Designing optical multi-band networks : polyhedral analysis and algorithms

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Abstract

A major challenge for nowadays telecommunication actors is to propose solutions to manage the traffic growth, and ensure a smart use of network resources. This can be possible by overlapping multi-band OFDM technology on an optical fibre infrastructure. A better and more flexible use of the wavelength capacity is then enabled by dividing each wavelength channel into smaller *sub-wavelengths* or *subbands*. Moreover, since it is necessary to meet user demand, OFDM multi-band networks must present enough capacity to carry the traffic evolution. One of the best ways to ensure a smart use of this new network infrastructure, is to provide an accurate answer in terms of resource planning, that is to guarantee that a sufficient number of resources are deployed so that traffic routing may be possible.

In this thesis we consider two problems related to the capacitated design of networks, using OFDM multi-band technology.

The first problem is associated with the capacitated design of single-layer networks, using some technical requirements of OFDM multi-band technology. We give an integer linear programming formulation for the problem and we study the polyhedra associated with arc-set restrictions of this problem. We describe two classes of valid inequalities and study the conditions under which they define facets for these polyhedra. We discuss the separation procedures for these inequalities and use them within an Branch-and-Cut algorithm to solve the problem.

Next, we investigate the multilayer version of capacitated network design in OFDM multi-band networks. We propose several integer linear programming formulations for the problem. The first one, namely cut formulation, is based on cut inequalities which are in exponential number. We conduct an investigation of the polyhedron associated with its feasible solutions. We identify several classes of valid inequalities and study their facial structure. We then discuss the related separation problems and devise a Branch-and-Cut algorithm to solve the problem. In particular, our approach embeds valid inequalities identified in both single-layer and multilayer contexts. Both approaches were used to solve random and realistic instances and provide results of a great interest for Orange Labs.

The second formulation for the problem is a compact formulation, which holds a polynomial number of constraints and variables. We use this formulation to perform a modeling approach based on paths, which yields two Branch-and-Price algorithms for the problem. The first algorithm deals with the routing associated with the physical and the virtual layers explicitly, while the second algorithm uses the interactions between both layers to get a unique pricing problem instead of two.

Key words : optical multi-band networks, network design, polytope, facet, Branch-and-Cut algorithm, Branch-and-Price algorithm.

Résumé

Un des enjeux majeurs pour les acteurs de l'industrie des télécommunications est de proposer des solutions afin de répondre au mieux à la croissance du trafic, et d'assurer une gestion intelligente des resources du réseau. Cela peut être possible en utilisant la technologie OFDM multi-bandes sur un réseau de fibres optiques. Cette technologie permet alors une utilisation plus flexible de la capacité offerte par les longueurs d'onde, du fait de la division de chacune de ces capacités en plusieurs entités indépendantes appelées *sous-bandes*. Par ailleurs, comme il est nécessaire de satisfaire la demande des usagers en trafic, les réseaux OFDM multi-bandes doivent présenter une capacité suffisante pour supporter l'évolution du trafic. L'un des meilleurs moyens d'assurer une utilisation astucieuse des infrastructures d'un réseau, est de fournir une réponse précise en terme de planification des resources. Il s'agit notamment de garantir qu'un nombre suffisant de resource est déployé afin que le routage du trafic soit possible.

Dans cette thèse, nous étudions deux problèmes liés à au dimensionnement de réseaux utilisant la technologie OFDM multi-bandes.

Nous nous intéressons d'abord à un problème de dimensionnement, dans le cas d'une seule couche de réseau, utilisant des contraintes techniques issues de la technologie OFDM multi-bandes. Nous donnons une formulation basée sur un programme linéaire en nombres entiers pour le problème et nous étudions le polyèdre associé à la restriction du problème sur un arc. Nous décrivons ensuite deux classes d'inégalités valides et examinons les conditions pour qu'elles définissent des facettes. Nous discutons la procédure de séparation pour ces inégalités et les intégrons dans un algorithme de coupes et branchements afin de résoudre le problème.

Nous étudions ensuite la version multi-couche du problème de dimensionnement dans les réseaux OFDM multi-bandes. Nous proposons plusieurs programmes linéaires en nombre entier pour formuler le problème. La première formulation, dite en coupes, est basée sur des contraintes de coupes, dont le nombre est exponentiel. Nous procédons à l'étude du polyèdre associé à ses solutions réalisables. Cette étude nous permet d'identifier plusieurs classes d'inégalités valides, dont nous examinons la structure faciale. Nous discutons ensuite des problèmes de séparation associés et élaborons un algorithme de coupes et branchements pour le problème. En particulier, notre approche intègre les inégalités valides issues de l'étude des versions mono-couche et multi-couche du problème.

La seconde formulation, dite compacte, possède un nombre polynomial de variables et de contraintes. Nous utilisons cette formulation afin de proposer une nouvelle approche de modélisation basée sur des chemins, qui induit deux algorithmes de génération de colonnes et branchements pour le problème. Le premier algorithme considère explicitement les niveaux de routages liés à chaque couche de réseau, tandis que le deuxième algorithme utilise implicitement les interactions entre les deux couches du réseau pour résoudre le problème.

Mots clés : réseaux optiques multi-bandes, conception de réseaux, polytope, facette, algorithme de coupes et branchements, algorithme de génération de colonnes et branchements.

Résumé long

Introduction

La demande des usagers en trafic a connu une croissance significative durant ces dernières décennies. De ce fait, les réseaux de télécommunication actuels atteignent déjà leurs limites, et il sera bientôt nécessaire d'accroître leur capacité de transport. En effet, l'avènement de nouveaux services, principalement dûs aux applications sur internet et aux contenus multimédias, nécessitent des infrastructures de réseau plus flexibles et avantageuses en terme de coûts. Afin de remédier à cette croissance explosive du trafic (estimée à 45% par an en moyenne [96]), les acteurs de l'industrie des télécommunications étudient de nouvelles technologies qui pourraient répondre au besoin d'augmenter la capacité tout en apportant la flexibilité nécessaire pour exploiter pleinement cette capacité.

Un réseau de télécommunication peut être perçu comme la supperposition de multiples couches, sur lesquelles différents services peuvent être fournis. En particulier, un réseau de fibres optiques est composé de deux couches : une couche physique et une couche virtuelle. La couche physique est constituée de fibres optiques, tandis que la couche virtuelle représente la technologie WDM (Wavelength Division Multiplexing). Un tel process est basé sur un ensemble d'équipements appelés multiplexeurs, interconnectés par des liens optiques, composés de plusieurs canaux optiques ou longueurs d'onde. Les deux couches communiquent, puisque les longueurs d'onde de la couche virtuelle utilisent les fibres optiques de la couche physique comme support pour transporter les demandes des usagers en trafic.

La technologie WDM est aujourd'hui utilisée pour transporter des informations sur de longues distances (régions métropolitaines, câbles sous-marins, etc.), avec des longueurs d'onde de 2.5, 10 ou même 40 Gbit/s, cependant, il n'est actuellement pas possible d'atteindre de telles distances avec des longueurs d'onde de plus grande capacité. En fait, l'existence de phénomènes physiques pouvant affecter les fibres optiques accentue la difficulté de mettre en place des longueurs d'onde de très grande capacité sur de longues distances (voir [28]). De récentes innovations dans le domaine des communications utilisant des fibres optiques, ont permis l'émergeance d'une technologie appelée Orthogonal Frequency Division Multiplexing (OFDM) multi-bandes. Les études concernant cette technologie ont montré que cette technologie pourrait permettre la transition des infrastructures basées sur le WDM vers de très grandes capacités (100 Gbit/s et plus pour chaque longueur d'onde), sur de longues distances. La technologie OFDM est basée sur la division de chaque canal optique en plusieurs entités indépendantes appelées sous-bandes. On parle alors de réseau optique OFDM multi-bandes.

Le but initial de ce travail était de répondre à certaines questions posées par les ingénieurs d'Orange Labs - France Telecom R&D, concernant la conception des réseaux utilisant la technologie OFDM optique. En particulier, nos résultats devraient permettre d'évaluer certains indicateurs de performance de la technologie OFDM et donner des outils d'aide à la décision pour le déploiement de cette technologie.

Les méthodes d'optimisation combinatoire, en particulier l'approche dite polyèdrale ont montré leur efficacité pour traiter des problèmes difficiles et ayant une combinatoire importante. Initiée par Edmonds dans le cadre du problème du couplage [44], cette technique consiste à réduire la résolution d'un problème d'optimisation combinatoire à celle d'un où plusieurs programmes linéaires. Il s'agit notamment de donner, une description complète (ou partielle) du polytope des solutions du problème considéré avec un système d'inégalités linéaires. L'approche polyèdrale a montré son efficacité sur plusieurs problèmes d'optimisation combinatoire tels que le Problème du Voyageur de Commerce, le Problème de Conception de Réseau, ainsi que le Problème de la Coupe Maximum.

Un aspect critique de l'émergeance des infrastructures multi-couches et multi-technologies est le déploiement et l'exploitation efficaces des ressources du réseau. Bien que les problèmes de conception de réseau sous-jacents aient été largement étudiés pour les réseaux composés d'une seule couche, ils constituent toujours des questions intéressantes dans le cadre des réseaux multi-couches. Ainsi, les problèmes de conception de réseau consistent en général à identifier le nombre de capacités modulaires à installer sur les liens du réseau afin de satisfaire une certaine demande en trafic. Dans le contexte des réseaux multi-couches, il faut considérer la relation entre les différentes couches, en plus des contraintes classiques du problème.

Dans cette thèse, nous considérons un problème de dimensionnement, pour des réseaux de télécommunication mono-couches et multi-couches, dans un contexte polyédral. Nous donnons plusieurs modèles pour les problèmes étudiés et examinons les propriétés des polyhèdres associés. Nous mettons en évidence la relation existant entre ces problèmes et d'autres problèmes classiques d'optimisation combinatoire. Nous décrivons des algorithmes de Branch-and-Cut et Branch-and-Price élaborés pour la résolution de ces problèmes. Une étude expérimentale est présentée pour chaque problème et plusieurs séries de tests sont conduits sur des instances réalistes et réelles de grand intérêt pour Orange Labs. Les résultats obtenus montrent de manière empirique l'efficacité de notre approche sur les instances considérées.

Dans ce qui suit nous présentons succintement le contenu de chaque chapitre.

Preliminaries and State-of-the-Art

Le premier chapitre est consacré à l'introduction de quelques notions préliminaires concernant l'optimisation combinatoire, les méthodes exactes en général et l'approche polyèdrale en particulier. Nous donnons notamment un aperçu des méthodes des plans sécants et de génération de colonnes, ainsi que des algorithmes de coupes et branchements, et de génération de colonnes et branchements. Nous donnons alors quelques définitions basiques sur la théorie des graphes et introduisons la terminologie et les notations utilisées dans ce manuscrit. Enfin, nous présentons un état de l'art sur les problèmes de conception et dimensionnement de réseaux. Dans le chapitre suivant nous présentons le contexte pratique ainsi que les enjeux technologiques de ce travail.

Multilayer Optical Networks

Ce chapitre préliminaire entend donner une brève esquisse de l'évolution des réseaux de télécommunication. Nous donnons quelques notions nécessaires pour la compréhension des contraintes techniques inhérentes à la définition des problèmes étudiés dans cette thèse. En particulier, nous donnons d'abord quelques éléments concernant les réseaux de télécommunication multi-couches. Nous présentons ensuite la technologie WDM et donnons un aperçu de l'architecture des réseaux optiques utilisant cette technologie. Nous introduisons enfin les nouveaux paradigmes qui permettront aux réseaux optiques d'évoluer vers plus de fléxibilité et une utilisation plus ingénieuse des ressources disponibles. Par ailleurs, quelques éléments concernant la technologie OFDM optique multi-bandes sont introduits. Enfin, nous fixons quelques hypothèses ainsi que la terminologie adoptée dans la thèse. Nous présentons dans les chapitre suivants les modèles et les approches proposées pour apphréhender les deux problèmes de dimensionnement de réseaux considérés.

Capacitated Network Design and Set Function Polyhedra

Nous considérons d'abord le problème de dimensionnement résultant de l'étude d'une seule couche de réseau. Etant donnée une couche de réseau optique composé d'un ensemble d'équipements interconnectés par des fibres optiques. Un ensemble de capacités modulaires ou modules peut être installeé sur les liens du réseau rendant ainsi possible la circulation du trafic sur ces liens. Chaque module induit un coût d'installation, impacté sur le lien qui le reçoit. Etant donné un ensemble de demandes de trafic, il s'agit de déterminer le nombre de capacités modulaires à installer sur les liens de la couche considée de sorte que chaque demande de trafic soit routée entre son origine et sa destination, et que le coût total soit minimum. Le problème sera désigné par Dimensionnement de Réseau Mono-Couche (Capacitated Single-Layer Network Design (CSLND) problem) afin de le différentier de la version multi-couche du problème de dimensionnement de réseaux, étudié dans les chapitres 5, 6 et 7. Par ailleurs, les contraintes de ce problème sont dues aux exigences techniques de la version multi-couche.

Nous proposons d'abord un programme linéaire en nombres entiers pour modéliser le problème. Cette formulation présente beaucoup de symétries, ce qui rend difficile la résolution efficace du problème par un algorithme de Branch-and-Bound basé sur ce modèle. Nous donnons alors une formulation alternative, dite *agrégée*, permettant de briser les symétries de la première formulation, et présentant ainsi une structure plus intéressante à étudier. Nous examinons ici les polyèdres associés à des relaxations simples du problème, notamment lorsqu'on se restreint à un seul lien du réseau. Le but étant d'étudier ces polyèdres et tirer profit de leur caractérisation partielle pour résoudre efficacement le problème CSLND. En d'autres termes, nous montrons dans ce chapitre que différents sous-problèmes, résultant d'une relaxation du problème CSLND sont en fait associé à la même classe de polyèdres. Ces problèmes sont appelés *fonctions*. Nous introduisons les polyèdres associés à une famille particulière de fonctions, dîtes *unitary step monotonically increasing* (usmi), puis nous étudions leur propriétés basiques. Nous dérivons deux familles d'inégalités, appelés *Min Set I* et *Min Set II*, qui sont valides pour tous les polyèdres appartenant à la classe considérée. Nous menons par ailleurs une investigation sur la structure faciale de ces inégalités et nous décrivons des conditions nécessaires et suffisantes pour qu'elle définissent des facettes des polyèdres étudiés.

Nous montrons que nos résultats polyèdraux restent valables quelque soit la fonction considérée (appartenant á la classe de fonctions usmi). Les procédures de séparation de ces inégalités peuvent notamment être similaires, mais nécessitent toutefois la prise en compte des spécificités de chaque fonction. Nous illustrons les résultats obtenus sur une application concernant la *fonction Bin-Packing*, qui est en réalité équivalent au problème CSLND restreint sur un lien, lorsque les demandes de trafic ne sont pas divisibles. En particulier, nos résultats concernant les inégalités Min Set I généralisent ceux donnés dans [27, 101, 10] concernant les inégalités c-strong. En outre, les deux familles d'inégalités Min Set I et Min Set II sont utilisées dans le cadre d'un algorithme de coupes et branchements permettant de résoudre le problème CSLND. Les procédures de séparation pour ces deux familles de contraintes ont été intégrées dans l'algorithme de coupes et branchements que nous avons implémenté. Le chapitre 4 est dédié aux aspects algorithmiques de cette implementation. En effet, dans ce chapitre nous montrons de manière empirique l'efficacité de l'approche que nous proposons et en particulier l'apport des contraintes valides proposées pour la résolution de CSLND.

Branch-and-Cut Algorithm for the CSLND problem

Nous décrivons ici l'algorithme de coupes et branchements que nous avons proposé pour la formulation agrégée du problème CSLND. Cet algorithme est basé sur les résultats théoriques introduits dans le chapitre 4. Nous donnons d'abord un aperçu du fonctionnement de cet algorithme, puis nous détaillons les procédures de séparation utilisées pour générer les inégalités de type Min Set I et Min Set II. L'objectif de ce chapitre est de présenter la mise en oeuvre de l'approche proposée dans le chapitre précédent et de donner un aperçu de l'efficacité des contraintes Min Set I et Min Set II en pratique. En particulier, une étude expérimentale est conduite et plusieurs séries de tests sont effectuées sur des instances réalistes provenant de la librairie SNDlib [1]. Cette étude a notamment permis de comparer les performances de l'algorithme de coupes et branchements et celles d'un algorithme de Branch-and-Bound basé sur la formulation compacte initiale du problème CSLND.

Nos résultats montrent très clairement que l'algorithme de coupes et branchement est beaucoup plus efficace que l'algorithme de Branch-and-Bound basé sur la formulation initiale. Les expérimentations montrent également que les inégalités valides Min Set I et Min Set II sont très efficaces en pratique pour le problème. Bien que l'efficience des contraintes de type Min Set I soit plus visible que celle des contraintes Min Set II, nous pouvons voir que les heuristiques de séparation développées pour ces contraintes fonctionnent bien, en particulier pour des instances correspondant à des réseaux peu denses. Enfin, nous montrons également grâce à ces résultats que la difficulté des instances traitées est très liée à la taille des demandes comparée à la capacité d'un module. Par ailleurs, cette propriété est aussi présente dans la version multi-couche du problème. Dans ce qui suit, nous étudions le problème de dimensionnement de réseau multi-couche et présentons plusieurs approches de modélisation et résolution pour ce problème. Par ailleurs, nous exploitons les inégalités valides issues de l'étude du problème CSLND pour la résolution du problème multi-couche.

Optical Multi-Band Network Design : polyhedral study

Nous nous intéressons ici au problème de dimensionnement d'un réseau optique multi-couche, utilisant la technologie OFDM multi-bandes. Etant donnée une couche physique de réseau composée d'un ensemble de noeuds de transmission, liés par des fibres optiques, et des demandes de trafic définiés par une origine, une destination et une quantité. On dispose d'un ensemble de capacités modulaires, appelées *sous-bandes OFDM*, à installer entre les noeuds de transmission, de sorte que la circulation du trafic soit possible. Chaque sous-bande possède une capacité et induit un coût d'installation, qui est impacté sur la fibre optique qui la reçoit. Si une ou plusieurs sous-bandes sont installées entre deux noeuds de transmission, on dit qu'il existe un lien virtuel entre ces noeuds. L'ensemble des liens virtuels ainsi que leux noeuds extrémités définissent la couche virtuelle du réseau optique. Le problème que nous étudions peut alors être défini comme suit. Nous souhaitons déterminer le nombre de sous-bandes OFDM à installer sur les liens du réseau, de sorte que toute les demandes soient routées, que chaque sous-bande utilisée soit associée à un chemin utilisant des fibres optiques, et que le coût total soit minimum. Nous appellerons ce problème Conception de Réseau Optique Multi-Bandes (Optical Multi-Band Network Design (OMBND) problem).

Nous proposons ici une approche de modélisation basée sur des coupes, et donnons une formulation en programme linéaire en nombres entiers ayant un nombre expoentiel de contraintes. Nous montrons d'abord que cette formulation est équivalente au problème OMBND. Nous examinons ensuite le polyèdre assocé à formulation en coupes ainsi que la structure faciale des contraintes de base. Nous dérivons alors d'autres familles d'inégalités valides, et décrivons les conditions nécéssaires aussi bien que les conditions suffisantes pour qu'elles définissent des facettes non triviales du polyèdre. Toutes les contraintes valides identifiées dans ce chapitre, ainsi que celles issues de l'étude du problème CSLND sont intégrées dans un algorithme de coupes et branchements, qui sera présenté dans le chapitre 6. En effet, nous discutons dans ce chapitre l'aspect algorithmique de l'étude présentée dans le chapitre 5. Par ailleurs, une étude expérimentale est également proposée dans ce chapitre, permettant d'avoir un aperçu sur l'efficacité, en pratique, des contraintes valides introduites.

Branch-and-Cut Algorithm for OMBND problem

Nous décrivons dans ce chapitre le cadre, notamment informatique, de notre algorithme. Ce chapitre est basé sur les résultats issus de l'investigation polyèdrale menée dans le chapitre précédent ainsi que celle présentée dans le Chapitre 3. En effet, l'ensemble des contraintes valides identifiée pour les deux problèmes CSLND et OMBND sont intégrées dans l'algorithme. Nous présentons d'abord les procédures de séparation que nous proposons afin de générer chaque famille d'inégalités valides. Nous donnons ensuite les détails de la mise en oeuvre et présentons les instances de réseaux considérées dans notre étude éxpérimentale. Enfin, des résultats expérimentaux sont donnés pour des instances réalistes issues de la librairie SNDlib, ainsi que pour des instances réelles fournies par Orange Labs.

Nos résultats montrent le gain apporté par les inégalités valides proposées comparé à la formulation de base. En particulier, les inégalités de type Min Set I, capacitated cutset inequalities et flow-cutset inequalities ont permis de réduire sensiblement le saut d'intégrité, et ainsi d'améliorer la qualité de la relaxation linéaire de la formulation en coupes. Les autres classes d'inégalités valides ont permis une augmentation moins significative des performances de l'algorithme. Cependant, nous pensons que des procédures de séparation plus sophistiquées permettraient d'obtenir le meilleur parti de ces inégalités sans que cela ne coûte trop cher en temps de calcul. Parallèlement, nos inégalités valides ont été utilisées au sein d'un second algorithme de coupes et branchements, basé sur une formulation compacte du problème, présentée au Chapitre 7. En effet, le recours à cette approche alternative permet de réduire le temps total dédié à la séparation des contraintes, puisque la formulation compacte possède un nombre polynomial de contraintes. Cette approche a permis de traiter de plus grandes instances, notamment les instances réelles d'Orange Labs, et ainsi d'obtenir de bonnes solutions pour ces instances en quelques heures de calcul. Dans ce qui suit, nous introduisons d'autres approches de modélisation du problème OMBND, basées sur des chemins. Nous développons deux procédures de génération de colonnes pour ces modèles, et les intégrons dans le cadre d'algorithmes de génération de colonnes et branchements.

Optical Multi-Band Network Design using paths

Dans ce chapitre, nous proposons une approche de résolution basée sur la génération de colonnes pour traiter le problème OMBND. Nous donnons d'abord une formulation compacte pour le problème. Nous avons proposé deux formulations utilisant des variables chemin pour le problème. La première formulation considère une approche de décomposition explicite et induit une procédure de génération de colonnes utilisant deux problèmes de pricing. Le second modèle, en l'occurrence une formulation de chemin agrégée, donne, elle, une décomposition implicite du problème. En effet, dans cette formulation, la couche virtuelle possède des informations sur la couche physique. Cette imbrication est possible grâce une nouvelle famille de variables avec une structure spécifique. Nous discutons les problèmes de pricing pour les deux formulations chemin, et nous montrons qu'ils se réduisent à un problème de plus court chemin. Nous proposons un algorithme de génération de colonnes et branchements pour résoudre chacune des formulations chemin, et comparons les deux approches à l'algorithme de Branch-and-Bound basé sur la formulation compacte. Quelques résultats numériques sont donnés pour illustrer l'efficacité de ces deux algorithmes.

Nos expérimentations montrent que l'approche basée sur la génération de colonnes est bien plus efficace que l'algorithme de Branch-and-Bound basé sur la formulation compacte. Par ailleurs, l'algorithme issu de la formulation chemin initiale donne généralement de meilleurs résultats que celui issu de la formulation agrégée, sur les instances testées. En effet, bien que ce dernier explore moins de noeuds dans l'arbre de branchements, il passe un temps non negligeable à générer des variables (pricing), en particulier au noeud racine. Cependant, à partir d'une certaine taille d'instance, les deux algorithmes éprouvent des difficultés á identifier une solution optimale pour le problème. Aussi, plusieurs perspectives intéressantes pourraient être consiérées afin d'améliorer les performances des deux algorithmes présentés dans ce chapitre. En fait, nous gagnerions, d'une part, à développer des stratégies de branchement plus élaborées afin de mieux gérer la taille de l'arbre de branchements concernant la formulation chemin initiale. D'autre part, un examen plus approfondi du problème de pricing pour la seconde formulation chemin (formulation agrégée) permettrait de mieux contrôler le processus de génération de colonnes et ainsi offrir un compromis entre le temps passé à "pricer" et celui dédié à l'exploration des noeuds de l'arbre de branchements.

Conclusion

Les résultats présentés dans cette thèse, notamment concernant le problème OMBND, peuvent servir à apporter des éléments de réponse pour le dimensionnement des réseaux utilisant la technologie OFDM. Plus généralement, les algorithmes proposés constituent des solutions génériques pour des problèmes de conception et dimensionnement de réseaux optiques qui se posent en pratique. Ces méthodes peuvent également être utilisées comme "outil référence" permettant d'évaluer la qualité d'une solution approchée obtenue à l'aide d'heuristiques ou mèta-heuristiques. Par ailleurs, nos résultats théoriques, en particulier concernant les *set functions polyhedra* peuvent être utilisés dans un cadre beaucoup plus général que le network design, en l'occurrence pour d'autres problèmes difficiles d'optimisation combinatoire.

Il y a plusieurs directions pertinentes dans lesquelles ce travail peut être poursuivi. En effet, en ce qui concerne la recherche d'inégalités valides pour le problème CSLND, nous considérons pour le moment une relaxation du problème sur un unique lien. Une extension naturelle de cette étude serait de considérer le polyèdre associé à la restriction du problème sur une coupe. En particulier, nous souhaitons comprendre comment les inégalités Min Set I et Min Set II se génélisent dans le contexte d'une coupe. Nous pensons que ces inégalités généralisées peuvent être utiles dans le cadre d'un algorithme de coupes et branchements. En ce qui concerne le problème OMBND, la plupart des efforts à faire doivent être investis dans l'amélioration des procédures de séparation pour une détection plus efficace des inégalités valides, qui soit également moins coûteuse en temps CPU. Par ailleurs, il serait également intéressant de proposer des heuristiques primales afin d'identifier plus facilement de bonnes solutions réalisables pour les algorithmes de coupes et branchements, aussi bien que les algorithmes de génération de colonnes et branchements. Enfin, nous souhaiterions également étudier d'autres versions du problème de dimensionnement de réseaux multi-couches, telle que la version robuste (avec incertitude sur les demandes, etc.). En effet, bien que la prise en compte des incertitudes sur les demandes de trafic ait déjà été bien étudiée pour les réseaux à une seule couche, à notre connaissance, il n'existe pas de travaux considérant la version robuste du problème de dimensionnement pour deux ou plusieurs couches de réseaux.

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Introduction

User demand in traffic has increased significantly during the last decades. Nowadays telecommunication networks are already reaching their limits, and it is necessary to upgrade their transport capacity. Indeed, the arising of new services, mainly driven by internet applications and multimedia contents, requires more flexible and cost-effective network infrastructures. To overcome this explosive growth of traffic (estimated at 45 % per year in average [96]), telecommunication industry actors investigate new technologies that provide a solution to the increasing capacity requirements, as well as the flexibility needed to use smartly this capacity.

Telecommunication networks can be seen as an overlapping of multiple layers, upon which different services may be furnished. In particular, optical fibers networks consists of two layers : a physical layer and a virtual layer. The physical layer is based on optical fibers, while the virtual layer supports the WDM (Wavelength Division Multiplexing) technology. Such a process is based on a set of devices referred to as multiplexers, interconnected by optical links, made of several wavelengths. Both layers are connected, as the wavelengths of the virtual layer use the optical fibers of the physical layer as a support to carry the customers traffic.

Although WDM technology is currently used to transport informations over long distances (metropolitan areas, submarine communications cables), with wavelength capacities of 2.5, 10 or 40Gb/s, it is not possible to reach similar distances with higher capacities. In fact, the existence of physical phenomena also called transmission impairments [28] that affect the optical fibers, highlights the difficulty of setting up higher capacitated wavelengths on long distances. Recent innovations in optical fibers comunications concerning a new technology called Multi-band Orthogonal Frequency Division Multiplexing (OFDM) have shown very promising results, and should enable the transition of WDM-based infrastructures to high capacitated wavelengths (100 Gb/s and more) over long distances. OFDM is based on the division of each available wavelength into many subwavelengths, also called subbands, this is known as Optical Multi-band OFDM network.

The initial purpose of this work was to answer some questions concerning the design of OFDM networks, suggested by Orange Labs - France Telecom R& D engineers. In particular, our results should enable to evaluate some performance indicators of the OFDM technology, and provide decision making tools for the deployment of this technology.

The combinatorial optimization tools, in particular the so called polyhedral method, have proved their efficiency to tackle hard combinatorial problems. Initiated by Edmonds in the context of the matching problem [44], this technique consists in reducing the resolution of a combinatorial problem to that of one or more linear programs. This is based, in particular, on giving a complete (or a partial) description of the polytope of solutions with a system of linear inequalities. The polyhedral approach has been proved to be very efficient when applied to many combinatorial optimization problems such as the Traveling Salesman Problem, the Network Design Problem and the Max-Cut Problem.

A critical aspect of emerging multilayer and multi-technology infrastructures is the efficient resources deployment and utilization. Despite the fact that underlying network design problems have been widely studied for single-layer networks, they still constitute very interesting issues in the context of multilayer networks. Thereby, network design problems consists in general to identify the number of modular capacities to install over the links in order to meet the traffic demand. In the context of multilayer networks, one has to consider the relationship between both layers, in addition to the classical constraints.

