The multi-terminal vertex separator problem : Total Dual Integrality and Polytope Composition

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Abstract

Let $G = (V \cup T, E)$ be a graph where $V \cup T$ is the set of vertices, with T a subset of distinguished vertices, called terminals, and E the set of edges. Given a weight function $w : V \to \mathbb{N} \setminus \{0\}$ associated with the nonterminal nodes, the multi-terminal vertex separator problem consists in partitioning $V \cup T$ into $k+1$ subsets $\{S, V_1, \dots, V_k\}$ such that there is no edge between two different subsets V_i and V_j, each V_i contains exactly one terminal and the weight of S is minimum. In this paper, we characterize the polytope of the solutions of this problem for two classes of the graph, and we show that the two linear systems are totally dual integral. Then, we study the polytope for the graphs that are decomposable by 1-node cutsets. We show that if G decomposes into $G_1, \dots, G_k$, then the polytope in G can be obtained from those in $\bar{G}_1, \dots, \bar{G}_k$, where $\bar{G}_1, \dots, \bar{G}_k$ are graphs related to $G_1, \dots, G_k$, respectively. We also derive a procedure for composing facets and give some algorithmic consequences for solving the problem in G from $\bar{G}_1, \dots, \bar{G}_k$.

Keywords: Polytope, vertex separator problem, total dual integral, facet, node cutset

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1. Introduction

Graph partitioning has become a flourishing area, and many optimization problems have been considered in the literature, especially on an important family of graph partitioning problems based on vertex deletion. These latter have several real applications like network security and strategic military planning, immunity networks (find the exact number of individuals to vaccinate, to reduce the transfer of the virus into a network). It also has applications in biology (to achieve maximum fragmentation of protein-protein interaction), image processing, VLSI conception, and parallel computing. The *k-way vertex cut problem* is one of the most studied versions in the literature. It consists, given a graph $G = (V, E)$ and a positive integer k, in finding a subset $S \subseteq V$ of minimum size such that $(V \setminus S, E)$ has at least k disjoint components. This problem is W[1]-hard with parameter k [10] but polynomial for bounded tree-width graphs [4]. In this paper, we consider a variant of the k-way vertex cut problem, called the multi-terminal vertex separator problem, defined as follows.

The multi-terminal vertex separator problem

Let $G = (V \cup T, E)$ be a simple graph where $V \cup T$ is the set of vertices and E the set of edges such that T is a subset of distinguished vertices called *terminals*. A *multi-terminal vertex separator* in G is a subset of vertices $S \subseteq V$ whose removal induces k disjoint components, each containing exactly one terminal. Given a weight function $w : V \to \mathbb{N} \setminus \{0\}$, the *multi-terminal vertex separator problem* (MTVSP for short) consists in finding a multi-terminal vertex separator in G of minimum weight. The MTVSP can also be seen as the problem of finding a vertex subset $S \subseteq V$ of minimum weight such that each path between each pair of terminals intersects S. Indeed, if S intersects all paths between each pair of terminals, the graph induced by $(V \cup T) \setminus S$ would have k components, each containing exactly one terminal.

Literature review

In the case where $k = 2$, the MTVSP can be solved in polynomial-time [2]. When $k \geq 3$, the authors in [7] show that the MTVSP is equivalent to the vertex cover problem known to be NP-hard. They also study the problem from a polyhedral point of view. They present several valid inequalities for the associated polytope and study the conditions for which they define facets. These results gave rise to an efficient Branch-and-Cut algorithm for the problem. In [9], Naves and Jost give a linear system for the MTVSP and characterize the class of graphs for which this system is total dual integral for any independent set T *i.e.,* the dual problem has an optimal integer solution for any weight vector.

Many variants of this problem have been considered in the literature. In [5], Cornaz et al., consider the balanced version of the k-way vertex cut problem such that the graph $(V\backslash S, E)$ has at least k disjoint subsets $W_1, \ldots, W_k$ and $|W_i| - |W_j| \leq q$ for all $i, j \in \{1, \ldots, k\}$ for some integer $q \geq 1$. they present a compact formulation for the problem and discuss it from a polyhedral point of view. However, when q represents the bound on the size of the components of $(V \setminus S, E)$, the problem is called *the q-separator problem.* This problem is the general form of the vertex cover problem ($q = 1$) and the dissociation number of a graph [12] (for $q = 2$ and unit weight). Thus, the problem is NP-hard. Oosten et al., in [11], propose an extended formulation for the problem; they investigate the related polytope and present several valid inequalities and facets. In [3], Ben-Ameur et al. study the complexity of the problem and show that it is polynomial for several classes of graphs. In [1, 8], the authors discuss the vertex separator problem that consists, given a simple graph $G = (V, E)$ and a positive integer q, in partitioning V into three subsets A, B and C such that no vertex in A is adjacent to a vertex in B, $\max\{|A|, |B|\} \leq q$ and $|C|$ is minimum. They study the facial structure of the associated polytope and develop a Branch-and-Cut algorithm. In [2], another variant of the k-separator problem is considered. Given a simple graph $G = (V, E)$ and two terminals $a, b \in V$, the problem here is to partition V into three subsets A, B and C such that $a \in A$, $b \in B$, no edge connecting A and B and the size of the cut induced by C is minimum. It is shown that this problem can be reduced to a minimum cut problem in an auxiliary graph, and then it can be solved in polynomial time.

Our contributions

In this paper, we consider the MTVSP in two special classes of graphs *the star trees* and *the clique stars*. These graphs, which do not belong to the classes characterized by Naves and Jost, are introduced in [7]. We give a complete description of the associated polytope in these classes of graphs and show that the related linear systems are totally dual integral. Moreover, we discuss the polytope in the graphs that are decomposable by one-node and terminal cutsets. When G decomposes into G_1 and G_2, we derive a linear system of inequalities which characterizes the multi-terminal vertex separator polytope of G from two linear systems related to G_1 and G_2. This approach is also used to derive a procedure for composing facets and give some algorithmic consequences for solving the problem from the pieces.

Organization of the paper

The paper is organized as follows. In Section 2, we give a complete description of the multi-terminal vertex separator polytope for the two classes of graph, star trees, and clique stars. Then, we show that the two linear systems are totally dual integral. In Section 3, we study the composition (decomposition) technique for the multi-terminal vertex separator polytope for the graphs decomposable by one-node cutsets and derive procedures for composing facets and algorithms. In Sections 4 and 5, we present a further composition procedure in graphs decomposable by terminal cutsets. The rest of this section is devoted to more notations and preliminaries.

Notations and definitions

We consider simple graphs $G = (V \cup T, E)$ where $V \cup T$ is the set of vertices, T is the set of terminals in G and E is the set of edges. We let $n = |V|$, $k = |T|$ and $m = |E|$. In the rest of this paper, and for the sake of convenience, we will refer to the multi-terminal vertex separator as *separator*. If $W \subseteq (V \cup T)$, we denote by $G[W]$ the subgraph of G induced by W. Given a vertex $v \in V \cup T$, we denote by $N(v) \subseteq (V \cup T) \setminus \{v\}$ the set of vertices adjacent to v and by $d_G(v)$ the size of $N(v)$, called *degree* of v, in G. We denote by $N(W) \subseteq (V \cup T) \setminus W$ the set of vertices adjacent to at least one vertex in W. For $v \in V$, we let $\delta(v)$ denote the set of edges incident to v and $\delta(W)$ the set of edges having exactly one vertex in W. $\delta(w)$ is called the cut induced by W. If $x \in \mathbb{R}^{V \cup T}$, we let $x(W) = \sum_{v \in W \cap V} x(v)$. For any subgraph H of G, we denote by $V(H)$, $T(H)$ and $E(H)$ the sets of nodes, terminals and edges, respectively, of H. The number of terminals in H is denoted by $k(H)$, *i.e.,* $k(H) = |T(H)|$.

Given an inequality $ax \leq b$, where $a \in \mathbb{R}^{V \cup T}$, the *support graph* of $ax \leq b$ is the subgraph induced by the vertices of variables having a non-zero coefficient in the inequality. A *path* P is a set of p distinct vertices $v_1, v_2, \ldots, v_p$ such that for all $i \in \{1, \ldots, p-1\}$, $v_i v_{i+1}$ is an edge. Vertices $v_2, \ldots, v_{p-1}$

are called *the internal vertices* of P. Given a path P between two terminals $t, t' \in T$ such that $P \cap T = \{t, t'\}$, the set of internal vertices of P will be called *a terminal path* and denoted by $P_{tt'}$. A terminal path is *minimal* if it does not strictly contain a terminal path. In Figure 1, terminal path $P_{t_i t_j}$ is not minimal, it contains $P_{tt'}$. (In all figures in this paper, the terminals are represented by triangles.)

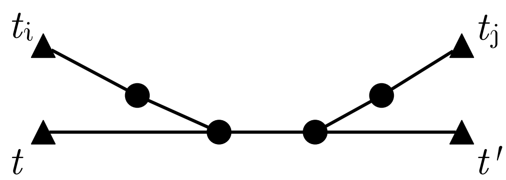


Figure 1: Minimal terminal path.

In what follows, we consider the following hypotheses.

1- There is no edge between two terminals. Otherwise, the problem has no solution.

2- For every two different terminals $t, t' \in T$, we have $N(t) \cap N(t') = \emptyset$. Otherwise, all vertices in $N(t) \cap N(t')$ must belong to the separator. In this case, we can remove these vertices from the graph.

3- For each vertex $v \in V$, there is at least one terminal path containing v. Otherwise, v cannot belong to a minimal separator, and in consequence, it can be deleted from the graph. Checking if a node v belongs to a terminal path can be done in polynomial time.

4- Graph G is connected. Otherwise, we consider the MTVSP in each component of the graph.

Given a graph $G = (V \cup T, E)$. Let $x \in \{0, 1\}^V$ be a vector of variables such that for a vertex $v \in V$, $x(v) = 1$ if v belongs to the separator and $x(v) = 0$ otherwise. Vector x^S is called the incidence vector of separator S.

The MTVSP is then equivalent to the following integer linear program [7]

$$\min \sum_{v \in V} w(v)x(v) \tag{1}$$

$$x(P_{tt'}) \geq 1 \qquad \forall P_{tt'} \in \Gamma, \tag{2}$$

$$x(v) \leq 1 \qquad \forall v \in V, \tag{3}$$

$$x(v) \geq 0 \qquad \forall v \in V, \tag{4}$$

$$x(v) \text{ integer} \qquad \forall v \in V. \tag{5}$$

where Γ is the set of all terminal paths in G. Let $P(G,T) = conv\{x \in \{0,1\}^V | \ x \text{ satisfies (2)}\}$ be the convex hull of the solutions of the MTVSP in G. $P(G,T)$ is called the *multi-terminal vertex separator polytope*. Under hypothesis (1) – (4), polytope $P(G,T)$ is full-dimensional [6].

A linear system $Ax \leq b$ is said to be *Totally Dual Integral* (TDI) if for all integer $c \in \mathbb{Z}^n$, the linear program $\min\{b^\top y : A^\top y = c; y \geq 0\}$ has an integer optimal solution, if a solution exists. If $Ax \leq b$ is TDI and b is integer, this implies that the polytope given by $Ax \leq b$ is integral.

2. $P(G,T)$ in star trees and clique stars

2.1. Star trees and clique stars

In [7], the authors introduce the following classes of graphs in the context of a polyhedral analysis of the MTVSP.

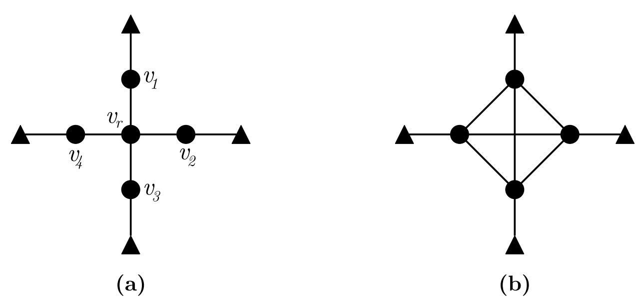
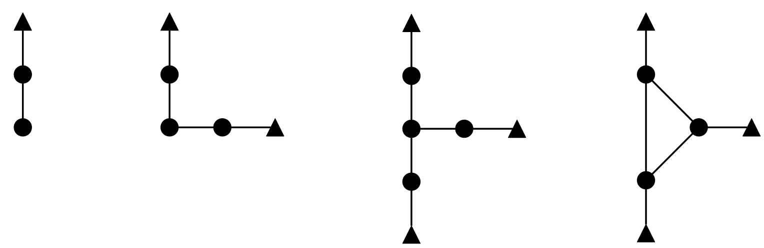


Figure 2: Star tree and clique star

A graph $H = (V(H) \cup T(H), E(H))$ is called a *star tree* if H consists of a rooted tree where the pending nodes are the terminal nodes of the tree, and all the other (nonterminal) nodes, different from the root node, are of degree two. A star tree H with q terminals $t_1, \dots, t_q$, can also be seen as the concatenation of q paths $P_{t_i},\ i = 1, \dots, q$ between the root v_r and $t_i,\ i = 1, \dots, q$. A star tree with 2 terminals is nothing but a terminal path.

A graph $Q = (V(Q) \cup T(Q), E(Q))$ is called a *clique star* if Q consists of a clique K_q on q vertices of $V(Q)$, q terminals $t_1, \dots, t_q$ (that is $T(Q) = \{t_1, \dots, t_q\}$) and q disjoints paths $P_{t_1}, \dots P_{t_q}$ such that each path P_{t_i} is between a different vertex of K_q and t_i. In what follows we will suppose that $q \geq 3$. If $q = 2$, then the clique star is nothing but a terminal path.
In the star trees and clique stars, the paths P_t are referred to as *branches*. Figure 2.(a) displays a star tree with 4 branches and Figure 2.(b) displays a clique star with 4 branches.



(a) (b) (c) (d)

Figure 3: Branches of the star trees and clique stars

Consider a graph $G = (V \cup T, E)$ that is a star tree (resp. clique star) of k terminals. If $k = 1$, then G is reduced to a single branch, see Figure 3(a). If $k = 2$, then G is reduced to a terminal path, see Figure 3(b). If $k \geq 3$, then G contains $\binom{k}{k'}$ star trees (resp. clique stars) as subgraphs with $k' \in \{1, \dots, k\}$ terminals. Let $\Pi(G)$ (resp. $\Theta(G)$) be the set of all star trees (resp. clique stars) subgraphs of G. Note that $\Pi(G)$ (resp. $\Theta(G)$) contains the star tree G (resp. the clique star G).

In this section, we will suppose that G (star tree or clique star) satisfies the following hypotheses.

