G' = (V, A')$. The arc set A' is obtained from A by adding, if necessary, arcs (u, v) and (v, u) between any pair of nodes u and v in V. Define the cost function $c^1_{uv} = 1$ for each arc $(u, v) \in A$ and $c^1_{uv} = 0$ for each arc $(u, v) \in A' \backslash A$. Observe that $c^1(C) = |\delta^+(C)|$ for any cut C in G'. The following claim shows that there exists a Fractional set for at least one budget bound.

Claim 2 *If the budget bound b satisfies $b = c^1(C^>)-1$, then there exists a* Fractional *set of nested $s-t$ cuts.*

Proof of Claim 2: Suppose on the contrary that there exists no Fractional set of nested $s-t$ cuts for $b = c^1(C^>)-1$. By Theorem 1, it follows taht the linear relaxation of the BMCP has no fractional minimal basic optimal solution. Consider the cost function

$$c^2_{uv} = \begin{cases} 0 \text{ if } (u, v) \in \delta^+(C^>) \cup \delta^+(C^<), \\ 1 \text{ otherwise.} \end{cases}$$

As $|\delta^+(C^<)| < |\delta^+(C^>)| - 2$, $c^1(C^<) < b - 1$. Suppose that the cuts $C^>$ and $C^<$ are not nested. The case where they are nested can be handled along the same line. By the submodularity of the cut functions $c^h(C) = \sum_{(u,v)\in\delta^+(C)} c^h_{uv}$ for any subset $C \subset V$ and $h = 1, 2$, we have

$$c^2(C^> \cap C^<) + c^2(C^> \cup C^<) \leq c^2(C^>) + c^2(C^<),$$

and

$$c^1(C^> \cap C^<) + c^1(C^> \cup C^<) \leq c^1(C^>) + c^1(C^<) < 2b.$$

By the nonnegativity of c^2 and the fact that $c^2(C^>) = c^2(C^<) = 0$, we obtain that

$$c^2(C^> \cap C^<) = c^2(C^> \cup C^<) = 0. \tag{17}$$

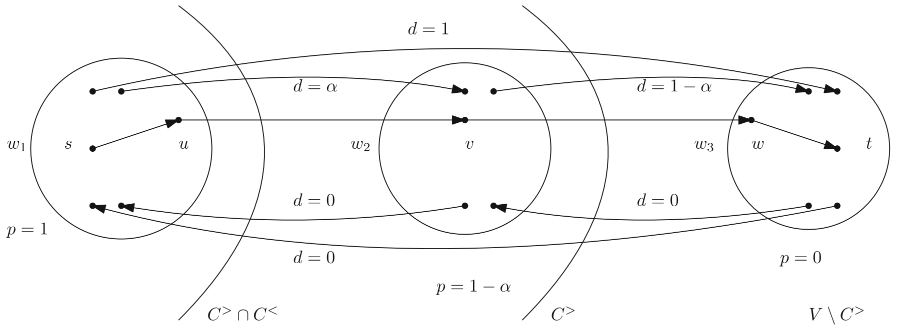


Fig. 10 Solution $(d, p) = \alpha(d^{C^> \cap C^<}, p^{C^> \cap C^<}) + (1-\alpha)(d^{C^>}, p^{C^>})$

Furthermore, not both cuts $C^> \cap C^<$ and $C^> \cup C^<$ have a c^1 cost at least equal to b. Suppose that $c^1(C^> \cap C^<) \leq b-1$. Consider the set $\mathcal{S} = \{C^> \cap C^<, C^>\}$ of nested $s-t$ cuts. There exists a scalar $0 < \alpha < 1$ such that $\alpha c^1(C^> \cap C^<) + (1-\alpha)c^1(C^>) = b$. Define the fractional solution (d, p) of $P(G', c, b)$ such that $(d, p) = \alpha(d^{C^> \cap C^<}, p^{C^> \cap C^<}) + (1-\alpha)(d^{C^>}, p^{C^>})$. See Fig. 10. We will show that $\mathcal{S}$ is a FRACTIONAL set and (d, p) corresponds to the minimal basic solution given in Definition 3. By (17), this implies that (d, p) is a minimal basic optimal solution of the linear relaxation and leads to a contradiction.