In this thesis, we study a capacitated network design problem for both single-layer and multilayer telecommunication networks, within a polyhedral context. We give several models for the considered problems and investigate the properties of the associated polyhedra. We highlight the relationship between these problems and other well-know combinatorial optimization problems. We devise Branch-and-Cut and Branch-and-Price algorithms for their resolution. We conduce several series of experiments on random, realistic and real networks, of great interest for Orange Labs. The obtained results show empirically the efficiency of our approaches.

This dissertation is organized as follows. In Chapter 1, we present basic notions of combinatorial optimization. This chapter also includes a state-of-the-art on communication network design problems. Chapter 2 introduces the practical context of the problems treated in the thesis. In this chapter some generalities on multilayer communication networks are given and emphasis is put on optical networks coordinating WDM and OFDM multi-band technologies. Chapters 3 and 4 concern the first considered capacitated network design problem, that is Capacitated Single-Layer Network Design (CSLND) problem. Chapter 5 discuss the multilayer version of the first problem, namely Optical Multi-Band Network Design (OMBND) problem, and study the associated polyhedron. Chapters 6 and 7 are dedicated to the algorithmic aspects related to two exact algorithms we developed to solve OMBND problem.

Chapter 1

Preliminaries and State-of-the-Art

Contents

This chapter is dedicated to the presentation of some preliminary notions concerning combinatorial optimization, exact approaches and polyhedra. In particular, we give an overview of cutting planes and column generation methods as well as Branch-and-Cut and Branch-and-Price algorithms. We then give some basic definitions in graph theory and introduce some notations and terminology that will be used throughout the dissertation. Finally, we give a state-of-the-art on the capacitated network design problem.

1.1 Combinatorial optimization

Combinatorial Optimization is a branch of operations research related to computer science and applied mathematics. Its purpose is the study of optimization problems where the set of feasible solutions is discrete or can be represented as a discrete one. Typically, the problems concerned with combinatorial optimization are those formulated as follows. Let $E = \{e_1, \ldots, e_n\}$ be a finite set called *basic set* where each element e_i is associated with a weight $c(e_i)$. Let $\mathcal{F}$ be a family of subsets of E. If $F \in \mathcal{F}$, then $c(F) = \sum_{e_i \in F} c(e_i)$ denotes the weight of F. The problem consists in identifying an element F^* of $\mathcal{F}$ whose weight is minimum or maximum. In other words,

$$\min(\mathit{or}\max)\{c(F) : F \in \mathcal{F}\}.$$

Such a problem is called *combinatorial optimization problem*. The set $\mathcal{F}$ represents the set of feasible solutions of the problem.

The term *combinatorial* refers to the discrete structure of $\mathcal{F}$. In general, this structure is represented by a graph. The term *optimization* signifies that we are looking for the best element in the set of feasible solutions. This set generally contains an exponential number of solutions, therefore, one can not expect to solve a combinatorial optimization problem by exhaustively enumerate all its solutions. Such a problem is then considered as a hard problem.

Various effective approaches have been developed to tackle combinatorial optimization problems. Some of these approaches are based on graph theory, while others use linear and non-linear programming, integer programming and polyhedral approach. Besides, several practical problems arising in real life, can be formulated as combinatorial optimization problems. Their applications are in fields as diverse as telecommunications, transport, industrial production planing or staffing and scheduling in airline companies. Over the years, the discipline got thus enriched by the structural results related to these problems. And, conversely, the progress made in computed science have made combinatorial optimization approaches even more efficient on real-world problems.

In fact, combinatorial optimization is closely related to algorithm theory and computational complexity theory as well. The next section introduces computational issues of combinatorial optimization.

1.2 Computational complexity

Computational complexity theory is a branch of theoretical computer science and mathematics, whose study started with works of Cook [35], Edmonds [43] and Karp [70]. Its objective is to give a classify a given problem depending on its difficulty. A plentiful literature can be find on this topic, see for example [51] for a detailed presentation of NP-completeness theory.

A *problem* is a question having some input parameters, and to which we aim to find an answer. A problem is defined by giving a general description of its parameters, and by listing the properties that must be satisfied by a solution. An *instance* of the problem is obtained by giving a specific value to all its input parameters. An *algorithm* is a sequence of elementary operations that allows to solve the problem for a given instance. The number of input parameters necessary to describe an instance of a problem is the *size* of that problem.

An algorithm is said to be polynomial if the number of elementary operations necessary to solve an instance of size n is bounded by a polynomial function in n. We define the class P as the class gathering all the problems for which there exists some polynomial algorithm for each problem instance. A problem that belongs to the class P is said to be "easy" or "tractable".

A *decision problem* is a problem with a *yes* or *no* answer. Let $\mathcal{P}$ be a decision problem and $\mathcal{I}$ the set of instances such that their answer is yes. $\mathcal{P}$ belongs to the class *class NP* (Nondeterministic Polynomial) if there exists a polynomial algorithm allowing to check if the answer is yes for all the instances of $\mathcal{I}$. It is clear that a problem belonging to the class P is also in the class NP. Although the difference between P and NP has not been shown, it is a highly probable conjecture.

In the class NP, we distinguish some problems that may be harder to solve than others. This particular set of problems is called *NP-complete*. To determine whether a problem is NP-complete, we need the notion of *polynomial reducibility*. A decision problem P_1 can be polynomially reduced (or transformed) into an other decision problem P_2, if there exists a polynomial function f such that for every instance I of P_1, the answer is "yes" if and only if the answer of $f(I)$ for P_2 is "yes". A problem $\mathcal{P}$ in NP is also NP-complete if every other problem in NP can be reduced into $\mathcal{P}$ in polynomial time. The Satisfiability Problem (SAT) is the first problem that was shown to be NP-complete (see [35]).

With every combinatorial optimization problem is associated a decision problem.

Furthermore, each optimization problem whose decision problem is NP-complete is said to be *NP-hard*. Note that most of combinatorial optimization problems are NP-hard. One of the most efficient approaches developed to solve those problems is the so-called polyhedral approach.

1.3 Polyhedral approach and Branch-and-Cut

1.3.1 Elements of polyhedral theory

The polyhedral method was initiated by Edmonds in 1965 [44] for a matching problem. It consists in describing the convex hull of problem solutions by a system of linear inequalities. The problem reduces then to the resolution of a linear program. In most of the cases, it is not straightforward to obtain a complete characterization of the convex hull of the solutions for a combinatorial optimization problem. However, having a system of linear inequalities that partially describes the solutions polyhedron may often lead to solve the problem in polynomial time. This approach has been successfully applied to several combinatorial optimization problems. In this section, we present the basic notions of polyhedral theory. The reader is referred to works of Schrijver [99] and [79].

We shall first recall some definitions and properties related to polyhedral theory.

Let n be a positive integer and $x \in \mathbb{R}^n$. e say that x is a *linear combination* of x_1, x_2, ..., $x_m \in \mathbb{R}^n$ if there exist m scalar λ_1, λ_2, ..., λ_m such that $x = \sum_{i \in 1}^{m} \lambda_i x_i$. If $\sum_{i=1}^{m} \lambda_i = 1$, then x is said to be a *affine combination* of x_1, x_2, ..., x_m. Moreover, if $\lambda_i \geq 0$, for all $i \in \{1, \ldots, m\}$, we say that x is a *convex combination* of x_1, x_2, ..., x_m.

Given a set $S = \{x_1, \ldots, x_m\} \in \mathbb{R}^{n \times m}$, the *convex hull* of S is the set of points $x \in \mathbb{R}^n$ which are convex combination of x_1, ..., x_m (see Figure 1.1), that is

$$conv(S) = \{x \in \mathbb{R}^n | x \text{ is a convex combination of } x_1, \ldots, x_m\}.$$

The points x_1, ..., $x_m \in \mathbb{R}^n$ are *linearly independents* if the unique solution of the system $\sum_{i=1}^{m} \lambda_i x_i = 0$ is $\lambda_i = 0$, for all $i \in \{1, \ldots, m\}$. They are *affinely independent* if the unique solution of the system

$$\sum_{i=1}^{m} \lambda_i x_i = 0, \sum_{i=1}^{m} \lambda_i = 1,$$

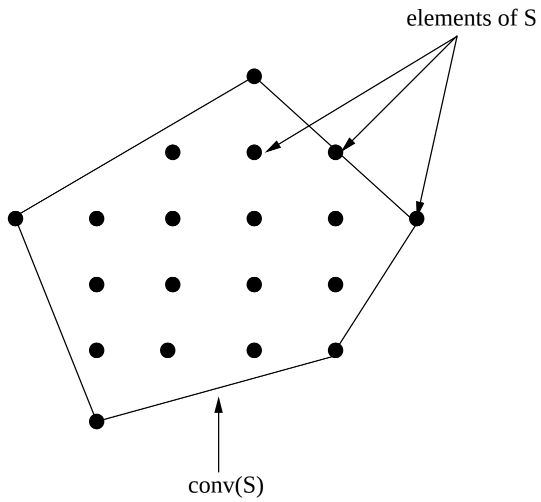


Figure 1.1: A convex hull

is $\lambda_i = 0$, $i = 1, \ldots, m$.

A *polyhedron* P is the set of solutions of a linear system $Ax \leq b$, that is $P = \{x \in \mathbb{R}^n | Ax \leq b\}$, where A is a m-row n-columns matrix and $b \in \mathbb{R}^m$. A *polytope* is a bounded polyhedron. A point x of P will be also called a *solution* of P.

A polyhedron P is said to be of *dimension* p if it has at most $p+1$ affinely independent solutions. We denote it by $dim(P) = p$. We also have that $dim(P) = n$ - $rank(A^=)$, where $A^=$ is the submatrix of A of *inequalities* that are satisfied with equality by all tje solutions of P (implicit equalities). The polyhedron P is full dimensional if $dim(P) = n$.

An inequality $ax \leq \alpha$ is *valid* for a polyhedron $P \subseteq \mathbb{R}^n$ if for every solution $\overline{x} \in P$, $a\overline{x} \leq \alpha$. This inequality is said to be *tight* for a solution $\overline{x} \in P$ if $a\overline{x} = \alpha$. The inequality $ax \leq \alpha$ is *violated* by $\overline{x} \in P$ if $a\overline{x} > \alpha$. Let $ax \leq \alpha$ be a valid inequality for the polyhedron P. $F = \{x \in P | ax = \alpha\}$ is called a *face* of P. We also say that F is a *face induced by* $ax \leq \alpha$. If $F \neq \emptyset$ and $F \neq P$, we say that F is a *proper* face of P. If F is a proper face and $dim(F) = dim(P) - 1$, then F is called a *facet* of P. We also say that $ax \leq \alpha$ induces a facet of P or is a *facet defining* inequality.

If P is full dimensional, then $ax \leq \alpha$ is a facet of P if and only if F is a proper face and there exists a facet of P induced by $bx \leq \beta$ and a scalar $\rho \neq 0$ such that $F \subseteq \{x \in P | bx = \beta\}$ and $b = \rho a$.

If P is not full dimensional, then $ax \leq \alpha$ is a facet of P if and only if F is a proper face and there exists a facet of P induced by $bx \leq \beta$, a scalar $\rho \neq 0$ and $\lambda \in \mathbb{R}^{q \times n}$ (where q is the number of lines of matrix $A^=$) such that $F \subseteq \{x \in P | bx = \beta\}$ and $b = \rho a + \lambda A^=$.

An inequality $ax \leq \alpha$ is *essential* if it defines a facet of P. It is *redundant* if the system $A'x \leq b'\}$ obtained by removing this inequality from $Ax \leq b$ defines the same polyhedron P. This is the case when $ax \leq \alpha$ can be written as a linear combination of inequalities of the system $A'x \leq b'$. A *complete minimal linear description* of a polyhedron consists of the system given by its facet defining inequalities and its implicit equalities.

A solution is an *extreme point* of a polyhedron P if and only if it cannot be written as the convex combination of two different solutions of P. It is equivalent to say that x induces a face of dimension 0. The polyhedron P can also be described by its extreme points. In fact, every solution of P can be written as a convex combination of some extreme points of P.

Figure 1.2 illustrates the main definitions given is this section.

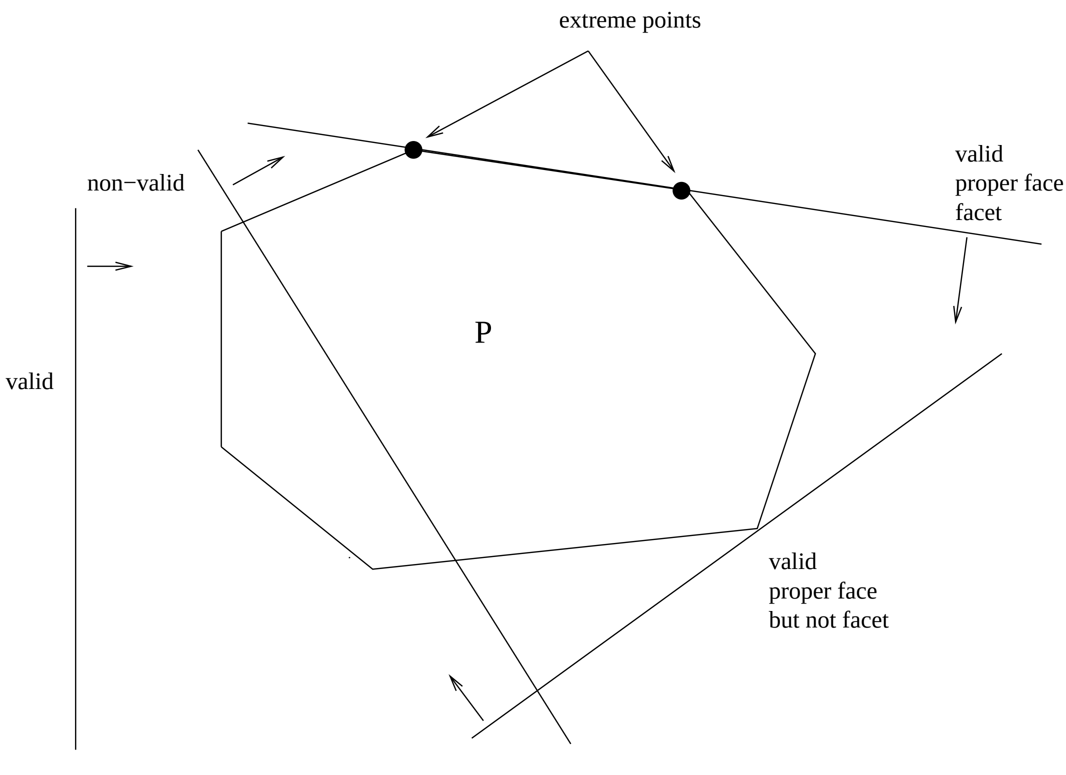


Figure 1.2: Valid inequality, facet and extreme points

1.3.2 Cutting plane method

Now let $\mathcal{P}$ be a combinatorial optimization problem, E its basic set, $c(.)$ the weight function associated with the variables of $\mathcal{P}$ and $\mathcal{S}$ the set of feasible solutions. Suppose that $\mathcal{P}$ consists in finding an element of $\mathcal{S}$ whose weight is maximum. If $F \subseteq E$, then the 0-1 vector $x^F \in \mathbb{R}^E$ such that $x^F(e) = 1$ if $e \in F$ and $x^F(e) = 0$ otherwise, is called the *incidence vector* of F. The polyhedron $P(\mathcal{S}) = conv\{x^S | S \in \mathcal{S}\}$ is the *polyhedron of the solutions* of $\mathcal{P}$ or *polyhedron associated with* $\mathcal{P}$. $\mathcal{P}$ is thus equivalent to the linear program $max\{cx | x \in P(\mathcal{S})\}$. Notice that the polyhedron $P(\mathcal{S})$ can be described by a set of a facet defining inequalities. And when all the inequalities of this set are known, then solving $\mathcal{P}$ is equivalent to solve a linear program.

Recall that the objective of the polyhedral approach for combinatorial optimization problems is to reduce the resolution of $\mathcal{P}$ to that of a linear program. This reduction induces a deep investigation of the polyhedron associated with $\mathcal{P}$. It is generally not easy to characterize the polyhedron of a combinatorial optimization problem by a system of linear inequalities. In particular, when the problem is NP-hard there is a very little hope to find such a characterization. Moreover, the number of inequalities describing this polyhedron is, most of the time, exponential. Therefore, even if we know the complete description of that polyhedron, its resolution remains in practice a hard task because of the large number of inequalities.

Fortunately, a technique called the *cutting plane method* can be used to overcome this difficulty. This method is described in what follows.

The cutting plane method is based on the so-called *separation problem*. This consists, given a polyhedron P of $\mathbb{R}^n$ and a point $x^* \in \mathbb{R}^n$, in verifying whether if x^* belongs to P, and if this is not the case, to identify an inequality $a^T x \leq b$, valid for P and violated by x^*. In the later case, we say that the hyperplane $a^T x = b$ separates P and x^* (see Figure).

Grötschel, Lovász and Schrijver [57] have established the close relationship between separation and optimization. In fact, they prove that optimizing a problem over a polyhedron P can be performed in polynomial time if and only if the separation problem associated with P can be solved in polynomial time. This equivalence has permitted an important development of the polyhedral methods in general and the cutting plane method in particular. More precisely, the *cutting plane* method consists in solving successive linear programs, with possibly a large number of inequalities, by using the following steps. Let $LP = \max\{cx, Ax \leq b\}$ be a linear program and LP' a linear program obtained by considering a small number of inequalities among $Ax \leq b$. Let

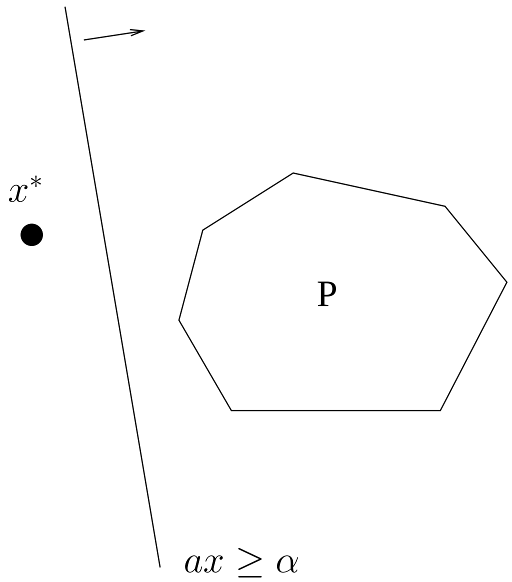


Figure 1.3: A hyperplan separating x^* and P

x^* be the optimal solution of the latter system. We solve the separation problem associated with $Ax \leq b$ and x^*. This phase is called the *separation phase*. If every inequality of $Ax \leq b$ is satisfied by x^*, then x^* is also optimal for LP. If not, let $ax \leq \alpha$ be an inequality violated by x^*. Then we add $ax \leq \alpha$ to LP' and repeat this process until an optimal solution is found. Algorithm 1 summarizes the different cutting plane steps.

Algorithm 1: A cutting plane algorithm

Data: A linear program LP and its system of inequalities $Ax \leq b$
Result: Optimal solution x^* of LP
Consider a linear program LP' with a small number of inequalities of LP;
Solve LP' and let x^* be an optimal solution;
Solve the separation problem associated with $Ax \leq b$ and x^*;
if *an inequality* $ax \leq \alpha$ *of* LP *is violated by* x^* **then**
  Add $ax \leq \alpha$ to LP';
  Repeat step 2 ;
end
else
  x^* is optimal for LP;
  return x^*;
end

Note that at the end, a cutting-plane algorithm may not succeed in providing an optimal solution for the underlying combinatorial optimization problem. In this case a *Branch-and-Bound algorithm* can be used to achieve the resolution of the problem, yielding to the so-called *Branch-and-Cut algorithm*.

1.3.3 Branch-and-Cut algorithm

Consider again a combinatorial optimization problem $\mathcal{P}$ and suppose that $\mathcal{P}$ is equivalent to $\max\{cx|Ax \leq b, x \in \{0,1\}^n\}$, where $Ax \leq b$ has a large number of inequalities. A Branch-and-Cut algorithm starts by creating a Branch-and-Bound tree whose root node corresponds to a linear program $LP_0 = max\{cx|A_0x \leq b_0, x \in \mathbb{R}^n\}$, where $A_0x \leq b_0$ is a subsystem of $Ax \leq b$ having a small number of inequalities. Then we solve the linear relaxation of $\mathcal{P}$ that is $LP = \{cx|Ax \leq b, x \in \mathbb{R}^n\}$ using a cutting plane algorithm whose starting from LP_0. Let x_0^* denote its optimal solution and $A_0'x \leq b_0'$ the set of inequalities added to LP_0 at the end of the cutting plane phase. If x_0^* is integral, then it is optimal. If x_0^* is fractional, then we perform a *branching phase*. This step consists in choosing a variable, say x^1, with a fractional value and adding two nodes P_1 and P_2 in the Branch-and-Cut tree. The node P_1 corresponds to the linear program $LP_1 = \max\{cx|A_0x \leq b_0, A_0'x \leq b_0', x^1 = 0, x \in \mathbb{R}^n\}$ and $LP_2 = \max\{cx|A_0x \leq b_0, A_0'x \leq b_0', x^1 = 1, x \in \mathbb{R}^n\}$. We then solve the linear program $\overline{LP}_1 = \max\{cx|Ax \leq b, x^1 = 0, x \in \mathbb{R}^n\}$ (resp., $\overline{LP}_2 = \max\{cx|Ax \leq b, x^1 = 1, x \in \mathbb{R}^n\}$) by a cutting plane method, starting from LP_1 (resp. LP_2). If the optimal solution of $\overline{LP}_1$ (resp. $\overline{LP}_2$) is integral then, it is feasible for $\mathcal{P}$. Its value is then a lower bound of the optimal solution of $\mathcal{P}$, and the node P_1 (resp. P_2) becomes a leaf of the Branch-and-Cut tree. If the solution is fractional, then we select a variable with a fractional value and add two children to the node P_1 (resp. P_2), and so on.

Note that sequentially adding constraints of type $x^i = 0$ and $x^i = 1$, where x^i is a fractional variable, may lead to an infeasible linear program at a given node of the Branch-and-Cut tree. Or, if it is feasible, its optimal solution may be worse than the best known lower bound of the problem. In both cases, that node is pruned from the Branch-and-Cut tree. The algorithm ends when all nodes have been explored and the optimal solution of $\mathcal{P}$ is the best feasible solution given by the Branch-and-Bound tree.

This algorithm can be improved by computing a good lower bound of the optimal solution of the problem before it starts. This lower bound can be used by the algorithm to prune the node which will not allow an improvement of this lower bound. This would permit to reduce the number of nodes generated in the Branch-and-Cut tree, and hence, reduce the time used by the algorithm. Furthermore, this lower bound may be improved by comparing at each node of the Branch-and-Cut tree a feasible solution when the solution obtained at the root node is fractional. Such a procedure is referred to as a *primal heuristic*. It aims to produce a feasible solution for $\mathcal{P}$ from the solution obtained at a given node of the Branch-and-Cut tree, when this later solution is fractional (and hence infeasible for $\mathcal{P}$). Moreover, the weight of this solution must be as best as possible. When the solution computed is better than the best known lower bound, it may significantly reduce the number of generated nodes, as well as the CPU time. Moreover, this guarantees to have an approximation of the optimal solution of $\mathcal{P}$ before visiting all the nodes of Branch-and-Cut tree, for example when a CPU time limit has been reached.

The Branch-and-Cut approach has shown a great efficiency to solve various problems of combinatorial optimization that are considered difficult to solve, such as the Travelling Salesman Problem [7]. Note a good knowledge of the polyhedron associated with the problem, together with efficient separation algorithms (exacts as well as heuristics), might help to improve the effectiveness of this approach. Besides, the cutting plane method is efficient when the number of variables is polynomial. However, when the number of variables is large (for example exponential), further methods, as column generation are more likely to be used. In what follows, we briefly introduce the outline of this method.

1.4 Column generation and Branch-and-Price

Compact formulations of combinatorial optimization problems often provide a weak linear relaxation. Those problems require then further formulations, whose linear relaxation is closer to the convex hull of feasible solutions. Those reformulations may have a huge number of variables, so that one can not consider them explicitly in the model. we describe a method that suits well to such reformulation, that is the so-called *column generation method*.

1.4.1 Column generation procedure

The column generation method is used to solve linear programs with a huge number of variables only by considering a few number among these variables. This method was pioneered by Dantzig and Wolfe in 1960 [37] in order to solve problems that could not be handled efficiently because of their size (CPU time and memory consumption).

Column generation is generally used either for problems whose structure is suitable for a *Dantzig-Wolfe decomposition*, or for problems with a large number of variables. Gilmore and Gomory [52, 53] used this method to solve a *cutting stock problem* belonging to the later class.

The overall idea of column generation is to solve a sequence of linear programs with a restricted number of variables (also referred to as columns). The algorithm starts by solving a linear program having a small number of variables, and such that a feasible solution for the original problem may be identified using this basis. At each iteration of the algorithm, we solve the so-called *pricing problem* whose objective is to identify the variables which must enter the current basis. These variables are characterized by a negative reduced cost. The reduced cost associated with a variable is computed using the dual variables associated with the constraints of the problem. We then solve the linear program obtained by adding the generated variables, and repeat the procedure until no variable with reduced cost can be identified by the pricing problem. In this case, the solution obtained from the last restricted program is optimal for the original model. The main step of column generation procedure is summarized in Algorithm 2.

Algorithm 2: A column generation algorithm

Data : A linear program MP (Master Problem) with a huge number of variables
Output : optimal solution x^* of MP

1: Consider a linear program RMP (Restricted Master Problem) including only a small subset of variables of the MP;
2: Solve RMP and let x^* be an optimal solution;
3: Solve the pricing problem associated with the dual variables obtained by the resolution of the RMP;
4: **If** there exists a variable x with a negative reduced cost then;
5: add x to RMP.
6: go to 2.
7: **else**
8: x^* is optimal for MP.
9: return x^*.

The column generation method can be seen as the dual of the cutting plane method since it adds columns (variables) instead of rows (inequalities) in the linear program. Furthermore, the pricing problem may be NP-hard. One can then use heuristic procedures to solve it. For more details on column generation algorithms, the reader is referred to [103, 40, 75].

1.4.2 Branch-and-Price algorithm

The solution obtained by a column generation procedure may not be integer. Therefore, to solve an integer programming problem, the column generation method has to be integrated within a Branch-and-Bound framework. This is known a *Branch-and-Price algorithm*. Branch-and-Price is similar to Branch-and-Cut approach, except that procedure focuses on column generation rather than row generation. In fact, generating variables (pricing) and adding inequalities (cutting plane) are complementary operations to strengthen the linear relaxation of a integer programming formulation.

The Branch-and-Price procedure works as follows. Each node of the Branch-and-Bound tree is solved by column generation, so that variables may be added to improve the linear relaxation of the current LP. The branching phase occurs when no columns price out to enter the basis and the solution of the linear program is not integer.

Branch-and-Price approaches have been widely used in the literature to solve large scale integer programming problems. The applications are in various fields, and even real life problems such as Cutting stock problem [6], Generalized Assignment Problem (GAP) [98], Airline Crew Scheduling [15], Multi-commodity Flow Problems [16], etc.

Note that, at each node of the Branch-and-Price tree, column generation may be combined with cutting plane approach, to tighten the LP relaxation of the problem. In this case, the algorithm is called *Branch-and-Cut-and-Price algorithm*. Such a method can be difficult to handle, since adding valid inequalities to the initial model may change the structure and complexity of the pricing problem. However, some successful applications of this algorithm can be found in the literature (see [95], [16] for instance).

1.5 Graph theory

In this section we will introduce some basic definitions and notations of graph theory that will be used throughout the chapters of this dissertation. For more details, we refer the reader to [99].

A graph is denoted $G = (V, E)$ where V is the set of vertices or nodes and E is the set of edges. If $e \in E$ is an edge with end initial end node u and terminal end node v, we may also use both notations uv or (u, v) to denote e. Given two node subsets T and T' of V, we denote by $[T, T']$ the set of edges such that their origins are in T and their destinations are in T'. We let $\overline{T}$ denote the subset $V \setminus T$. If $T' = \overline{T}$, then $[T, T']$

is called a cut, and will be denoted by $\delta(T)$. Similarly, we denote by $\delta(T)$ the set of edges having their origins in $\overline{T}$ and destinations in T.

The graphs considered here are directed, finite, loopless and may include multiple arcs.

A *directed graph* or *digraph* is denoted $G = (V, A)$ where V is the set of vertices or nodes and A is the set of arcs. If $a \in A$ is an arc with origin node u and destination node v, we may also use both notations uv or (u, v) to denote a. The graph G is said to be *complete* if there exists an arc between each pair of nodes (u, v). Given two node subsets T and T' of V, we denote by $[T, T']$ the set of arcs such that their origins are in T and their destinations are in T'. We let $\overline{T}$ denote the subset $V \setminus T$. If $T' = \overline{T}$, then $[T, T']$ is called a *directed cut* or *dicut*, and will be denoted by $\delta^+(T)$. Similarly, we denote by $\delta^-(T)$ the set of arcs having their origins in $\overline{T}$ and destinations in T (see Figure 1.4).