1. The number of terminals is at least three.

2. Each branch of a star tree contains at least one internal vertex.

For otherwise, by [9] the linear system $(2)-(4)$ is TDI.

2.2. Polytopes

In what follows, we give complete descriptions of $P(G,T)$ for the star trees and clique stars.

2.2.1. $P(G,T)$ for star trees

First, we give the following remark.

Remark *For star trees, the polytope given by* $(2)-(4)$ *is not integral.*

Proof Consider a star tree $H=(V(H)\cup T(H),E(H))$ with at least one internal vertex in each branch. Let $\overline{x}\in[0,1]^{V(H)}$ be defined as follows

$$\overline{x}(v)=0.5 \qquad \forall v\in N(v_r),$$

$$\overline{x}(v)=0 \qquad \forall v\in V(H)\setminus N(v_r).$$

Vector $\overline{x}$ is an extreme point of $P(H,T(H))$. Among all tight inequalities for $\overline{x}$ we can select $|V(H)|$ linearly independent equalities that are tight for $\overline{x}$. As it can be easily seen, we can select

- $k(H)$ inequalities of type (2) (inequalities induced by the terminal path between a fixed $t\in T(H)$ and all $t'\in T(H)\setminus\{t\}$ plus any other terminal path), and
- $|V(H)\setminus k(H)|$ inequalities of type (4),

that are tight for $\overline{x}$. This constitutes a non-singular equality system for which $\overline{x}$ is the unique solution. ■

In [7], the authors introduce a class of valid inequalities for $P(G,T)$ induced by star trees.

Theorem 2.2.1 ([7]) *Given a graph $G=(V\cup T,E)$, inequalities*

$$x(V(H)\setminus\{v_r\})+(k(H)-1)x(v_r)\geq k(H)-1 \quad \forall H\in\Pi(G) \qquad (6)$$

are valid for $P(G,T)$.

Inequalities (6) are called *star tree inequalities*. Recall that the terminal paths are star trees with two terminals. Thus inequalities (2) are special cases of (6). Also remark that inequalities (6) associated with the star trees of $\Pi(G)$ with one terminal are dominated by trivial inequalities. Let $\tilde{P}(G,T)$ be the polytope given by inequalities (6) together with the trivial inequalities (3) and (4).

Now we state the main result of this section.

Theorem 2.2.2 *If $G = (V \cup T, E)$ is a star tree then $P(G,T) = \tilde{P}(G,T)$.*

Proof Let us assume the contrary, and let $x \in \mathbb{R}^{|V|}$ be a fractional extreme point of $\tilde{P}(G,T)$. Suppose $|V|$ is minimum, that is to say, $\tilde{P}(G,T)$ is integral for all star tree of n' vertices with $n' < |V|$. Thus there exists a subset $\Pi_1(G)$ of $\Pi(G)$ and two subsets V_1, V_2 of V such that x is in the unique solution of the following system $S(x)$,

$$x(V(H) \setminus \{v_r\}) + (k(H)-1)x(v_r) = k(H)-1 \quad \forall H \in \Pi_1(G), \tag{7}$$

$$x(v) = 1 \quad \forall v \in V_1, \tag{8}$$

$$x(v) = 0 \quad \forall v \in V_2 \tag{9}$$

such that $|\Pi_1(G)| + |V_1| + |V_2| = |V|$.

We have the following claims.

Claim 1 *For all $v \in V \setminus \{v_r\}$, $x(v) > 0$.*

Proof of Claim 1. Let us assume, on the contrary, that there exists a vertex $v \in V \setminus \{v_r\}$ such that $x(v) = 0$. Since G is a star tree, v has exactly two neighbors, $v_1, v_2 \in V \cup T$. Let $G' = (V' \cup T', E')$ be the star tree obtained from G by deleting v and adding an edge v_1v_2. Let $x' \in [0,1]^{|V'|}$ be the restriction of x on V' and $S'(x)$ be the system of equalities obtained from $S(x)$ by deleting equality $x(v) = 0$ and variable $x(v)$ from all other equalities involving $x(v)$. This new system is related to $\tilde{P}(G',T')$. Solution x' is an extreme point of $\tilde{P}(G',T')$. In fact, x' is the unique solution of the (non-singular) system. Since x' is fractional and $|V'| < |V|$, this contradicts the minimality of $|V|$. $\Diamond$

Claim 2 *For each branch P_t of G, $x(P_t \setminus \{t\}) \leq 1$.*

Proof of Claim 2. Suppose there exists a branch P_t of G such that $x(P_t \setminus \{t\}) > 1$. Let G' be the star tree obtained from G by deleting the vertices of $P_t \setminus \{v_r\}$. Let $ax \geq b$ be the star tree inequality induced by G. Then, $a'x \geq b-1$ is the star tree inequality induced by G' where a' is the restriction of a on $V(G')$ with $a'(v_r) = a(v_r) - 1$. As a consequence, we obtain that x violates the star tree inequality induced by G', a contradiction. ◊

Claim 3 $x(v_r) < 1$.

Proof of Claim 3. Suppose $x(v_r) = 1$. From Claim 2, it follows that $x(v) = 0$ for all $v \in V \setminus \{v_r\}$, a contradiction with the fact that x is fractional. ◊

Claim 4 *Each branch P_t of G contains at most one internal vertex.*

Proof of Claim 4. Let us assume, on the contrary, that there exists a branch P_t of G that contains two internal vertices v_1 and v_2. From Claims 1 and 2, it follows that $0 < x(v_1) < 1$ and $0 < x(v_2) < 1$. Let $x^1, x^2 \in \mathbb{R}^{|V|}$ be the solutions given by

$$x^1(v) = \begin{cases} x(v) & \text{if } v \in V \setminus \{v_1\}, \\ x(v) + \epsilon & \text{if } v = v_1. \end{cases}$$

and

$$x^2(v) = \begin{cases} x(v) & \text{if } v \in V \setminus \{v_2\}, \\ x(v) - \epsilon & \text{if } v = v_2. \end{cases}$$

where ϵ is a positive scalar sufficiently small.
As $x(v_1)$ and $x(v_2)$ only appear in equations of type (7) of system $S(x)$, and v_1, v_2 either both belong to a star tree or none of them does, it follows that x^1 and x^2 are both solutions of $S(x)$. As $x^1 \neq x^2$, this contradicts the extremality of x. ◊

Claim 5 *If there exists a branch P_t such that $x(P_t \setminus \{t\}) < 1$, then all the support graphs of equalities* (7) *contain branch P_t.*

Proof of Claim 5. Let $ax = b$ be an equality (7) of system $S(x)$ whose support graph does not contain $P_t \setminus \{v_r\}$. Thus, the star tree inequality $x(P_t \setminus \{t\}) + ax \geq b + 1$ is violated by x, contradicting the feasibility of x. ◊

Remark 2.1 *If there exists a branch P_t with $x(P_t \setminus \{t\}) < 1$, then all other branches contain at least one internal node. Otherwise, there must exist a terminal path inequality* (2) *between t and another terminal violated by x.*

Claim 6 $x(v_r) > 0$.

Proof of Claim 6. Suppose on the contrary, that $x(v_r) = 0$. Since x is fractional, there must exist a branch P_t and a vertex $v_1 \in P_t \setminus \{v_r\}$ such that $0 < x(v_1) < 1$. Moreover, there must exist an equation of type (7) involving $x(v_1)$. As the coefficient of $x(v_1)$ in this equation is 1 and the right-hand side is integer, there must also exist another vertex $v_2 \in V \setminus \{v_r, v_1\}$ such that $0 < x(v_2) < 1$. From Claim 5, $x(v_1)$ and $x(v_2)$ appear together in all equations (7). Now as we did in the proof of Claim 4, we can construct two different solutions x^1, x^2 of $S(x)$ such that $x = \frac{1}{2}(x^1 + x^2)$, and get a contradiction. $\diamondsuit$ From Claims 1, 2, 3 and 6 it follows that $0 < x(v) < 1$ for

all $v \in V$. Now we distinguish two cases

a. There exists a branch P_t such that $x(P_t \setminus \{t\}) < 1$.
From Claim 5, P_t belongs to all support graphs of equalities (7). Moreover, from Claim 4 and Remark 2.1, every branch different from P_t contains exactly one internal vertex. Let $\overline{x} \in \mathbb{R}^V$ be given by

- $\overline{x}(v) = x(v)$ for all $v \in P_t \setminus \{v_r\}$
- $\overline{x}(v_r) = x(v_r) + \epsilon$
- $\overline{x}(v) = x(v) - \epsilon$ for all $v \in V \setminus (P_t \cup \{v_r\})$

Clearly, any equality (7) remains tight for $\bar{x}$. It then follows that vector $\overline{x}$ satisfies all equalities of $S(x)$, contradicting the fact that x is the unique solution of $S(x)$.

b. For each branch P_t, $x(P_t \setminus \{t\}) = 1$.
By Claims 3 and 4, it follows that each branch contains exactly one internal node. Therefore, for all $v \in V \setminus \{v_r\}$, $x(v) = 1 - x(v_r)$. Since x is fractional, there exists at least one equality (7) in System $S(x)$,

$$x(V(H) \setminus \{v_r\}) + (k(H) - 1)x(v_r) = k(H) - 1.$$

By replacing $x(v)$ by $1 - x(v_r)$, for all $v \in V(H) \setminus \{v_r\}$, we obtain

$$|V(H) \setminus \{v_r\}| - |V(H) \setminus \{v_r\}|x(v_r) + (k(H) - 1)x(v_r) = k(H) - 1.$$

Since $|V(H) \setminus \{v_r\}| = k(H)$, we have

$$k(H) - k(H)x(v_r) + (k(H) - 1)x(v_r) = k(H) - 1.$$

This implies that $x(v_r) = 1$, contradicting Claim 3. ■

2.2.2. $P(G,T)$ in clique stars

First, we give the following remark

Remark 2.2 *In clique stars, the polytope given by (2) − (4) is not integral.*

Proof Consider a clique star $Q = (V(Q) \cup T(Q), E(Q))$ defined by q branches and a clique K_q. Let x be a solution given by

$$x(v) = 0.5 \qquad \forall v \in K_q$$

$$x(v) = 0 \qquad \forall v \in V(Q) \setminus K_q$$

It is not hard to see that x is the unique solution of a system of $k(Q)$ equations coming from terminal path inequalities and $|V(Q)| - k(Q)$ trivial equations. ■

Let $G = (V \cup T, E)$ be a graph. In [7], it is shown that the following inequalities are valid for $P(G,T)$.

$$x(V(Q)) \geq \quad k(Q) - 1 \qquad \forall Q \in \Theta(G) \tag{10}$$

Recall that, $\Theta(G)$ is the set of clique stars in G.

Note that the terminal paths are clique stars with two terminals. Thus inequalities (2) are special cases of (10). Also note that inequalities (10) associated with clique stars having one terminal are redundant with respect to trivial inequalities.

Let $P^*(G,T)$ be the polytope given by inequalities (10) and the trivial inequalities.

Theorem 2.2.3 *If $G = (V \cup T, E)$ is a clique star, then $P(G,T) = P^*(G,T)$*

Proof Let us assume the contrary and let x be a fractional extreme point of $P^*(G,T)$. Suppose that $|V|$ is minimum (i.e., for all clique star $G' = (V' \cup T', E')$ with $|V'| < |V|$, $P^*(G',T')$ is integral). Thus x is the unique solution of an equation system $S(x)$ of the form

$$\begin{aligned} x(V(Q)) &= k(Q) - 1 \quad && \forall Q \in \Theta_1 && (11)\\ x(v) &= 1 && \forall v \in V_1 && (12)\\ x(v) &= 0 && \forall v \in V_2 && (13)\\ & && && (14) \end{aligned}$$

such that $|\Theta_1| + |V_1| + |V_2| = |V|$, $\Theta_1 \subseteq \Theta(G)$, $V_1 \subseteq V$ and $V_2 \subseteq V$. Let K_q be the clique of G on $q = |T|$ nodes. We have the following Claims. The two first claims are given without proof. They are similar to proofs given in Theorem 2.2.2.

Claim 1 *For all $v \in V \setminus K_q$, $x(v) > 0$.*

Claim 2 *For every branch P_t, $x(P_t \setminus \{t\}) \leq 1$.*

Claim 3 *For all $v \in K_q$ not adjacent to a terminal, $x(v) > 0$.*

Proof of Claim 3. Suppose that there exists a vertex $v \in K_q$ of a branch P_t, adjacent to a vertex $u \in P_t \setminus \{v\}$ and $x(v) = 0$. Let Q be the clique star obtained from G by contracting edge uv. Let $\overline{x} \in [0,1]^{V(Q)}$ be the restriction of x on $V(Q)$ and $S'(x)$ the system of equations obtained from $S(x)$ by deleting equation $x(v) = 0$ and variable $x(v)$ from the other equations of the system. Clearly, $\overline{x}$ satisfies all equalities of $S'(x)$, which implies that $\bar{x}$ is an extreme point of $P(Q, T(Q))$. As $\overline{x}$ is fractional, this contradicts the minimality of $|V|$. $\Diamond$

Claim 4 *The support graphs of equations* (11)*, that are terminal paths, contain at most* 2 *nodes of K_q.*

Proof of Claim 4. Let us assume the contrary and suppose there is an equation $ax = b$ of (11) induced by a terminal path $P_{tt'}$ containing 3 internal nodes of K_q. Then, there must exist $t'' \in T \setminus \{t, t'\}$ such that $P_{t''} \cap P_{tt'} = \{v\}$ for some $v \in K_q$. From Claims 2 and 3, $x(P_{t''} \setminus \{t'', v\}) < 1$. Therefore $ax + x(P_{t''} \setminus \{v\}) < b+1$ is a clique star inequality violated by x, contradicting the feasibility of x. $\Diamond$

Claim 5 *Each vertex of K_q is adjacent to a terminal.*

Proof of Claim 5. Suppose there exists a branch P_t containing $v \in K_q$ and an internal vertex u. From Claims 1, 2 and 3, we have that $0 < x(u) < 1$ and $0 < x(v) < 1$, which implies that $x(u)$ and $x(v)$ belong to at least one equation (11). Moreover, since u and v belong to the same branch, from Claim 4 variables $x(u)$ and $x(v)$ appear together in equalities (11). Hence, there must exist $\epsilon > 0$ such that $x' \in [0,1]^V$, obtained from x by adding ϵ to $x(v)$ and subtracting ϵ from $x(u)$, is a feasible solution of $S(x)$. But this yields a contradiction with the fact that x is a unique solution of $S(x)$. $\diamondsuit$

As x is fractional, from Claim 5, there must exist $u \in K_q$ such that $0 < x(u) < 1$. Since all coefficients of the variables in $S(x)$ are 0 or 1, it follows that, there exists another vertex $v \in K_q$ such that $0 < x(v) < 1$. Now, we show that the variables of u and v appear together in the equations of $S(x)$. Suppose that there exists an equality (11), associated with a clique star $Q \in \Theta_1$, containing, say, u but not v. This yields

$$x(V(Q)) = k(Q) - 1.$$

Let Q' be the clique star obtained from Q by adding the branch P_t containing v. As by Claim 2, $x(P_t \setminus \{t\}) < 1$, the clique star inequality associated with Q' is violated by x, a contradiction. Let $x' \in \mathbb{R}^V$ given by

$$x'(w) = x(w) \text{ for all } w \in V \setminus \{u, v\},$$

$$x'(u) = x(u) + \epsilon,$$

$$x'(v) = x(v) - \epsilon.$$

Clearly, y is a solution for $S(x)$. As $x \neq y$, this yields a contradiction with the fact that x is a unique solution of $S(x)$, which ends the proof of the theorem. ∎

2.3. *Total Dual Integrality*

In what follow, we discuss TDI-ness descriptions for $P(G,T)$ in star trees and clique stars.