Let us first show that (d, p) is a minimal solution of $P(G', c, b)$. Suppose on the contrary that (d, p) is dominated by a solution (d', p') of $P(G', c, b)$, that is to say, $d' \leq d$ and $d'_{u_0v_0} < d_{u_0v_0}$ for some arc $(u_0, v_0) \in A'$. For every arc $(u, v) \in A'$ such that both ends u and v either belong to $C^> \cap C^<$, $C^> \setminus (C^> \cap C^<)$ or $V \setminus C^>$, we have $0 \leq d'_{uv} \leq d_{uv} = 0$, and hence $d'_{uv} = 0$. Since G' is complete and symmetric, any arc $(u, v) \in A'$ whose extremities lie in different sets $C^> \cap C^<$, $C^> \setminus (C^> \cap C^<)$, $V \setminus C^>$ belongs to a $s-t$ path. For instance, consider the $s-t$ path $P = s, u, v, w, t$ where $u \in C^> \cap C^<$, $v \in C^> \setminus (C^> \cap C^<)$, and $w \in V \setminus C^>$. By the previous argument, we have $d'_{su} = d'_{wt} = 0$. Since (d', p') is in $P(G', c, b)$ and dominates (d, p), we have

$$\begin{aligned}
d'_{uv} + d'_{vw} &= \sum_{(i,j)\in P} d'_{ij} \\
&\geq \sum_{(u_j, u_{j+1})\in P} p'_{u_j} - p'_{u_{j+1}} \text{ (by constraint (2b))} \\
&= p'_s - p'_t = 1.
\end{aligned}$$

and

$$\begin{aligned}
\sum_{(i,j)\in P} d'_{ij} \leq \sum_{(i,j)\in P} d_{ij} \\
&= d_{uv} + d_{vw} \\
&= \alpha d_{uv}^{C^> \cap C^<} + (1-\alpha) d_{vw}^{C^>} \\
&= \alpha + (1-\alpha) = 1.
\end{aligned}$$

This implies that $\sum_{(i,j)\in P} d'_{ij} = 1$. If $d'_{uv} < d_{uv}$ or $d'_{vw} < d_{vw}$, then $\sum_{(i,j)\in P} d'_{ij} < 1$ and this leads to a contradiction. Therefore, $d'_{uv} = d_{uv}$ and $d'_{vw} = d_{vw}$. By using the same argument, one can show that for any arc $(u, v) \in A'$ such that its extremities lie in different node sets $C^> \cap C^<$, $C^>\backslash(C^> \cap C^<)$, and $V\backslash C^>$, we have $d'_{uv} = d_{uv}$. However, this contradicts the fact that (d', p') dominates (d, p).

Next, we show that (d, p) is a basic solution of $P(G', c, b)$. By contradiction, suppose that there exists two solutions (d^1, p^1) and (d^2, p^2) in $P(G', c, b)$ such that $(d, p) = \frac{1}{2}(d^1, p^1) + \frac{1}{2}(d^2, p^2)$. Since G' is complete and symmetric, then for every node $u \in C^> \cap C^<$ there exist arcs (s, u) and (u, t) such that $d_{su} = \frac{1}{2}d^1_{su} + \frac{1}{2}d^2_{su} = 0$ and $d_{ut} = \frac{1}{2}d^1_{ut} + \frac{1}{2}d^2_{ut} = 1$. This implies that $d^1_{su} = d^2_{su} = 0$. By Constraints (2b), we have $d^1_{su} \geq 1 - p^1_u$, $d^2_{su} \geq 1 - p^2_u$, $d^1_{ut} \geq p^1_u$, and $d^2_{ut} \geq p^2_u$. Therefore, $p^1_u \geq 1$ and $p^2_u \geq 1$ for every node $u \in C^> \cap C^<$. If $p^1_u > 1$ or $p^2_u > 1$, then $d_{ut} = \frac{1}{2}d^1_{ut} + \frac{1}{2}d^2_{ut} > 1$, a contradiction. Consequently, $p_u = p^1_u = p^2_u = 1$ for every node $u \in C^> \cap C^<$. By using the same argument, one can show that $p_u = p^1_u = p^2_u = 1 - \alpha$ for every node $u \in C^>\backslash(C^> \cap C^<)$, and $p_u = p^1_u = p^2_u = 0$ for every node $u \in V\backslash C^>$. Therefore