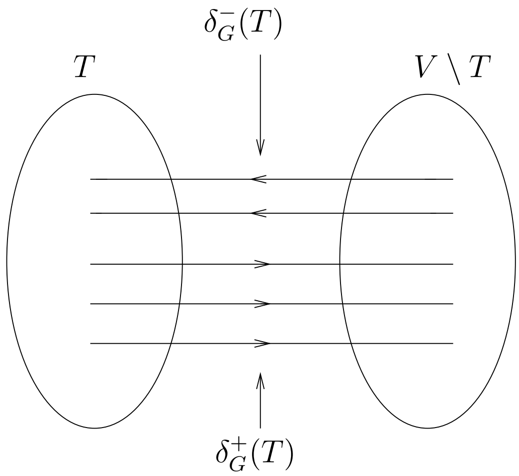


Figure 1.4: Directed cuts

If $T = \{u\}$, where u is a node of V, then we denote by $\delta^+(u)$ and $\delta^-(T)$ the directed cuts $\delta^+(\{u\})$ and $\delta^-(\{u\})$, respectively. Arcs of $\delta^+(u)$ and $\delta^-(u)$ are said to be *incidents* to u. If s and t are two nodes of G such that $s \in T$ and $t \in \overline{T}$, we may refer to $\delta^+(T)$ and $\delta^-(\overline{T})$ as *st-dicuts* of G.

G is said to be a *bidirected graph* if for each arc uv of A, there also exists an arc vu in A. Two arcs a, a' are called *parallel arcs* if $a = uv = a'$ (they have the same origin and destination nodes). They are said to be *antiparallel* if $a = uv$ and $a' = vu$. A pair (u, v) occurring more than once in A is called a *multiple arc*. We may refer to each occurrence of (u, v) as a *copy* of arc uv. If $a \in A$ is a multiple arc, then we let the pair (a, i), $i \in \mathbb{Z}_+$, denote the i^{th} copy of a.

Let $G' = (V', A')$ be a subgraph of G, with $V' \subseteq V$ and $A' \subseteq A$. If $c(.)$ is a weight function associates with an arc $a \in A$ the weight $c(a)$, then the total weight of G' is $c(A') = \sum_{a \in A'} c(a)$.

In what follows, we will use the graph as a subscript. In other words, we will write $\delta^+_G(T)$, $\delta^-_G(T)$, whenever the considered graphs may not be clearly deduced from the context.

We define a *path* in a directed graph G as an alternate sequence of of arcs $(u_1, a_1, u_2, \ldots, u_l, a_l, u_{l+1})$, with $a_i = (u_i, u_{i+1})$, for i = 1, ..., l. u_1 and u_{l+1} will be called endnodes of the path. A path is denoted either by its node sequence $(u_1, \ldots, u_{l+1})$, or by its arc sequence $(a_1, a_2, \ldots, a_l)$. Throughout this manuscript, we will use the notation $\{a_1, \ldots, a_l\}$ to designate a path. We will use the notation $\{(a_1, i_1), (a_2, i_2), \ldots, (a_l, i_l)\}$ to designate a path in a graph with multiple arcs. This notations specifies the copy of each arc used in the path.

Given a directed graph $G = (V, A)$. G is said to be *connected* if for every pair of nodes (u, v) there exists at least one path between u and v. Let s, t be two nodes of V, then two st-paths are *arc-disjoint* if they have no arc in common. If each arc of G is assigned a capacity, we define an *st-flow* as a nonnegative real-valued function on the arcs of G, satisfying the "flow conservation law" and such that the flow on an arc does not exceed the capacity of that arc.

Note that for seek of clarity, and all along the subsequent chapters, we will represent edges instead of antiparallel arcs in the figures showing bidirected graphs.

1.6 State-of-the-art on network design problems

Network Design has become a flourishing area and many problem variants have been considered in the literature. In this section we discuss two important families of problems related with network design field. We first introduce a general and widely studied design problem arising in telecommunication networks. Then, we present a version of this problem that takes into account the evolution of networks architecture towards a multilayer structure.

1.6.1 The network design problem

Network planning problems have several applications, specifically in telecommunication industry. They consists in choosing the capacities to be installed on the network links, so that they can carry the traffic demand flowing in the network. We assume that demands between pairs of origin and destination of a given network are input data. Assume that a set of modular capacities are available. The capacities have a limited value, and their installation yields a certain cost, which is positive. The network design problem is then to determine the number of capacities to set up on the network, so that the traffic demands can be met and the total cost is minimum.

Let $G = (V, E)$ be a finite and undirected graph, where V represent the set of network nodes and E is the set of edges. We denote by Q the set of commodities or demands. For every $k \in Q$, $O(k)$ denotes the origin node of k, $D(k)$ its destination node, and u_k its amount of traffic. Let b_{ij} be the cost of installing a modular capacity C on edge ij. Let us denote by f^k_{ij} the flow of commodity k using the edge ij from i to j. Let y_{ij} be an integer variable that is the number of capacities of size C installed on edge ij.

The Network Design Problem is then equivalent to the following mixed integer programming formulations

$$\min \sum_{ij \in E} b_{ij} y_{ij}$$

$$\sum_{j \in V} f^k_{ij} - \sum_{j \in V} f^k_{ji} = \begin{cases} -u_k, & if\ i = O(k), \\ u_k, & if\ i = D(k), \\ 0, & otherwise, \end{cases} \quad \forall i \in V, \forall k \in Q, \tag{1.1}$$

$$\sum_{k \in Q} (f^k_{ij} + f^k_{ji}) \le C y_{ij}, \quad \forall ij \in E, \tag{1.2}$$

$$y_{ij} \ge 0, \quad \forall ij \in E, \tag{1.3}$$

$$y_{ij} \text{ entier }, \quad \forall ij \in E, \tag{1.4}$$

$$f^k_{ij}, f^k_{ji} \ge 0, \quad \forall ij \in E, \forall k \in Q. \tag{1.5}$$

Inequalities (1.1) are called *flow conservation constraints*. Inequalities (1.2) are *capacity constraints*. Inequalities (1.3) and (1.5) are nonnegativity constraints, while (1.4) are the integrity constraints.

This problem is also referred to as *Network Loading Problem* or *Capacitated Network Design* (CND) *Problem*, and has been investigated in many works. In [77], Magnanti, Mirchandani and Vachani study two relaxations of CND problem. The first one is induced by the restriction of the problem to a single edge of the graph. They introduce a new class of valid inequalities, namely the arc-residual capacity inequalities. The second subproblem restricts the graphs to three nodes. This restriction allows to introduce further valid inequalities, namely 3-partition inequalities.

Bienstock and Muratore [22] the CND problem with survivability requirements. They have considered the cutset polyhedron associated with the problem, and studied its extreme points. They described several lifting procedures to derive general facet defining inequalities for this polyhedron.

Further versions of the problem have been studied. In fact, Magnanti, Mirchandani, and Vachani [77] study an extension of CND to the case of two facilities. In particular, they consider low capacity type and high capacity type. Moreover, traffic demands may be known in advance or submitted to uncertainty, the later is known as the Robust Network Design problem [84, 19]. In this work, we assume that the facilities have the same capacity and that traffic demands are reliably estimated, since several telecommunication operators use forecast traffic matrix for the design of their networks. The commodities are said to be *unsplittable* if their traffic value can not be divided along several paths, a unique path is then associated with each commodity for its routing. They are said to be *splittable* otherwise. In [71] authors analyse the network design problem with survivability requirements. They examine some of the models proposed in the literature for this problem as well as the methods developed to solve them.

1.6.2 The multilayer network design problem

More recently, the evolution of telecommunication networks has led some authors to turn themselves towards problems related to multilayer networks. In its most general form, the multilayer network design can be defined as follows [83].

Definition 1 *Given a multilayer network where each layer is represented by a graph $G_i = (V_i, E_i)$, and a traffic matrix given in the last layer, such that*

(i) nodes in layer $i+1$ are a subset of nodes in layer i, that is to say $V_{i+1} \subseteq V_i$,

(ii) an edge $e \in E_{i+1}$ corresponds to a path in layer i between its endpoints,

(iii) commodities are routed in the last layer,

(iv) capacities installed on layer i + 1 define demands for layer i,

We wish to determine the capacities to be installed over edges of all layers, so that the traffic is routed and the total cost is minimum.

Actually, the problem of designing layered networks have been studied first by Dahl and Stoer in [49]. Authors wish to set up a set of virtual links referred to as "pipes" on the physical layer. They propose an integer linear programming formulation based on cut constraints for the problem. They study the associated polytope and provide several classes of valid inequalities that define facets under some conditions which are described. Authors also provide a cutting planes based algorithm embedding their theoretical results.

Earlier works on this topic address the problem of designing virtual layer over an existing infrastructure. They take into account engineering constraints such as traffic multiplexing and assignment of wavelengths to the virtual links. In [111, 62], authors give decompositions of the problem in several subproblems solved sequentially. In [61], authors provide a heuristic approach to solve SDH over WDM network design. They develop several procedures based on greedy algorithms, random start heuristic as well as a metaheuristic based on a GRASP (greedy randomized adaptive search procedure) algorithm.

Additional works consider exact methods for different variants of the multilayer network design. In fact, in [87], Orlowski et *al.* propose an cutting plane approach for solving two-layer network design problems, using different MIP-based heuristic allowing to find good solutions early in the Branch-and-Cut tree. Belotti et *al.* [17] investigate the design of multilayer networks in the context of MPLS networks. They propose a mathematical programming formulation based on paths, that takes into account technical operations in MPLS technology for processing traffic demands, called *statistical traffic multiplexing*. They apply a Lagrangian relaxation working with a column generation procedure to solve their model. We also cite a more recent work of Raghavan and Stanojević [94] that study the two-layer network design arising in WDM optical networks. They consider the non-splittable traffic demands and propose a path based formulation for the problem. They provide an exact Branch-and-Price algorithm which solves simultaneously the WDM topology design and the traffic routing. In [88], Orlowski et *al.* address the problem of planning multilayer SDH/WDM networks. They consider the minimum cost installation of link and node hardware for both layers, under various practical constraints such as heterogeneity of traffic bit-rates, node capacities and survivability issues. They propose a mixed integer programming formulation and develop a Branch-and-Cut algorithm using strong inequalities, from the single-layer network design problem, to solve it. In [48], Fortz and Poss study the multi-layered network design problem. They propose a Branch-and-Cut algorithm to solve a capacity formulation based on the so-called metric inequalities, enhancing the results obtained by Knippel and Lardeux in [73] for the same formulation. In [83], Mattia studies two versions of the two-layer network design problem. The author was particularly interested in capacity formulations for both versions and investigates the associated polyhedron. Some polyhedral results are provided for both versions of the problem, specifically proving that tight metric inequalities [11] define all the facets of the considered polyhedra. The author show how to extend these polyhedral results to an arbitrary number of layers. In [26], Borne et *al.* study the problem of designing an IP-over-WDM network with survivability against failures of the links. They conduce a polyhedral study of the problem and give several facet defining valid inequalities, and propose a Branch-and-Cut algorithm to solve the problem. Further results on survivability in multilayer network design can be found in [100], where author highlight the close relationship between the design of survivable network and the Steiner travelling salesman problem. Several formulations are proposed for the problem and exact algorithms are developed to solve them.

The capacitated single-layer network design has receive a lot of attention in the literature, and the associated polyhedron was studied in details. Yet the investigation of capacitated multilayer network design problems received only a limited attention, specifically in a polyhedral point of view. In this thesis we consider the dimensioning aspect in both single-layer and multilayer network design problems. Unlike the previously cited works, we consider here that the commodities can not be split along several routing paths or even several facilities of the same path. This assumption, together with additional requirements related to OFDM multi-band technology, further complicates both problems. In the following chapter, we address a variant of the capacitated network design problem.

Chapter 2

Multilayer Optical Networks

Contents

This preliminary chapter, intends to give a brief outline on the evolution of telecommunication networks. We seek for giving here some key notions to understand the technical requirements that fall within the definition of problems studied in this thesis. In particular, we first give some elements concerning multilayer optical networks. We then focus on optical WDM technology and the infrastructures used in optical fibre based networks. We finally introduce new paradigms that will guide the evolution of WDM networks towards greater flexibility.

2.1 Optical networks : a layered structure

Telecommunication networks have continually evolved since their introduction. This evolution has been mainly driven by the increase and diversification of user traffic and services. Emerging paradigm for present and future telecommunication networks are based on a multilayer representation of the networks, where different technologies are able to provide various services to the customers. Each layer has a specific functionality and provides a service to the layer above.

The transmission of information between the different layers is governed by various protocols. A protocol can be defined as a formal description of the conventions and rules that are used by a layer to manage data traffic and ensure the interactions with the other layers. These protocols have been classified by the ISO (International Standardization Organization), that proposed a model with seven layers, called the OSI (Open Systems Interconnection) model (see Figure 2.1). Even though this model constitutes

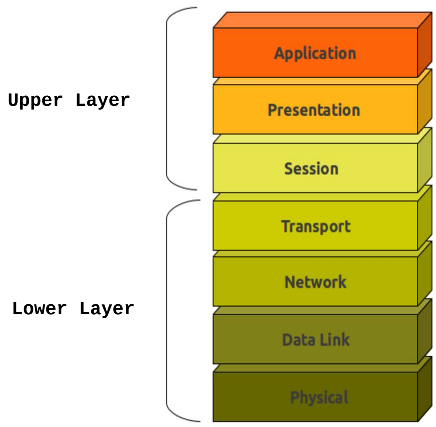


Figure 2.1: Reference model OSI

a reference allowing to understand the earlier multilayer network representations, it remains relevant only in a theoretical point of view. In practice, there are generally less than seven layers, and each layer may ensure several functionalities.

Basically, most commons architectures are composed by an IP layer overlying and ATM, which is itself placed on a SDH support. The IP (Internet Protocol) layer is used as a platform for users' applications, ATM (Asynchronous Transfer Mode) for traffic engineering, flow control and carrying different QoS (Quality of Service) support. ATM flows are then sent on SDH (Synchronous Digital Hierarchy), and finally over WDM (Wavelength Division Multiplexing) fibres.

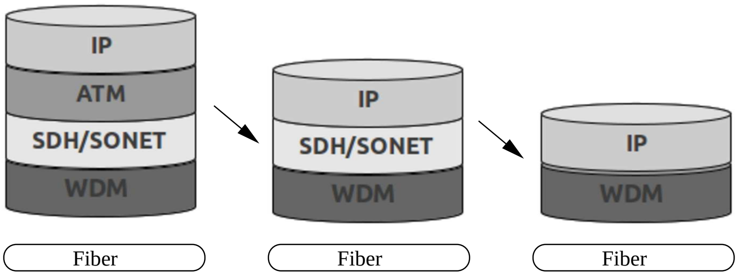


Figure 2.2: Towards IP-over-WDM architecture

This architecture results from a progressive evolution, yet it suffers from a deficiency in flexibility to cope with the constant growth of traffic [74]. It has also been proposed an overlaying concept based on transmitting IP over MPLS (Multiple Label Switching Protocol). IP over MPLS flows are sent on SDH, which is itself sending flows on WDM fibre. Note that the first and second approaches are widely deployed on nowadays networks. However, a third approach appears progressively as the solution to which today's networks will converge. This solution consists in employing IP directly over WDM, and thus to get profit from the huge capacity of optical fibres to soak up the traffic generated by IP layer. Although this solution seems to be the most efficient, it requires that either IP or WDM have the capability to manage all the restoration functions carried by SDH layer, as well as traffic engineering functions of ATM layer, in previous models.

To overcome this difficulty, telecommunication operators had the idea to use the control protocols such as MPLS (Multi-Protocol Label Switching) and the GMPLS (Generalized-MPLS) [80]. These protocols allow the implementation of the traffic engineering in the IP layer (at the packet level) and in the optical layer (at the wavelength level), and hence the ATM can be removed from the network. Similarly, many functions of the SDH can be transferred to the optical layer. However, some functionalities such as processing data can not be moved down to the WDM layer, thus a restricted SDH layer ensured some necessary functionalities must remain. The three approaches of IP-over-WDM networks can be seen in Figure 2.2.

2.2 Optical WDM networks

2.2.1 Architecture

An optical telecommunication network is composed by a set of devices interconnected by a set of links so as to enable data exchange between the nodes. Transmission and reception of information of different type is made according to well defined rules. In general, optical telecommunication networks are maid of three parts. The first one, referred to as *access network* links the user (customer, company, etc.) to the network. The length of links connecting the user to the first interface of the network does not exceed a few kilometers. The second part of the network is called *metropolitan* or *backhaul network*, and it covers a distance of few tens of kilometers. Its role is to aggregated and route the traffic to the third part of the network, namely the *core network*. The core network is the central part of telecommunication networks. It interconnects all the the metropolitans networks, and provides various services (internet, VoD, etc.) to the users who are connected by the access networks. This part of the network carries out the greatest traffic amount by using most efficient technologies allowing to support important traffic rates on long distances. Since all the metropolitan networks are connected to the core network, the nodes of metropolitan networks are linked to those of the core network via optical fibres that may reach a huge transmission capacity.

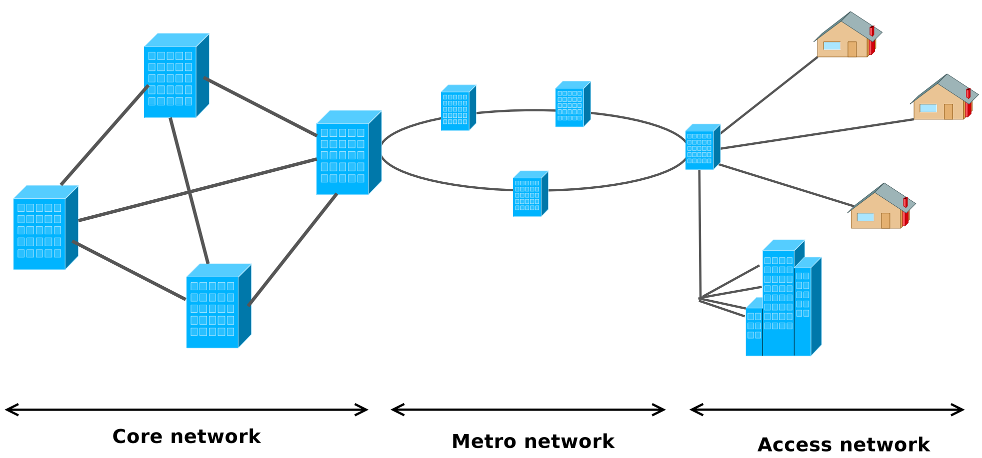


Figure 2.3: Optical network architecture

2.2.2 WDM technology

Telecommunication operator that manages a core network use to provide point-to-point connexions to its users. This connexions result from the aggregation of several low bit-rate data streams, so that they have enough traffic to make a good use of the large capacity offered by optical fibres. In what follows, we introduce the definitions and terminology related to the fibre based communication networks, as well as their operating principle.

2.2.2.1 Optical fibres

Optical fibre is a flexible, transparent fibre made of high quality of glass or plastic. It can work as a waveguide to transmit light between the two ends of the fibre, by using refraction properties. In fact, when a light beam strikes the surface at an angle between two environments that are more or less transparent, it splits in two. The first part is reflected while the second one is refracted, that is to say, transmitted in the other medium when changing direction. This principle is used to guide light along an optical fibre.

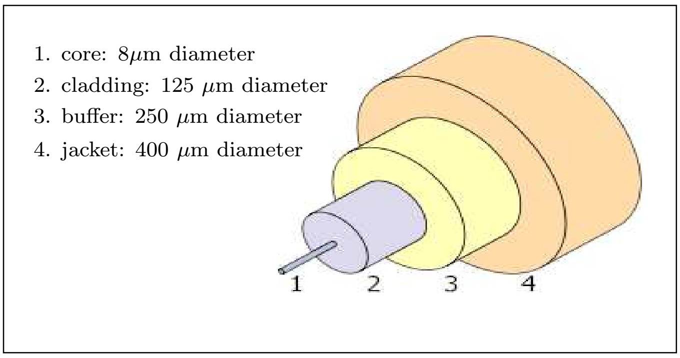


Figure 2.4: The structure of a typical single-mode fibre

Optical fibres typically include a transparent core surrounded by a transparent cladding material with lower index of refraction. Light is kept in the core by total internal reflection. Fibres that support many propagation paths or transverse modes are called Multi-Mode Fibres (MMF) while those that only support a single mode are called Single-Mode Fibres (SMF). This property makes the optical fibres widely used in fibre-optic communications. Moreover, from an electromagnetic point of view, optical fibres are quite immune to interference. As a consequence, they constitute a very good choice for high-speed transmissions. Another advantage of the optical fibre is safety aspect. Indeed, it is very difficult to connect a listening cable to an optical fibre and such an operation results in a significant drop of signal, whose cause can be easily localized.

It is possible to use several wavelengths within the same optical fibre in order to send different signals simultaneously. Indeed, each fibre can carry many independent channels, each one using a different wavelength of light, this is known as *wavelength-division multiplexing (WDM)*.

2.2.2.2 WDM transmission system

Data transmission in telecommunication networks using optical fibres is mainly based on wavelength division multiplexing technology. Indeed, optical networks consists in a set of nodes interconnected by several cables, each one containing up to tens of optical fibres.

Thanks to WDM, several distinct wavelengths may share the same optical fibre, and then to perform high bit-rate data stream transmission without being subject to interferences. The nodes have the capability to multiplex or to combine a number of optical carrier signals onto a single optical fibre by using different wavelengths (*i.e.* colors) of laser light. This technique enables bidirectional communications over one strand of fibre, as well as multiplication of capacity.

In current core networks, WDM divides the large bandwidth available in an optical fibre into several tens of wavelengths, each one having a transport capacity of 10 Gbit/s, 40 Gbit/s or even 100 Gbit/s.

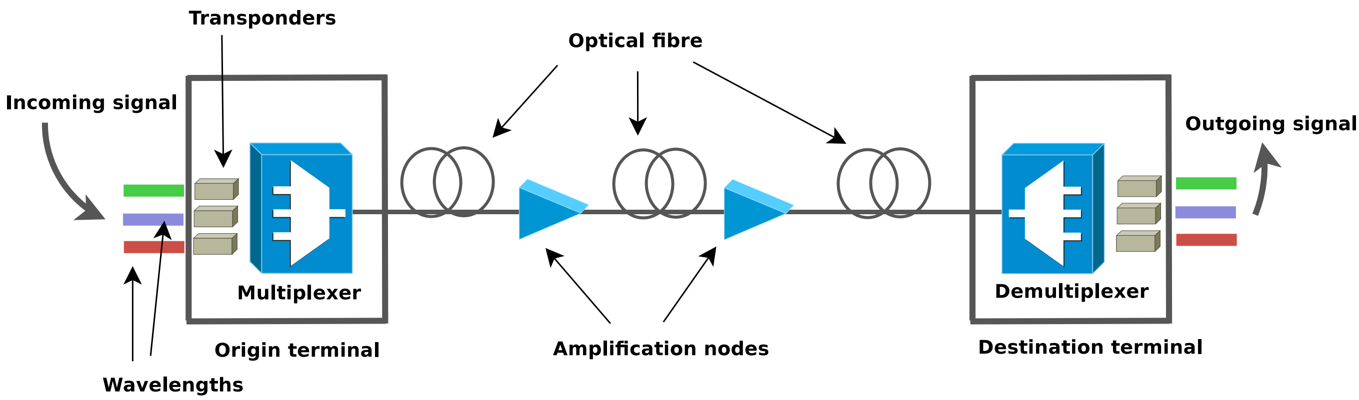


Figure 2.5: A typical WDM system

Overall, a WDM system holds two terminal nodes. Those nodes includes several *transponders* that are in the interface of emission and reception of optical signal in WDM systems. Indeed, at each node, a *transmitter* sends data on a specific wavelength. Then, a *multiplexer* packs the wavelengths together in order to form a unique signal that is transmitted along a single optical fibre to the destination terminal. A *demultiplexer* installed on the destination node does the inverse work. In fact, it is responsible for splitting the signal, and returns each receiver the corresponding wavelength. A wavelength established between two terminals is somehow a *virtual link*, as it connects directly two nodes of the networks that are not necessarily neighbours (not linked by the same optical fibre). This virtual links may also be referred to as a *lightpath*.

Transmission on long distances (*long haul* WDM) may require using additional devices, referred to as *optical amplifiers*, since the signal may suffers from attenuation. In general, amplifiers devices are installed each 100 kilometers in average. Although the optical fibre offer a huge bandwidth capacity, the limitations in terms of transmission possibilities come essentially from node architecture and functionalities.

Figure 2.5 shows an example of WDM system with three wavelengths (respectively depicted in green, purple and red). Each wavelength is carried out by a transponder device, and the three wavelengths are processed by the multiplexer device so as to form a unique signal, transmitted via the outgoing fibre to the destination terminal. The signal is then demultiplexed and gave back to the receiver transponders, which directs each stream to the remaining routing sections.

The installation of optical amplifiers will not be considered here since it does not affect the studied problems. Moreover, one can take into account the wavelength routing cost, which can be expressed in terms of length of fibre based path associated with each used wavelength.

2.2.2.3 Traffic grooming

The heterogeneity of data streams granularities raises the question of an efficient filling of wavelengths. In fact, these low-rate traffic request may range from a few megabits up to the full wavelength capacity. Moreover, any used wavelength induces a cost mainly related to the transponders responsible for its emission and reception, as well as the routing. Thus, make the best use of set up devices and transmitted wavelengths is one the most relevant issues in optical networking. Since multiplexing and demultiplexing are predominant features in nodes of WDM networks, it is possible to use this property to efficiently grooming low bit-rate data streams into wavelengths. In other words, *traffic grooming* can be seen as multiplexing, demultiplexing and switching low rate traffic streams onto high capacity lightpath [42]. Despite the fact that traffic grooming improves the wavelength utilization in the network, it also further complicates the architecture of nodes.

Observe that traffic grooming operations together with wavelength multiplexing yields a specific multi-layer like structure. In practice, traffic streams are transmitted via the new channels that are ligthpaths, while each wavelength needs a physical media support, which is the optical fibre. This structure suggests two levels of routing. Indeed, data traffic need to be routed using ligthpaths from their origins to their destinations. On an other hand, each ligthpath corresponds to a wavelength that has to be routed from the transmitter terminal, to the receiver terminal, by using the optical fibres. We speak about physical routing and virtual routing, since the former uses optical fibres and is related to wavelengths, while the latter is based on lightpaths and concerns the traffic data streams.

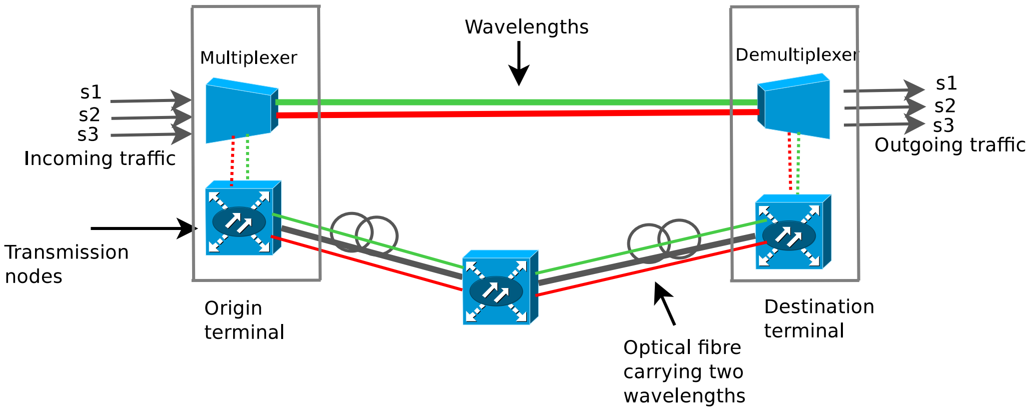


Figure 2.6: Levels of routing

Figure 2.6 depicts a bi-layer representation of a WDM system. In this example, three traffic streams $s1$, $s2$ and $s3$, are groomed thanks to a multiplexer within two wavelengths, represented in green and red, respectively. The traffic streams are carried by the two lightpaths from their origin terminal to their destination terminal, where they are separated and sent back to their final receiver. Both wavelengths are put together in the same optical fibre along a routing path having two sections. This example clearly shows that there are two levels of grouping traffic streams, as well as two routing levels.