2.3.1. TDI systems for star trees

Consider a star tree $G = (V \cup T, E)$. Given two star trees H_i and H_j of G, we denote by $H^{\cap}_{i,j}$ the star tree of G whose branches are those in common in H_i and H_j, and by $H^{\cup}_{i,j}$ the star tree whose branches belong either to H_i or to H_j such that

$$k(H^{\cup}_{i,j} = k(H_i) + k(H_j) - k(H^{\cap}_{i,j}) \text{ if } k(H^{\cap}_{i,j}) \geq 1,$$

$$k(H^{\cup}_{i,j}) = k(H_i) + k(H_j) - 1 \text{ if } k(H^{\cap}_{i,j}) = 0.$$

Note that in the second case (i.e., $k(H^{\cap}_{i,j}) = 0$), $H^{\cup}_{i,j}$ may be not unique.

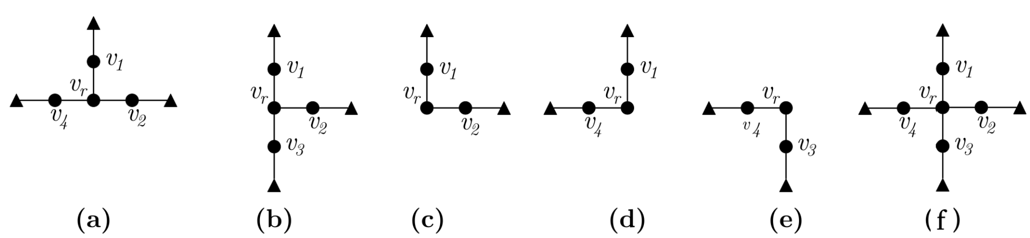


Figure 4: Star trees, subgraphs of the star tree in Figure 2.(a)

To illustrates the above notations, if H_i is the graph in Figure 4.(c) and H_j the graph in Figure 4.(e), then $H^{\cup}_{i,j}$ is the graph in Figure 4.(a) or the graph in Figure 4.(b) and $H^{\cap}_{i,j}$ does not exist. If H_i is the graph in Figure 4.(a) and H_j the graph in Figure 4.(b), then $H^{\cup}_{i,j}$ is the graph in Figure 4.(f) and $H^{\cap}_{i,j}$ is the graph in Figure 4.(c).

Let P^G be the linear program given by objective function (1), trivial inequalities (4) and inequalities (6), that is

$$\min \sum_{v \in V} w(v)x(v)$$
$$x(V(H) \setminus \{v_r\}) + (k(H) - 1)x(v_r) \geq k(H) - 1, \quad \forall H \in \Pi(G),$$
$$x(v) \geq 0 \qquad \forall v \in V.$$

Let $\Pi^v(G)$ be the set of star trees in $\Pi(G)$ containing vertex $v \in V$. Let $y \in \mathbb{R}_+^{\Pi(G)}$ be the dual variable vector associated with inequalities (6). The dual D^G of P^G can be given as follows

$$\max \quad \sum_{H\in\Pi(G)} (k(H)-1)y(H)$$

$$\sum_{H\in\Pi^v(G)} y(H) \le w(v), \quad \forall v \in V\setminus\{v_r\}, \tag{15}$$

$$\sum_{H\in\Pi(G)} (k(H)-1)y(H) \le w(v_r), \tag{16}$$

$$y(H) \ge 0, \qquad \forall H \in \Pi(G). \tag{17}$$

We notice that D^G consists in selecting $y(H) \in \mathbb{R}^+$ times star tree $H \in \Pi(G)$ respecting the vertex capacities w. This problem is called *the packing problem* of star trees.

For all $l \in \{1,\dots,k\}$, let Π_l be the set of star trees in $\Pi(G)$ with l branches.

We say that a dual solution y of D^G *dominates* a dual solution $\overline{y}$ if there exists $s \in \{0,\dots,k\}$ such that the following hold.

1. $\sum_{H\in\Pi_l} \overline{y}(H) = \sum_{H\in\Pi_l} y(H), \qquad$ for all $l \in \{s+1,\dots,k\}$,
2. $\sum_{H\in\Pi_s} \overline{y}(H) < \sum_{H\in\Pi_s} y(H)$.

A solution y is said to be *maximal optimal* if y is optimal and dominates any other optimal solution of D^G.

Claim 1 *There exists a maximal optimal solution of D^G.*

Proof of Claim 1. Let S_k^{max} be the set of optimal solutions of D^G with the highest value $\sum_{H\in\Pi_k} y(H)$ over all optimal solutions of D^G, *i.e.,*

$$S_k^{max} = \{y \;\mid \sum_{H\in\Pi_k} y(H) \ge \sum_{H\in\Pi_k} \bar{y}(H) \;\; \forall \;\; y \text{ and } \bar{y} \;\text{ solutions of } D^G\}$$

For $l \in \{1,\dots,k-1\}$, let S_l^{max} be the set of solutions in S_{l+1}^{max} with the highest value $\sum_{H\in\Pi_l} y(H)$ over all solutions of S_{l+1}^{max}, *i.e.,*

$$S_l^{max} = \{y \;\mid\; \sum_{H\in\Pi_l} y(H) \ge \sum_{H\in\Pi_l} \bar{y}(H) \;\; \forall \;\; y \text{ and } \bar{y} \in S_{l+1}^{max}\}$$

It is easy to see that S_1^{max} is not empty, *i.e.,* $|S_1^{max}| \geq 1$, and all its solutions are maximal optimal. $\diamondsuit$

Through the following claims, we present some properties of maximal optimal solutions of D^G. Given two star trees H_1 and H_2 of G, H_1 is said to be *completely included* in H_2 ($H_1 \subset H_2$), if $V(H_1) \subseteq V(H_2)$ and $T(H_1) \subseteq T(H_2)$.

Let y be a maximal optimal solution of D^G.

Claim 2 *Given a pair of star trees H_i and $H_j \in \Pi(G)$ such that $y_{H_i} > 0$ and $y_{H_j} > 0$, then either $H_i \subset H_j$ or $H_j \subset H_i$.*

Proof of Claim 2. Let us assume the contrary, suppose that $y_{H_j} > y_{H_i} > 0$ and none of the star trees is completely included in the other. Thus by definition, $k(H_{i,j}^{\cup}) \geq k(H_i) + k(H_j) - 1$. Let $\beta = k(H_{i,j}^{\cup})$ and $\alpha = \max\{k(H_{i,j}^{\cap}), 1\}$. We will show that there exists an optimal solution $\overline{y} \in \mathbb{R}_+^{\Pi(G)}$ that dominates y. Let $\overline{y}$ be defined as follows

$$\overline{y}_{H_{i,j}^{\cup}} = y_{H_{i,j}^{\cup}} + y_{H_i}$$

$$\overline{y}_{H_i} = 0$$

$$\overline{y}_{H_j} = y_{H_j} - y_{H_i}$$

$$\overline{y}_{H_{i,j}^{\cap}} = y_{H_{i,j}^{\cap}} + y_{H_i}$$

$$\overline{y}_H = y_H \qquad \forall H \in \Pi(G) \setminus \{H_i, H_j, H_{i,j}^{\cap}, H_{i,j}^{\cup}\}.$$

First, we show that $\overline{y}$ satisfies all inequalities (15)-(19). We distinguish four cases, $v \in V(H_{i,j}^{\cap}) \setminus \{v_r\}$, $v \in V(H_i) \setminus V(H_{i,j}^{\cap})$, $v \in V(H_j) \setminus V(H_{i,j}^{\cap})$ and $v = v_r$.

Each vertex $v \in V(H_{i,j}^{\cap}) \setminus \{v_r\}$ belongs to $V(H_i), V(H_j), V(H_{i,j}^{\cap})$ and $V(H_{i,j}^{\cup})$. Since the coefficient of each vertex of $V(H_{i,j}^{\cap}) \setminus \{v_r\}$ is 1 in inequalities (15), by adding y_{H_i} to the variable of $H_{i,j}^{\cup}$ and to the one of $H_{i,j}^{\cap}$, and by subtracting y_{H_i} from the variable of H_i and from the one of H_j, we obtain a solution still feasible for D^G. It then follows that inequalities (15) and (19), associated with the vertices of $V(H_{i,j}^{\cap}) \setminus \{v_r\}$, remain satisfied by $\overline{y}$.

Each vertex $v \neq v_r$ of $V(H_i) \setminus V(H_{i,j}^{\cap})$ (resp. $V(H_j) \setminus V(H_{i,j}^{\cap})$) only belongs to $V(H_i)$ (resp. $V(H_j)$) and possibly to $V(H_{i,j}^{\cup})$. Thus, since the coefficient of each vertex of $V(H_i) \setminus V(H^{\cap}_{i,j})$ (resp. $V(H_j) \setminus V(H^{\cap}_{i,j})$ is 1 in inequalities (15), by adding y_{H_i} to the variable of $H^{\cup}_{i,j}$ and by subtracting y_{H_i} from the variable of H_i (resp. H_j), inequalities (15) and (19) associated with the vertices of $V(H_i) \setminus V(H^{\cap}_{i,j})$ (resp. $V(H_j) \setminus V(H^{\cap}_{i,j})$) remain satisfied by $\overline{y}$.

It remains to show that inequality (16), associated with v_r, is satisfied by $\overline{y}$. Let us remark that the left-hand side of inequality (16) associated with v_r is exactly the function that we want to maximize. Thus, showing that y and $\overline{y}$ have equal objective values will prove both the optimality and the feasibility of $\overline{y}$.

As $\overline{y}_H = y_H$ for all $H \in \Pi(G) \setminus \{H_i, H_j, H^{\cap}_{i,j}, H^{\cup}_{i,j}\}$, to prove the optimality of $\overline{y}$ we just need to prove that

$$(k(H_i)-1)\overline{y}_{H_i} + (k(H_j)-1)\overline{y}_{H_j} + (\beta-1)\overline{y}_{H^{\cup}_{i,j}} + (\alpha-1)\overline{y}_{H^{\cap}_{i,j}} =$$
$$(k(H_i)-1)y_{H_i} + (k(H_j)-1)y_{H_j} + (\beta-1)y_{H^{\cup}_{i,j}} + (\alpha-1)y_{H^{\cap}_{i,j}}$$

Since $\overline{y}_{H_i} = 0$, $\overline{y}_{H_j} = y_{H_j} - y_{H_i}$, $\beta = k(H_i) + k(H_j) - \alpha$, it follows that

$$0+(k(H_j)-1)(y_{H_j}-y_{H_i})+(k(H_i)+k(H_j)-\alpha-1)(y_{H^{\cup}_{i,j}}+y_{H_i})+(\alpha-1)(y_{H^{\cap}_{i,j}}+y_{H_i})$$

$$= (k(H_j)-1)y_{H_j} + (k(H_i)-1)y_{H_i} + (\beta-1)y_{H^{\cup}_{i,j}} + (\alpha-1)y_{H^{\cap}_{i,j}}$$

Thus $\overline{y}$ is a feasible optimal solution. Since $y_{H_i} > 0$, it follows that $\overline{y}_{H^{\cup}_{i,j}} > y_{H_i}$ which implies that $\overline{y}$ dominates y, contradicting its maximality. $\Diamond$

Corollary 2.3.1 *For $l \in \{1, \ldots, k\}$, there is at most one star tree H among those of Π_l, with $y_H > 0$.*

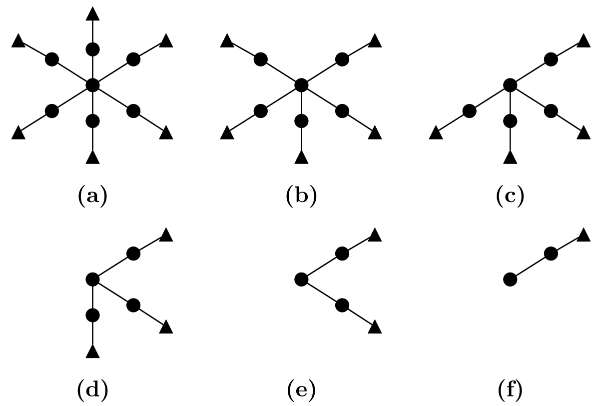


Figure 5: An example of a maximal optimal solution structure.

Figure 5 illustrates the structure of the maximal optimal solution y (each star tree is included in another).

Proposition 2.3.1 *If y is a maximal optimal solution, for all $H \in \Pi(G)$, one of the following holds:*

there exists $v \in V(H) \setminus \{v_r\}$ such that $\sum_{l=k(H)}^{k} \sum_{H' \in \Pi_l} y_{H'} = w_v$,

$$\sum_{l=k(H)}^{k} \sum_{H' \in \Pi_l} (k(H') - 1) y_{H'} = w_{v_r}.$$

Proof Let us suppose the contrary, that is to say there exists $H \in \Pi(G)$, such that

for all $v \in V(H) \setminus \{v_r\}$, $\sum_{l=k(H)}^{k} \sum_{H' \in \Pi_l} y_{H'} < w_v$, and

$$\sum_{l=k(H)}^{k} \sum_{H' \in \Pi_l} (k(H') - 1) y_{H'} < w_{v_r}.$$

Let y' be the vector obtained from y by adding $\epsilon > 0$ to y_H and setting $y'_{H'} = 0$ for all H' with $k(H') < k(H)$ such that

$$\epsilon = \min\{\frac{w_{v_r} - \sum_{l=k(H)}^{k} \sum_{H' \in \Pi_l} (k(H') - 1) y_{H'}}{k(H) - 1}, \min_{v \in V(H) \setminus \{v_r\}} \{w(v) - \sum_{l=k(H)}^{k} \sum_{H'' \in \Pi_l} y_{H''}\}\}.$$

Clearly, y' is a feasible solution of D^G that dominates y. This contradicts the fact that y is maximal optimal, and the result follows. ■

Theorem 2.3.1 *For a star tree $G = (V \cup T, E)$, the linear system given by* (4) *and* (6) *is TDI.*

Proof It is enough to show that there exists an integer optimal solution of D^G for any $w \in \mathbb{Z}_+^V$. Let y be a maximal optimal solution of D^G.