$$p_u = p^1_u = p^2_u \text{ for every node } u \in V. \tag{18}$$

Let $G'' = (V'', A'')$ denote the minor digraph with three supernodes w_1, w_2, and w_3 obtained by merging all the nodes in $C^> \cap C^<$, $C^>\backslash(C^> \cap C^<)$, and $V\backslash C^>$. Let (d'', p''), (d'^1, p'^1), and (d'^2, p'^2) denote the restrictions of solutions (d, p), (d^1, p^1), and (d^2, p^2), respectively, to G''. By (18), we have $p''_{w_1} = p'^1_{w_1} = p''^2_{w_1} = 1$, $p''_{w_2} = p'^1_{w_2} = p''^2_{w_2} = 1 - \alpha$, and $p''_{w_3} = p'^1_{w_3} = p''^2_{w_3} = 0$. For any arc (u, v) with both ends in $C^> \cap C^<$, $C^>\backslash(C^> \cap C^<)$, or $V\backslash C^>$, we have $0 = d_{uv} = \frac{1}{2}d^1_{uv} + \frac{1}{2}d^2_{ut}$ and thus $d^1_{uv} = d^2_{ut} = 0$. Since $(d, p) = \frac{1}{2}(d^1, p^1) + \frac{1}{2}(d^2, p^2)$, this implies that $(d'', p'') = \frac{1}{2}(d'^1, p'^1) + \frac{1}{2}(d'^2, p'^2)$.
Any arc joining nodes w_1 and w_2 in G'' corresponds to an arc in $\delta^+(C^> \cap C^<)$ in G and thus $d''_{w_1w_2} = \alpha = p''_{w_1} - p''_{w_2}$. Similarly, any arc joining nodes w_2 and w_3 in G'' corresponds to an arc in $\delta^+(C^>)$ in G and thus $d''_{w_2w_3} = 1-\alpha = p''_{w_2} - p''_{w_3}$. Observe that any arc joining nodes w_1 and w_3 in G'' corresponds to an arc in $\delta^+(C^>)$ and $\delta^+(C^> \cap C^<)$ in G and thus $d''_{w_1w_3} = 1 = p''_{w_1} - p''_{w_3}$. Since the arcs joining w_2 and w_1 or joining w_3 and w_2 in G'' do not belong to $\delta^+(C^> \cap C^<) \cup \delta^+(C^>)$, it follows that $d''_{w_iw_j} = 0$ if $i > j$. See Fig. 10. Therefore, any arc $(w_i, w_j) \in A''$ satisfies either $d''_{w_iw_j} = p''_{w_i} - p''_{w_j}$ and $d''_{w_iw_j} > 0$ or $d''_{w_iw_j} = 0 > p''_{w_i} - p''_{w_j}$. This implies that (d'', p'') satisfies a system of $|A''| + |V''|$ linearly independent equations:

- $d''_{w_iw_j} = p''_{w_i} - p''_{w_j}$ for each arc in G'' joining w_i and w_j such that $i < j$,
- $d''_{w_iw_j} = 0$ for each arc in G'' joining w_i and w_j such that $i > j$,
- $p''_{w_1} = 1$, $p''_{w_3} = 0$, and
- $\sum_{(w_i,w_j)\in A''} c^1_{w_iw_j} d''_{w_iw_j} = b$,

where the last equation follows from the choice of α as $c^1(d'') = c^1(d) = \alpha c^1(C^> \cap C^<) + (1-\alpha)c^1(C^>) = b$. This implies that (d'', p'') is a basic feasible solution of the linear relaxation and this contradicts the fact that $(d'', p'') = \frac{1}{2}(d'^1, p'^1) + \frac{1}{2}(d'^2, p'^2)$.
□

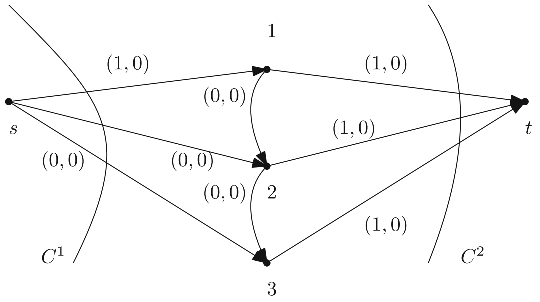


Fig. 11 The set formed by the cuts C^1 and C^2 is FRACTIONAL

Assume that there exists an algorithm $\mathcal{A}$ to check whether there exists a FRACTIONAL set of nested $s-t$ cuts. Call $\mathcal{A}$ for each bound value $b = |A| - 1, \ldots, 1$. Note that for $b \geq |\delta^+(C^>)|$, there exists no FRACTIONAL set. Otherwise, let $\mathcal{S} = \{C_1, C_2\}$ denote a FRACTIONAL set formed by two nested $s-t$ cuts such that one of them, say C_2, is infeasible. We have $|\delta^+(C_2)| = c^1(C_2) > b \geq |\delta^+(C^>)|$, a contradiction. Therefore, by Claim 2, the first time where $\mathcal{A}$ returns a FRACTIONAL set of nested $s-t$ cuts, say $\{C_3, C_4\}$, corresponds to the case $b = |\delta^+(C^>)| - 1$. The infeasible cut, say C_4, satisfies $|\delta^+(C_4)| = c^1(C_4) > b$ and thus $|\delta^+(C_4)| = |\delta^+(C^>)|$. This shows that C_4 is a maximum $s-t$ cut. This shows that algorithm $\mathcal{A}$ solves the directed maximum $s-t$ cut problem. ◻

The reduction given in the proof of Theorem 2 constructs an instance of the BMCP problem from the directed maximum $s-t$ cut problem. In this construction, a FRACTIONAL set of nested $s-t$ cuts is obtained from the minimum and maximum $s-t$ cuts for the c^1 cost function, denoted by $C^<$ and $C^>$ respectively. Note that these cuts may be intersecting. By using the submodularity of the cut function, we show that we can uncross them to obtain a FRACTIONAL set.

Figure 11 depicts an instance of the BMCP problem obtained by adding some arcs to the example given in Fig. 9. In this case, the condition of Corollary 2 does not hold any more. Observe that the minimum and maximum $s-t$ cuts $C^< = C^1$ and $C^> = C^2$ for the c^1 cost function are nested and form a FRACTIONAL set. In contrast with the example given in Fig. 9, $(d, p) = \frac{1}{2}(d^{C^1}, p^{C^1}) + \frac{1}{2}(d^{C^2}, p^{C^2})$ is a minimal basic feasible solution.

4 Conclusion

In this paper we have addressed the minimum $s-t$ cut problem with budget constraints. We have given a necessary and sufficient condition for the optimal minimal basic solutions of the linear relaxation to be integer. We have also shown that the problem of recognizing whether or not this is the case is NP-hard.

In order to develop a cutting plane based algorithm for the problem, it would be of interest to identify some classes of valid inequalities. For this, one can take profit of the structural properties of the minimal basic solutions given in this paper. This will be our future research line on the problem.

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