2.2.2.4 Transparency in WDM networks

The traffic using WDM systems are submitted to a set of operations that aim to process its different streams so as to get a signal transmitted more efficiently. Such operations require that the signal goes through *Optical-Electrical-Optical* (O/E/O) conversions at every node of the WDM system. More precisely, the optical signal (including one or several wavelengths) is systematically converted to an electrical signal, each time it goes through a node in the network. In this kind of networks, WDM layer is only used to transport point-to-point data. The O/E/O conversions are often costly and power consuming. Thus, networking actors have introduced a node architecture having the capability to process signal including traffic streams only at their origin or destination terminals. In other words, this new type of nodes avoids O/E/O conversions at intermediate nodes for traffic streams. Thus, at a given node, the incoming signal or one that reaches its destination are subjects to O/E/O conversions, while the remaining signal passes through or *by-passes* the current node. Such node is known as *transparent node*, and by extension, a WDM network using this technique is also called *transparent WDM network*.

The basic network element in a transparent network is a device called *Optical Add/Drop Multiplexer* (OADM). *Add* and *drop* here refer to the capability of the device to add one or more new wavelength channels to an existing multi-wavelength WDM signal, and/or to drop (remove) one or more channels, passing those signals to another network path.

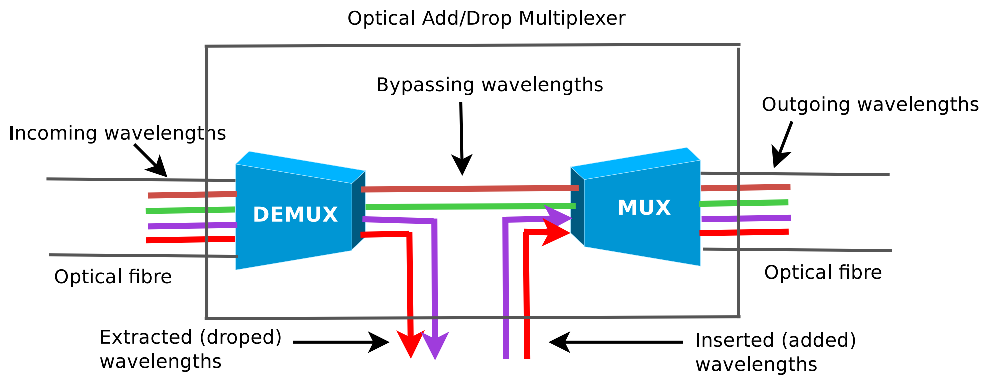


Figure 2.7: Optical Add/Drop Multiplexer

Figure 2.7 shows a typical architecture of OADM device. In this figure are represented four wavelength using the node in different ways. In fact, two wavelengths respectively represented in purple and red, are dropped at this node and replaced by two further wavelengths having the same colours. The brown and green wavelengths, in turn bypass the node without being subject to any O/E/O conversion.

Note that next-generation OADM, called Reconfigurable Optical Add Drop Multiplexer (ROADM), has the extra flexibility so that adding wavelengths or changing the wavelengths destination becomes easy and even possible to perform remotely. This capability provides a full control over the capacity of transparent WDM networks.

In what follows, we briefly survey some solutions being studied to reach even more flexibility in transparent WDM networks, and enhance the efficiency in wavelength utilization.

2.3 Towards more flexibility in the optical layer

The increase in number as well as transport capacity of wavelengths, is closely related to the growth of traffic in optical networks. Actually, current WDM systems may ensure transmission of about a hundred of different wavelengths, each one having a capacity of 10 Gbit/s to 40 Gbit/s. Telecommunication operators are even prepare the deployment of 100 Gbit/s capacitated wavelengths on some optical networks [64, 23]. This important growth requires that a trade-off is identified between flexibility in processing data at nodes of the network from one part, and the cost plus power consumption from the other part. Furthermore, it should be pointed out that the signal carried by high capacitated wavelengths (100 Gbit/s and more) may suffer from some form of alterations over long distances. This is explained by the existence of physical phenomena that might affect signal travelling on long distances [28].

Recent advances in networking have enabled the advent of a new technology called *optical multi-band Orthogonal Frequency Division Multiplexing* (OFDM) as an answer to the challenges highlighted above. This technology offers the possibility to being able to perform processing within a traffic stream transported by a given wavelength. Such operation can be done without leaving the optical domain (without O/E/O conversions). Next section is devoted to give a short presentation of this technology.

2.3.1 Optical Multi-Band OFDM

OFDM is a technology that has been initially developed for the wireless transmissions like mobile communication. Its utilization to the optical fibre networks has received an increasing attention in recent years. Optical multi-band OFDM can be defined as a multicarrier modulation technique in which the traffic is carried over many lower rate subcarriers. In other words, each WDM channel spectrum is divided into smaller independent entities called OFDM *sub-wavelengths* or *subbands* [64, 24] each one having a set of sub-carriers. These subbands may be used to transport traffic and can be processed independently from other each other, without using O/E/O conversions. To this end, OFDM transponder generates just enough spectral resource to carry the incoming signal. Such process enables a better filling of WDM channels since resources can be provisioned elastically by allocating a required number of subbands, in accordance to the traffic stream bit-rate [72].

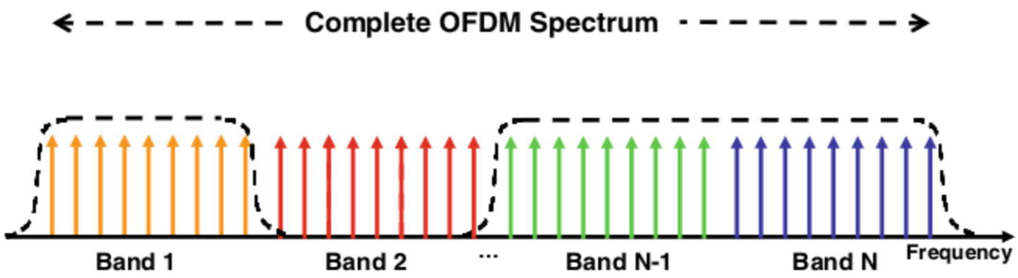


Figure 2.8: Principle of OFDM

Figure 2.8 [109] shows an illustration of the division of a WDM channel spectrum into multiple OFDM subbands, denoted Band1 to BandN, each one being composed of several subcarriers. It is then possible to attribute one a several subbands to the incoming signal for its transmission. Since the subbands are transmitted and processed independently from each other, this allows to use the same wavelength to transport data streams that do not necessarily have the same origin and destination terminals. Furthermore, it provides a granularity smaller than one of the WDM channel, and this property avoids wasting bandwidth.

It should be noted that the architecture of the multi-band OFDM transponder remains complex because it requires several single band generation and reception [24]. Besides, several architectures are currently under review, and some patents have already been proposed for this technology. Overall, it appears from the investigations on optical multi-band OFDM that it is a promising technology that may carry out the evolution of optical networks towards deployment of very high capacitated WDM systems on long distances.

In Figure 2.9 is shown a ROADM with an incoming fibre that includes two wavelengths, respectively represented in green and purple. Both wavelengths are divided

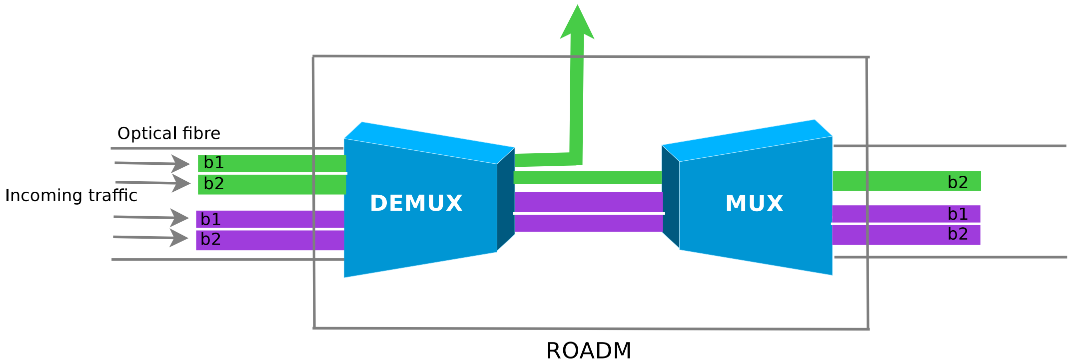


Figure 2.9: ROADM function

into two subbands, denoted b1 and b2. Note that subband b1 (respectively b2) of green wavelength and subband b1 (respectively b2) of purple wavelength are not equivalent, since they do not correspond to the same resource in the spectrum. In this example, there are four data streams incoming to the ROADM, each one uses a specific subband. The traffic stream carried by subband b1 of the green wavelength reaches its destination at this node, and is extracted (dropped), while the remaining subband (b2) of the wavelength together with b1 and b2 in wavelength purple, bypasses the ROADM.

2.3.2 Further solutions

Parallel investigations have been conducted on further technological solutions seeking to get more flexibility without paying too much in processing data (essentially due to O/E/O conversions). One of these technologies is called *Optical Burst Switching* (OBS), and where incoming data are assembled into basic units referred to as *bursts* that are then transported over the optical network. OBS ensures a division of wavelength different from one performed in in OFDM. In fact, it provides the division of the wavelength in the time domain at the optical layer. We refer the reader to [110], [91], and references therein, for more details on this technology.

Some works also focus on a technology called SLICE (Spectrum-sliced Elastic Optical Path Network). This process allows to adapt the capacity of the wavelength to size of data stream to be transported. This can be possible thanks to specific transponders, that have the capability to generate an optical signal using the minimum spectral resources to allow the transmission of data stream from its origin node to its terminal node [23]. SLICE introduces un new concept of elasticity that offers more flexibility in the optical layer, specifically in terms of bandwidth allocation. Note that this technology does not allow to have a granularity smaller than the wavelength, since it is not possible to perform processing on "portions" of optical channel. Additional informations on this technology can be found in [65].

2.4 Terminology and assumptions

In this thesis we consider optical WDM networks using the technology multi-band OFDM. We will consider given a set of ROADMs compatible with OFDM technology. Moreover, since subbands of a wavelength can be used independently from each other, we do not longer mention the wavelengths. The subbands here play the role of ligth-paths since they connect two nodes that are not specially linked to the same fibre. In order to allow an effective occupation of WDM channels, we assume that the cost of a subband increases with its index. In other words, it is more relevant to fill the WDM channel progressively in practice. Besides, two subbands with same index coming from two wavelengths of the same color can not be associated with the same optical fibre. Indeed, since a subband corresponds to a specific resource in the wavelength channel, it can not be associated twice with an optical fibre. This constraint will be referred to as *disjunction constraint*.

We deal here with dimensioning aspects of optical networks using OFDM technology. Note that, although WDM technology is considered in practice in the physical layer, we assume here that it is a virtual layer. In fact, the physical layer is the layer composed by optical fibres and transmission nodes, and the WDM layer is to be determined. We mean by installing a subband on a link setting up two transponders at the ends of this link, that generate the subband. We further suppose that all the subbands installed on a virtual link are carried by the same WDM system. In other words, a unique pair of ROADMs at terminal nodes may generate all the subbands needed to carry the traffic on this link. Besides, we consider that installing a capacity on a link of the network is equivalent to set up a subband on this link. We assume that data stream can not be split along several routing paths, or even several subbands in the same WDM system. Finally, we will differentiate traffic routing and subband routing since the former uses lightpaths while the later uses optical fibres.

2.5 Concluding remarks

In this section we have introduced some elementary notions concerning multi-layer in general, and optical WDM networks in particular. We have focused on optical WDM networks, and showed that these networks can be seen as the superposition of two-layers: the physical layer (fibre layer), and the virtual layer (WDM layer). More precisely, we have presented the multi-band OFDM technology and its principle. In the sequel, we will consider two optimization problems related to these optical WDM networks. The first problem focus on the virtual (WDM) layer dimensioning and does not take into account physical (fibre) layer. In fact, this first problem attempts to be very generic and will use only some technical requirements of OFDM technology such as non splittable traffic assumption. The second problem is related to multi-layer optical network design. It considers the dimensioning of virtual layer in terms of number of required subbands, taking into account the physical layer.

Chapter 3

Capacitated Network Design and Set Function Polyhedra

Contents

In this chapter we study the Capacitated Single-Layer Network Design (CSLND) problem. We first present an integer linear programming formulation for the problem. Then we consider the polyhedron associated with a simple relaxation of this problem, namely arc-set polyhedron. We highlight the relationship between this relaxation and a classical combinatorial optimization problem: the bin-packing problem. We use this relationship to provide new classes of valid inequalities, and describe necessary and *sufficient conditions for these inequalities to define facets. The identified inequalities are then embedded within a Branch-and-Cut framework to solve the CSLND problem. Some computational experiments are presented in Chapter 4 to empirically test the efficiency of our approach.*

3.1 Capacitated network design problem

Network design problems are becoming one of the major economic issues for nowadays telecommunications industry. The Capacitated Network Design (CND) problem can be defined as follows. Given a network with a set of commodities, we want to select the minimum cost capacitated facilities to install over the links of this network such that all the commodities may be routed simultaneously. We consider a variant of the classical capacitated network design problem that can be defined as follows. Given an optical network, composed by optical devices interconnected by fibre links. Each link holds two optical fibres, so that it can be used in both directions independently. A set of modules with the same capacity can be installed on the links of the network. Each module installation yields a positive cost, impacted on the link concerned. Given a set of traffic demands (commodities), each one defined by an origin device, a destination device and an amount of traffic to route between both devices. Note that, throughout the chapter, we will use either "modules" or "copies" to designate the modular capacities installed on a the links of the network.

We wish to determine the number of modules to set up on the network so that the commodities can be routed from their origins to their destinations, and the total cost is minimum. This problem will precisely be referred to as *Capacitated Single-Layer Network Design* (CSLND) problem, to differentiate it from a multilayer version of the Capacitated Network Design problem, discussed later in this manuscript (Chapters 5, 6 and 7). In fact, CSLND is nothing but a relaxation of this multilayer network design problem. Besides, the constraints and specificities of CSLND problem come from the technical requirements related to its multilayer version.

The earlier results on the CND problem and the associated polyhedron can be find in [77, 78], where authors study a single commodity multifacility network design problem. Some of their results are generalized by Bienstock and Günlük in [21] and extended to the case of multicommodity network design using two type of facilities. The CND problem is also studied in [13, 30, 11, 93] under splittable traffic assumption. Several polyhedral results are presented for the problem and cutting planes based approaches are developed in all the referenced works. More recently, some authors have focused on the *Multi-layer Network Design* problem (see for instance [49, 48, 83]).

We are interested in the polyhedra associated with simple relaxations of CSLND restricted to some link of the network. The idea behind this is to investigate those polyhedra and take advantage of their partial characterization to solve CSLND problem efficiently. Some studies have already shown the effectiveness of such approach for solving network design problems.

In fact, Magnanti et al. [77] study the restriction of CSLND on one arc for two facilities and splittable flow assumption. Pochet and Wolsey [92] study the polyhedron of a single-arc network design problem with an arbitrary number of facilities and splittable flow assumption. Brockmüller et al. [27] and van Hoesel [101] investigate the CSLND restricted to one edge (the edge capacity problem). They study the integer knapsack problem arising from this relaxation then introduce the so-called c-strong inequalities and give necessary and sufficient conditions for these inequalities to define facets. In [101], authors give conditions under which the facets of edge capacity polytope define also facets for the CSLND polytope. In [10], Atamtürk and Rajan study both splittable and unsplittable CSLND arc-set polyhedra by considering the existing capacity of the arc. They give a linear-time separation procedure for the residual capacity inequalities and show its effectiveness for the splittable CSLND. They also use the c-strong inequalities and derive a second class of valid inequalities for the unsplittable CSLND problem. Similar approach have also been used to study cut-set polyhedra associated with the CSLND in [8] and CSLND with survivability constraints in [22].

Our contribution

The objective of this chapter is to study the polyhedra associated with the arc-set CSLND problem. We show that many different subproblems, arising as relaxations of our problem are in fact associated with the same polyhedron. We refer to these subproblems as *functions*. We introduce the polyhedra associated with a general class of functions called *unitary step monotonically increasing* functions, and we study their basic properties. We provide two classes of inequalities called *Min Set I* and *Min Set II*, that are valid for each considered function, and we describe general separation procedures for these inequalities. We give necessary and sufficient conditions for these inequalities to define facets for the considered polyhedra. Our polyhedral results remain the same for every considered function, and the separation procedures are still available by integrating the specificities of each function. We give an application to the *bin-packing function*, that is in fact equivalent to the arc-set CSLND with unsplittable flow. In particular, our results for Min Set I inequalities generalize those provided in [27, 101, 10] for c-strong inequalities, and both inequalities Min Set I and Min Set II are used within a Branch-and-Cut algorithm to solve efficiently CSLND problem. The rest of the chapter is organized as follows. In this section we briefly describe the CSLND problem and its restriction to a single arc. In section 3.2 we introduce the set-functions polyhedra and study their basic properties. We then present the so-called Min Set I and Min Set II inequalities, and investigate their facial structure. In section 3.3, we give and application of our polyhedral results to the bin-packing function, and we show the interest of such application for the CSLND problem.

3.1.1 Compact formulation for CSLND

In terms of graphs, the problem can be presented as follows. Consider a bi-directed graph $G = (V, A)$ that represents an optical network. Each node $v \in V$ corresponds to an optical device and each arc $a = ij \in A$ corresponds to an optical fibre. If an arc ij exists in A, then ji also belongs to A. Let K be a set of commodities. Each commodity $k \in K$ has an origin node $o_k \in V$, a destination node $d_k \in V$ and a traffic $D^k > 0$ that has to be routed between o_k and d_k. Suppose given a set of available modules, denoted by W having the same capacity C. Assume without loss of generality that $D^k \leq C$, for all $k \in K$. A module $w \in W$ installed on an arc ij is a copy of that arc, and yields a cost denoted c_{ij}. Every module w can carry one or many commodities, but a commodity can not be split on several modules. This specificity makes impossible the aggregation of commodities having the same source and destination nodes to reduce the size of the problem. Thus, there might be several different commodities with the same origin and destination nodes.

The CSLND problem is to determine a minimum cost assignment of the modules to the arcs of G so that a routing path is associated with each commodity from its origin to its destination.

Let $y \in \mathbb{R}^{|A||W|}$ such that, for each arc $ij \in A$ and for each module $w \in W$,

$$y_{ij}^{w} = \begin{cases} 1, & \text{if } w \text{ is installed on } ij, \\ 0, & \text{otherwise.} \end{cases}$$

and let $x \in \mathbb{R}^{|K||A||W|}$ such that, for each $k \in K$, $w \in W$ and $ij \in A$,

$$x_{ij}^{kw} = \begin{cases} 1, & \text{if } k \text{ uses the module } w \textit{ on arc } ij \text{ for its routing,} \\ 0, & \text{otherwise.} \end{cases}$$

The CSLND problem is then equivalent to the following integer linear programming formulation:

$$\min \sum_{ij \in A} \sum_{w \in W} c_{ij} y_{ij}^{w}$$

$$\sum_{w \in W} \sum_{j \in V} x_{ji}^{kw} - \sum_{w \in W} \sum_{j \in V} x_{ij}^{kw} = \begin{cases} 1, & if\ i = d_k, \\ -1, & if\ i = o_k, \\ 0, & \text{otherwise} \end{cases}, \qquad \begin{array}{c} \forall k \in K, \\ \forall i \in V, \end{array} \tag{3.1}$$

$$\sum_{k \in K} D^k x_{ij}^{kw} \leq C y_{ij}^{w}, \qquad \forall w \in W, \forall ij \in A, \tag{3.2}$$

$$0 \leq x_{ij}^{kw} \leq 1, x_{ij}^{kw} \in \{0,1\}, \qquad \forall k \in K, \forall w \in W, \forall ij \in A, \tag{3.3}$$

$$0 \leq y_{ij}^{w} \leq 1, y_{ij}^{w} \in \{0,1\}, \qquad \forall w \in W, \forall ij \in A. \tag{3.4}$$

Equalities (3.1) are the flow conservation constraints, they require that a unique path between o_k and d_k is associated with each commodity k. Inequalities (3.2) are the capacity constraints for each installed module. They also ensure that the capacity installed on arc ij is large enough to carry the commodities using this arc. (3.3) and (3.4) are the trivial and integrity constraints.

This problem as well as capacitated network design variants is known to be NP-hard even for special cases (see Bienstock et al. [30] and Chopra et al. [31]. Thus, it is difficult to solve CSLND to optimality using Branch-and-Bound, even for small instances.

3.1.2 Aggregated formulation

Suppose now that G consists of nodes i, j connected by a single edge ij. Then the CSLND problem here, is to determine the number of modules to install over ij, in such a way that each commodity using ij is assigned to at most one module and the total cost is minimum. Consider the polyhedron:

$$P_{ij} := conv\{(x,y) \in \{0,1\}^{|K| \times |W|} \times \{0,1\}^{|W|} :$$
$$\sum_{k \in K} D^k x_{ij}^{kw} \leq C y_{ij}^{w}\ \forall w \in W, \sum_{w \in W} x_{ij}^{kw} \leq 1\ \forall k \in K\}$$

P_{ij} is the convex hull of CSLND problem restricted to ij. Note that the polyhedron P_{ij} has many symmetric solutions and does not present a suitable structure to investigate. In fact, there are few chances that such an investigation can bring any relevant information to help in solving CSLND problem. To overcome this difficulty, we will introduce a new *aggregated* model that does not specify which copy of the arc ij is used for the routing of a commodity k. Indeed, the idea is just to determine the number of modules that have to be installed on ij, so that each commodity can be assigned to one of these modules.

We will define the following additional decision variables. Let $y \in \mathbb{Z}^+$ such that for each arc $ij \in A$, $y_{ij} = \sum_{w \in W} y_{ij}^w$ is the number of modules installed on ij. Let $x \in \mathbb{R}^{|K||A|}$ such that for each commodity $k \in K$, and for each arc $ij \in A$, $x_{ij}^k = \sum_{w \in W} x_{ij}^{kw}$, and

$$x_{ij}^k = \begin{cases} 1, & \text{if } k \text{ uses some module of the arc } ij \text{ for its routing,} \\ 0, & \text{otherwise.} \end{cases}$$

The CSLND problem can then be formulated using the following ILP:

$$\min \sum_{ij \in A} c_{ij} y_{ij}$$

$$\sum_{j \in V} x_{ji}^k - \sum_{j \in V} x_{ij}^k = \begin{cases} 1, & if\ i = d_k, \\ -1, & if\ i = o_k, \\ 0, & \text{otherwise}\ , \end{cases} \qquad \begin{matrix} \forall k \in K, \\ \forall i \in V, \end{matrix} \tag{3.5}$$

$$\sum_{k \in K} D^k x_{ij}^k \leq C y_{ij}, \qquad \forall ij \in A, \tag{3.6}$$

$$0 \leq x_{ij}^k \leq 1, \qquad \forall k \in K, \forall ij \in A, \tag{3.7}$$

$$x_{ij}^k \in \{0, 1\}, y_{ij} \in \mathbb{Z}^+, \qquad \forall k \in K, \forall ij \in A. \tag{3.8}$$

As in formulation (3.1)-(3.3), equalities (3.5) are the flow conservation constraints for each commodity of K. Inequalities (3.6) will be called *aggregated capacity constraints*. They ensure that the overall capacity of the modules installed over ij is not exceeded by the commodities flowing along ij, $ij \in A$. (3.7) and (3.8) are the trivial and integrity constraints.

Proposition 3.1 *Every solution of compact formulation* (3.1)-(3.4) *is a solution of aggregated formulation.*

Proof. Trivial.

Proposition 3.2 *A solution for the aggregated formulation* (3.5)-(3.8) *is not necessary feasible for the compact formulation.*

Proof. To prove this proposition, we give a counterexample.

Consider an instance of CSLND problem given by a three nodes graph and 3 available modules per arc. There are five commodities denoted k_1 to k_5, with traffic amount 4, 3, 6, 6, 6 (see Figure 3.1), while the capacity of each module is $C = 10$. Figure 3.1 shows a feasible solution for the aggregated formulation (3.5)-(3.8). Let us denote by $(\overline{x}, \overline{y})$ this solution. Then we can describe its entries as follows. $\overline{y}_{12} = \overline{y}_{23} = 1$ while $\overline{y}_{13} = 2$. The commodities are routed using path given in Figure 3.1. We can see for example that k_1 uses the arc 12, while k_3, k_4, and k_5 use the arc 13.

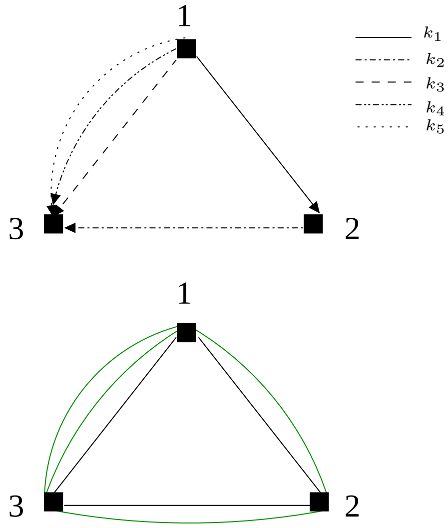


Figure 3.1: Example of solution for aggregated formulation

The solution $(\overline{x}, \overline{y})$ clearly satisfies all the constraints of the aggregated formulation (3.5)-(3.8). However, it is not feasible for the compact formulation (3.1)-(3.4). In fact, k_3, k_4 and k_5 are routed on arc 13, since the overall capacity installed on this arc allows this packing $(6 + 6 + 6 < 10 + 10)$. Yet this solution is not feasible for the compact formulation since no two commodities among k_3, k_4 and k_5 might fit together in one module.

This implies that solution described in Figure 3.1 can not induce a feasible solution for the compact formulation (3.1)-(3.4).

In order to ensure a feasible solution for compact formulation by considering aggregated formulation, we should add the following constraint:

$$(x_{ij}^k, y_{ij}) \in Q_{ij}, \quad \forall ij \in A, \forall k \in K, \tag{3.9}$$

where

$$Q_{ij} := conv\{(x,y) \in \{0,1\}^{|K|} \times \mathbb{Z}^+ : x_{ij}^k = \sum_{w\in W} x_{ij}^{kw}, y_{ij} \geq \sum_{w\in W} y_{ij}^w,$$
$$\sum_{k\in K} D^k x_{ij}^{kw} \leq C y_{ij}^w \ \forall w \in W, x_{ij}^{kw} \in \{0,1\}, y_{ij}^w \in \{0,1\}, \forall k \in K, \forall w \in W\}$$

Q_{ij} is the projection on (x_{ij}^k, y_{ij}) of the polyhedron P_{ij}. Observe that the symmetric solutions of P_{ij} will project on a single point, and Q_{ij} would then be more suitable to investigate.

Polyhedron Q_{ij}, $ij \in A$ belongs to a more general class of polyhedra, associated with simple structured relaxations that may be considered for the CSLND problem. In what follows, we introduce a family of functions inducing some of these relaxations, and we show that we can get benefit from the characteristics of underlying polyhedra to better understand the related CSLND problem.

3.2 Set function polyhedra

Let E be a base set with n elements and let $f : 2^E \longrightarrow \mathbb{Z}_+$ be a set function over E. Let S be a subset of E. The incidence vector of S, denoted $x^S \in \{0,1\}^n$, is such that for each element $i \in E$

$$x_i^S = \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

By abuse of notation, we may write $f(x^S)$ to designate $f(S)$.

Definition 2 *A function f defined on a subset of elements $S \subseteq E$ with integer values is called monotonically increasing function if*

$$f(S \cup \{s\}) - f(S) \geq 0, \forall S \subseteq E, \forall s \in E \setminus S,$$

A combinatorial interpretation of such a function is that adding any element to the subset S may induce an increase of the function value.

Definition 3 *The function f is said to be unitary step monotonically increasing if* *(i)* $f(\emptyset) = 0$,

(ii) $f(S \cup \{s\}) - f(S) \in \{0, 1\}, \forall S \in E, \forall s \in E \setminus S$.

In other words, given a subset of element $S \subseteq E$, a function f is said to be unitary step monotonically increasing if adding any element s to the initial subset S yields an increase of at most one in the value of f.

Given a set function $f : \{0,1\}^n \longrightarrow \mathbb{Z}_+$. We define the convex hull of incidence vectors $(x^S, f(x^S))$, for all $S \subseteq \{0,1\}^n$ as follows:

$$P_f := conv\{(x, y) \in \{0,1\}^n \times \mathbb{Z}_+ : y = f(x) + \lambda(0, 0, 0, \ldots, 1), \lambda \geq 0\},$$

That is to say

$$P_f := conv\{(x, y) \in \{0,1\}^n \times \mathbb{Z}_+ : y \geq f(x)\},$$

The optimization problem associated with P_f may then be written as follows

$$\min\{y - \sum_{i \in E} c_i x_i : (x, y) \in P_f\},$$

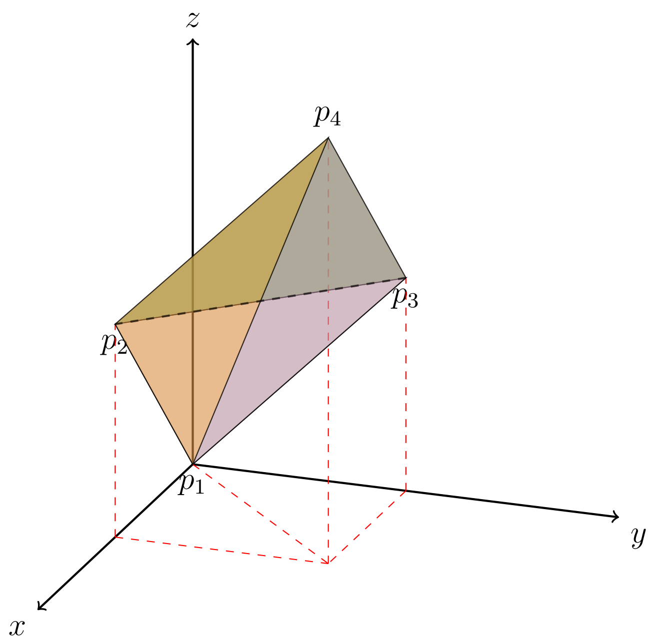
where $c \in \mathbb{R}^n$ is a vector of coefficients such that a coefficient $c_i > 0$ is associated with each element $i \in E$. We will refer to this problem as *set function problem*. Furthermore, given a subset of elements $S \subseteq E$, we will define the solution of a set function problem as the pair (x^S, y^S), such that associated incidence vector $(x^S, y^S \geq f(x^S)) \in P_f$. Besides, we let $x(S)$ be equal to $\sum_{i \in S} x_i$.