Claim 1 *If y is fractional, then there exists exactly one star tree $H \in \Pi(G)$ such that y_H is fractional.*

Proof of Claim 1. Suppose that there exist two different star trees $H_i, H_j \in \Pi(G)$, such that y_{H_i} and y_{H_j} are fractional. Suppose that $k(H_j)$ is maximum, that is to say, for all $H \in \Pi(G)$ with more than $k(H_j)$ terminals, y_{H_j} is integer. From Corollary 2.3.1, it follows that $k(H_j) > k(H_i)$. We distinguish two cases:

1. There exists a vertex $v \in V(H_j) \setminus \{v_r\}$ such that $\sum_{l=k(H_j)}^{k} \sum_{H \in \Pi_l} y_H = w(v)$. As $k(H_j)$ is maximum, y_H is integer for any star tree H with $k(H) \in \{k(H_j)+1, \ldots, k\}$. Moreover, by Corollary 2.3.1, it follows that H_j is the only tree in H with $k(H_j)$ positive value. Since $w(v)$ is integer, it follows that y_{H_j} is integer, a contradiction with y_{H_j} fractional.

2. For all vertex $v \in V(H_j) \setminus \{v_r\}$, we have $\sum_{l=k(H_j)}^{k} \sum_{H \in \Pi_l} y_H < w(v)$. From Proposition 2.3.1, it follows that $\sum_{l=k(H_j)}^{k} \sum_{H' \in \Pi_l} (k(H') - 1) y_{H'} = w(v_r)$. By Constraints (16) it follows that $y_H = 0$ for any star tree $H \in \Pi(G)$ with $k(H) \leq k(H_j) - 1$ terminals, a contradiction with y_{H_i} is fractional.

Thus if y is fractional, there exists exactly one star tree $H_j \in \Pi(G)$ such that y_{H_j} is fractional. ♢

Claim 2 *If y is fractional, then there exists another optimal solution which is integer.*

Proof of Claim 2. From Claim 1, since y is fractional, there exists exactly one star tree $H \in \Pi(G)$ such that y_H is fractional. We distinguish two cases

- If $k(H) = 1$, then let $\overline{y} \in \mathbb{R}_+^{\Pi(G)}$ be the solution obtained from y by setting $y_H = 0$. Solution $\overline{y}$ is an integer feasible optimal solution.

- If $k(H) \geq 2$, then as y is maximal optimal and $w(v_r)$ is integer, by Proposition 2.3.1 we have that $(k(H)-1)y_H$ is integer. Let $\epsilon = y_H - \lfloor y_H \rfloor$. Thus $(k(H)-1)\epsilon$ is integer. For an arbitrary chosen star tree $H' \in \Pi_{\epsilon.(k(H)-1)+1}$, let $y' \in \mathbb{R}_+^{\Pi(G)}$ be the solution defined as follows

  $y'_H = y_H - \epsilon,$

  $y'_{H'} = y_{H'} + 1,$

  $y'_{H''} = 0$ for all $H'' \in \Pi(G) \setminus \{H, H'\}$.

  As It is easy to check that y' is feasible, integer and has the same objective value as y.

Therefore, y' is an integer optimal solution for D^G. ◊

By Claim 2, the proof is complete. ■

As a consequence, we obtain the following *min-max* relation:

Corollary 2.3.2 *In star trees, the minimum number of vertices covering all terminal paths is equal to the maximum packing of star trees.*

Then we deduce the following algorithm for D^G.

Algorithm 1: An exact algorithm for solving D^G

Data: A star tree $G = (V \cup T, E)$, a vertex capacities vector w
Result: Maximal optimal solution y

1 **begin**
2   **for** $(t \in T)$ **do**
3         $w'(P_t) = \min_{u \in P_t \setminus \{v_r\}} \{w(u)\}$;
4   **if** $(w'(P_t)=0)$ **then**
5   Delete branch P_t from G;
6   **while** $(k(G) > 1$ *and* $w(v_r) > 0)$ **do**
7         $y_G = \min\{\lfloor \frac{w(v_r)}{k(G)-1} \rfloor, \min_{t \in T}\{w'(P_t)\}\}$;
8   **for** $(t \in T)$ **do**
9             $w'(P_t) = w'(P_t) - y_G$;
10         $w(v_r) = w(v_r) - y_G(k(G) - 1)$;
11   **for** $(t \in T)$ **do**
12   **if** $(w'(P_t)=0)$ **then**
13   Delete branch P_t from G;

Corollary 2.3.3 *For the star trees, the multi-terminal vertex separator problem can be solved in a $O(|V| + \frac{|T|\times(|T|-1)}{2})$-time.*

Proof In loop 2-5 we iterate over each branch and we iterate over each node of the branch to get the minimum weight w'. This is done in $O(|V|)$. The second loop 6-13 has $|T|$ iterations. At each iteration, at least one branch is removed. Due to loops 8-9 and 11-13, the whole loop is done in $O(\frac{|T|\times(|T|-1)}{2})$. ■

2.3.2. TDI system for clique stars

Consider a clique star $G = (V \cup T, E)$.

Let $\tilde{P}^G$ be the linear program defined by variable vector x, objective function (1) and inequalities (4) and (10). Let $y \in \mathbb{R}_+^{\Theta(G)}$ be the dual variable vector associated with inequalities (10). The dual $\tilde{D}^G$ of $\tilde{P}^G$ is given by

$$\max \sum_{Q\in\Theta(G)} (k(Q)-1)y(Q)$$

$$\sum_{Q\in\Theta^v(G)} y(Q) \le w(v), \qquad \forall v \in V, \tag{18}$$

$$y(Q) \ge 0, \qquad \forall Q \in \Theta(G). \tag{19}$$

Let us remark that $\tilde{D}^G$ consists in selecting $y(Q) \in \mathbb{R}^+$ times clique stars $Q \in \Theta(G)$ respecting the vertex capacities w. This problem is called *the packing problem* of clique stars. Let $\Theta^v(G)$ be the set of clique stars in $\Theta(G)$ containing vertex $v \in V$. For all $l \in \{1,\dots,k\}$, let Θ_l be the set of clique stars in $\Theta(G)$ with l branches.

We say that a dual solution $y \in \mathbb{R}_+^{\Theta(G)}$ dominates a dual solution $\overline{y} \in \mathbb{R}_+^{\Theta(G)}$ if there exists $s \in \{1,\dots,k\}$ satisfying the following:

1. $\sum_{Q\in\Theta_l} \overline{y}_Q = \sum_{Q\in\Theta_l} y_Q, \qquad$ for all $l \in \{s+1,\dots,k\}$
2. $\sum_{Q\in\Theta_s} \overline{y}_Q < \sum_{Q\in\Theta_s} y_Q,$

A solution y is said to be *maximal optimal* if y is optimal and dominates any other optimal solution of $\tilde{D}^G$. We have the following claims.

Claim 1 *There exists a maximal optimal solution of $\tilde{D}^G$.*

Proof of Claim 1. The proof is similar to Claim 1 of Subsection 2.3.1. ♢
Let $y \in \mathbb{R}_+^{\Theta(G)}$ be a maximal optimal solution.

Given two clique stars Q_1 and Q_2 of G, Q_1 is said to be *completely included* in Q_2, if $V(Q_1) \subseteq V(Q_2)$ and $T(Q_1) \subseteq T(Q_2)$.

Claim 2 *For every two different clique stars $Q_i, Q_j \in \Theta(G)$, such that $y_{Q_i} > 0$ and $y_{Q_j} > 0$, either Q_i is completely included in Q_j or Q_j is completely included in Q_i.*

Proof of Claim 2. The proof is similar to Claim 2 of Subsection 2.3.1. ♢

Corollary 2.3.4 *For $s \in \{1, \ldots, k\}$, there exists at most one clique star Q among the clique stars of Θ_s with $y_Q > 0$.*

Claim 3 *For all $Q_j \in \Theta(G)$, there exists a vertex $v \in V(Q_j)$ such that $\sum_{l=k(Q_j)}^{k} \sum_{Q \in \Theta_l} y_Q = w(v)$.*

Proof of Claim 3. The proof is similar to Proposition 2.3.1. $\Diamond$

Theorem 2.3.2 *For a clique star $G = (V \cup T, E)$, the linear system given by inequalities (4) and (10) is TDI.*

Proof Using Corollary 2.3.4 and Claim 3, the maximal optimal solution of $\tilde{D}^G$ can be obtained by packing the clique star with the highest number of terminals, until the capacity of one vertex is saturated. Then we delete branches with at least one saturated vertex, and we repeat the operations until the graph becomes a branch. Then the proof is complete, and the linear system of $\tilde{P}^G$ is TDI. ■

As a consequence, we obtain the following *min-max* relation:

Corollary 2.3.5 *In clique stars, the minimum number of vertices covering all terminal paths is equal to the maximum packing of clique stars.*

Therefore, we deduce an algorithm to solve $\tilde{D}^G$.

Algorithm 2: An exact algorithm for solving $\tilde{D}^G$

Data: Graph $G = (V \cup T, E)$ that is a clique star, a weight vector w
Result: A maximal optimal solution y

begin
  for $(t \in T)$ **do**
    $w'(P_t) = \min_{u \in P_t}\{w(u)\}$;
    if $(w'(P_t){=}0)$ **then**
      Delete branch P_t from G;
  while $(k(G) > 1)$ **do**
    $y_G = \min_{\forall t \in T}\{w'(P_t)\}$;
    for $(t \in T)$ **do**
      $w'(P_t) = w'(P_t) - y_G$;
    for $(t \in T)$ **do**
      if $(w'(P_t){=}0)$ **then**
        Delete branch P_t from G;

We also have the following consequence, its proof is similar to that of Corollary 2.3.3.

Corollary 2.3.6 *For the clique stars, the multi-terminal vertex separator problem can be solved in a* $O(|V| + \frac{|T|\times(|T|-1)}{2})$*-time.*

3. Composition of polyhedra by 1-sums

In what follows we study a composition (decomposition) technique for the multi-terminal vertex separator polytope in graphs that are decomposable by one-node cutsets (1-sums). If G decomposes into G_1 and G_2, we show that the multi-terminal vertex polytope of G can be described from two linear systems related to G_1 and G_2. As a consequence, we obtain a procedure to construct this polytope in graphs that are recursively decomposed.

Given a graph $G = (V \cup T, E)$ and two subgraphs of G, say $G_1 = (V_1 \cup T_1, E_1)$ and $G_2 = (V_2 \cup T_2, E_2)$. Graph G is called a *k-sum* of G_1 and G_2 if $V = V_1 \cup V_2$, $T = T_1 \cup T_2$, $|T_1 \cap T_2| = 0$, $|V_1 \cap V_2| = \alpha$ and subgraph $(V_1 \cap V_2, E_1 \cap E_2)$ is complete. Set $V_1 \cap V_2$ is a *k-node* cutset of G. k-node is any set of k nodes that by removal the graph is disconnected.

In what follows we investigate a polytope composition procedure for graphs that are 1-sums.

Let $G = (V \cup T, E)$ be the 1-sum of $G_1 = (V_1 \cup T_1, E_1)$ and $G_2 = (V_2 \cup T_2, E_2)$. Let $V_1 \cap V_2 = \{u\}$ such that u is not adjacent to a terminal. Let $\widetilde{G}_i = (\widetilde{V}_i \cup \widetilde{T}_i, \widetilde{E}_i)$ be the graph obtained from G_i, for $i = 1, 2$, by adding a node w_i, a terminal q_i and edges $q_i w_i, w_i u$. Figure 6 illustrates graphs G, $\widetilde{G}_1$ and $\widetilde{G}_2$.

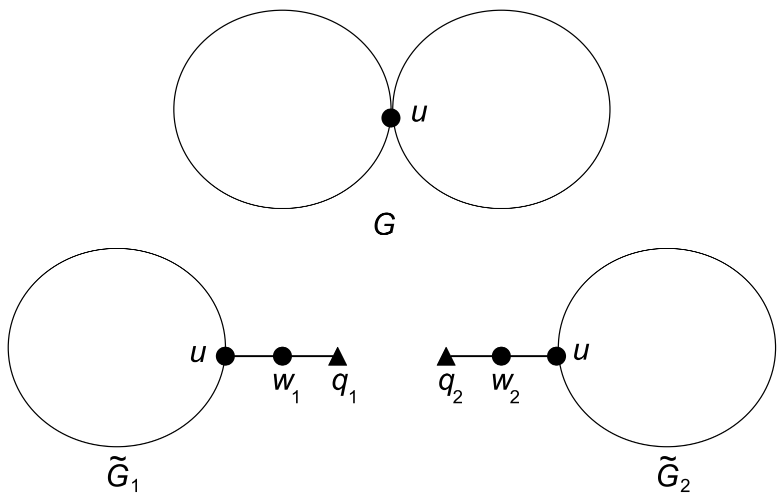


Figure 6: Composition (decomposition) of graphs

In $\widetilde{G}_i$, the multi-terminal vertex separator polytope $P(\widetilde{G}_i, \widetilde{T}_i)$ is completely described by a minimal linear inequality system of the form

$$\sum_{v \in \widetilde{V}_i} a^i_j(v)x(v) \geq \alpha^i_j \quad \forall j \in L^i \tag{20}$$

$$x(v) \leq 1 \quad \forall v \in \widetilde{V}^i$$

$$x(v) \geq 0 \quad \forall v \in \widetilde{V}^i$$

where L^i is the set of indices.

3.1. Structural properties

In what follows we shall study some structural properties of the non trivial facet defining inequalities (20) of $P(\widetilde{G}_i, \widetilde{T}_i)$. These will be used later for the composition of polyhedra.

Lemma 3.1.1 *$a_l^i(v) \geq 0$ for all $v \in \widetilde{V}_i$.*

Proof Let us assume the contrary, and suppose there exists a vertex $v_0 \in \widetilde{V}_i$ such that $a_l^i(v_0) < 0$. As (20) is different from a bound inequality, there should exists a vertex separator S that does not contain v_0 and $\sum_{v \in \widetilde{V}_i \setminus \{v_0\}} a_l^i(v) x^S(v) = \alpha_l^i$. Let $S^{v_0} = S \cup \{v_0\}$. Set S^{v_0} is also a separator for $\widetilde{G}_i$. Therefore $\sum_{v \in \widetilde{V}_i \setminus \{v_0\}} a_l^i(v) x^S(v) + a_l^i(v_0) < \alpha_l^i$, a contradiction. ■

Lemma 3.1.2 *$a_l^i(u) \geq a_l^i(w_i)$.*

Proof Since (20) defines a facet of $P(\widetilde{G}_i, \widetilde{T}_i)$ different from a nonnegativity inequality, there must exist a separator S such that x^S satisfies (20) with equality and $w_i \in S$. If $u \in S$, then $S \setminus \{w_i\}$ is a separator for $\widetilde{G}_i$, yielding $a_l^i(w_i) = 0 \leq a_l^i(u)$. If $u \notin S$, then let $\overline{S} = (S \setminus \{w_i\}) \cup \{u\}$. Since $\overline{S}$ is a separator for $\widetilde{G}_i$, this implies that $a_l^i(u) \geq a_l^i(w_i)$. ■

From Lemmas 3.1.1, 3.1.2, the linear inequality system describing $P(\widetilde{G}_i, \widetilde{T}_i)$ can be given as follows

$$\sum_{v \in V_i \setminus \{u\}} a_j^i(v) x(v) \geq \alpha_j^i \quad \forall j \in L_1^i, \tag{21}$$

$$\sum_{v \in V_i \setminus \{u\}} a_j'^i(v) x(v) + x(u) \geq \alpha_j'^i \quad \forall j \in L_2^i, \tag{22}$$

$$\sum_{v \in V_i} b_j^i(v) x(v) + x(w_i) \geq \beta_j^i \quad \forall j \in L_3^i, \tag{23}$$

$$x(v) \leq 1 \qquad \forall v \in \widetilde{V}^i, \tag{24}$$

$$x(v) \geq 0 \qquad \forall v \in \widetilde{V}^i. \tag{25}$$

Where L_1^i (resp. L_2^i, resp. L_3^i) is the set of inequalities whose support does not intersect $\{w_i, u\}$, (resp. does not intersect w_i but contains u) (resp. contains u and w_i).

3.2. Composition of polyhedra

In what follows we derive a system of inequalities that describes $P(G,T)$. To this end, we first give the following lemmas.

Lemma 3.2.1 *For $j \in L_3^i$, inequality*

$$\sum_{v \in V_i} b_j^i(v)x(v) - x(u) \geq \beta_j^i - 1$$

is valid for $P(G,T)$.

Proof Consider a separator S for graph G. We distinguish two cases

$u \in S$: let $S_i = (S \cap V_i)$. It is clear that S_i is a separator of $\widetilde{G}_i$. We then have $\sum_{v \in V_i} b_j^i(v)x^{S_i}(v) \geq \beta_j^i$. Since $-x^{S_i}(u) = -1$, it follows that $\sum_{v \in V_i} b_j^i(v)x^{S_i}(v) - x^{S_i}(u) \geq \beta_j^i - 1$.

$u \notin S$: we claim either $(S \cap V_1)$ is a separator for $\widetilde{G}_1$ or $(S \cap V_2)$ is a separator for $\widetilde{G}_2$. Suppose that this is not the case. Then there should exist a terminal q'_1 of G_1 and a terminal q'_2 of G_2 and two terminal paths $P_{q_1 q'_1}$ and $P_{q_2 q'_2}$ of $\tilde{G}_1$ and $\tilde{G}_2$, respectively, such that $S \cap P_{q_1 q'_1} = \emptyset = S \cap P_{q_2 q'_2}$. Hence $P_{q'_1 q'_2} = (P_{q_1 q_2} \setminus \{w_1, q_1\}) \cup (P_{q_2 q'_2} \setminus \{w_2, q_2\})$ is a terminal path of G such that $P_{q'_1 q'_2} \cap S = \emptyset$, which is impossible. Suppose for instance, that $(S \cap V_2)$ is a separator for $\widetilde{G}_2$ and let $S_1 = (S \cap V_1) \cup \{w_1\}$ and $S_2 = (S \cap V_2)$. Now, sets S_1 and S_2 are both separators for $\widetilde{G}_1$ and $\widetilde{G}_2$ respectively. Thus

for $\widetilde{G}_1$, we have $\sum_{v \in V_1} b_j^1(v)x^{S_1}(v) + x^{S_1}(w_1) \geq \beta_j^1$ and $x^{S_1}(w_1) = 1$. Then $\sum_{v \in V_1} b_j^1(v)x^{S_1}(v) \geq \beta_j^1 - 1$, and as $x^{S_1}(u) = 0$, we obtain that

$$\sum_{v \in V_1} b_j^1(v)x^{S_1}(v) - x^{S_1}(u) \geq \beta_j^1 - 1.$$

for $\widetilde{G}_2$, we have $\sum_{v \in V_2} b_j^2(v)x^{S_2}(v) \geq \beta_j^2$ and since $-x^{S_2}(u) \geq -1$, this yields

$$\sum_{v \in V_2} b_j^2(v)x^{S_2}(v) - x^{S_2}(u) \geq \beta_j^2 - 1.$$

■

Lemma 3.2.2 *For $j \in L_3^i$, inequality*

$$\sum_{v \in V_i} b_j^i(v)x(v) - x(u) \geq \beta_j^i - 1$$

is valid for $P(\widetilde{G}_i, \widetilde{T}_i)$ for $i = 1, 2$.

Proof Suppose that $i = 1$. The proof for $i = 2$ is similar. Let S^1 be a separator of $\widetilde{G}_1$. We distinguish two cases

Case 1. $u \in S^1$ or ($u \notin S^1$ and $w_1 \notin S^1$)
Set $S' = S^1 \cup (V_2 \setminus \{u\})$ is clearly a separator for G. From Lemma (3.2.1), it follows that

$$\sum_{v \in V_1} b_j^1(v)x^{S'}(v) - x^{S'}(u) \geq \beta_j^1 - 1.$$

Since $S^1 = S' \cap (V_1 \cup \{w_1\})$, we get

$$\sum_{v \in V_1} b_j^1(v)x^{S^1}(v) - x^{S^1}(u) \geq \beta_j^1 - 1.$$

Case 2. $u \notin S^1$ and $w_1 \in S^1$
This implies that, $x^{S^1}(u) = 0$ and $x^{S^1}(w_1) = 1$. As

$$\sum_{v \in V_1} b_j^1(v)x^{S^1}(v) + x^{S^1}(w_1) \geq \beta_j^1,$$

We obtain that

$$\sum_{v \in V_1} b_j^1(v)x^{S^1}(v) - x^{S^1}(u) \geq \beta_j^1 - 1.$$

And the lemma follows. ■

Theorem 3.2.1 *If $ax \geq \alpha$ is a valid inequality for $P(\widetilde{G}_i, \widetilde{T}_i)$ with $a(w_i) = 0$, then there exists an inequality $\overline{a}x \geq \overline{\alpha}$ valid for $P(\widetilde{G}_i, \widetilde{T}_i)$ that dominates $ax \geq \alpha$ and is a linear combination of inequalities* (21), (22) *and* (24) *associated with vertices of V_i.*

Proof Let A' be the system given by inequalities (21) − (23) and (24) associated with vertices of V_i. Let P' be the program

$$\min\{ax | x \text{ solution of } A'\},$$

and P'' the program

$$\min\{ax | x \in P(\widetilde{G}_i, \widetilde{T}_i)\}.$$

An optimal solution, say x_0, of P'' is a feasible solution for P'. Since $a(w_i) = 0$, we can assume that $x_0(w_i) = 1$.

Let $r \in \mathbb{R}^{L^i_1}, z \in \mathbb{R}^{L^i_2}, \theta \in \mathbb{R}^{L^i_3}, \lambda \in \mathbb{R}^{V_i}$ associated with inequalities (21), (22), (23) and (24) associated with vertices of V_i, respectively. The dual D' of P' can be written as

$$\max \quad \sum_{j \in L^i_1} \alpha^i_j r_j + \sum_{j \in L^i_2} \alpha'^i_j z_j + \sum_{j \in L^i_3} \beta^i_j \theta_j - \sum_{v \in V_i} \lambda(v)$$

$$\sum_{j \in L^i_1} a^i_j(v) r_j + \sum_{j \in L^i_2} a'^i_j(v) z_j + \sum_{j \in L^i_3} b^i_j(v)\theta_j - \lambda(v) \leq a(v) \qquad \forall v \in \widetilde{V}_i \setminus \{u, w_i\} \tag{26}$$

$$\sum_{j \in L^i_2} z_j + \sum_{j \in L^i_3} b^i_j(u)\theta_j - \lambda(u) \leq a(u) \tag{27}$$

$$\sum_{j \in L^i_3} b^i_j(w_i)\theta_j \leq a(w_i) \tag{28}$$

$$r^i_j \geq 0 \qquad \forall j \in L^i_1 \tag{29}$$

$$z^i_j \geq 0 \qquad \forall j \in L^i_2 \tag{30}$$

$$\theta^i_j \geq 0 \qquad \forall j \in L^i_3 \tag{31}$$

$$\lambda(v) \geq 0 \qquad \forall v \in V_i \tag{32}$$

As x_0 is a feasible solution of P', dual D' admits an optimal solution $(r^*, z^*, \theta^*, \lambda^*)$. From Lemma 3.1.1, all coefficients in inequality (28) are nonnegative. As $a(w_i) = 0$, we deduce that $\theta^*_j = 0$ for all $j \in L^i_3$. Let

$$\overline{a}(v) = \sum_{j \in L^i_1} a^i_j(v) r^*_j + \sum_{j \in L^i_2} a'^i_j(v) z^*_j - \lambda^*(v) \qquad , \text{for all } v \in \widetilde{V}_i \setminus \{u, w_i\},$$

$$\overline{a}(u) = \sum_{j \in L_2^i} z_j^* - \lambda^*(u),$$

$$\overline{a}(w_i) = 0,$$

$$\overline{\alpha} = \sum_{j \in L_1^i} \alpha_j^i r_j^* + \sum_{j \in L_2^i} \alpha_j'^i z_j^* - \sum_{v \in V_i} \lambda^*(v).$$

We can see that $\overline{a}x \geq \overline{\alpha}$ is a combination of inequalities (21), (22) and inequalities (24) associated with vertices of V_i using coefficients (r^*, z^*, λ^*). Thus $\overline{a}x \geq \overline{\alpha}$ is valid for $P(\widetilde{G}_i, \widetilde{T}_i)$.

From the inequalities of D' we notice that $\overline{a}(v) \leq a(v)$ for all vertex $v \in \widetilde{V}_i$. Since $ax \geq \alpha$ and $\overline{\alpha} = \min\{ \quad ax \quad | \quad x$ solution of $A'\}$, it follows that $\overline{\alpha} \geq \alpha$. Thus, inequality $\overline{a}x \geq \overline{\alpha}$ dominates $ax \geq \alpha$. ■

Consider the following system of inequalities

$$\sum_{v \in V_i \setminus \{u\}} a_j^i(v)x(v) \geq \alpha_j^i \qquad \forall i = 1,2, \quad \forall j \in L_1^i \qquad (33)$$

$$\sum_{v \in V_1 \setminus \{u\}} a_j'^i(v)x(v) + x(u) \geq \alpha_j'^i \qquad \forall i = 1,2, \quad \forall j \in L_2^i \qquad (34)$$

$$\sum_{p=1}^{2} \sum_{v \in V_i} b_{j_p}^p(v)x(v) - x(u) \geq \sum_{p=1}^{2} \beta_{j_p}^p - 1 \qquad \forall j_1 \in L_3^1, \forall j_2 \in L_3^2 \qquad (35)$$

$$x(v) \leq 1 \qquad \forall v \in V \qquad (36)$$

$$x(v) \geq 0 \qquad \forall v \in V \qquad (37)$$

In what follows we will show that $P(G,T)$ is completely described by inequalities $(33) - (37)$. Inequalities (35) are called *mixed inequalities.*

Lemma 3.2.3 *Inequality*

$$\sum_{v \in V_i} b_j^i(v)x(v) - x(u) \geq \beta_j^i - 1$$

is valid for the polytope given by inequalities $(33) - (37)$ *and it is dominated by inequality* $\overline{a}x \geq \overline{\alpha}$ *that is obtained from a linear combination of inequalities* $(33) - (34)$ *and inequalities* (36) *associated with vertices of* V_i*.*

Proof From Lemma 3.2.2 inequality

$$\sum_{v \in V_i} b_j^i(v)x(v) - x(u) \geq \beta_j^i - 1$$

is valid for $P(\widetilde{G}_i, \widetilde{T}_i)$. As the coefficient of w_i is 0, from Theorem 3.2.1, this inequality is dominated by an inequality $\overline{a}x \geq \overline{\alpha}$ that is a combination of inequalities (21), (22) and inequalities (24) associated with vertices of V_i. These inequalities correspond to (33) $-$ (34) and (36) associated to i. Thus $\overline{a}x \geq \overline{\alpha}$ is also valid for the polytope given by inequalities (33) $-$ (37) and our lemma holds. ■

Lemma 3.2.4 *The mixed inequalities are valid for $P(G, T)$.*

Proof Consider a mixed inequality associated with $j_1 \in L_3^1$ and $j_2 \in L_3^2$. Consider a separator S for graph G. We distinguish two cases.

Case 1. $u \in S$: Let $S_1 = (S \cap V_1)$ and $S_2 = (S \cap V_2)$. It is clear that S_1 and S_2 are both separators for $\widetilde{G}_1$ and $\widetilde{G}_2$, respectively. Thus by (23) we have

$$\sum_{v \in V_1} b_{j_1}^1(v)x^{S_1}(v) \geq \beta_{j_1}^1$$

$$\sum_{v \in V_2} b_{j_2}^2(v)x^{S_2}(v) \geq \beta_{j_2}^2$$

Therefore $\sum_{v \in V_1} b_{j_1}^1(v)x^{S_1}(v) + \sum_{v \in V_2} b_{j_2}^2(v)x^{S_2}(v) - x^{S_1}(u) \geq \beta_{j_1}^1 + \beta_{j_2}^2 - 1$, and inequality (35) is satisfied for S.