Example

Consider a simple set function given by $g : \{0,1\}^2 \times \mathbb{Z}_+ \longrightarrow \mathbb{R}$ such that $z = g(x, y) = x + y$. Figure 3.2 shows a set of solutions $(x, y, z = g(x, y)) \in \{0,1\}^2 \times \mathbb{Z}_+$.

These solutions are denoted p_1, p_2, p_3 and p_4, and we can see that $g(x, y)$ is a monotonically step increasing function. In fact, adding any non negative element to the solution induces an increasing of at most 1 in the value of $g(x, y)$. Then the convex hull of solutions p_1 to p_4 is given in the figure 3.2.

In what follows we will study the properties of polyhedra associated with general set functions. We will introduce two classes of valid inequalities, namely *Min Set I* and *Min Set II*, and discuss some necessary and sufficient conditions for these inequalities to define facets for any polyhedron having the form of P_f, where f is a set function.



$$M = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \end{array}$$

Figure 3.2: Polyhedron associated with $g(x, y)$

3.2.1 Properties of P_f for general f

3.2.1.1 Dimension

Theorem 3.3 *The polyhedron P_f is full dimensional.*

Proof. We shall exhibit $n+2$ solutions p_i, $i = 1, \ldots, n+2$, whose incidence vectors (x^{S_i}, y^{S_i}) are affinely independent. First, consider the solutions $(x^{S_i}, f(S_i))$ induced by the subsets $S_i = \{i\}$, for $i \in E$. Moreover, consider the solutions $(x^{\emptyset}, 1)$ and $(x^{\emptyset}, 0)$. It can be easily seen that these $n+2$ solutions are affinely independent. The solutions defined above are given in the matrix M_1 described thereafter.

$$M_1 = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_n \\ S_1 \\ S_0 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & \ldots & x_n & y \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 1 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 1 & \ldots & 0 & 1 \\ & & & & & \\ 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \end{array}$$

We can easily remark that the system formed by

$$\sum_{i=1}^{n+2} \lambda_i x_i = 0,$$
$$\sum_{i=1}^{n+2} \lambda_i = 0.$$

admits a unique solution $\lambda_i = 0$, for $i = 1, \ldots$, n + 2. It follows that the $n + 2$ rows contained in M_1 are affinely independent, thus P_f is full dimensional. □

Proposition 3.4 $r^* = (0, 0, ..., 0, 1)$ *is an extreme ray of* P_f.

Proof. Let $\mathcal{F}$=$\{\lambda r^*, \lambda \in \mathbb{R}_+\}$ be a face of the polyhedron P_f. We will show that the dimension of $\mathcal{F}$ is one. Consider the $m \times n$ matrix A and a vector b of $\mathbb{R}^m$ such that

$$P_f = \{(x, y) \in \{0, 1\}^n \times \mathbb{Z}_+ : Ax \leq b\}$$

Let $\overline{A}$ be the matrix containing the rows A_i of A such that $A_i r = 0$ for any ray r of P_f. We can see that $r^* = (0, 0, \ldots, 0, 1)$ verifies

$$x_i = 0, i \in E,$$

therefore, *rank*$(\overline{A}) = |E| = n$, and it follows that the dimension of $\mathcal{F}$ is

$$dim(\mathcal{F}) = n + 1 - rank(\overline{A}) = n + 1 - n = 1$$

hence, r^* is an extreme ray of P. □

In the sequel, we will use the following definition of P_f

$$P_f := conv\{(x, y) \in \{0, 1\}^n \times \mathbb{Z}_+ : y \geq f(x)\},$$

In what follows, we will be interested in the facial structure of P_f. In particular we study the trivial inequalities $x_i \geq 0$, and $x_i \leq 0$, for all $i \in E$, before introducing further facet defining valid inequalities.

3.2.1.2 Trivial inequalities

Theorem 3.5 *For $i \in E$, $x_i \geq 0$ defines a facet of P_f.*

Proof. Denote by $\mathcal{F}_i$ the face induced by inequality $x_i \geq 0$, that is

$$\mathcal{F}_i = \{(x, y) \in P_f : x_i = 0\},$$

Similarly to proof of Theorem 3.3, we can identify $n + 1$ solutions whose incidence vectors belong to P_f and also to $\mathcal{F}_i$. First consider the solutions $(x^{S_j}, 1)$, where $S_j = \{j\}$, for $j \in E \setminus \{i\}$. Also consider the solutions $(x^{\emptyset}, 1)$ and $(x^{\emptyset}, 0)$. Clearly, all these solutions are in P_f and in $\mathcal{F}$.

The $n + 1$ solutions described above are summarized in matrix M_2.

$$M_2 = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_{n-1} \\ S_n \\ S_0 \end{array} \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & \ldots & x_i & \ldots & x_n & y \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & \ldots & 0 & 1 \\ 0 & 1 & 0 & \ldots & 0 & \ldots & 0 & 1 \\ 0 & 0 & 1 & \ldots & 0 & \ldots & 0 & 1 \\ & & & & & & & \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \end{pmatrix} \end{array}$$

It is easy to see that the $n + 1$ rows of the matrix M_2 induces affinely independent incidence vectors, which completes the proof. □

Theorem 3.6 *For $i \in E$, $x_i \leq 1$ defines a facet of P_f.*

Proof. Let us denote by $\mathcal{F}_i$ the face induced by inequality $x_i \leq 1$, that is

$$\mathcal{F}_i = \{(x, y) \in P_f : x_i = 1\},$$

Consider the subsets S_j of E such that $S_j = \{j, i\}$, for $j \in E \setminus \{i\}$. Clearly, the solution $(x^{S_j}, 2)$ for $j \in E \setminus \{i\}$ belongs to P_f and also to $\mathcal{F}$. Moreover, the solutions $(x^E, |E|$

+ 1) also belong to P_f and to $\mathcal{F}$. Let us denote by M_3 the matrix containing the n + 1 solutions described above. M_3 is given as follows:

$$M_3 = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_{n-1} \\ S_{\tilde{i}} \\ E \end{array} \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & \dots & x_{\tilde{i}} & \dots & x_n & y \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \dots & 1 & \dots & 0 & 2 \\ 0 & 1 & 0 & \dots & 1 & \dots & 0 & 2 \\ 0 & 0 & 1 & \dots & 1 & \dots & 0 & 2 \\ & & & & & & & \\ 0 & 0 & 0 & \dots & 1 & \dots & 1 & 2 \\ 0 & 0 & 0 & \dots & 1 & \dots & 0 & 2 \\ 1 & 1 & 1 & \dots & 1 & \dots & 1 & |E|+1 \end{pmatrix} \end{array}$$

We can see that these n + 1 solutions are affinely independent. □

In what follows, we will show that all the non-trivial facets of the polyhedron P_f have non negative coefficients.

Theorem 3.7 *All the non-trivial facets of P_f are of the form $\sum_{i \in E} \pi_i x_i \leq \pi_0 y + p$, where p in a non negative integer parameter, $\pi_i, \pi_0 \geq 0$, and $\pi_i \leq \pi_0$, for all $i \in E$.*

Proof. We will first show that $\pi_i \geq 0$, for all $i \in E$. For this, assume that there exists an element $j \in E$ such that $\pi_j < 0$.

As $\sum_{i \in E} \pi_i x_i \leq \pi_0 y + p$ is different from $x_j \geq 0$, there must exist a subset of elements, say $S \subseteq E$, containing j and $y \in \mathbb{Z}_+$ such that the vector (x^S, y) belongs to P_f, and $\sum_{i \in E} \pi_i x_i^S = \pi_0 y + p$.

Consider the subset $S' = S \setminus \{j\}$ and the solution $(x^{S'}, y)$. Hence, $\sum_{i \in E} \pi_i x_i^{S'} \leq \pi_0 y + p$. In addition, since $\sum_{i \in E} \pi_i x_i^{S'} = \sum_{i \in E} \pi_i x_i^S$ - π_j, it follows that $\sum_{i \in E} \pi_i x_i^S - \pi_j \leq \pi_0 y + p$. As $\sum_{i \in E} \pi_i x_i^S = \pi_0 y + p$, we obtain $-\pi_j \leq 0$, which is a contradiction. Hence, $\pi_i \geq 0$, for all $i \in E$.

Now we shall show that $\pi_i \leq \pi_0$, for all $i \in E$.

Suppose that $\pi_j > \pi_0$, for some $j \in E$. As $\sum_{i \in E} \pi_i x_i \leq \pi_0 y + p$ is different from $x_j \leq 1$, there is a set $S \subseteq E \setminus \{j\}$ and $y \in \mathbb{Z}_+$ such that $\sum_{i \in E} \pi_i x_i = \pi_0 y + p$. Now consider the subset $\tilde{S} = S \cup \{j\}$ and consider the solution $(x^S,\, y{+}1)$. Then we have $\sum_{i\in E} \pi_i x^{\tilde{S}} \leq \pi_0 y + p$ for this solution. However, as $\sum_{i\in E} x_i^{S'} = \sum_{i\in E\setminus\{j\}} \pi_i x_i^S + \pi_j$, and $\pi_0 y^{S'} + p = \pi_0 y^S + \pi_0 + p$, we get $\sum_{i\in E\setminus\{j\}} \pi_i x_i^{S'} + \pi_j \leq \pi_0 y^S + \pi_0 + p$, since $\sum_{i\in E\setminus\{j\}} \pi_i x_i^S = \pi_0 y^S + p$, it implies that $\pi_j \leq \pi_0$ which is a contradiction. Hence, $0 \leq \pi_i \leq \pi_0$, for all $i \in E$ and the proof is complete. □

Remark 3.8 *Let S be a subset of E, then a non-trivial inequality of the format $\sum_{i\in S} \pi_i x^i \leq \pi_0 y + p$, is valid if and only if $p \geq \pi(S)$ - $\pi_0 f(S)$.*

In what follows we present two families of valid inequalities. We describe some conditions under which these inequalities may define facets for polyhedron P_f.

3.2.2 Min Set I inequalities

Proposition 3.9 *Let $f : \{0,1\}^n \longrightarrow \mathbb{Z}_+$ be a unitary step monotonically increasing function. Let S be a subset of E and p a non negative integer such that $p = |S| - f(S)$. Then, the following inequality*

$$\sum_{i\in S} x_i \leq y + p, \tag{3.10}$$

is valid for P_f.

Proof. Let S'' be a subset of elements of E, and define $S' = S'' \cap S$. Consider the solution induced by S'', whose incidence vector is denoted $(x(S''), f(S'')) \in P_f$. As the function f is unitary step monotonically increasing, the following is true

$$|S| - |S'| \geq f(S) - f(S'),\ \text{ for all } S' \subseteq S,$$

This implies that $|S'| \leq f(S') + |S| - f(S)$ and by the same way $|S'| \leq f(S'') + |S| - f(S)$ that may be obtained by substituting the point $(x(S''), f(S''))$ in the inequality (3.10). □

Theorem 3.10 *Given a subset $\tilde{S}$ of E and p a non negative integer parameter. Inequality $\sum_{i\in\tilde{S}} x_i \leq y + p$, define a facet of P_f if and only if the following conditions hold*

(i) $f(\tilde{S} \cup \{s\}) = |\tilde{S}| - p$, *for all* $s \in E \setminus \tilde{S}$,

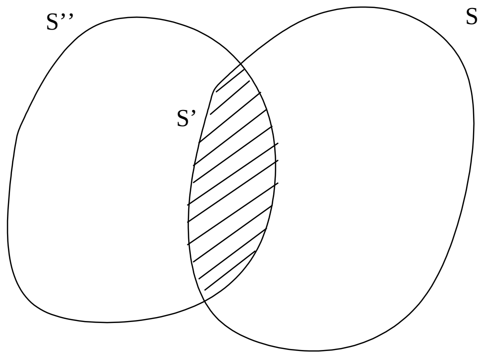


Figure 3.3: Proof of Proposition 3.9

(ii) $f(\tilde{S} \setminus \{\overline{s}\}) = |\tilde{S}| - p - 1$, *for all* $\overline{s} \in \tilde{S}$.

Proof. *Necessity*

(i) First, it is clear that $f(S \cup \{s\}) \leq f(S) + 1$, for all $s \in E \setminus S$. Suppose that there exists an element s of $E \setminus S$ such that $f(S \cup \{s\}) \leq |S| - p$. Then the inequality (3.10) with respect to $S \cup \{s\}$ can be written as

$$\sum_{i \in S \cup \{s\}} \leq y + |S| - f(S) = y + p, \tag{3.11}$$

However, (3.11) dominates (3.10), and therefore the latter cannot define a facet.

(ii) Clearly, $f(S \setminus \{s\}) \geq f(S) - 1$, for all $s \in E \setminus S$. Suppose there exists $s \in E \setminus S$, such that $f(S \setminus \{s\}) = |S| - p$. Then, inequality (3.10), with respect to $S \setminus \{s\}$ can be written

$$\sum_{i \in S \setminus \{s\}} \leq y + (|S| - 1 - f(S \setminus \{s\})) = y + |S| - 1 - f(S) = y + p - 1,$$

Inequality (3.10) can be obtained as a linear combination of the inequality above and $x_s \leq 1$. Therefore, it cannot define a facet.

Sufficiency

Assume now that conditions (i) and (ii) of Theorem 3.10 are fulfilled. We will denote by $\mathcal{F}$ the face induced by inequality (3.10). That is

$$\mathcal{F} = \{(x, y) \in P_f : \sum_{i \in S} x_i = y + p\}$$

We will exhibit $n + 1$ subsets of E, solutions of $\mathcal{F}$, and whose incidence vectors are affinely independent. First consider the solution $p_0 = (x^S, f(S))$. Clearly, $p_0 \in \mathcal{F}$. Now let us consider the solutions $p_s = (x^{S \cup \{s\}}, f(S))$, for $s \in E \setminus S$. As by (i), $f(S \cup \{s\}) = f(S)$, we have that p_s is a solution of P_f and also of $\mathcal{F}$. Finally, consider the solutions $p_s = (x^{S \setminus \{s\}}, f(S)$ - 1) for all $s \in S$. By (ii), it follows that p_s, for $s \in S$, is a solution of P_f. Moreover, p_s satisfies (3.10) with equality, and then it is also a solution of $\mathcal{F}$. Now, one can easily see that p_0, p_s for $s \in E \setminus S$, p_s for $s \in S$ are affinely independent. $\square$

3.2.3 Min Set II inequalities

Proposition 3.11 *Let $f : \{0, 1\}^n \longrightarrow \mathbb{Z}_+$ be a unitary step monotonically increasing function. Let S be a subset of E, p and q two non negative integers, with $q \geq 2$. Then, the inequality*

$$\sum_{i \in S} x_i \leq qy + p, \tag{3.12}$$

is valid for P_f if $p \geq |S'| - qf(S')$, for all $S' \subseteq S$.

Proof. Let S' be a subset of S. By summing trivial inequalities $x_i \leq 1$ over S', we get $\sum_{i \in S'} x_i \leq |S'|$ which is valid. On the other hand, by definition of the polyhedron P_f, we have that $y \geq f(S')$, for all $S' \subseteq S$. As $q \geq 0$, it then follows that, $q(y - f(S')) \geq 0$. Thus

$$\sum_{i \in S} x_i^{S'} = \sum_{i \in S'} x_i^{S'} \leq |S'| + q(y - f(S')) = qy + |S'| - qf(S') \leq qy + p,$$

yielding the validity of (3.12). $\square$

Theorem 3.12 *Given a subset of elements $S \subseteq E$, two non negative integers $q \geq 2$ and p. The inequality*

$$\sum_{i \in \tilde{S}} x_i \leq qy + p \tag{3.13}$$

defines a facet of P_f, if the following hold.

(i) There exists an integer $r \in \mathbb{Z}_+$, $p \leq r \leq |S|-1$, *such that for all* $S' \subseteq S$ *with* $|S'| = r$, $f(S') = \frac{|S'|-p}{q}$,

(ii) for all $s \in E \setminus S$, *there exists* $S' \subseteq S$ *such that* $f(S') = \frac{|S'|-p}{q} = f(S' \cup \{s\})$,

Proof. We will denote by $\mathcal{F}$ the face induced by inequality (3.13), i.e.

$$\mathcal{F} = \{(x, y) \in P_f : \sum_{i \in S} x_i = qy + p\}$$

Suppose that conditions (i) and (ii) hold. We will exhibit $n+1$ solutions of $\mathcal{F}$ that are affinely independent. Consider a subset S' of S such that $|S'| = r$. As by (i), $p \leq r \leq |S|-1$, $S' \neq \emptyset$ $S \emptyset S'$. Let e' and $\overline{e}'$ be elements of S' and $S \setminus S'$, respectively.

Consider the sets $S_e = S_e = (S' \setminus \{e'\}) \cup \{e\}$ for all $e \in S \setminus S'$ and $S_e = (S' \setminus \{e\}) \cup \{\overline{e}'\}$ for all $e \in S'$. Clearly, by (i), the solutions $(x^S,\ f(S))$, $(x^{S_e},\ f(S_e))$, $e \in S$ all belong to $\mathcal{F}$.

Next, for each $e \in E \setminus S$, by (ii) there exists $S'_e \subseteq S$ such that $f(S'_e) = \frac{|S'_e|-p}{q} = f(S'_e) \cup \{e\}$. Hence, the solutions $(x^{S'_e \cup \{e\}},\ f(S'_e \cup \{e\}))$ for all $e \in E \setminus S$ all belong to $\mathcal{F}$. Finally, consider the solution $(x^S,\ f(S) = \frac{|S|-p}{q})$ which is also in $\mathcal{F}$. Now, it is not hard to see that these solutions constitute a set of $n+1$ affinely independent points. $\square$

In the next section, we will study an application that illustrates well how our results for general set functions, are still valid for a specific function.

3.3 Bin-packing function

Given m items (demands) and n bins. We denote by D^k the weight of the item k, $k \in \{1, 2, ..., m\}$ and C is the capacity of each bin. The *Bin-Packing problem* (BPP) consists in assigning each item to one bin so that the total weight in each bin does not exceed C and the number of bins used is minimum [81].

We assume, without loss of generality, that the weights D^k and the capacity C are positive integer and $D^k \leq C$, $\forall k \in K$. Moreover, we can assume that the number of available bins n is large enough so a feasible packing exists for the m items.

The Bin-Packing Problem belongs to the class of NP-hard problems [51] and many approaches have been proposed to solve it during the three last decades. Most of the algorithms described in the literature are approximation algorithms and relatively complete part of them can be found in the survey of Coffman et al [34].

There exists less references on the exact algorithms developed to solve the Bin-Packing Problem to optimality. We cite Martello and Toth [81] that developed a branch-and-bound algorithm, based on an integer linear programming formulation. They also provided lower bounds and dominance criteria [82] for the BPP and evaluated them through a concept of worst-case performance. More recently, many linear programming formulations have been introduced to model the BPP and most of them are reviewed in the very good survey in [39], where Valério de Carvalho highlights the similarities between the Bin-Packing Problem and the One-Dimension Cutting Stock Problem and compares the presented LP formulations. In [38], Valério de Carvalho introduces an arc-flow formulation for the Bin-Packing problem and proposes an equivalent path formulation obtained by applying a Dantzig-Wolfe decomposition. He proposes a column generation procedure embedded within a branch-and-bound algorithm. Vanderbeck [104] and Vance [102] also proposes branch-and-price algorithms to solve the bin-packing and the one-dimensional cutting stock problems. In particular, Vanderbeck discussed some branching schemes and cutting planes in order to strengthen the formulation and improve the efficiency of his branch-and-bound algorithm. A cutting plane approach combined with column generation is developed in [18] for the case of multiple stock lengths in the one-dimensional cutting stock problem, which is closely related to bin-packing problem. Several works also focus on computing good lower bounds for bin-packing problem (see [33, 46, 81]).

We will denote by $N = \{1, 2, ..., n\}$ the set of available bins, and $K = \{1, 2, ..., m\}$ the set of items. Let us introduce the binary decision variable y_j, $j \in N$, that takes the value 1 if the bin j is used, and 0 otherwise. Let x_j^k, $k \in K$, $j \in N$ be a binary decision variable that takes the value 1 if the item k is assigned to the bin j, and 0 otherwise.

The Bin-Packing Problem is equivalent to the following integer linear programming formulation, given by Kantorovitch [67] in 1939 and also used by Martello and Toth later in [81] for their branch-and-bound approach:

$$\min \sum_{j \in N} y_j$$

$$s.t:$$

$$\sum_{k \in K} D^k x_j^k \leq C y_j, \qquad \forall j \in N, \tag{3.14}$$

$$\sum_{j \in N} x_j^k = 1, \qquad \forall k \in K, \tag{3.15}$$

$$0 \leq y_j \leq 1, \quad y_j \in \{0,1\}, \qquad \forall j \in N, \tag{3.16}$$

$$0 \leq x_j^k \leq 1, \quad x_j^k \in \{0,1\}, \forall k \in K, \forall j \in N. \tag{3.17}$$

In this formulation, there are $n \times (1 + m)$ binary decision variables and a polynomial number of constraints. The objective is to minimize the number of open bins needed to carry all of the items. Inequalities (3.14) are the capacity constraints of each bin while equalities (3.15) ensure that each item is assigned to exactly one bin. (3.16) and (3.17) are the trivial and integrity constraints. Note that a lower bound can be obtained by replacing (3.16) and (3.17) by

$$0 \leq y_j \leq 1, \qquad \forall j \in N, \tag{3.18}$$

$$0 \leq x_j^k \leq 1, \forall k \in K, \forall j \in N. \tag{3.19}$$

and (3.14)-(3.15)-(3.18)-(3.19) is the linear relaxation of the formulation (3.14)-(3.17).

Proposition 3.13 (Martello and Toth [81]) *The lower bound provided by the linear programming relaxation of this model is equal to* $\lceil \frac{\sum_{k \in K} D^k}{C} \rceil$

Proof. A valid solution to the linear relaxation formulation is $x_j^k = 1$ for $k = j$, $x_j^k = 0$ $\forall k \in K, j \in N$ such that $k \neq j$, and $y_k = \frac{D^k}{C}$, $\forall k \in K$. The corresponding value of the objective function is $\frac{\sum_{k \in K} D^k}{C}$. As the number of bins should be integer, the lower bound is equal to the smallest integer grater or equal to $\frac{\sum_{k \in K} D^k}{C}$. □

We will denote by P the convex hull of the solutions of the bin-packing problem. That is to say,

$$P := conv\{(x, y) \in \{0,1\}^{m \times n} \times \{0,1\}^n : (3.14) - (3.15) \text{ are satisfied}\}.$$

Due to the large number of possible assignments of items to the bins, there exists a large number of x_j^k variable values. Then, several equivalent solutions may arise and need to be checked during the exploration of branch-and-bound tree, which often makes the process time consuming. In other words, this formulation suffers from symmetry as one can arbitrary permute the bins [63]. For this reason, the polyhedron P should not be convenient to investigate. We will then consider further *aggregated* decision variables:

$$x^k = \begin{cases} 1, & \text{if the item k is assigned to some bin,} \\ 0, & \text{otherwise} \end{cases} \tag{3.20}$$

and the variable $y \in \mathbb{Z}_+$ being the number of bins used to pack the items of K. Note that an item is said to be *satisfied* if it is assigned to some bin (no matter which). We also give the relationship between the original variables and the aggregated ones:

$$\begin{aligned} x^k &= \sum_{j \in N} x_j^k, \qquad \forall k \in K, \\ y &= \sum_{j \in N} y_j. \end{aligned}$$

In what follows, we will study the polyhedron associated with BPP, using the aggregated variables. In particular, we will show how results provided for general set functions may be applied for bin-packing problem.

3.3.1 Associated Polyhedron

Consider the polyhedron Q defined as follows

$$Q := conv\{(x,y) \in \{0,1\}^m \times \mathbb{Z}^+ : x^k = \sum_{j \in N} x_j^k, y \geq \sum_{j \in N} y_j,$$
$$\sum_{k \in K} D^k x_j^k \leq C y_j, \forall j \in N, x_j^k \in \{0,1\}, y_j \in \{0,1\}, \forall k \in K, \forall j \in N\}$$

Q is the projection on (x^k, y) of the polyhedron P. We denote by $BP(S)$ the solution of the Bin-Packing problem for a subset of items S of K. In other words, $BP(S)$ is the minimum number of bins needed to carry the objects of S. We let x^S denote the incidence vector of S. By the same way, given a vector $x \in \{0,1\}^m$, we denote by $S(x)$ the subset of items induced by x. That is to say, $S(x) = \{k \in K, x^k = 1\}$. Then we provide an alternative definition of Q:

$$Q := conv\{(x,y) \in \{0,1\}^{|K|} \times \mathbb{Z}^+ : y \geq BP(S(x))\}$$

This polyhedron is associated with a problem that will be referred to as *Bin-Packing Function* (BPF). In what follows, we will study the dimension of the polyhedron Q and give the conditions under which inequalities Min Set I (3.10) and Min Set II (3.12) remain facet defining for Q.

Theorem 3.14 *Q is full dimensional.*

Proof. We will exhibit $m+2$ solutions whose incidence vectors are affinely independent. Let us introduce the m solutions S_k, $k \in K$, such that one bin is used to satisfy the item k, while the other items are not satisfied. Consider the incidence vector associated with each S_k, given by $(0, \dots, x^k = 1, 0, \dots, y = 1)$, $k \in K$. We denote by S_{k_1,k_2}, the solution defined as follows. Suppose that three bins are used to pack two among the m items, namely k_1 and k_2. Then, incidence vector of S_{k_1,k_2} is given by $(0, \dots, x^{k_1} = 1, x^{k_2} = 1, 0, \dots, y = 3)$. Consider now the solution S_0 where no item is satisfied and no bin is open. The associated incidence vector is then given by $(0, .., 0)$. It is clear that S_0, S_{k_1,k_2}, and S_k, $k \in K$, belong to polyhedron Q and their incidence vectors are affinely independent. Hence, the results follows. $\square$

3.3.1.1 Trivial Inequalities

Theorem 3.15 *For $\tilde{k} \in K$, inequality $x^{\tilde{k}} \geq 0$ defines a facet of Q.*

Proof. Let us denote by $\mathcal{F}_{\tilde{k}}$ the face induced by $x^{\tilde{k}} \geq 0$.

$$\mathcal{F}_{\tilde{k}} = \{(x, y) \in \{0, 1\}^m \times \mathbb{Z}_+ : x^{\tilde{k}} = 0\}$$

We can exhibit $m+1$ solutions of $\mathcal{F}_{\tilde{k}}$ having their incidence vectors affinely independent.

Let S_k, $k \in K$ be the solution corresponding to $x^k = 1$ for some $k \in K \setminus \{\tilde{k}\}$ while $x^{\tilde{k}} = 0$. The incidence vectors associated with S_k are $(0, 0, 0, ..., x^{\tilde{k}} = 0, 0, 0, ..., x^k = 1, 0, .., y = 1)$, for all $k \in K \setminus \{\tilde{k}\}$.

Now let us denote by S_{k_1,k_2} that consists in satisfying two among the m items, $k_1, k_2 \in K \setminus \{\tilde{k}\}$, by using three bins. $(x^{S_{k_1,k_2}}, y^{S_{k_1,k_2}})$ is then given by $(0, \dots, x^{k_1} = 1, x^{k_2} = 1, \dots, x^{\tilde{k}} = 1, \dots, y = 3)$.

We consider also S_0 with the associated incidence vector $(0, 0, ..., 0)$ where no item is satisfied. The incidence vectors associated with solutions described above are affinely independent, and the result follows. $\square$

Theorem 3.16 *For $\tilde{k} \in K$, inequality $x^{\tilde{k}} \leq 1$ defines a facet of Q*

Proof. We will denote by $\mathcal{F}_{\tilde{k}}$ the face induced by $x^{\tilde{k}} \leq 1$.

$$\mathcal{F}_{\tilde{k}} = \{(x, y) \in \{0, 1\}^m \times \mathbb{Z}_+ : x^{\tilde{k}} = 1\}$$

Similarly to the previous proof, we can identify $m + 1$ solutions having their incidence vectors in $\mathcal{F}$.