Case 2. $u \notin S$: then either $(S \cap V_1)$ is a separator for $\widetilde{G}_1$ or $(S \cap V_2)$ is separator for $\widetilde{G}_2$ or both. For otherwise, as it has been shown before in Lemma 3.2.1, there would exist a terminal path in G not covered by S, which is impossible. Suppose, for instance, that $(S \cap V_2)$ is a separator for $\widetilde{G}_2$, and let $S_1 = (S \cap V_1) \cup \{w_1\}$ and $S_2 = (S \cap V_2)$. Clearly, sets S_1 and S_2 are separators for $\widetilde{G}_1$ and $\widetilde{G}_2$, respectively. By (23) it follows that

$$\sum_{v \in V_1} b_i^1(v)x^{S_1}(v) + x^{S_1}(w_1) \geq \beta_{j_1}^1,$$

$$\sum_{v \in V_2} b_i^2(v)x^{S_2}(v) \geq \beta_{j_2}^2.$$

Since $x^{S_1}(w_1) = 1$ and $x(u) = 0$, this implies that

$$\sum_{v \in V_1} b_i^1(v) x^{S_1}(v) + \sum_{v \in V_2} b_i^2(v) x^{S_2}(v) \geq \beta_{j_1}^1 + \beta_{j_2}^2 - 1.$$

■

Let $Q(G,T)$ be the polytope given by inequalities (33) – (37).

Lemma 3.2.5 *All integer points of $Q(G,T)$ belong to $P(G,T)$.*

Proof We need to prove that every integer point of $Q(G,T)$ satisfies all terminal path inequalities induced by terminal paths of G. Let us assume the contrary, that there exists an integer point $x \in Q(G,T)$ that does not belong to $P(G,T)$. Clearly, x satisfies all terminal path inequalities of $P(\widetilde{G}_i, \widetilde{T}_i)$ in which $x(w_i)$ is not involved, for $i = 1, 2$. Let $S = \{v \in V | x(v) = 1\}$. As x does not belong to $P(G,T)$ there must exist a terminal path $P_{t_1 t_2}$ in G with, say $t_1 \in T_1$ and $t_2 \in T_2$ not intersecting S. Hence $u \notin P_{t_1 t_2}$. Let $P_{t_i q_i} = (P_{t_1 t_2} \cap V_i) \cup \{w_i, q_i\}$ be the terminal path in $\widetilde{G}_i$ between t_i and q_i. The mixed inequality coming from the terminal path inequalities induced by $P_{t_1 q_1}$ and $P_{t_2 q_2}$ looks as

$$\sum_{v \in P_{t_1 q_1} \setminus \{w_1\}} x(v) + \sum_{v \in P_{t_2 q_2} \setminus \{w_2\}} x(v) - x(u) \quad \geq 1$$

is violated by x, a contradiction. ■

Theorem 3.2.2 *$P(G,T) = Q(G,T)$.*

Proof Let us assume the contrary, and let x be a fractional extreme point of $Q(G,T)$. Let $y_1, \ldots, y_s$ (resp. $z_1, \ldots, z_p$) be the extreme points of $\tilde{P}(\tilde{G}_1, \tilde{T}_1)$ (resp. $\tilde{P}(\tilde{G}_2, \tilde{T}_2)$). Then, we distinguish two cases.
Case 1: $x(u) = 1$. Let $x_1 \in [0,1]^{\widetilde{V}_1}$ and $x_2 \in [0,1]^{\widetilde{V}_2}$ be two vectors given by.

$x_i(v) = x(v)$ for all $v \in V_i$

$x_i(w_i) = 0$

As $x(u) = 1$, from Lemma 3.2.1, it follows that $\sum_{v \in V_i} b_j^i(v)x(v) \geq \beta_j^i$ is satisfied by x_1 and x_2 for $i = 1, 2$, respectively. Therefore, $x_1 \in P(\widetilde{G}_1, \widetilde{T}_1)$ and $x_2 \in P(\widetilde{G}_2, \widetilde{T}_2)$. Thus there exists $\lambda \in [0,1]^s$ and $\mu \in [0,1]^p$ such that

$$x_1 = \sum_{i=1}^{s} \lambda_i y_i,$$

$$x_2 = \sum_{j=1}^{p} \mu_j z_j,$$

$$\lambda_i \geq 0, \text{ for all } i \in \{1, \ldots, s\} \text{ and } \sum_{i=1}^{s} \lambda_i = 1,$$

$$\mu_j \geq 0, \text{ for all } j \in \{1, \ldots, p\} \text{ and } \sum_{j=1}^{p} \mu_j = 1.$$

Moreover, as $x_1 \neq 0 \neq x_2$, there must exist $i_0 \in \{1, \ldots, s\}$ and $j_0 \in \{1, \ldots, p\}$ such that, $\lambda_{i_0} > 0$ and $\mu_{j_0} > 0$, respectively. Each non-mixed inequality which is tight for x is also tight for x_1 or x_2. Also note that, each inequality tight for x_1 (reps. x_2) is also tight for y_i, $i \in \{1, \ldots, s\}$ (resp. z_j, $j \in \{1, \ldots, p\}$). As $x(u) = 1$, it follows that $y_i(u) = 1$ for all $i \in \{1, \ldots, s\}$ and $z_j(u) = 1$ for all $j \in \{1, \ldots, p\}$. Let $x^* \in [0,1]^V$ be the vector

$$x^*(v) = y_{i_0}(v), \quad \forall v \in V_1,$$

$$x^*(v) = z_{j_0}(v), \quad \forall v \in V_2 \setminus \{u\}.$$

We have that each non-mixed inequality tight for x is tight for x^*. Now, we need to prove that each mixed inequality tight for x is also tight for x^*.

Let $j_1 \in J_1$ and $j_2 \in J_2$ such that the corresponding mixed inequality tight for x, that is

$$\sum_{v \in V_1} b_{j_1}^1(v)x(v) + \sum_{v \in V_2} b_{j_2}^2(v)x(v) - x(u) = \beta_{j_1}^1 + \beta_{j_2}^2 - 1. \tag{38}$$

As $x_i \in P(\widetilde{G}_i, \widetilde{T}_i)$ for $i = 1, 2$, we have that

$$\sum_{v \in V_1} b_{j_1}^1(v)x_1(v) + x_1(w_1) \geq \beta_{j_1}^1,$$

$$\sum_{v \in V_2} b_{j_2}^1(v)x_2(v) + x_2(w_2) \geq \beta_{j_2}^2,$$

Thus from Lemma 3.2.2, inequalities

$$\sum_{v \in V_1} b^1_{j_1}(v)x(v) - x(u) \geq \beta^1_{j_1} - 1,$$
$$\sum_{v \in V_2} b^1_{j_2}(v)x(v) - x(u) \geq \beta^2_{j_2} - 1.$$

are valid for $P(\widetilde{G}_1, \widetilde{T}_1)$ and $P(\widetilde{G}_2, \widetilde{T}_2)$, respectively. Since $x(u) = 1$, it follows that

$$\sum_{v \in V_1} b^1_{j_1}(v)x(v) \geq \beta^1_{j_1},$$
$$\sum_{v \in V_2} b^1_{j_2}(v)x(v) \geq \beta^2_{j_2}.$$

By (38), we obtain that these inequalities are tight for x,

$$\sum_{v \in V_1} b^1_{j_1}(v)x(v) = \sum_{v \in V_1} b^1_{j_1}(v)x_1(v) = \beta^1_{j_1},$$
$$\sum_{v \in V_2} b^2_{j_2}(v)x(v) = \sum_{v \in V_2} b^2_{j_2}(v)x_2(v) = \beta^2_{j_2}.$$

Therefore

$$\sum_{v \in V_1} b^1_{j_1}(v)y_{i_0}(v) = \beta^1_{j_1},$$
$$\sum_{v \in V_2} b^2_{j_2}(v)z_{j_0}(v) = \beta^2_{j_2}.$$

Implying that

$$\sum_{v \in V_1} b^1_{j_1}(v)y_{i_0}(v) + \sum_{v \in V_2} b^2_{j_2}(v)z_{j_0}(v) = \beta^1_{j_1} + \beta^2_{j_2},$$

and hence

$$\sum_{v \in V_1} b^1_{j_1}(v)x^*(v) + \sum_{v \in V_2} b^2_{j_2}(v)x^*(v) - x^*(u) = \beta^1_{j_1} + \beta^2_{j_2} - 1.$$

Thus each inequality tight for x is also tight for x^*. Since $x \neq x^*$, this contradicts the extremality of x.

Case 2: $x(u) < 1$.
We will distinguish two cases,

Case 2.1: There is no mixed inequality tight for x.
Let $x_i \in [0,1]^{\widetilde{V}_1}$ be the vector given by

$$x_i(v) = x(v) \text{ for all } v \in V_i \text{ and}$$

$$x_i(w_i) = 1$$

for $i = 1, 2$. All terminal path inequalities induced by terminal paths between two terminals of T_i, are satisfied by x. Thus they are satisfied by x_i. As $x_i(w_i) = 1$, all terminal path inequalities related to terminals of T_i are also satisfied by x_i for $i = 1, 2$. Thus $x_1 \in P(\widetilde{G}_1, \widetilde{T}_1)$ and $x_2 \in P(\widetilde{G}_2, \widetilde{T}_2)$. Therefore there exist $\nu \in [0,1]^s, \mu \in [0,1]^p$ such that

$$x_1 = \sum_{j=1}^{s} \nu_j y_j,$$

$$x_2 = \sum_{j=1}^{p} \mu_j z_j,$$

$$\nu_j \geq 0, \text{ for all } j \in \{1, \dots, s\} \text{ and } \sum_{j=1}^{s} \nu_j = 1,$$

$$\mu_j \geq 0, \text{ for all } \forall j \in \{1, \dots, p\} \text{ and } \sum_{j=1}^{p} \mu_j = 1.$$

Since $x(u) < 1$, there must exist $i_0 \in \{1, \dots, s\}$ and $j_0 \in \{1, \dots, p\}$ such that, $y_{i_0}(u) = z_{j_0}(u) = 0$. Observe that each non-mixed inequality tight for x is tight for x_1 or x_2. Let $x^* \in [0,1]^V$ be the vector given by

$$x^*(v) = y_{i_0}(v) \quad \forall v \in V_1,$$

$$x^*(v) = z_{j_0}(v) \quad \forall v \in V_2 \setminus \{u\}.$$

We have that any inequality tight for x_1 and x_2 is also tight for y_{i_0} and z_{j_0}. Thus, each non-mixed inequality tight for x is also tight for x^*. In consequence, x and x^* satisfy the same equation system. As $x \neq x^*$, this is a contradiction.

Case 2.2: There is at least one mixed inequality tight for x.
Consider a mixed inequality associated with $j_1 \in J_1$ and $j_2 \in J_2$, tight for x,

$$\sum_{v \in V_1} b^1_{j_1}(v)x(v) + \sum_{v \in V_2} b^2_{j_2}(v)x(v) - x(u) = \beta^1_{j_1} + \beta^2_{j_2} - 1 \tag{39}$$

Claim 1 *There exist* $0 \leq \lambda \leq 1$ *and* $0 \leq \epsilon < 1$ *such that*

$$\sum_{v \in V_1} b^1_{j_1}(v)x(v) = \beta^1_{j_1} - 1 + \lambda$$

$$\sum_{v \in V_2} b^2_{j_2}(v)x(v) = \beta^2_{j_2} - \lambda + \epsilon$$

Proof of Claim 1. From Lemma 3.2.3, the following inequality

$$\sum_{v \in V_1} b^1_{j_1}(v)x(v) - x(u) \geq \beta^1_{j_1} - 1$$

is satisfied by x. Then, by (39), we obtain that

$$\sum_{v \in V_2} b^2_{j_2}(v)x(v) \leq \beta^2_{j_2}$$

From Lemma 3.2.3, we also have

$$\beta^2_{j_2} - 1 \leq \sum_{v \in V_2} b^2_{j_2}(v)x(v) - x(u) \leq \beta^2_{j_2}.$$

Therefore, there exists $1 \geq \lambda \geq 0$ such that

$$\sum_{v \in V_2} b^2_{j_2}(v)x(v) - x(u) + \lambda = \beta^2_{j_2}.$$

Let $\epsilon = x(u)$, thus

$$\sum_{v \in V_2} b^2_{j_2}(v)x(v) = \beta^2_{j_2} - \lambda + \epsilon$$

From equality (39) we then obtain

$$\sum_{v \in V_1} b^1_{j_1}(v)x(v) = \beta^1_{j_1} - 1 + \lambda.$$

And, the claim follows. $\diamondsuit$ Now consider solutions $x_1 \in [0,1]^{\widetilde{V}_1}$ and $x_2 \in [0,1]^{\widetilde{V}_2}$ such that

$x_1(v) = x(v) \quad \forall v \in V_1,$

$x_1(w_1) = 1 - \lambda,$

and

$x_2(v) = x(v) \quad \forall v \in V_2,$

$x_2(w_2) = \lambda - \epsilon.$

Claim 2 $x_i \in P(\widetilde{G}_i, \widetilde{T}_i), \qquad$ *for* $i = 1, 2.$

Proof of Claim 2. We prove that $x_1 \in P(\widetilde{G}_1, \widetilde{T}_1)$. The proof of $x_2 \in P(\widetilde{G}_2, \widetilde{T}_2)$ is similar.
It is clear that x_1 satisfies all inequalities (21)-(22). Assume, on the contrary that for some $j \in L_3^1$,

$$\sum_{v \in V_1} b_{j_3}^1(v) x_1(v) + x_1(w_1) < \beta_{j_3}^1.$$

From Claim 1, it then follows that

$$\sum_{v \in V_1} b_{j_3}^1(v) x_1(v) + x_1(w_1) + \sum_{v \in V_2} b_{j_2}^2(v) x(v) < \beta_{j_3}^1 + \beta_{j_2}^2 - \lambda + \epsilon.$$

Since $x_1(w_1) = 1 - \lambda$ and $x(u) = \epsilon$, this yields

$$\sum_{v \in V_1} b_{j_3}^1(v) x(v) + \sum_{v \in V_2} b_{j_2}^2(v) x(v) - x(u) < \beta_{j_3}^1 + \beta_{j_2}^2 - 1.$$

But this contradicts the fact that $x \in Q(G, T)$. $\Diamond$ Since $x_1 \in P(\widetilde{G}_1, \widetilde{T}_1)$ and

$x_2 \in P(\widetilde{G}_2, \widetilde{T}_2)$, there exist $\lambda \in [0, 1]^s$ and $\mu \in [0, 1]^p$ such that

$x_1 = \sum_j^s \lambda_j y_j,$

$x_2 = \sum_j^p \mu_j z_j,$

$\lambda_j > 0$, for all $i \in \{1, \ldots, s\}$,

$\mu_j > 0$, for all $\forall j \in \{1, \dots, p\}$.

Since $x_1(u) < 1$ and $x_2(u) < 1$, there must exist y_{i_0} and z_{j_0} such that $y_{i_0}(u) = z_{j_0}(u) = 0$.
Let $x^* \in [0,1]^V$ be given by

$$x^*(v) = y_{i_0}(v) \quad \forall v \in V_1,$$

$$x^*(v) = z_{j_0}(v) \quad \forall v \in V_2 \setminus \{u\}.$$

Each non-mixed inequality for $j \in (L_1^1 \cup L_2^1)$ (resp. $j \in (L_1^2 \cup L_2^2)$) tight for x is tight for x_1 (resp. x_2). Therefore it is also tight for x^*.

Claim 3 *Each mixed inequality is tight for x is tight for x^*.*

Proof of Claim 3. Consider a mixed inequality for, say $j' \in L_3^1$ and $j'' \in L_3^2$ tight for x, that is

$$\sum_{v \in V_1} b_{j'}^1(v)x(v) + \sum_{v \in V_2} b_{j''}^2(v)x(v) - x(u) = \beta_{j'}^1 + \beta_{j''}^2 - 1 \tag{40}$$

By Claim 1, there exists $0 \le \lambda'' \le 1$ and $0 \le \epsilon < 1$ such that

$$\sum_{v \in V_1} b_{j'}^1(v)x(v) = \beta_{j'}^1 - 1 + \lambda'' \tag{41}$$

$$\sum_{v \in V_2} b_{j''}^2(v)x(v) = \beta_{j''}^2 - \lambda'' + \epsilon \tag{42}$$

We have that $\lambda'' \ge \lambda$. For otherwise, as $x(u) = \epsilon$, by summing equality (41) and the second equality of Claim 1 we obtain a mixed inequality violated by x. We have the following facts.

Fact 1 $\lambda'' = \lambda$.

Proof of Fact 1. If $j' = j_1$ and $j'' = j_2$ then the claim holds. Therefore, suppose that $j' \ne j_1$ or $j'' \ne j_2$ and $\lambda'' > \lambda$. From equality (41), it then follows that

$$-\sum_{v \in V_1} b_{j'}^1(v)x(v) < -\beta_{j'}^1 + 1 - \lambda.$$

By summing the above inequality together with equality (40), this yields

$$\sum_{v \in V_2} b^2_{j''}(v)x(v) - x(u) < \beta^2_{j''} - \lambda.$$

And by summing the above inequality together with the first equality of Claim 1, we obtain

$$\sum_{v \in V_1} b^1_{j_1}(v)x(v) + \sum_{v \in V_2} b^2_{j''}(v)x(v) - x(u) < \beta^1_{j_1} + \beta^2_{j''} - 1,$$

a contradiction and the claim follows. ♢

Fact 2 *The inequalities corresponding to j' and j'' are tight for x_1 and x_2, respectively.*

Proof of Fact 2. We have that

$$\sum_{v \in V_1} b^1_{j'}(v)x(v) = \sum_{v \in V_1} b^1_{j'}(v)x_1(v) = \beta^1_{j'} + \lambda - 1$$
$$\sum_{v \in V_2} b^2_{j''}(v)x(v) = \sum_{v \in V_2} b^2_{j''}(v)x_2(v) = \beta^2_{j''} - \lambda + \epsilon$$

As $x_1(w_1) = 1 - \lambda$ and $x_2(w_2) = \lambda - \epsilon$, it follows that

$$\sum_{v \in V_1} b^1_{j'}(v)x_1(v) + x_1(w_1) = \beta^1_{j'} \tag{43}$$
$$\sum_{v \in V_2} b^2_{j''}(v)x_2(v) + x_2(w_2) = \beta^2_{j''} \tag{44}$$

Therefore, the statement holds.♢ Now by summing equalities (43) and (44), this yields

$$\sum_{v \in V_1} b^1_{j'}(v)x_1(v) + \sum_{v \in V_2} b^2_{j''}(v)x_2(v) + x_2(w_2) + x_1(w_1) = \beta^1_{j'} + \beta^2_{j''}$$

Since $x_1(w_1) = 1 - \lambda$ and $x_2(w_2) = \lambda - \epsilon$, this yields

$$\sum_{v \in V_1} b^1_{j'}(v)x^*(v) + \sum_{v \in V_2} b^2_{j''}(v)x^*(v) - x^*(u) = \beta^1_{j'} + \beta^2_{j''} - 1$$

Therefore, any inequality tight for x is also tight for x^*. ◊

As $x \neq x^*$, this yields a contradiction with the fact that x is an extreme point of $Q(G, T)$. ◊

■

3.3. Facet composition

In this section, we will present some results for the facets of the multi-terminal vertex separator polytope.

Given a graph $G = (V \cup T, E)$, and let H be a subgraph of G. We first give a lemma that is easily seen to be true.

Lemma 3.3.1 *If $ax \geq \alpha$ is valid for $P(H, T(H))$ then it is also valid for $P(G, T)$.*

Lemma 3.3.2 *Let S be a separator of graph $\widetilde{G}_i$. If x^Ssatisfies an inequality* (23) *with equality, then $|S \cap \{w_i, u\}| \leq 1$.*

Proof Let us assume the contrary, that is $\{w_i, u\} \subseteq S$ and x^S satisfies an inequality (23) with equality. Let $\overline{S} = S \setminus \{w_i\}$. It is clear that $\overline{S}$ is a separator for $\widetilde{G}_i$. It follows that inequality (23) is violated by $\overline{S}$, which is impossible. ■

Theorem 3.3.1 *If an inequality of type* (23) *defines a facet for $P(\widetilde{G}_1, \widetilde{T}_1)$ and an inequality* (23) *defines a facet for $P(\widetilde{G}_2, \widetilde{T}_2)$, then the associated mixed inequality* (35) *defines a facet for $P(G, T)$.*

Proof. Since the two inequalities (23) define facets for $P(\widetilde{G}_i, \widetilde{T}_i)$, $i = 1, 2$, let $\mathcal{S}_i$ be the set of $|\widetilde{V}_i|$ affinely independent solutions satisfying inequality (23) with equality, for $i = 1, 2$. From Lemma 3.3.2, there is no solution in $\mathcal{S}_i$ containing both w_i and u, for $i = 1, 2$. Let $n_1^i, n_2^i, n_3^i \in \mathbb{N}$ defined as follows

n_1^i : the number of solutions in $\mathcal{S}_i$ containing u and not w_i,

n_2^i : the number of solutions in $\mathcal{S}_i$ containing w_i and not u,

n_3^i : the number of solutions in $\mathcal{S}_i$ containing neither u and nor w_i,

where $n_1^i + n_2^i + n_3^i = |\widetilde{V}_i|$, for $i = 1, 2$.

We need to construct $|V|$ solutions affinely independent satisfying (35) with equality. Since w_1 and w_2 do no appear in G and u appears once, we need only $|\widetilde{V}_1| + |\widetilde{V}_2| - 3$ solutions. For this, we are going to compose solutions from S_1 with solutions from S_2 to form the desired solutions. Consider a solution

$\overline{S}_1 \in \mathcal{S}_1$ containing u. For each solution $S_2 \in \mathcal{S}_2$ containing u, consider the solution $(\overline{S}_1 \cup S_2)$. Let A_1 be the set of all these solutions. Consider a solution $\overline{S}_2 \in \mathcal{S}_2$ containing u. For each solution $S_1 \in \mathcal{S}_1$ containing u, we construct a new solution $(S_1 \cup \overline{S}_2)$. Let A_2 be the set of all these solutions. Clearly, all the solutions in $A_1 \cup A_2$ are solutions of the face defined by (35). Moreover, there is one solution in A_1 which is the same as a solution of A_2. Thus $A = A_1 \cup A_2$ contains $n_1^1 + n_1^2 - 1$. These solutions are affinely independent. Consider now a solution $\widehat{S}_1 \in \mathcal{S}_1$ containing neither u, nor w_1. For each solution $S_2 \in \mathcal{S}_2$ containing w_2, we construct a new solution $(\widehat{S}_1 \cup S_2)$. Let B_1 be the set of all these solutions. And consider one solution $\widehat{S}_2 \in \mathcal{S}_2$ containing neither u, nor w_1. For each solution $S_1 \in \mathcal{S}_1$ containing w_1, we construct a new solution $(S_1 \cup \widehat{S}_2)$. Let B_2 be the set of all these solutions. We have that $B = B_1 \cup B_2$ contains $n_2^1 + n_2^2$ affinely independent solutions. Note that solutions $\widehat{S}_1$ and $\widehat{S}_2$ exist. In fact, if for instance $\widehat{S}_1$ does not exist, then each solution of S_1 contains exactly one node among u and w_1 and in consequence, its incidence vector satisfies the equation $x(u) = x(w_1) = 0$. Thus, the equality induced by (23) is equivalent to $x(u) = x(w_1)$. This contradicts Lemma 3.1.2. Consider one solution $\tilde{S}_1 \in \mathcal{S}_1$ containing w_1. For each solution $S_2 \in \mathcal{S}_2$ containing neither u, nor w_2, we construct a new solution $(\tilde{S}_1 \cup S_2)$. Let C_1 be the set of all these solutions. And consider a solution $\tilde{S}_2 \in \mathcal{S}_2$ containing w_2. For each solution $S_1 \in \mathcal{S}_1$ containing neither u, nor w_1, we consider a new solution $(S_1 \cup \tilde{S}_2)$. Let C_2 be the set of all these solutions. Clearly, there is a solution in C_1, similar to a solution in B_1 and one solution in C_2 which is the same as in B_2. Thus $C = (C_1 \cup C_2) \setminus (B_1 \cup B_2)$ contains $n_3^1 + n_3^2 - 2$ solutions affinely independent.

Therefore, $A \cup B \cup C$ contains a set of $|\widetilde{V}_1| + |\widetilde{V}_2| - 3$ solutions affinely independent satisfying inequality (35) with equality. ■

Now we present two valid inequalities, which are the generalization of the clique star and the terminal cycle inequalities [6].

3.3.0.1 General clique star inequality

A general clique star $H = (V' \cup T', E)$ is given by a composition of one clique star and several star trees. Thus, H contains one clique $K_f \subseteq V'$ on f vertices and q terminals. Figure 7 displays a general clique star of 14 terminals.

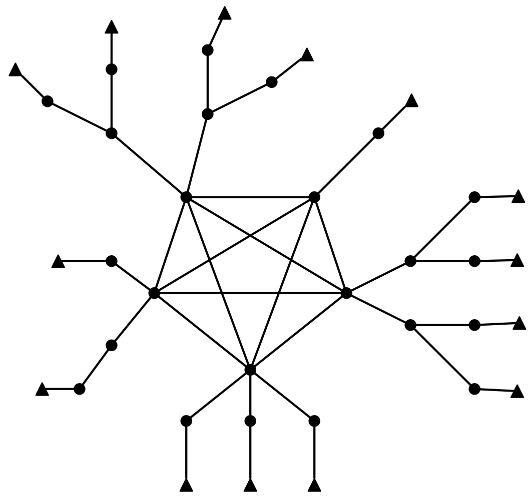


Figure 7: General clique star

Lemma 3.3.3 *Given a general clique star* $H = (V' \cup T', E)$ *subgraph of* $G = (V \cup T, E)$*, the following inequality*

$$\sum_{v \in K_f} (d_H(v) - (f-1))x(v) + \sum_{v \in V' \setminus K_f} (d_H(v) - 1)x(v) \geq q - 1$$

is valid for $P(G,T)$*.*

3.3.0.2 General terminal cycle inequality

Given a general terminal cycle $H = (V' \cup T', E)$ is given by a composition of one terminal cycle and several star trees. Thus, H contains one cycle $C_f \subseteq V'$ on f vertices and has q terminals. Let $C' \subset C_f$ be the subset of vertices in C_f of degree greater or equal to 3 in H. Figure 8 displays a general terminal cycle of 14 terminals.

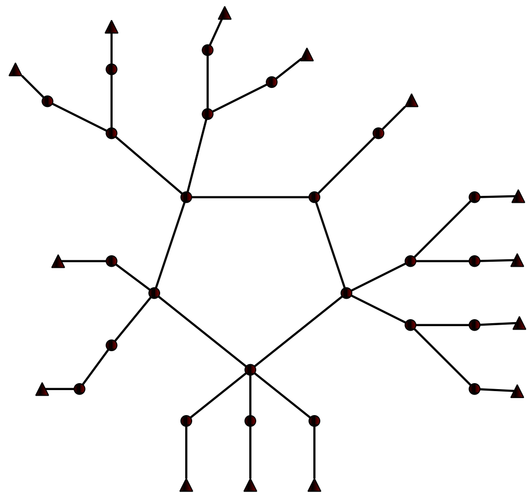


Figure 8: General terminal cycle

Lemma 3.3.4 *For a general terminal cycle $H = (V' \cup T', E)$ subgraph of $G = (V \cup T, E)$, the following inequality*

$$\sum_{v \in C'} (d_H(v) - 2)x(v) + \sum_{v \in V' \setminus C'} (d_H(v) - 1)x(v) \geq q - \lfloor \frac{|C'|}{2} \rfloor$$

is valid for $P(G, T)$.

3.4. Algorithmic aspect

In this section, we will discuss the algorithmic consequences of the composition/decomposition studied before. If G is the 1-sum of two graphs G_1 and G_2, then the multi-terminal vertex separator problem can also be decomposed. We will show that the MTVSP in G can be reduced to the MTVSP in $\widetilde{G}_2$ with respect to a weight system obtained by solving the problem in $\widetilde{G}_1$.

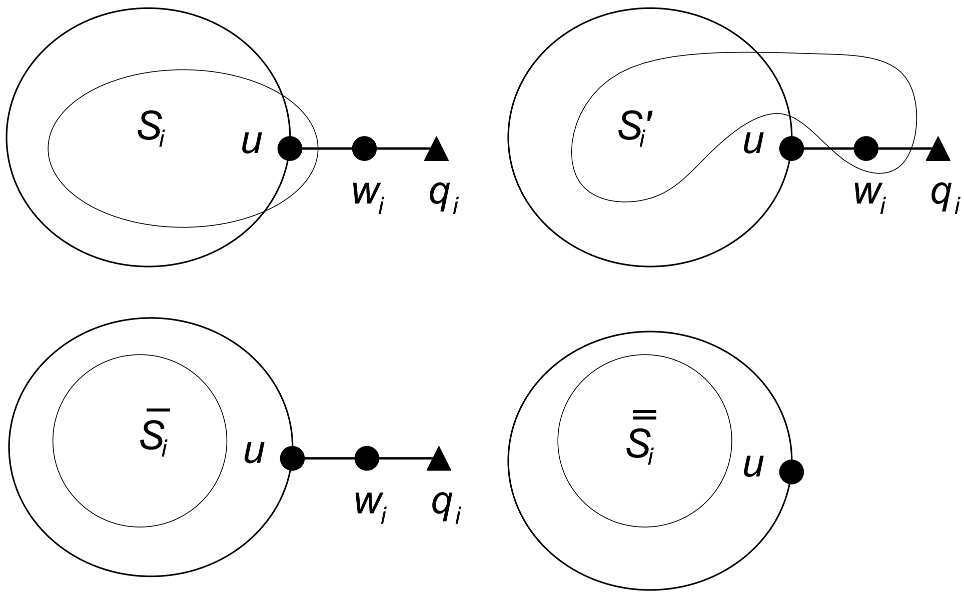


Figure 9: Illustration of the separators S_i, S_i', $\bar{S}_i$ and $\tilde{S}_i$.