First consider the solution $S_{\tilde{k}}$ that is to open a unique bin, used to pack the item $\tilde{k}$. $S_{\tilde{k}}$ clearly induces a feasible solution, and $(x^{S_{\tilde{k}}}, y^{S_{\tilde{k}}}) \in \mathcal{F}_{\tilde{k}}$.

Next, we will exhibit $m-1$ solutions having their incidence vectors in both Q and $\mathcal{F}_{\tilde{k}}$. Consider the solutions S_k where $\tilde{k}$ and some additional item $k \in K \setminus \{k\}$ are satisfied. We set y^{S_k} to 3, for all $k \in K \setminus \{\tilde{k}\}$. In other words, 3 bins are open to pack two items, in each solution S_k, $k \in K \setminus \{\tilde{k}\}$. It is easy to see that S_k induce feasible solutions. Moreover, their incidence vectors are given by $(0, 0, 0, ..., x^k = 1, 0, x^{\tilde{k}}.., y = 3)$, with $k \in K \setminus \{\tilde{k}\}$, and belong to $\mathcal{F}_{\tilde{k}}$.

Now let us consider the solution S_m where a bin is assigned to each item of K. In other words, (x^{S_k}, y^{S_k}) is such that $x^k = 1$, for all $k \in K$ and $y = BP(K)$. This solution is obviously feasible and (x^{S_k}, y^{S_k}) belongs to the face $\mathcal{F}_{\tilde{k}}$. Furthermore, the incidence vectors of solutions previously described are affinely independent. Therefore, the proof is complete. □

3.3.2 Valid inequalities

In this section we will adapt the results obtained for Min Set I and Min Set II inequalities in the context of Bin-Packing Function, and we will show the relationship between this function and CSLND problem.

3.3.2.1 Min Set I Inequalities

Proposition 3.17 *Given a subset $S \subseteq K$ and a non negative integer $p \in \mathbb{Z}^+$, inequality*

$$\sum_{k \in S} x^k \leq y + p, \tag{3.21}$$

is valid for Q if and only if $p \geq |S| - BP(S)$.

Proof. *Sufficiency*

Suppose that $p \geq |S| - BP(S)$. Then by definition of polyhedron Q, we have $y \geq BP(S) \geq |S| - p$. Hence, we have $|S| \leq y + p$. Summing up trivial inequalities $x^k \leq 1$ over subset S yields $\sum_{k \in S} x^k \leq |S|$. In consequence, $\sum_{k \in S} x^k \leq y + p$. Thus inequality (3.21) is valid for Q.

Necessity

Suppose now that $p < |S| - BP(S)$. Then consider the solution that consists in using $BP(S)$ bins to pack all the items of S. Its incidence vector is given as follows. $x^k = 1$, if $k \in S$, and 0 otherwise, while $y = BP(S)$. This solutions is obviously feasible. However, it is cut off by (3.21). □

Theorem 3.18 *Let $\tilde{S}$ be a subset of K and p a non negative integer parameter. Inequality* (3.21) *induced by $\tilde{S}$ and p defines a facet of Q if and only if the following conditions hold*

1) $BP(\tilde{S}) = |\tilde{S}| - p$*,*

2) $BP(\tilde{S} \cup \{\tilde{s}\}) = |\tilde{S}| - p$*, where $\tilde{s}$ is the largest element of $K \setminus \tilde{S}$,*

3) $BP(\tilde{S} \setminus \{\overline{s}\}) \leq |\tilde{S}| - p - 1$*, where $\overline{s}$ is the smallest element of $\tilde{S}$.*

Proof. *Necessity*

We show that (i), (ii) and (iii) are necessary conditions for (3.21) to define facets.

(i) Suppose that inequality (3.21) induced by $\tilde{S}$ and p defines a facet of Q. Then, there must exist a solution, say $(\tilde{x}, \tilde{y})$, such that $\sum_{k \in \tilde{S}} \tilde{x}^k = \tilde{y} + p$. We have, by definition of polyhedron Q that $\tilde{y} \geq BP(\tilde{S})$. Thus, $BP(\tilde{S}) \leq \sum_{k \in \tilde{S}} \tilde{x}^k - p$, and then

$$BP(\tilde{S}) \leq |\tilde{S}| - p \tag{3.22}$$

Furthermore, the validity condition of (3.21) states that

$$BP(\tilde{S}) \geq |\tilde{S}| - p \tag{3.23}$$

Hence, by (3.22) and (3.23), we conclude that $BP(\tilde{S}) = |\tilde{S}| - p$.

(ii) Assume now that there exists an element $\widetilde{s}$ of $K \setminus \tilde{S}$ such that $BP(\tilde{S} \cup \{\widetilde{s}\}) \leq |\tilde{S}| - p$. Then, Min Set I inequality induced by $\tilde{S}$ and p is dominated by another constraint, namely

$$\sum_{k \in \tilde{S} \cup \{\widetilde{s}\}} x^k \leq y + p$$

In consequence, (3.21) can not be a facet of Q.

(iii) If $BP(\tilde{S} \setminus \{\overline{s}\}) \geq |\tilde{S}| - p$, we can see that (3.21) is dominated by

$$\sum_{k \in \tilde{S} \setminus \{\overline{s}\}} x^k \leq y + p$$

and $x^k \leq 1$, for all $k = \overline{s}$. Thus (3.21) can not define facets for Q.

Sufficiency

Assume now that conditions (i), (ii) and (iii) of Theorem 3.18 are satisfied. Let $\mathcal{F}$ be the face induced by inequality $\sum_{k \in \tilde{S}} x^k \leq y + p$, where

$$\mathcal{F} = \{(x, y) \in \{0, 1\}^n \times \mathbb{Z}_+ : \sum_{k \in \tilde{S}} x^k = y + p\}$$

We shall exhibit $m+1$ solutions denoted by S_k, $k \in \{1, ..., m+1\}$ of Q that also belong to $\mathcal{F}$. The construction of these $m + 1$ solutions is quite similar to proof of Theorem (3.10).

First consider the solution S_1 where we use $BP(\tilde{S})$ to pack $\tilde{S}$ items. The corresponding incidence vector is composed by the following entries. $x^k = 1$, for all $k \in \tilde{S}$, and $y = BP(\tilde{S}) = |\tilde{S}| - p$, by condition (i). It is clear that this solution is feasible, and its incidence vector is in $\mathcal{F}$.

Now we will provide $|K \setminus \tilde{S}|$ further solutions of Q that also belong to $\mathcal{F}$. Consider the solutions S_i, $i \in K \setminus \tilde{S}$ defined as follows. We add an item of $K \setminus \tilde{S}$, say i, to the solution, and we still use $BP(\tilde{S})$ bins to pack $\tilde{S} \cup \{i\}$. Condition (ii) ensures that such a solution is feasible by and, by conditions (i), it also belongs to $\mathcal{F}$.

Finally, we will construct the $|\tilde{S}|$ remaining by removing any item from $\tilde{S}$. The number of bins needed to packs $\tilde{S} \setminus \{i\}$, $i \in \tilde{S}$ is $|\tilde{S}| - p - 1$, since by condition (iii), the value of $BP(\tilde{S})$ decreases even if the smallest item of $\tilde{S}$ is removed from this subset. These solutions are clearly feasible, and, by conditions (i) and (iii), they belong to $\mathcal{F}$.

M_6 denotes a $(m+1) \times (m+1)$ matrix containing the incidence vectors of solutions described above.

$$
M_6 = \begin{array}{c} \\ S_1 \\ S_2 \\ \vdots \\ S_{|K \setminus \overline{S}|+1} \\ S_{|K \setminus \overline{S}|+2} \\ \vdots \\ S_{|K|+1} \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} x^1 & x^2 & x^3 & \dots & x^{|\overline{S}|} & x^{|\overline{S}|+1} & \dots & x^{|K|} & & y \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & \dots & 1 & 0 & \dots & 0 & & |\overline{S}| - \overline{p} \\
1 & 1 & 1 & \dots & 1 & 1 & \dots & 0 & & |\overline{S}| - \overline{p} \\
 & & & & & & & & & \\
1 & 1 & 1 & \dots & 1 & 0 & \dots & 1 & & |\overline{S}| - \overline{p} \\
1 & 1 & 1 & \dots & 0 & 0 & \dots & 0 & & |\overline{S}| - \overline{p} - 1 \\
 & & & & & & & & & \\
0 & 1 & 1 & \dots & 1 & 0 & \dots & 0 & & |\overline{S}| - \overline{p} - 1
\end{pmatrix}
\end{array}
$$

We can easily check that the incidence vectors of S_k, $k \in \{1, 2, ..., m+1\}$ are affinely independent. Hence, the proof is complete. □

3.3.2.2 Min Set II Inequalities

Proposition 3.19 *Let S be a subset of K, and p and q, two non negative integer parameters such that $q \geq 2$. Then, the inequality*

$$\sum_{k \in S} x^k \leq qy + p, \tag{3.24}$$

is valid for Q if and only if $p \geq (|S'| - qBP(S'))$, for all $S' \subseteq S$.

Proof. *Sufficiency*

Suppose that the inequality (3.24) is valid for Q. Then, by definition of polyhedron Q, we have that $y \geq BP(S) \geq BP(S')$, for all $S' \subseteq S$. Multiplying both sides of this inequality by q yields

$$qy \geq qBP(S) \geq qBP(S'), \quad \forall S' \subseteq S, \tag{3.25}$$

Besides, summing the trivial inequalities $x^k \leq 1$, over any S' gives

$$\sum_{k \in S'} x^k \leq |S'|, \tag{3.26}$$

By doing (3.25) - (3.26), we get

$$\sum_{k \in S'} x^k - qy \leq |S'| - qBP(S),$$

which is equivalent to

$$\sum_{k \in S'} x^k \leq qy + |S'| - qBP(S'), \forall S' \subseteq S,$$

and it follows $p \geq |S'| - qBP(S')$, for all $S' \subseteq S$.

Necessity

Assume now that there exists some subset $S' \subseteq S$ such that $p < |S'| - qBP(S')$. Then the solution having

$$x^k = \begin{cases} 1 & \text{if } k \in S', \\ 0 & \text{otherwise} \end{cases}$$

$$y = BP(S'),$$

belongs to Q but is cut off by inequality (3.24). Indeed, we would have $|S'| - qBP(S') \leq p$ which is a contradiction. $\square$

Example of Min Set II inequality that defines a facet Consider a set K of six items with sizes 12, 9, 8, 7, 3 and 2. Assume that each available bin has a capacity of 15. Then, the following inequality

$$x^1 + \quad x^5 + x^6 - 2 \times y \leq 0, \tag{3.27}$$

defines a facet of Q. In fact, we can exhibit $|K| + 1$ solutions of Q whose incidence vector are also in the face induced by (3.27). The matrix M_7 contains those affinely independent incidence vectors.

$$M_7 = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} x^1 & x^2 & x^3 & x^4 & x^5 & x^6 & y \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

3.3.3 CSLND using Bin-Packing function

Let us consider again the restriction of CSLND problem on one arc $ij \in A$. Recall that the polyhedron Q_{ij}, associated with this relaxation is defined as follows

$$Q_{ij} := conv\{(x,y) \in \{0,1\}^{|K|} \times \mathbb{Z}^+ : x_{ij}^k = \sum_{w \in W} x_{ij}^{kw}, y_{ij} \geq \sum_{w \in W} y_{ij}^w,$$
$$\sum_{k \in K} D^k x_{ij}^{kw} \leq C y_{ij}^w \ \forall w \in W, x_{ij}^{kw} \in \{0,1\}, y_{ij}^w \in \{0,1\}, \forall k \in K, \forall w \in W\}$$

Observe that Q_{ij}, $ij \in A$ is equivalent to polyhedron Q. Indeed, if an item is associated with a commodity, and a bin is associated with a module, then an instance of restricted CSLND problem can be obtained from an instance of Bin-Packing problem. In other words, CSLND problem restricted to one arc reduces to Bin-Packing problem.

Thus it is clear that the following formulation is equivalent to CSLND problem

$$\min \sum_{ij \in A} c_{ij} y_{ij}$$

$$\sum_{j \in V} x_{ji}^k - \sum_{j \in V} x_{ij}^k = \begin{cases} 1, & if\ i = d_k, \\ -1, & if\ i = o_k, \\ 0, & otherwise, \end{cases} \qquad \begin{matrix} \forall k \in K, \\ \forall i \in V, \end{matrix} \tag{3.28}$$

$$\sum_{k \in K} D^k x_{ij}^k \leq C y_{ij}, \qquad \forall ij \in A, \tag{3.29}$$

$$(x_{ij}^k, y_{ij}) \in Q, \qquad \forall k \in K, \forall ij \in A. \tag{3.30}$$

Where Q is the Bin-Packing function polyhedron. In consequence, facets of Q can be very useful to solve CSLND problem.

3.4 Concluding remarks

In this chapter, we considered the capacitated single-layer network design. We focused our attention on the arc-set polyhedron associated with this problem. We studied a set of functions that are, in fact, relaxations of the considered problem, when it is restricted to an arc. We investigated the basic properties of this polyhedron and derived new classes of valid inequalities. We then described necessary and sufficient conditions for theses inequalities to define facets. We presented an application of these results to the Bin-Packing function. In particular, our results concerning set functions polyhedra generalize those presented in [10] for the unsplittable traffic assumption. The identified valid inequalities were used to devise a branch-and-cut algorithm for the capacitated single-layer network design problem. The later was implemented to solve randomly generated instances, using realistic network topologies. The chapter 4 is dedicated to the algorithmic aspect of this implementation. We show in this chapter some preliminary experiments for the considered instances, that confirm the effectiveness of the identified valid inequalities.

Chapter 4

Branch-and-Cut Algorithm for the CSLND problem

Contents

In this chapter, we devise a Branch-and-Cut algorithm for the aggregated formulation of CSLND problem. This algorithm is based on the theoretical results presented in the previous chapter. First, we give an outline of the algorithm. Then, we describe the separation procedures used for some valid inequalities. Our objective is to discuss the algorithmic implementation of the cuts introduced in the polyhedral study and to give an insight of their effectiveness in practice. In particular, we test our approach on a set of random and realistic instances.

4.1 Branch-and-Cut algorithm

4.1.1 Overview

Here, we describe the framework of our algorithm. Suppose that we are given a bi-directed graph $G = (V, A)$ and a weight vector $c \in \mathbb{R}_+^A$ associated with the arcs of G. Let K be a set of commodities to be routed on G and W a set of available facilities per arc. To start the optimization, we consider the following linear program, LP$_{initial}$, given by the flow conservation constraints and the aggregated capacity constraints associated with the arcs of G, together with the trivial inequalities, that is

$$\min \sum_{(i,j)\in A} c_{ij} y_{ij}$$

$$\sum_{j\in V} x_{ji}^k - \sum_{j\in V} x_{ij}^k = \begin{cases} 1, & if i = d_k, \\ -1, & if i = o_k, \\ 0, & otherwise, \end{cases} \qquad \forall k \in K, \forall i \in V,$$

$$\sum_{k\in K} D^k x_{ij}^k \leq C y_{ij}, \qquad \forall ij \in A,$$

$$0 \leq x_{ij}^k \leq 1, \qquad \forall k \in K, \forall ij \in A,$$

$$y_{ij} \geq 0, \qquad \forall ij \in A.$$

Let us denote by $(\overline{x}, \overline{y}) \in \mathbb{R}^{K\times A} \times \mathbb{R}^A$ the optimal solution of this relaxation of the CSLND problem. Note that $(\overline{x}, \overline{y})$ is feasible for the problem if it is integer and it satisfies all Min Set I inequalities. Usually, this is not the case. Then, at each iteration of the Branch-and-Cut algorithm, one has to generate further valid inequalities. This procedure is known as the *separation problem*, and consists, given a class of valid inequalities in deciding whether if there exists an inequality violated by the current solution. The identified inequalities are added to the current linear program, that is solved again. This procedure is repeated until no violated inequality may be found. The final solution if then optimal for the linear relaxation of the aggregate formulation. If the solution is integral, then it is optimal for CSLND problem. Otherwise, we create two new problems by branching on a fractional variable. The separation routine is then performed at each node of the Branch-and-Cut tree until the optimal solution is found. The Branch-and-Cut algorithm uses the classes of inequality previously introduced and their separation is performed in the following order

1. Min Set I inequalities

2. Min Set II inequalities

The algorithm 3 summarizes the principal steps of the Branch-and-Cut algorithm.

Algorithm 3: Branch-and-cut algorithm

Data : a graph $G = (V, A)$, a set of commodities K, a set of available facilities W, and a cost vector $c \in \mathbb{R}^A$.
Output : optimal solution of CSLND problem, or best feasible upper bound.

1: LP $\leftarrow$ LP$_{initial}$
2: solve the linear program LP.
   let $(\overline{x}, \overline{y})$ be the optimal solution of LP.
3: **If** $(\overline{x}, \overline{y})$ is feasible for CSLND problem **then**
   $(\overline{x}, \overline{y})$ is an optimal solution. STOP
4: **If** constraints (Min Set I and Min Set II) violated by $(\overline{x}, \overline{y})$ are found **then**
   add them to LP.
   go to 2.
5: **else**
   create two sub-problems by branching on a fractional variable.
6: return the best solution for all the sub-problems.

In our Branch-and-Cut algorithm, we apply the following separation strategy. At each separation procedure, we can add more than one violated inequality if there is any. Also we move to the separation of a new class of inequalities only if no additional inequality can be identified in the current class. Note that the cutting plane is a global procedure, applied to all the nodes of the Branch-and-Cut tree. This allows to get the best possible lower bound. In what follows, we describe the separation algorithm used to identify violated inequalities introduced above. We devised heuristic procedures for the separation of both classes of valid inequalities. First, let us introduce the test used to check whether if a solution $(\overline{x}, \overline{y}) \in \mathbb{R}^{K \times A} \times \mathbb{R}^A$ is feasible for CSLND problem.

4.1.2 Feasibility test

The basic constraints of the aggregated formulation (3.5)-(3.6) are not enough to guarantee that a solution $(\overline{x}, \overline{y})$, even integer, is feasible. In fact, this solution has to satisfy all the Min Set I inequalities. We have considered a feasibility test that verifies if $(\overline{x}, \overline{y})$ is feasible or not. This tests consists in checking, for each arc $a \in A$, if $\overline{y}_a \geq BP(S)$, or not. Here, S is the subset of commodities using a. In other words,

$S = \{k \in K, \overline{x}_a^k > 0.1\}$. Note that this test requires, for each arc, and given a subset of commodities S, the computation of $BP(S)$. For this test, we perform $BP(S)$ computation by solving the ILP formulation (3.14)-(3.17) using a branch-and-bound algorithm.

4.1.3 Separation of Min Set I inequalities

In this section we discuss the separation problem of Min et I inequalities. This problem consists, given a fractional solution $(\overline{x}, \overline{y})$, in finding a Min Set I inequality (3.21) that is violated by $(\overline{x}, \overline{y})$, or showing that no such inequality exists. Namely, one has to identify a subset of commodities $S \subseteq K$, and a non negative integer parameter p that induces a valid Min Set I inequality

$$\sum_{k \in K} x_a^k \leq y_a + p.$$

for some arc $a \in A$. As the validity condition requires the computation of $BP(S)$ value for each subset S, we have to embed the resolution of the bin packing problem within the separation process.

In other words, for a particular arc $a \in A$, one has to produce a subset S that ensures the validity condition and decide whether the induced Min Set I inequality is violated or not. To formulate this separation problem, we will introduce a binary decision variable denoted α^k, $k \in K$, that takes the value 1 if the commodity k belongs to S, and 0 otherwise. Note that $S = \{k \in K : \alpha^k = 1\}$ and $|S| = \sum_{k \in K} \alpha^k$. The separation problem is then equivalent to the following :

$$\max \sum_{k \in K} \overline{x}_{ij}^k \alpha^k - \overline{y}_{ij} - p$$

$$s.t :$$

$$BP(S) \geq \sum_{k \in K} \alpha^k - p \tag{4.1}$$

$$0 \leq \alpha^k \leq 1, \alpha^k \in \{0,1\}, \forall k \in K. \tag{4.2}$$

The objective function states that we are looking for the most violated constraint while the inequality (4.1) ensures that this constraint is valid. Observe that inequality (4.1) requires the computation of $BP(S)$ within the separation process, which further complicates the problem. In what follows, we show that the separation problem associated with Min Set I inequalities is not in the class NP.

Proposition 4.1 *The separation problem* (4.1)-(4.2) *does not belong to the class NP*

Proof. An instance of the separation problem (4.1)-(4.2) is defined as follows

Instance

- a set of commodities K. Each commodity $k \in K$ has a traffic value $D^k > 0$,
- a set of available facilities W. Each facility has a capacity $C > 0$, $D^k \leq C$, for all $k \in K$,
- a "gain" given by the vector $(\overline{x}, \overline{y})$, corresponding to the current fractional solution. We can consider that a gain $\overline{x}^k$ is associated to each commodity $k \in K$,
- an integer parameter $p = |S| - BP(S)$.

The separation problem (4.1)-(4.2) is to decide whether there is a subset of commodities $S \subseteq K$ that maximizes the total gain, such that the smallest number of facilities needed to pack all the commodities of S is greater or equal than $|S| - p$. Now consider a solution $\tilde{\alpha} \in \{0, 1\}^{|K|}$ of the formulation (4.1)-(4.2), and the corresponding subset of commodities $\tilde{S}$. One has to solve the bin packing problem in order to check if $\tilde{\alpha}$ is feasible for (4.1)-(4.2). As the bin packing problem is NP-hard, it is not possible to answer to the question above by using a polynomial algorithm. Hence, the separation problem does not belong to the class NP. □

As the separation problem for Min Set I inequalities does not belong to the class NP, one can not aim to perform an exact separation of this class. In the next section, we will present a heuristic procedure that we devised. The idea of this routine is to consider the separation of a relaxed version of Min Set I inequalities, which is "easier" to handle.

4.1.3.1 Heuristic separation

For each arc $a \in A$, we look for a Min Set I inequality (3.21) that is violated by the current fractional solution $(\overline{x}, \overline{y})$. Let a be an arc of A, and S be a subset of commodities in K. Consider the inequality

$$\sum_{k \in S} x_a^k \leq y_a + p_r \tag{4.3}$$

where p_r is a non negative integer such that $p_r = |S|$ - $\sum_{k\in S}\frac{D^k}{C}$. Notice that p_r is obtained by replacing $BP(S)$ by the trivial lower bound $\sum_{k\in S}\frac{D^k}{C}$, in $p = |S|$ - $BP(S)$. Separating this new class of inequalities allows to easily exhibit a subset S which might induce a violated Min Set I inequality. Besides, given an arc $a \in A$, and a fractional solution $(\overline{x}, \overline{y})$ of (3.5)-(3.6), the separation problem associated with inequality (4.3) is equivalent to the following integer linear program:

$$\max Z = \sum_{k\in K}(\overline{x}_a^k + \frac{D^k}{C} - 1)\alpha^k - \overline{y}_a \tag{4.4}$$

$$0 \leq \alpha^k \leq 1, \alpha^k \in \{0,1\}, \qquad \forall k \in K. \tag{4.5}$$

The function (4.4) is maximised using a simple greedy procedure working as follows. Given a fractional solution $(\overline{x}, \overline{y})$, we start with an empty set S, then we check for each $k \in K$ if adding the commodity k to S increases the value of Z. This greedy procedure allows to iteratively build a subset S that maximizes the function Z (4.4). We use the greedy algorithm given above within a heuristic separation described in

Algorithm 4: Greedy procedure SEP_MSI

Data : a fractional solution $(\overline{x}, \overline{y})$, a set of commodities K, an arc $a \in A$.
Output : a subset of commodities S that might induce a violated Min Set I inequality.

1: $S = \emptyset$,
2: $Z^* \leftarrow -\overline{y}_a$,
3: **Forall** $k \in K$ **do**
    $Z \leftarrow (\overline{x}_a^k + \frac{D^k}{C} - 1) + Z^*$,
    If $Z > Z^*$ **then**
        $Z^* \leftarrow Z$,
        $S = S \cup \{k\}$,
4: return S.

Algorithm 5, that may be presented as follows. Our separation algorithm consists first in determining, for each arc $a \in A$, a subset of commodities S using the greedy procedure described in Algorithm 4. Based on subset S, we give an approximate value of the parameter p by using Fekete and Shepers's dual feasible function [46] to find a good lower bound of $BP(S)$. In fact, this bound is computed using the so-called *dual feasible functions*. These functions have been used first by Johnson [66] then by Lueker in [76] to derive lower bounds for bin-packing problems (see [33] for detailed description of dual feasible functions used in the literature to obtain either lower bounds or valid inequalities within a cutting planes context). In particular we use the class of lower bounds introduced by Fekete and Schepers in [46], that is $L_*^{(p)}$ with $p = 2$. This class of bounds allows to strengthen the elementary bounds L_1 and L_2 given by Martello and Toth in [81, 82]. We will denote this function by $f_{FS}(S)$. We finally check if the identified subset S and parameter p produce a violated Min Set I inequality and add the obtained constraint to the current linear program if so. Fekete and Shepers's function

Algorithm 5: Heuristic separation of inequalities (3.21)

Data : fractional solution $(\overline{x}, \overline{y})$
Output : a set $\mathcal{S}$ of Min Set I inequalities violated by $(\overline{x}, \overline{y})$
$\mathcal{S} \leftarrow \emptyset$;
Forall $a \in A$ **do**
    $S_a = \text{SEP\_MSI}(\overline{x}, \overline{y}, a)$,
    `/* the set of commodities that may induce a violated Min Set I */`
    `/* inequality for` a
    Compute the parameter $p = |S| - f_{FS}(S)$
    If $\sum_{k \in S_a} \overline{x}_a^k - \overline{y}_a > p$ **then**
    `/* there is a violated Min Set I inequality */`
        Denote $\mathcal{S}_a$ this inequality;
        $\mathcal{S} \leftarrow \mathcal{S}$;
return the identified violated Min Set I inequalities S;
`/*` $\mathcal{S} = \emptyset$ `if no violated Min Set I inequalities are detected */`

$f_{FS}(S)$ that gives a lower bound of $BP(S)$ can be computed in $\mathcal{O}(|K|log(|K|))$. In fact, the computational effort consists in sorting the commodities by traffic amount. As the operation is iterated for each arc of A, our separation procedure is running in $\mathcal{O}(m|K|log(|K|))$, where $m = |A|$. However, if the commodities are sorted by traffic amount, then we have a complexity of $\mathcal{O}(m|K|)$.

4.1.4 Separation of Min Set II inequalities

Our separation algorithm for Min Set II inequalities (3.24) consists first in identifying a subset of commodities S that induces a violated constraint, and satisfies certain conditions. The former step is performed by using the greedy procedure described in Algorithm 4. We consider the separation of inequalities with $q = 2$ and $p = 0$. The validity condition for this inequality requires the verification of all subsets of S. This number may be very large ($2^{|S|}$ subsets of S), we only look at sets with at most 4 elements. In such a way that the subsets of S are thus not so many, and it possible to verify in a relatively small time if the validity condition is satisfied. We then check if the corresponding Min Set II inequality is violated by the current fractional solution, and we add it to the current linear program if so.

4.2 Computational study

4.2.1 Implementation's feature

We now briefly describe the hardware and software tools used for implementations as well as the instances considered for the experiments, before giving the numerical results. The Branch-and-Cut algorithm depicted in the previous section has been implemented in C++ using CPLEX 12.5 Callable Library [2] as a linear solver and to handle the Branch-and-Cut framework. Our algorithm was tested on a Bi-Xeon quad-core E5507 2.27GHz with 8Go of RAM, running under Linux distribution. Finally, we have fixed a CPU time limit of five hours.

All the experiments for this algorithm have been conduced on SNDlib based instances with two types of traffic matrices : randomly generated and realistic. Both types are described in the following sections.

4.2.2 Description of instances

The experiments given here have been obtained by considering instances from a library dedicated to the optimization of telecommunication networks, namely SNDlib [1]. The set of considered instances includes instances with randomly generated commodities and realistic commodities. Both classes of instances are characterized by the number of nodes V, the number of arcs A, the number of available facilities denoted W, and the number of commodities K.

4.2.2.1 Instance with randomly generated traffic

These instances are based on data from *polska*, *nobel_us*, *newyork*, *geant*, *ta1*, *norway* and *pioro40* instances. The set of nodes and arcs in G correspond to those of SNDlib instances for bidirected link instances, while each edge of undirected SNDlib instances induces two inversely directed arcs in our instances. We associate with each arc a length that is rounded euclidean distance between the arc's vertices. Moreover, a facility settled on an arc induces a cost equivalent to the length of this arc. The available facilities are supposed to have the same capacity and their number is fixed to ten facilities per arc, for all the instances ($W = 10$). Concerning the traffic matrices, we randomly generate the origin node, the destination node and the traffic amount of each commodity. The commodities traffic is uniformly distributed in $]\epsilon, C]$, where $\epsilon = 0.2C$. Finally, we generate five examples of each previously described instance, and we give the average results obtained for each instance.