Given a vertex weight vector $c \in \mathbb{R}^V$, for a graph $\widetilde{G}_i$, let $\{s_i, s_i', \overline{s}_i, \tilde{s}_i\} \in \mathbb{N}^4$,
(resp. $\{\omega_i, \omega_i', \overline{\omega}_i, \tilde{\omega}_i\} \in \mathbb{N}^4$) such that

$s_i + 1$ (resp. ω_i), is the size (resp. weight) of the minimum multi-terminal vertex separator S_i in graph $\widetilde{G}_i$ that contains u but not w_i.

$s_i' + 1$ (resp. ω_i'), is the size (resp. weight) of the minimum multi-terminal vertex separator S_i' in graph $\widetilde{G}_i$ that contains w_i and not u.

$\overline{s}_i$ (resp. $\overline{\omega}_i$) is the size (resp. weight) of the minimum multi-terminal vertex separator $\overline{S}_i$ in graph $\widetilde{G}_i$ containing neither u nor w_i.

$\tilde{s}_i$ (resp. $\tilde{\omega}_i$) is the size (resp. weight) of the minimum multi-terminal vertex separator $\tilde{S}_i$ in graph $G_i = \widetilde{G}_i \setminus \{w_i, q_i\}$ not containing u.

Figure 9 illustrates the separators S_i, S_i', $\bar{S}_i$ and $\tilde{S}_i$. To force a vertex to be in the separator we can delete the vertex from the graph. To force a vertex v to be out of the separator we can set $c(v) = +\infty$.

Let $\overline{c}_2 \in \mathbb{R}^{\widetilde{V}_2}$ be the weight vector associated with $\widetilde{V}_2$ such that

$\overline{c}_2(u) = \omega_1 - s_2\frac{\tilde{\omega}_1}{\overline{s}_2}$

$\overline{c}_2(w_2) = \overline{\omega}_1 - s'_2\frac{\tilde{\omega}_1}{\overline{s}_2}$

$\overline{c}_2(v) = c(v) + \frac{\tilde{\omega}_1}{\overline{s}_2} \quad \forall v \in \widetilde{V}_2 \setminus \{u, w_2\}$

Clearly, $\overline{s}_2 > 0$, since $\bar{S}_i$ contains at least one node. Let $S_2^* \in \{S_2, S'_2, \overline{S}_2\}$ be the separator having the minimum weight ω_2^* in $\tilde{G}_2$ with respect to $\bar{c}_2$. Clearly, since we add the same value $\frac{\tilde{\omega}_1}{\overline{s}_2}$ to the weight of each vertex $v \in V_2 \setminus \{u\}$, the minimum separator in $\widetilde{G}_2$ under vertex weight vector $\overline{c}_2$ is either S_2, S'_2 or $\overline{S}_2$.

Consider the solution S^* given by

$$S^* = \begin{cases} S_1 \cup S_2^* & \text{If } u \in S_2^* \text{ and } w_2 \notin S_2^*, \qquad i.e., S_2^* = S_2 \\ (\overline{S}_1 \cup S_2^*) \setminus \{w_2\} & \text{If } w_2 \in S_2^* \text{ and } u \notin S_2^*, \qquad i.e., S_2^* = S'_2 \\ \tilde{S}_1 \cup S_2^*, & \text{If } u, w_2 \notin S_2^* \qquad\qquad\qquad i.e., S_2^* = \overline{S}_2. \end{cases}$$

Lemma 3.4.1 *Vertex set S^* is a minimum multi-terminal vertex separator in G whose weight is $\omega^* = \omega_2^*$.*

Proof We distinguish three cases for the state of separator S^*,

1) S^* contains u.

2) S^* does not contain u and $(S^* \cap V_1)$ intersects all terminal paths between q_1 and terminals of T_1 in graph $\widetilde{G}_1$.

3) S^* does not contain u and there exists a terminal path between q_1 and a terminal of T_1 in graph $\widetilde{G}_1$ that does not intersect $(S^* \cap V_1)$.

Thus the weight of S^* is

$$\omega^* = \min(\omega_1 + \omega_2 - c(u), \overline{\omega}_1 + \omega'_2 - c(w_2), \tilde{\omega}_1 + \overline{\omega}_2)$$

It follows that

$$\omega^* = \min((\overline{c}_2(u) + s_2\frac{\tilde{\omega}_1}{\overline{s}_2}) + \omega_2 - c(u), (\overline{c}_2(w_2) + s'_2\frac{\tilde{\omega}_1}{\overline{s}_2}) + \omega'_2 - c(w_2), \tilde{\omega}_1 + \overline{\omega}_2)$$

We know that

$$s_2\frac{\tilde{\omega}_1}{\overline{s}_2} + \omega_2 - c(u) = \omega_2^* - \overline{c}_2(u)$$

$$s_2'\frac{\tilde{\omega}_1}{\overline{s}_2} + \omega_2' - c(w_2) = \omega_2^* - \overline{c}_2(w_2)$$

$$\overline{c}_2(\overline{S}_2) = w_2^* = \overline{s}_2\frac{\tilde{\omega}_1}{\overline{s}_2} + c(\overline{S}_2) = \tilde{\omega}_1 + c(\overline{S}_2) = \tilde{\omega}_1 + \overline{\omega}_2$$

This implies $\omega^* = \omega_2^*$. ■

Let us remark that if we can solve the MTVSP in polynomial time in $\widetilde{G}_1$ and $\widetilde{G}_2$, then we can solve the MTVSP in polynomial time in G. Indeed, as we can solve the MTVSP in polynomial time in $\widetilde{G}_1$, then we can compute all the following parameters : $s_i, s_i', \overline{s}_i, \tilde{s}_i, \omega_i, \omega_i', \overline{\omega}_i, \tilde{\omega}_i$ and we can obtain sets $S_i, S_i', \overline{S}_i, \tilde{S}_i$ in polynomial time.

4. Composition of polyhedra by terminal-sum

Consider two graphs $G_1 = (V_1 \cup T_1, E_1)$ and $G_2 = (V_2 \cup T_2, E_2)$ and let $T_1' = \{t_1^1, \ldots, t_q^1\} \subseteq T_1$ and $T_2' = \{t_1^2, \ldots, t_q^2\} \subseteq T_2$ be two subsets of q terminals. $G = (V \cup T, E)$ is called a *terminal-sum* of $G_1 = (V_1 \cup T_1, E_1)$ and $G_2 = (V_2 \cup T_2, E_2)$ if it is obtained by merging each terminal $t_i^1 \in T_1$ with terminal $t_i^2 \in T_2$, for all $i \in \{1, \ldots, q\}$. Figure 10 illustrates the graphs G, G_1 and G_2.

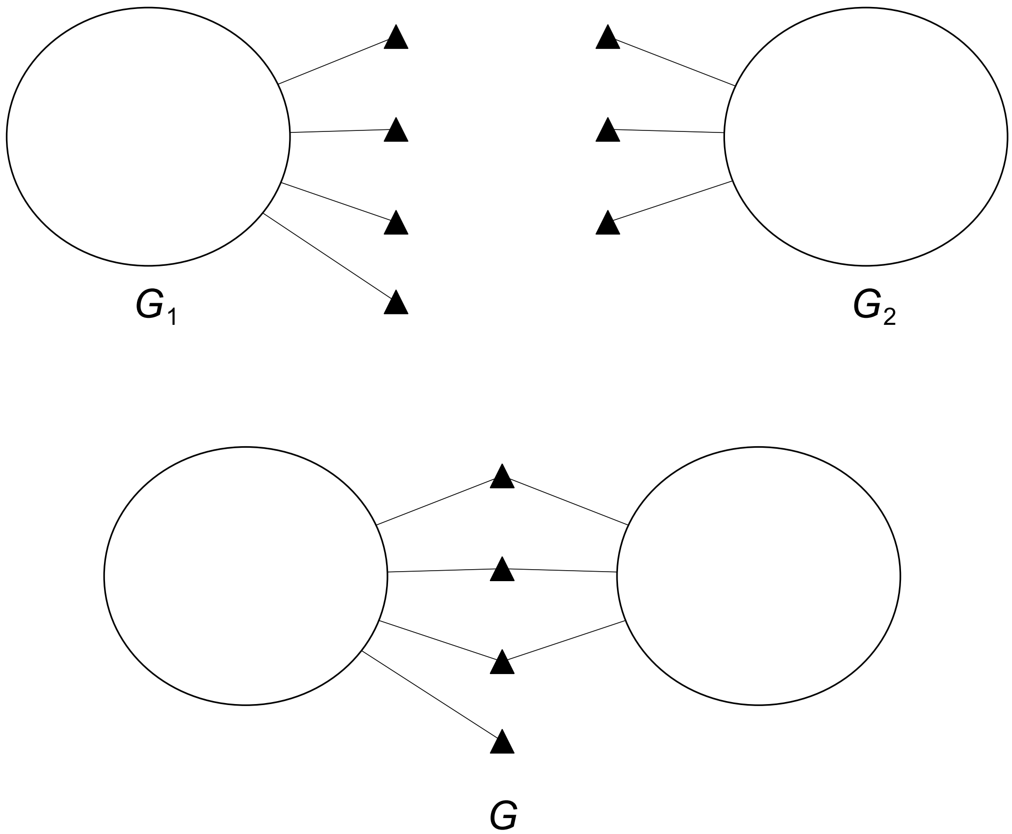


Figure 10: Terminal-sum where $q = 3$.

The multi-terminal vertex separator polytope for graph G_i is completely described by the following system of inequalities A_i

$$\sum_{v \in V_i} a_l^i(v)x(v) \geq \alpha_l^i \quad \forall l \in L^i \tag{45}$$

$$x(v) \leq 1 \quad \forall v \in \widetilde{V}_i$$

$$x(v) \geq 0 \quad \forall v \in \widetilde{V}_i$$

where L^i is a set of indices, for $i = 1, 2$.

Theorem 4.0.1 *The multi-terminal vertex separator polytope for G is completely described by a system of inequalities A obtained by juxtaposition of inequalities* (45) *associated with G_1, inequalities* (45) *associated with G_2 and the trivial inequalities and identification of the common variables.*

Proof Let us assume the contrary, let x^* be a fractional extreme point of the polytope given by the inequalities of A. Vector x^* satisfies a subset of inequalities $A^* \subseteq A$ with equality, where $|A^*| = |V_1| + |V_2|$. There is no equality in A^* containing two variables associated with two vertices, one in G_1 and the other in G_2. Equalities of A^* can be partitioned into 2 subsets A_1^* and A_2^*, one associated with G_1 and the other one associated with G_2. Since x^* is an extreme point, there must exists $|V|$ inequalities of A satisfied with equality. Since $V_1 \cap V_2 = \emptyset$, there must exist $A_1' \subseteq A_1^*$ and $A_2' \subseteq A_2^*$ such that $|A_1'| = |V_1|$ and $|A_2'| = |V_2|$. Let x_i^* be the restriction of x^* on V_i for $i = 1, 2$. Clearly, x_i^* satisfies all inequalities of A_i' with equality. Since x^* is fractional, it follows that, either x_1^* is a fractional extreme point in $P(G_1, T_1)$ or x_2^* is a fractional extreme point in $P(G_2, T_2)$, a contradiction. ■

Theorem 4.0.2 *If the linear systems A_1 and A_2 are TDI, then the linear system given by the juxtaposition of A_1 and A_2 is also TDI.*

Proof Let y_i^* be the integer optimal solution of the dual problem associated with A_i. Since the dual problem associated with G_i is TDI, y_i^* should exist. Thus, clearly $y^* = [y_1^*, y_2^*]$ is a feasible integer solution for the dual problem of the juxtaposition of A_1 and A_2. Moreover, it should be optimal. Suppose this is not the case, and let $\overline{y}$ be an optimal solution of the dual problem of the juxtaposition of A_1 and A_2. Let $\overline{y}_i$ be the restriction of $\overline{y}$ on V_i. Clearly, $\overline{y}_i$ is a feasible solution for the dual problem of A_i. Thus y_1^* or y_2^* is not optimal, a contradiction. ■

Lemma 4.0.1 *Let S_i^* be the optimal solution in G_i, for $i = 1, 2$. Then vertex separator $S^* = S_1^* \cup S_2^*$ is the minimum multi-terminal vertex separator in G.*

5. Conclusion

In this paper, we have considered the multi-terminal vertex separator problem. We have characterized the MTVS polytope for two classes of graphs the star trees and the clique stars. We have also given a linear system for each class of graph that is total dual integral.

Moreover, we have studied a composition (decomposition) technique for the multi-terminal vertex separator polytope in graphs that are decomposable by one-node cutsets. As a consequence, we have provided a procedure to construct this polytope in graphs that are recursively decomposed. We have also studied two further polytope compositions, and have presented the algorithmic aspect resulting from these polytope compositions.

We also have presented a procedure for generating new valid inequalities by composition of valid inequalities. We have shown that if G is a 1-sum of G_1 and G_2, a mixed inequality composed of a facet of $P(G_1, T_1)$ and a facet of $P(G_2, T_2)$ is a facet for G.

Now some questions may arise. First, it would be interesting to characterize the graphs for which the terminal path, star tree, clique star, terminal cycle and terminal tree inequalities suffice to describe the multi-terminal vertex separator polytope. Then, it would also be interesting to characterize the graphs for which the composition of their associated TDI linear systems, is TDI. Moreover, a natural question that may be posed is to extend the composition technique to graphs composed by k-sums, that is by identifying two cliques on k nodes, for some $k \geq 2$. In particular, in which cases, the facet composition introduced in this paper may apply when the graph is obtained by k-sum from two pieces. Finally, it would be good to study the possibility to extend the results in this paper, in particular the valid inequalities and the composition technique, to other variants of the vertex separator problem

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