4.2.2.2 Instance with realistic traffic

The realistic instances considered here are based on SNDlib network topologies as well as instances described above. We used topologies derived from data of *abilene*, *atlanta*, *nobel_germany*, *france*, *nobel_eu*, *india35*, *cost266* and *zib54*. Again, V corresponds to the set of nodes of SNDlib instances, and A is either obtained from the set of edges of SNDlib undirected link instances, or equivalent for the bi-directed link instances. We assume that we can install up to five facilities having the same capacity, on arcs of each instance. Furthermore, concerning the commodities, we choose the K most important commodities according to the traffic amount for each instance.

4.2.3 Data preprocessing

We describe here a simple preprocessing operation we have performed on our instances data to enhance the solution process. The idea of this operation comes from some techniques used for computing lower bounds for bin-packing. In fact, each commodity which is not compatible with any other commodity into the same subband is increased to the subband capacity. In other words, commodities that are too large to fit with any other commodity are increased to fill completely the capacity of a subband. More formally, let us denote by k a commodity of this class. Then, if $D^k + D^{k'} \geq C$, for all $k' \in K$, then $D^k = C$. In particular, this may help ton increase the bound on design variables y, that are closely related to the value of ratio $\frac{D^k}{C}$, for all $k \in K$ (see Table 4.3 for further details on the difficulty of an instance).

4.2.4 Computational results

In this section we present some experiments obtained by using our Branch-and-Cut approach on both classes of instances, previously described. The results are reported in the tables given in what follows. The entries of the various tables are:

V	:	number of nodes in G,
A	:	number of arcs,
K	:	number of commodities,
NmsI	:	number of generated Min Set I inequalities,
NmsII	:	number of generated Min Set II inequalities,
nodes	:	number of nodes in the Branch-and-Cut tree,
o/p	:	number of problem solved to optimality over number of tested instances (only for instances with randomly generated traffic),
Gap	:	the relative error between the best upper bound (optimal solution if the problem has been solved to optimality) and the lower bound obtained at the root node of the Branch-and-Cut tree
TT	:	total CPU time in h:m:s
TTsep	:	CPU time spent in performing the constraints separation.

The instances whose CPU time reaches 5 hours are not solved to optimality. For those instances, the gap is indicated in italic.

4.2.4.1 The effectiveness of Branch-and-Cut algorithm

Our first series of experiments concerns a subset of instances with randomly generated commodities. Those instances were handled using a Branch-and-Bound algorithm to solve the compact formulation (3.1)-(3.3), and a Branch-and-Cut algorithm to solve the aggregated formulation (3.5)-(3.6) and by considering valid inequalities. The idea behind these experiments is to bring out the efficiency of valid inequalities introduced in the previous chapter. The results are summarized in Table 6.1. In Table 6.1 are presented three parameters that usually make it possible to compare the performance of two approaches, namely the gap, the number of nodes of the Branch-and-Bound (respectively Branch-and-Cut) tree, and the CPU time for computation. It appears from Table 6.1 that the aggregated formulation with valid inequalities performs better than the compact formulation for all the instances. In fact, we can notice that the Branch-and-Cut approach allows to solve to optimality all tested instances in a very short time (less than 5 minutes for all the instances except for *newyork*_20_1, *ta1*_16_1, *ta1*_18_1, and *ta1*_20_1), whereas the Branch-and-bound performs worse results for all the instances according to this criterion. See for example *newyork*_10_1 which is solved to optimality instantly by Branch-and-Cut while it could not be solve to optimality within 5 hours using the Branch-and-Bound. Indeed, a significant number of instances could not be solved to optimality within 5 hours using the Branch-and-Bound algorithm. Also, we can see that the gap is more important for some instances using Branch-and-Cut algorithm. However, it is clear that the number of Branch-and-Bound tree's nodes is much more greater than one of Branch-and-Cut, for almost all the instances. The instance *newyork*_8_1, for example, required 10206 nodes in the Branch-and-Bound tree to be solved to optimality, while the Branch-and-Cut explored only 157 nodes.

The results presented in Table 6.1 clearly shows the gain provided by using the valid inequalities introduced in the previous chapter, within a Branch-and-Cut framework. Indeed, our approach allows to solve efficiently the CSLND problem and requires less CPU time and fewer explored nodes than the Branch-and-Bound approach.

We give hereafter the results of the experiments for SNDlib instances with realistic and randomly generated traffic matrices.

4.2.4.2 Instances with randomly generated traffic matrices

Our second series of experiments concerns instances based on SNDlib topologies, with randomly generated commodities. The instances considered here have graphs with 12 up to 40 nodes and 36 up to 178 arcs, while the number of commodities varies from 4 to 40 (4 to 20 for *pioro*). The results are reported in Table 4.2. Note that instances used in Table 6.1 are a part of those used in Table 4.2. We can see from Table 4.2 that, in average 34 among 46 families of instances have been solved to optimality within the fixed time limit (i.e. Opt = 5/5). Also remark that for only 6 families of instances, the Branch-and-Cut could not provide any optimal solution within 5 hours. Moreover, 5 over 33 families of instances solved to optimality have a gap value greater than 30%. The remaining instances have a gap that may reach 66% (instance *pioro*_4). These observations together with the gap value raise the question of what makes an instance difficult, outside of its size.

In order to answer this question, we have made some experiments on a family of instances with different traffic amount. We have considered the topology of *atlanta* network which has 15 nodes and 44 arcs, while the commodities ranges from 5 to 50. For each instance size we have generated five types of commodities. In fact, the first type of commodities are generated in interval $]0, \frac{C}{4}]$, the second type of commodities are

Table 4.1: Aggregated formulation versus Compact formulation

				Compact formulation			Aggregated formulation (B&C)				
Instance	|V|	|A|	|K|	Gap	Nodes	TT	Gap	Nodes	TT	NmsI	NmsII
polska_2_1	12	36	2	0.00	1	0:00:00	0.00	1	0:00:00	2	0
polska_4_1	12	36	4	18.71	1829	0:00:28	21.21	12	0:00:00	44	0
polska_6_1	12	36	6	0.00	1	0:00:00	2.41	9	0:00:00	86	0
polska_8_1	12	36	8	27.68	1076	0:05:26	21.68	56	0:00:02	143	2
polska_10_1	12	36	10	14.04	919	0:02:56	12.28	71	0:00:05	384	0
polska_12_1	12	36	12	14.15	977	0:02:19	11.33	100	0:00:05	499	3
polska_14_1	12	36	14	*18.83*	83500	5:00:00	18.70	367	0:00:22	619	4
polska_16_1	12	36	16	13.48	1241	0:03:56	12.17	249	0:00:24	840	2
polska_18_1	12	36	18	12.32	55299	1:35:48	14.08	421	0:00:50	1372	11
polska_20_1	12	36	20	*27.33*	16777	5:00:00	23.11	596	0:01:49	692	7
nobel_us_2_1	14	42	2	22.37	182	0:00:09	58.83	18	0:00:00	46	0
nobel_us_4_1	14	42	4	30.98	1734	0:02:23	39.57	14	0:00:00	46	0
nobel_us_6_1	14	42	6	0.00	1	0:00:01	25.55	17	0:00:00	109	0
nobel_us_8_1	14	42	8	13.29	4978	0:23:03	22.35	64	0:00:03	169	0
nobel_us_10_1	14	42	10	*29.08*	82768	5:00:00	29.02	114	0:00:01	158	0
nobel_us_12_1	14	42	12	*11.75*	71534	5:00:00	28.90	278	0:00:26	291	2
nobel_us_14_1	14	42	14	*27.47*	55221	5:00:00	22.50	302	0:00:48	556	14
nobel_us_16_1	14	42	16	*6.57*	74429	5:00:00	20.28	281	0:00:46	580	21
nobel_us_18_1	14	42	18	*7.48*	64164	5:00:00	22.34	1001	0:03:45	757	27
nobel_us_20_1	14	42	20	*10.79*	13291	5:00:00	22.92	295	0:00:49	434	6
newyork_2_1	16	98	2	0.00	1	0:00:00	0.19	5	0:00:00	22	0
newyork_4_1	16	98	4	38.60	11354	1:12:00	16.95	24	0:00:00	78	0
newyork_6_1	16	98	6	25.12	10822	1:21:01	20.28	58	0:00:03	113	0
newyork_8_1	16	98	8	35.91	10206	1:33:28	27.65	157	0:00:24	226	2
newyork_10_1	16	98	10	12.65	1360	0:17:06	1.82	10	0:00:01	88	1
newyork_12_1	16	98	12	13.94	974	0:07:24	1.84	13	0:00:01	109	1
newyork_14_1	16	98	14	21.19	489	0:06:15	8.01	46	0:00:05	268	3
newyork_16_1	16	98	16	22.17	1859	0:16:00	5.96	47	0:00:05	246	2
newyork_18_1	16	98	18	21.65	5663	0:20:04	10.08	288	0:00:59	905	10
newyork_20_1	16	98	20	*28.18*	8032	5:00:00	18.41	7776	2:01:37	2250	70
geant_2_1	22	72	2	0.00	1	0:00:00	0.00	1	0:00:00	6	0
geant_4_1	22	72	4	23.60	2617	0:04:46	22.27	7	0:00:00	41	0
geant_6_1	22	72	6	15.09	865	0:01:36	5.48	5	0:00:00	22	0
geant_8_1	22	72	8	13.23	2284	0:14:15	28.82	29	0:00:02	144	1
geant_10_1	22	72	10	8.28	659	0:00:45	1.96	12	0:00:00	80	0
geant_12_1	22	72	12	16.28	4885	0:13:57	11.91	21	0:00:00	158	0
geant_14_1	22	72	14	5.13	1522	0:03:13	6.07	28	0:00:01	182	0
geant_16_1	22	72	16	3.22	1216	1:01:10	5.62	15	0:00:00	133	0
geant_18_1	22	72	18	0.26	28	0:00:21	6.51	29	0:00:01	207	4
geant_20_1	22	72	20	*12.34*	17900	5:00:00	10.49	211	0:00:52	438	4
ta1_2_1	24	102	2	0.00	1	0:00:01	22.23	9	0:00:00	38	0
ta1_4_1	24	102	4	23.81	2021	0:12:18	32.99	57	0:00:01	76	0
ta1_6_1	24	102	6	17.80	11190	1:51:06	26.17	66	0:00:05	199	1
ta1_8_1	24	102	8	20.69	12434	2:33:57	32.13	405	0:00:50	391	2
ta1_10_1	24	102	10	7.47	2084	0:14:10	6.83	55	0:00:08	282	0
ta1_12_1	24	102	12	8.88	11453	0:32:03	22.89	301	0:00:22	378	4
ta1_14_1	24	102	14	*21.84*	25186	5:00:00	29.65	898	0:02:05	770	2
ta1_16_1	24	102	16	*8.14*	23252	5:00:00	26.94	2450	0:09:25	1423	9
ta1_18_1	24	102	18	*8.52*	20941	5:00:00	25.48	4188	0:26:33	1724	31
ta1_20_1	24	102	20	*30.91*	12088	5:00:00	27.23	27257	4:42:36	2307	58

Table 4.2: Branch-and-Cut results for SNDlib instances with random traffic

Instance	V	A	K	NmsI	NmsII	Gap	Opt	nodes	TT
polska_4	12	36	4	50.2	0	16.81	5/5	11	0:00:00
polska_6	12	36	6	70.6	0	10.78	5/5	11	0:00:00
polska_8	12	36	8	143.6	1.2	14.40	5/5	47.2	0:00:01
polska_10	12	36	10	247.6	3.2	14.61	5/5	66.4	0:00:03
polska_15	12	36	15	533.2	3.6	16.63	5/5	205.8	0:00:25
polska_20	12	36	20	970.8	39.8	17.78	5/5	835.2	0:03:37
polska_30	12	36	30	3294.2	149.2	15.06	4/5	4818	1:14:15
polska_40	12	36	40	7596.8	388.4	16.43	1/5	17778.6	4:02:08
nobel_us_4	14	42	4	72.8	0.8	29.25	5/5	29.6	0:00:00
nobel_us_6	14	42	6	90	0.4	23.23	5/5	21.6	0:00:00
nobel_us_8	14	42	8	158.2	0.8	26.13	5/5	54.4	0:00:01
nobel_us_10	14	42	10	123.6	0.4	21.69	5/5	54	0:00:01
nobel_us_15	14	42	15	795.2	26.4	26.98	5/5	3056.2	0:06:03
nobel_us_20	14	42	20	1609.2	37	26.15	5/5	6399.2	0:29:31
nobel_us_30	14	42	30	3566.8	62.4	22.35	4/5	8436.6	1:17:57
nobel_us_40	14	42	40	8723.6	129.8	*25.37*	0/5	14616	5:00:00
newyork_4	16	98	4	43	0	8.71	5/5	11.8	0:00:00
newyork_6	16	98	6	157.4	2	23.02	5/5	94	0:00:03
newyork_8	16	98	8	281.2	4.8	19.70	5/5	184.6	0:00:12
newyork_10	16	98	10	271.2	2.4	10.61	5/5	110.6	0:00:10
newyork_15	16	98	15	598	11.4	12.66	5/5	527.4	0:01:33
newyork_20	16	98	20	1993.6	28.2	14.09	5/5	3778.4	0:34:21
newyork_30	16	98	30	4683.4	101.2	*15.78*	0/5	17894.6	5:00:00
newyork_40	16	98	40	8994.8	99.4	*60.53*	0/5	10812.2	5:00:00
geant_4	22	72	4	45.6	0	25.16	5/5	12.2	0:00:00
geant_6	22	72	6	47.8	0	7.63	5/5	8	0:00:00
geant_8	22	72	8	129.2	0.8	27.70	5/5	35.6	0:00:02
geant_10	22	72	10	155.6	0.6	16.65	5/5	35.6	0:00:03
geant_15	22	72	15	312.8	3	14.27	5/5	98.2	0:00:17
geant_20	22	72	20	353	1	13.11	5/5	98.2	0:00:23
geant_30	22	72	30	1496.6	21.6	12.34	5/5	686.8	0:07:15
geant_40	22	72	40	3111.2	37.2	12.60	5/5	2096.4	0:54:45
ta1_4	24	110	4	96	0	35.72	5/5	47	0:00:01
ta1_6	24	110	6	165.4	0.4	29.49	5/5	59.2	0:00:04
ta1_8	24	110	8	319.6	1.6	27.09	5/5	381.8	0:01:03
ta1_10	24	110	10	415.6	7.6	21.03	5/5	778	0:02:04
ta1_15	24	110	15	1120.4	78.4	27.32	4/5	8788.4	1:08:46
ta1_20	24	110	20	1920	49.4	25.25	3/5	10870	2:45:12
ta1_30	24	110	30	4570	69.6	*25.88*	0/5	9886.8	5:00:00
ta1_40	24	110	40	9187.2	117.8	*52.85*	0/5	13739.8	5:00:00
pioro_4	40	178	4	174.4	0	66.01	5/5	418.4	0:00:07
pioro_6	40	178	6	217.4	0.4	56.49	5/5	365.8	0:00:13
pioro_8	40	178	8	786.4	11.2	59.67	5/5	8678.8	0:11:42
pioro_10	40	178	10	884	9	51.72	5/5	5719.6	0:10:36
pioro_15	40	178	15	2471.4	82.8	56.60	2/5	46315.2	3:51:36
pioro_20	40	178	20	3426.4	87.4	*55.55*	0/5	41249.8	5:00:00

Instance	V	A	K	NmsI	NmsII	Gap	nodes	TT	TT(sep)
atlanta5_1	15	44	5	41	0	46.77	71	0:00:01	0
atlanta5_2	15	44	5	69	0	16.43	34	0:00:01	0
atlanta5_3	15	44	5	32	0	0.00	1	0:00:01	0
atlanta5_4	15	44	5	55	0	0.89	7	0:00:01	0
atlanta5_5	15	44	5	39	0	0.46	5	0:00:01	0
atlanta10_1	15	44	10	157	0	55.86	833	0:00:27	0
atlanta10_2	15	44	10	120	0	33.33	100	0:00:08	0
atlanta10_3	15	44	10	83	0	4.48	4	0:00:01	0
atlanta10_4	15	44	10	207	0	8.94	32	0:00:02	0
atlanta10_5	15	44	10	193	1	27.60	83	0:00:03	0
atlanta15_1	15	44	15	411	0	50.14	538	0:00:44	0
atlanta15_2	15	44	15	1837	456	29.31	31724	2:33:09	73
atlanta15_3	15	44	15	665	0	16.19	109	0:00:20	0
atlanta15_4	15	44	15	148	0	2.08	17	0:00:01	0
atlanta15_5	15	44	15	988	72	21.97	2169	0:06:17	5
atlanta20_1	15	44	20	1063	207	57.31	9571	0:24:50	16
atlanta20_2	15	44	20	3107	643	*31.82*	59587	5:00:00	67
atlanta20_3	15	44	20	864	0	11.85	92	0:00:12	1
atlanta20_4	15	44	20	662	0	4.41	66	0:00:08	1
atlanta20_5	15	44	20	2749	354	26.01	39985	3:01:31	108
atlanta25_1	15	44	25	1450	123	*48.74*	50604	5:00:00	48
atlanta25_2	15	44	25	2262	332	*33.62*	33482	5:00:00	103
atlanta25_3	15	44	25	2340	0	17.88	634	0:09:02	6
atlanta25_4	15	44	25	1596	0	5.41	199	0:02:10	4
atlanta25_5	15	44	25	1661	88	20.22	6576	0:29:54	26.92
atlanta30_1	15	44	30	1316	159	*44.24*	31804	5:00:00	52
atlanta30_2	15	44	30	3211	355	*36.88*	23474	5:00:00	85
atlanta30_3	15	44	30	4302	0	15.98	1134	0:11:27	6
atlanta30_4	15	44	30	2100	0	5.84	266	0:01:23	0
atlanta30_5	15	44	30	2769	205	17.82	16437	1:52:31	69.96
atlanta35_1	15	44	35	2316	148	*43.19*	30825	5:00:00	44
atlanta35_2	15	44	35	4124	463	*36.32*	17560	5:00:00	153
atlanta35_3	15	44	35	3997	0	18.26	1643	0:46:12	24
atlanta35_4	15	44	35	2322	0	4.61	325	0:06:28	8
atlanta35_5	15	44	35	5230	845	*25.22*	34494	5:00:00	100
atlanta40_1	15	44	40	2211	598	*43.92*	34685	5:00:00	88
atlanta40_2	15	44	40	4255	704	*28.40*	14758	5:00:00	147
atlanta40_3	15	44	40	9489	0	*23.78*	4386	5:00:00	73
atlanta40_4	15	44	40	3622	0	4.76	383	0:12:48	13
atlanta40_5	15	44	40	5117	414	*15.83*	28951	5:00:00	157
atlanta50_1	15	44	50	2857	134	*46.08*	22886	5:00:00	97
atlanta50_2	15	44	50	3955	643	*52.87*	11473	5:00:00	163
atlanta50_3	15	44	50	13741	0	*25.23*	2966	5:00:00	77
atlanta50_4	15	44	50	10283	0	5.46	2201	4:19:52	98
atlanta50_5	15	44	50	9053	546	*16.75*	17879	5:00:00	130

Table 4.3: The hardness of CSLND instances

in $]\frac{C}{4}, \frac{C}{2}]$, the third one in $]\frac{C}{2}, \frac{3}{4}C]$, then in $]\frac{3}{4}C, C]$, and the last type was generated in $]\frac{C}{4}, \frac{3C}{4}]$. For example, if $C = 100$, then commodities will be generated in the following intervals $]0, 25]$, $]25, 50]$, $]50, 75]$, $]75, 100]$ and $]25, 75]$. An instance name is followed by the extension 1, 2, 3, 4 or 5 according to the interval that contains its traffic demands. The results are show in Table 4.3. It appears clearly from Table 4.3 that all the instances with traffic in intervals $]\frac{C}{2}, \frac{3}{4}C]$ and $]\frac{3}{4}C, C]$ are solved to optimality within the time limit, except for *atlanta*_40_3 and *atlanta*_50_3. These two instances are also the only ones among the 18 instances whose traffic is generated in intervals 3 and 4, to have a gap greater than 20%. Moreover, we can easily observe that instances whose commodities are generated in intervals 1 and 2 have the worse results in terms of gap and size of the Branch-and-Cut tree. In fact, the instances whose traffic is generated in interval 1 are clearly the most difficult to solve, followed by those of intervals 2, 5, 3 and 4, in decreasing order of difficulty. Those observations are consistent with most of the works concerning bin-packing problem (see for example [46] and references therein), which state that instances with large commodities are easier to solve. However, although interval 5 has less small commodities than interval 1, so less chances to fill the modules gap, instances are not more difficult to solve. All these remarks lead us to conclude that the hardness of an instance is closely related to ratio between commodities traffic and facilities capacity ($\frac{D^k}{C}$, $k \in K$).

Note also that the number of generated Min Set I inequalities is significantly higher than the number of generated Min Set II inequalities. This means that Min Set II inequalities are more likely to improve the efficiency of Branch-and-Cut in terms of number of explored nodes in the tree. Although the separation procedure for Min Set II inequalities can be enhanced, we do not expect them to be as effective as Min Set I inequalities. This can be explained by the structure of random instances as Min Set II inequalities seem to be more helpful for instances with small commodities.

4.2.4.3 Realistic instances

Our last series of experiments concerns realistic instances based on SNDlib topologies. The tested instances have graphs with 12 up to 54 nodes and sets of 2 to 45 commodities (6 to 45 for instances *abilene* and *atlanta*). We have treated 92 instances and the results obtained are divided into two tables, namely Table 7.3 and Table 7.4. It appears from Table 7.3 and Table 7.4 that 70 among the 92 tested instances were solved to optimality within the fixed time limit. The remaining instances, are generally those having more than 30 commodities and/or more than 35 nodes. In addition, note that 60 instances could be solved to optimality in less than 15 minutes. We can remark that, the gap values are slightly better than those obtained for the instances with randomly generated traffic (see Table 4.2). However, it seems that realistic traffic based instances are as challenging as randomly generated traffic ones. Also we can see that CPU time dedicated to the separation is very short regarding to the number of valid inequalities that are generated. Indeed, using good lower bound instead of integer programming to solve the bin-packing problem within separation routines helped to make it faster and more efficient.

Similarly to instances with randomly generated commodities, the difficulty of realistic SNDlib-based instances is closely related to the nature of commodities (amount of traffic compared to the capacity of facilities), and to the size of the graph as well (number of nodes and number of arcs). Yet this justifies why two instances outwardly equivalents in size are not handled with the same ease by the Branch-and-Cut algorithm. Moreover, it must be pointed out that valid inequalities are more likely to be efficient for the instances with sparse graphs. Actually, in those graphs, the commodities routing paths would be longer, so more commodities would have to share the same arcs and "bin-packing effect" is significant. For example, we can compare instances *nobel_germany* in Table 7.3 with instances *newyork* in Table 4.2, that are similar in terms of number of nodes. It is clear that the class of instances *nobel_germany* are solved more easily than instances *newyork*. In fact, the graph of latter instances is strongly meshed. This induces a heaviest model (in terms of number of variables) but also more possibilities in the routing of commodities. In consequence, this does not encourage the emergence of valid inequalities that are violated.

4.3 Concluding remarks

In this chapter, we have presented the results provided by our Branch-and-Cut algorithm, devised and implemented to solve CSLND problem. We have first given an overview of the algorithm and discussed some aspects of the separation problems associated with two classes of valid inequalities. In particular, we have proposed heuristic procedures to generate both Min Set I (3.21) and Min Set II (3.24) inequalities.

Our computational experiments have shown that the Branch-and-Cut approach is much more efficient than a Branch-and-Bound on the compact formulation to solve the problem. They have also shown that Min Set I and Min Set II inequalities are very effective for the problem. We could also see that our heuristics to separate Min Set I and Min Set II inequalities performs well, especially for instances based on sparse graphs. These experiments also illustrated the fact that CSLND problem is easier to solve when the traffic demands are not so small in comparison with facilities capacity.

Instance	|V|	|A|	|K|	NmsI	NmsII	Gap	Nodes	TT	TTsep
abilene	12	30	6	15	0	5.75	32	0:00:00	0
abilene	12	30	8	68	0	10.05	9	0:00:00	0
abilene	12	30	10	112	0	15.38	22	0:00:00	0
abilene	12	30	12	199	0	21.59	22	0:00:00	0
abilene	12	30	14	173	4	17.52	427	0:00:07	0
abilene	12	30	16	450	1	24.49	151	0:00:05	0
abilene	12	30	18	688	3	26.71	354	0:00:21	0
abilene	12	30	20	620	4	25.25	231	0:00:14	0
abilene	12	30	25	1273	8	18.61	462	0:00:51	1
abilene	12	30	30	1801	9	20.48	1362	0:02:21	3
abilene	12	30	40	3994	576	19.13	10246	0:45:03	73
abilene	12	30	45	5781	451	16.86	10005	2:00:09	35
atlanta	15	44	6	47	0	0.36	4	0:00:00	0
atlanta	15	44	8	181	26	43.31	545	0:00:05	0
atlanta	15	44	10	76	1	4.61	27	0:00:00	0
atlanta	15	44	12	205	35	43.92	379	0:00:00	0
atlanta	15	44	14	268	0	45.69	140	0:00:05	0
atlanta	15	44	16	305	0	33.94	264	0:00:14	1
atlanta	15	44	18	820	654	48.94	23977	0:40:40	20
atlanta	15	44	20	297	17	10.17	127	0:00:08	0
atlanta	15	44	25	1209	383	18.64	3781	0:10:40	9
atlanta	15	44	30	1443	305	13.72	4145	0:13:23	13
atlanta	15	44	40	3771	484	45.61	15569	5:00:00	22
nobel_germany	17	52	6	27	6	0.78	8	0:00:00	0
nobel_germany	17	52	8	24	1	1.5	7	0:00:00	0
nobel_germany	17	52	10	37	1	1.52	8	0:00:00	0
nobel_germany	17	52	12	49	0	12.22	5	0:00:00	0
nobel_germany	17	52	14	57	0	21.25	12	0:00:00	0
nobel_germany	17	52	16	62	0	21.55	17	0:00:00	0
nobel_germany	17	52	18	91	1	15.29	31	0:00:00	0
nobel_germany	17	52	20	400	10	13.47	80	0:00:04	0
nobel_germany	17	52	25	632	20	9.31	123	0:00:11	0
nobel_germany	17	52	30	325	18	33.06	345	0:00:19	0
nobel_germany	17	52	40	1088	130	30.57	0:45:03	232	6
nobel_germany	17	52	45	703	44	34.00	721	0:01:01	2
france	25	90	2	31	0	57.86	16	0:00:00	0
france	25	90	4	77	0	48.05	27	0:00:00	0
france	25	90	6	123	1	44.10	37	0:00:00	0
france	25	90	8	190	0	42.06	72	0:00:03	0
france	25	90	10	296	10	39.43	229	0:00:11	0
france	25	90	12	404	12	37.39	332	0:00:22	0
france	25	90	14	467	19	33.84	324	0:00:32	0
france	25	90	16	713	63	28.52	1933	0:03:55	2
france	25	90	18	1017	103	25.44	2010	0:04:57	4
france	25	90	20	1223	137	27.80	4610	0:14:57	8
france	25	90	25	1748	68	17.06	3903	0:19:21	13
france	25	90	30	4585	951	35.04	26412	5:00:00	64
france	25	90	40	5763	1154	33.06	18025	5:00:00	69
france	25	90	45	7865	1497	61.83	18521	5:00:00	84

Table 4.4: Branch-and-Cut results for realistic instances (1)

Table 4.5: Branch-and-Cut results for realistic instances (2)

Instance	|V|	|A|	|K|	NmsI	NmsII	Gap	Nodes	TT	TTsep
india35	35	160	2	25	0	33.81	23	0:00:00	0
india35	35	160	4	49	0	44.7	454	0:00:06	0
india35	35	160	6	54	0	35.69	452	0:00:10	0
india35	35	160	8	171	1	43.05	487	0:00:22	0
india35	35	160	10	234	1	39.05	93	0:00:09	0
india35	35	160	12	608	47	40.00	1585	0:04:18	3
india35	35	160	14	864	118	39.08	3927	0:13:60	9
india35	35	160	16	1126	139	35.31	5004	0:19:43	15
india35	35	160	18	1783	487	*45.11*	17553	5:00:00	77
india35	35	160	20	2349	286	*51.05*	12724	5:00:00	78
india35	35	160	25	3793	405	*66.69*	16037	5:00:00	114
india35	35	160	30	2779	419	*75.71*	10017	5:00:00	106
india35	35	160	40	3402	665	*72.93*	6747	5:00:00	166
india35	35	160	45	3789	438	*66.2*	5879	5:00:00	171
cost266	37	102	2	57	0	48.36	20	0:00:00	0
cost266	37	102	4	57	0	39.43	17	0:00:00	0
cost266	37	102	6	51	0	31.14	15	0:00:00	0
cost266	37	102	8	214	2	32.83	81	0:00:12	0
cost266	37	102	10	168	10	36.40	106	0:00:04	0
cost266	37	102	12	389	14	34.79	204	0:00:18	0
cost266	37	102	14	544	29	30.79	483	0:01:14	1
cost266	37	102	16	934	69	37.71	2291	0:07:25	4
cost266	37	102	18	1109	157	36.66	3081	0:14:71	7
cost266	37	102	20	414	751	37.67	4808	1:19:30	35
cost266	37	102	25	2680	780	38.9	11252	5:00:00	91
cost266	37	102	30	2523	646	42.48	9710	5:00:00	85
cost266	37	102	40	4224	689	58	6505	5:00:00	164
cost266	37	102	45	3599	794	68.62	6439	5:00:00	168
zib54	54	160	2	42	0	40.92	13	0:00:00	0
zib54	54	160	4	161	0	42.76	104	0:00:05	0
zib54	54	160	6	344	3	44.2	269	0:00:36	1
zib54	54	160	8	539	25	44.46	485	0:02:15	1
zib54	54	160	10	869	111	46.62	975	0:03:83	4
zib54	54	160	12	1747	493	52.92	4957	1:24:00	25
zib54	54	160	14	2500	392	53.98	12818	5:00:00	54
zib54	54	160	16	3208	131	58.83	16591	5:00:00	46
zib54	54	160	18	3686	129	56.95	19299	5:00:00	49
zib54	54	160	20	4050	913	60.52	17227	5:00:00	49
zib54	54	160	25	5474	558	66.46	13841	5:00:00	57
zib54	54	160	30	4816	565	64.2	11809	5:00:00	58
zib54	54	160	40	3264	246	81.46	10289	5:00:00	85
zib54	54	160	45	5245	367	68.89	8446	5:00:00	93

Chapter 5

Optical Multi-Band Network Design : polyhedral study

Contents

In this chapter we consider the optical multi-band network design problem from a polyhedral point of view. We first present the problem and give a linear programming *formulation to model it. We then introduce further valid inequalities for the associated polytope and describe necessary and sufficient conditions for these inequalities to define facets. In chapter 6, we discuss the algorithmic aspect of this study. We devise separation heuristics for the valid inequalities and embed them within a branch-and-cut algorithm. We show some numerical experiments that give an insight of the practical efficiency of the valid inequalities.*

5.1 Presentation of OMBND problem

5.1.1 General Statement

Consider an optical multi-band OFDM network that consists in an OFDM/WDM network over a fiber layer. The OFDM/WDM layer is called virtual layer and the fiber layer is called physical layer as well. The OFDM/WDM layer is composed of ROADMs (Reconfigurable Optical Add-Drop Multiplexer) devices which are interconnected by virtual link. A virtual link may receive one or many OFDM subbands. Note that, although a subband is said to be installed over a virtual link, it is in fact generated by a pair of ROADMs at the extremities of the link. The physical layer is composed of several transmission nodes interconnected by physical links. Each physical link contains two optical fibers, so that the traffic can be transported in both directions. The physical and virtual layers are communicating via an interface referred to as OEO (Optical-Electrical-Optical) interface.

Each ROADM in the virtual layer is associated with a transmission node in the physical layer. And every link in the virtual layer carries one or several subbands. We suppose that there exists a link between each pair of ROADMs in the virtual layer, as one or many subbands may be installed between any pair of devices. Each subband installed over a virtual link is assigned a path in the physical layer. A link in the physical layer can be associated with several subbands. However, due to technical aspects of OFDM technology, a physical link can be associated at most once with a set up subband. In practice, one or many ROADMs may be installed upon a transmission node. However, we assume without loss of generality that all the subbands installed over a virtual link are produced by a unique pair of ROADMs set up on the extremities of this link. In addition, establishing a subband yields a certain cost, which is the cost of ROADMs that generate this subband. We assume that we have a traffic matrix, where each element is a point-to-point traffic demand that may correspond to a given service, internet application or a multimedia content. This traffic demand has a value

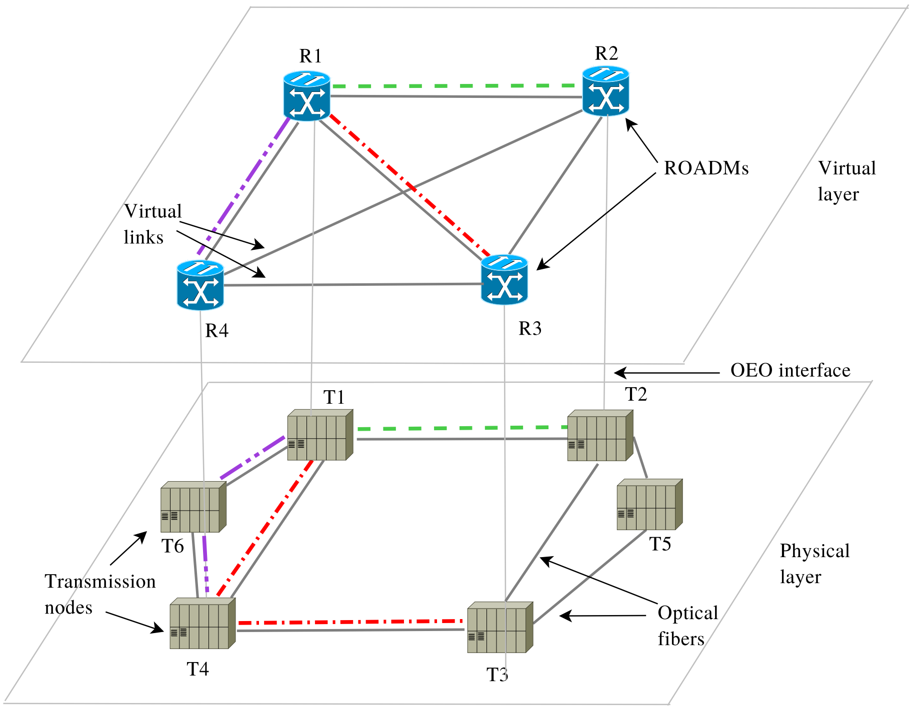


Figure 5.1: Example of multilayer network

that is an amount of informations measured in Mb/s or Gb/s.

The Figure 5.1.1 shows a bilayer network. The virtual layer includes four ROADMs denoted R_1, R_1, R_3 and R_4, while physical layer contains six transmission nodes denoted T_1 to T_6. We can see that R_1, R_2, R_3 and R_4 are connected to T_1, T_2, T_3 and T_4 via OEO interfaces. In addition, there exists a link between each pair of installed ROADMs. Remark that nodes R_5 and R_6 have not been represented as they do not carry any ROADM. Furthermore, three subbands are represented in the figure, respectively set up on the links (R_1, R_2), (R_1, R_3) and (R_1, R_4). The traffic using these virtual links is in fact transmitted through paths made of optical fibres in the physical layer. Indeed, the link (R_1, R_2) is associated with the path (T_1, T_2), while (R_1, R_3) is assigned the path (T_1, T_4), (T_4, T_3) and (R_1, R_4) is physically routed by (T_1, T_6), (T_6, T_4). It should be pointed out that there are two levels of routing in such networks. The traffic is routed using subbands installed on the virtual links, and the subbands themselves may be seen as demands for the physical layer. Thus, when given those two layers of network and a traffic matrix, one may determine the set of virtual links that will receive the subbands, and the set of physical links involved in the routing of those subbands, and establish the traffic commodities routing.

In this context, we are interested in a problem related to the design of OFDM/WDM networks. Thereby, assume that we are given an optical fiber layer, an OFDM/WDM layer and a traffic matrix. The *Optical Multi-band Network Design (OMBND)* problem consists in determining the number of subbands to be installed over the virtual links, and their physical path as well, so that the traffic can be routed on the virtual layer and the design is cost-efficient.

5.1.2 Notations and examples

In terms of graphs, the problem can be presented as follows. We associate with the virtual layer, a directed graph $G_1 = (V_1, A_1)$. G_1 is a complete graph where V_1 is the set of nodes and A_1 is the set of arcs. Each node $v \in V_1$ corresponds to a ROADM and each arc $e \in A_1$ corresponds to a virtual link between a pair of ROADMs. In addition, G_1 is a bi-directed graph, i.e. there exists two arcs $(u, v) \in A_1$ and $(v, u) \in A_1$, connecting each pair of nodes u and v of V_1. Consider the directed graph $G_2 = (V_2, A_2)$ that represents the physical layer of the optical network. V_2 denotes the set of nodes and A_2 is the set of arcs. Each node $v' \in V_2$ corresponds to a transmission node and each arc $a \in A_2$ corresponds to an optical fibre. Every node u in V_1 has its corresponding node u' in V_2. The graph G_2 is such that if there exists an arc (u', v') between two nodes u' and v' of V_2, then (v', u') is also in A_2. In this way, the link can be used in both directions between u' and v'.

Suppose that we have $n \in \mathbb{Z}^+$ available subbands. We denote by $W = \{1, 2, ..., n\}$, the set of indices associated with these subbands. Every subband $w \in W$ has a certain capacity C and a cost $c(w) > 0$, $w \in W$. Moreover, a subband installed over an arc $e \in A_1$ is a copy of this arc. Each pair (e, w) such that w is installed over the arc $e = (u, v)$, is associated with a path in G_2 connecting nodes u' and v'. Let $\Delta_{ew} \subset A_2$ be a subset of arcs containing this path. The same path in G_2 may be assigned to several subbands of W. Nevertheless, an arc $a \in A_2$ can be associated at most once with a given subband w. This comes from an engineering restriction that will be called disjunction constraint. In other words, if the subband w is installed p times, $p \in \mathbb{Z}^+$ over different arcs $e_1, \ldots, e_p$ of A_1, then the pairs (e_i, w), $i = 1, \ldots, p$, have to be assigned p paths in G_2 that are arc-disjoint.

Now let K be a set of commodities in G_1. Each commodity $k \in K$ has an origin node $o_k \in V_1$, a destination node $d_k \in V_1$ and a traffic value $D_k > 0$. We suppose, without loss of generality that $D^k \leq C$, for all $k \in K$. Note that a pair of nodes (u, v) may correspond to several pairs (o_k, d_k), $k \in K$, since there might exist several commodities whose origin is u and destination is v. A path in G_1 has to be assigned to each commodity $k \in K$ connecting its origin node o_k and its destination node d_k. Let $\mathcal{C}^k \subset A_1$ be a subset of arcs containing this path. Every section of a routing path uses the subbands installed over the arcs of A_1. Thereby, we will say that a pair (e, w), $e \in A_1$, $w \in W$ is used by a commodity k, if w is installed on e and (e, w) is involved in the routing of k. Furthermore, several commodities are allowed to use the same subband w, if its capacity allows it. However, one commodity can not be split into several subbands or several paths. Note that some extra arcs might be associated to k, in addition to its routing, but they are not materially used by k. Similarly, a subband may be installed on an arc of G_1 without being used for routing any commodity. Note that the total traffic flowing along an arc must be at most the overall capacity installed on this arc.

Definition 4 *Optical Multi-Band Network Design (OMBND): Given two bi-directed graphs G_1 and G_2, a set of installable subbands W, the installation cost $c(w)$ for each subband w, and a set of commodities K, we wish to determine the subbands to be installed over the arcs of G_1 such that*

(i) the commodities can be routed in G_1,

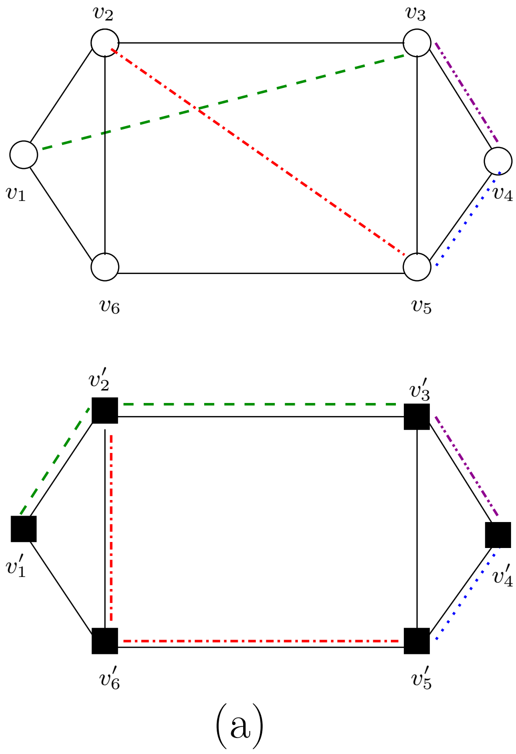
(ii) a path in G_2 is associated with each installed subband,

(iii) the total cost is minimum.

In addition to the design cost, we will impact a physical routing cost $b^{ew}(a)$ for every arc of $V_2 \times V_2$ involved in the routing of a pair (e,w) such that w is installed on e.

In what follows, we will assume that $G_2 = (V_2, A_2)$ is also a complete graph. This is a relevant assumption, since the problem when G_2 is not complete can reduce to the case when G_2 is complete. Indeed, it is possible to introduce a weigh system that penalizes the utilization of a fictive arc (an arc (u', v') such that $(u', v') \notin A_2$). Then, one can write an adequate objective function and obtain a solution using the initial arcs of A_2, whenever this is possible. To do this, it suffices to associate a large cost with the fictive arcs of G_2. Let b be this cost function.

$$b^{ew}(a) = \begin{cases} 1, & \text{if } a \in A_2, \\ M, & \text{if } a \in V_2 \times V_2 \setminus A_2. \end{cases}$$



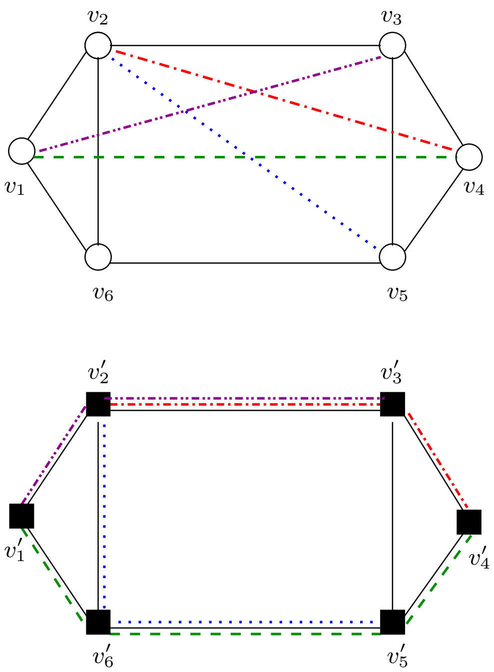
(arc,subband)	path in G_2
$\{(v_1, v_3), w_g\}$	$(v_1', v_2'), (v_2', v_3')$
$\{(v_3, v_4),\ w_p\}$	(v_3', v_4')
$\{(v_2, v_5),\ w_r\}$	$(v_2', v_6'), (v_6', v_5')$
$\{(v_5, v_4), w_b\}$	(v_5', v_4')

commodity	path in G_1
$v_1 - v_3$	$\{(v_1, v_3), w_g\}$
$v_1 - v_4$	$\{(v_1, v_3), w_g\}$, $\{(v_3, v_4),\ w_p\}$
$v_2 - v_4$	$\{(v_2, v_5),\ w_r\}$, $\{(v_5, v_4), w_b\}$
$v_2 - v_5$	$\{(v_2, v_5),\ w_r\}$

Figure 5.2: Feasible solution for OMBND problem (a)

where M is a large integer value. We also make the assumption that the number of available subbands is sufficiently large (polynomial in the size of the instance). In this way, one can install as much subbands as possible and easily obtain a feasible solution. Note that such an assumption is possible in practice because the number of subbands per fiber is significantly large regarding to the number of commodities. Figures 5.2 and 5.3 depict two feasible solutions for an instance of OMBND problem. This instance is composed by two graphs G_1 and G_2 corresponding to a bilayer network. The virtual layer contains six nodes denoted v_1 to v_6, while the physical layer holds six nodes denoted v_1' to v_6'. We can see here that each virtual node v_i, $i \in \{1, \ldots, 6\}$, is associated with a physical node v_i', $i \in \{1, \ldots, 6\}$. Only a subset of arcs is shown to allow a clearer reading of the figure. Consider four commodities whose origin-destination nodes are $v_1 - v_3$, $v_1 - v_4$, $v_2 - v_4$, and $v_2 - v_5$, and with the traffic values 5, 19, 20 and 5 Gb/s, respectively. We suppose given a set of four available subbands denoted w_g, w_r, w_b and w_p, each one having a capacity of 25 Gb/s.

Two solutions are given in Figure 5.2 and Figure 5.3. First, solution (a) consists in installing subbands w_g, w_r, w_p and w_b respectively on the arcs (v_1, v_3), (v_2, v_5), (v_3, v_4) and (v_5, v_4). Both routing of commodities and pairs (arc, subband) are summarized in Figure 5.2. For example, $\mathcal{C}^1 = \{(v_1, v_3)\}$ and $\mathcal{C}^2 = \{(v_1, v_3), (v_3, v_4)\}$ while $\Delta_{(v_1,v_3),w_g}$



(arc,subband)	path in G_2
$\{(v_1, v_3), w_p\}$	$(v'_1, v'_2), (v'_2, v'_3)$
$\{(v_1, v_4), w_g\}$	$(v'_1, v'_6), (v'_6, v'_5), (v'_5, v'_4)$
$\{(v_2, v_4),\ w_r\}$	$(v'_2, v'_2), (v'_3, v'_4)$
$\{(v_2, v_5),\ w_b\}$	$(v'_2, v'_6), (v'_6, v'_5)$

commodity	path in G_1
$v_1 - v_3$	$\{(v_1, v_3), w_p\}$
$v_1 - v_4$	$\{(v_1, v_4), w_g\}$
$v_2 - v_4$	$\{(v_2, v_4),\ w_r\}$
$v_2 - v_5$	$\{(v_2, v_5),\ w_b\}$

(b)

Figure 5.3: Feasible solution for OMBND problem (b)

$= \{(v'_1, v'_2), (v'_2, v'_3)\}$ and $\Delta_{(v_3,v_4),w_p} = \{(v'_3, v'_4)\}$. Indeed, the first commodity is routed along the path $\{(v_1, v_3)\}$ using the subband w_g and the pair $\{(v_1, v_3), w_g\}$ is itself associated with path $\{(v'_1, v'_2), (v'_2, v'_3)\}$ in G_2, and so on.

Figure 5.3 shows a second feasible solution with a different configuration of routing for the commodities and subbands. In this solution, subbands w_g, w_r, w_b and w_p are installed on arcs (v_1, v_4), (v_2, v_4), (v_2, v_5) and (v_1, v_3), respectively. Note that, in this solution, each commodity k is associated with a routing path corresponding to the arc (o_k, d_k). In addition, all the installed subbands G_1 are assigned paths in G_2. Note that both solutions (a) and (b) are feasible for the problem. However, solution (a) seems to be cost-efficient in comparison to solution (b). In fact, in solution (b), the cost impacted by physical routing of the subbands is higher than in solution (a).

Note that, if the subband w_r was used in solution (b) instead of subband w_p (if w_r was installed on both arcs (v_1, v_3) et (v_2, v_4)), then solution (b) would reduce to solution (c1) (see Figure 5.4) which is infeasible. In fact, arc (v'_2, v'_3) is associated twice with subband w_r, which makes the disjunction constraint violated. An alternative routing is given in solution (c2) (Figure 5.4) which is feasible.

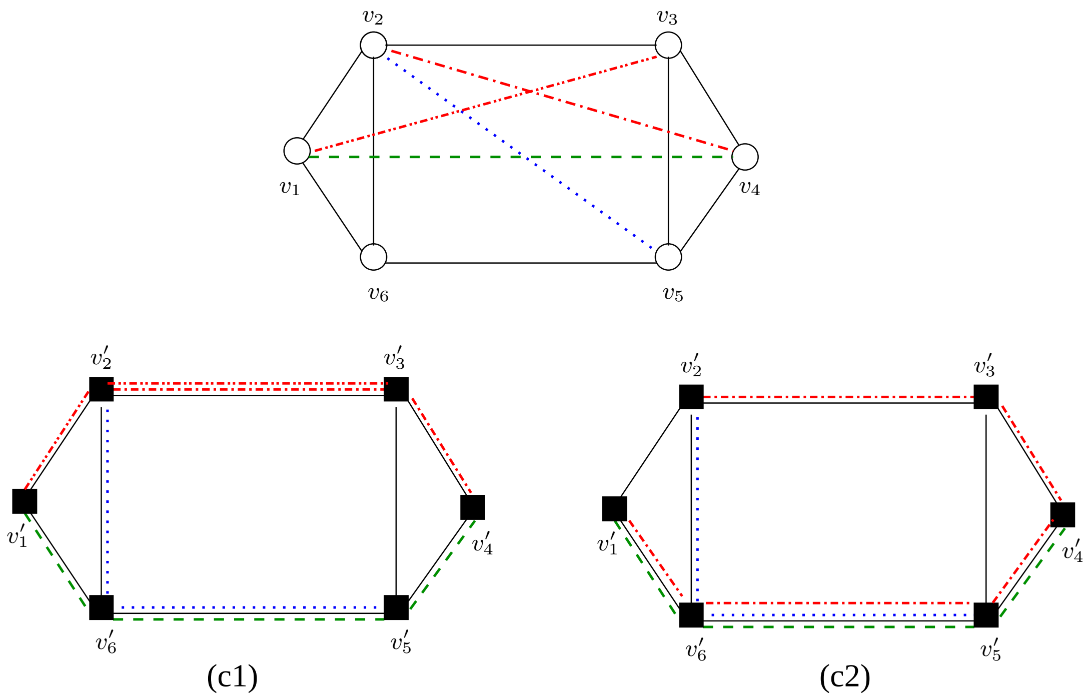


Figure 5.4: Infeasible solution for OMBND

5.2 Cut Formulation

In what follows we will first introduce some necessary notations, in order to give an integer linear programming formulation to OMBND problem. Let $T \subset V_1$ be a subset of nodes. We denote by $\delta^+_{G_1}(T)$ (resp. $\delta^-_{G_1}(T)$), the directed cut induced by T in G_1. In other words, $\delta^+_{G_1}(T)$ (resp. $\delta^-_{G_1}(T)$) is the set of arcs of A_1 having their initial node (resp. terminal node) in T and their terminal node (resp. initial node) in $V_1 \setminus T$. The cut $\delta^+_{G_1}(T)$ is defined as follows :

$$\delta^+_{G_1}(T) = \{e = (u, v) \in A_1 \text{ with } u \in T \text{ and } v \notin T\}$$

By the same way, we introduce $T \subset V_2$, as a subset of nodes in G_2. Let us define the directed cut $\delta^+_{G_2}(T)$ (resp. $\delta^-_{G_2}(T)$) as a subset of arcs having their initial node (resp. terminal node) in T and their terminal node (resp. inital node) in $V_2 \setminus T$. The cut $\delta^+_{G_2}(T)$ is defined as follows :

$$\delta^+_{G_2}(T) = \{a = (u', v') \in A_2 \text{ with } u' \in T \text{ and } v' \notin T\}$$

Now we will present an integer linear programming formulation using three sets of

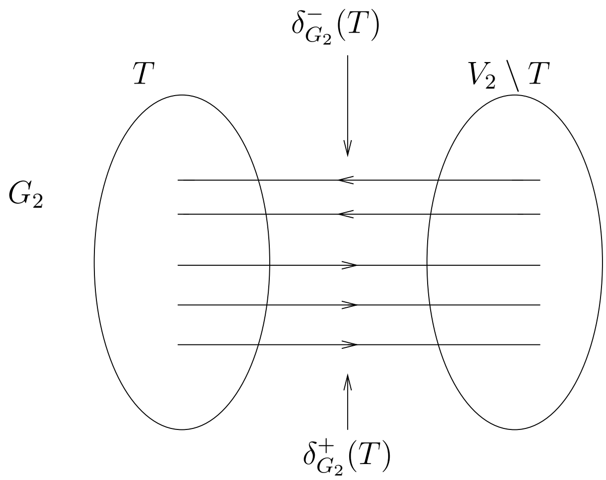


Figure 5.5: Directed cut in G_2

variables. First, the *design variables* y give the subbands selected for installation on the arcs of G_1 and that can be used to route the commodities. The second family of variables are *routing variables* for the subbands denoted z, they allow to associate a path in G_2 to each pair (e, w), $e \in A_1$, $w \in W$. The last family of variables, denoted x, are routing variables for the commodities.

Let $y \in \mathbb{R}^{A_1 \times W}$ be a variable such that, for each arc $e \in A_1$ and for each subband $w \in W$

$$y_{ew} = \begin{cases} 1, & \text{if } w \text{ is installed on } e, \\ 0, & \text{otherwise.} \end{cases}$$

let $z \in \mathbb{R}^{A_1 \times W \times A_2}$ be such that for each arc $e \in A_1$, for each subband $w \in W$ and for each arc $a \in A_2$

$$z_{ewa} = \begin{cases} 1, & \text{if } a \text{ belongs to the path in } G_2 \text{ associated with pair } (e, w), \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, let $x \in \mathbb{R}^{K \times A_1 \times W}$ be such that for each commodity $k \in K$, for each arc $e \in A_1$ and for each subband $w \in W$

$$x_{kew} = \begin{cases} 1, & \text{if } k \text{ uses } (e, w) \text{ for its routing,} \\ 0, & \text{otherwise.} \end{cases}$$

An instance of OMBND is defined by the quadruplet (G_1, G_2, K, C). Let $\mathcal{S}(G_1, G_2, K, C)$ denote the set of feasible solutions of OMBND problem, associated with an instance (G_1, G_2, K, C). A vector (x, y, z) associated with a solution of $\mathcal{S}(G_1, G_2, K, C)$ satisfies the following inequalities:

$$\min \sum_{e\in A_1}\sum_{w\in W} c(w)y_{ew} + \sum_{e\in A_1}\sum_{w\in W}\sum_{a\in A_2} b^{ew}(a)z_{ewa}$$

$$\sum_{e\in\delta^+_{G_1}(T)}\sum_{w\in W} x_{kew} \geq 1, \qquad \begin{array}{l}\forall k\in K, \forall T\subset V_1,\\ \emptyset\neq T\neq V_1, o_k\in T, d_k\notin T,\end{array} \tag{5.1}$$

$$\sum_{k\in K} D^k x_{kew} \leq C y_{ew}, \qquad \forall e\in A_1, \forall w\in W, \tag{5.2}$$

$$\sum_{a\in\delta^+_{G_2}(T)} z_{ewa} \geq y_{ew}, \qquad \begin{array}{l}\forall e=(u,v)\in A_1, \forall w\in W,\\ \forall T\subset V_2, \emptyset\neq T\neq V_2, u'\in T, v'\notin T,\end{array} \tag{5.3}$$

$$\sum_{e\in A_1} z_{ewa} \leq 1, \qquad \forall w\in W, \forall a\in A_2, \tag{5.4}$$

$$x_{kew}\in\{0,1\}, 0\leq x_{kew}\leq 1, \qquad \forall k\in K, \forall e\in A_1, \forall w\in W, \tag{5.5}$$

$$y_{ew}\in\{0,1\}, 0\leq y_{ew}\leq 1, \qquad e\in A_1, \forall w\in W, \tag{5.6}$$

$$z_{ewa}\in\{0,1\}, 0\leq z_{ewa}\leq 1, \qquad \forall e\in A_1, \forall w\in W, \forall a\in A_2. \tag{5.7}$$

Inequalities (5.1) are the *cut constraints*. They will also be referred to as *connectivity constraints*. They ensure that a path in G_1 exists for each commodity k between nodes o_k and d_k. Inequalities (5.2) are the *capacity constraints* for each subband installed over an arc of G_1. They ensure that the flow using the subband w on arc e does not exceed the capacity of w. They also ensure that the overall capacity installed on arc e is large enough to carry the traffic using e. Inequalities (5.3) are the *subband connectivity constraints*. The guarantee, for each pair (e, w) where w is installed on $e = (u, v)$, that a path in G_2 is associated with (e, w) between nodes u' and v'. Inequalities (5.4) are referred to as *disjunction constraint*. They express the fact that each subband can be associated at most once to an arc in G_2. Finally, inequalities (5.5)-(5.7) are the *trivial constraints*.

Theorem 5.1 *The set* $\{(x, y, z) \in \{0,1\}^{(K+1+A_2)\times A_1\times W} : (x, y, z)$ *satisfies* $(5.1) - (5.4)\}$ *corresponds to the convex hull of incidence vectors of solutions in* $\mathcal{S}(G_1, G_2, K, C)$.

Proof. The incidence vector of any solution of OMBND problem clearly satisfies inequalities (5.1)-(5.7). Let $(\tilde{x}, \tilde{y}, \tilde{z})$ be a vector of $\{0,1\}^{(K+1+A_2)\times A_1\times W}$ that does not induce a feasible solution of OMBND problem. Suppose that $(\tilde{x}, \tilde{y}, \tilde{z})$ satisfies inequalities (5.1) and inequalities (5.3)-(5.7). We will show that at least one inequality (5.2) is violated by $(\tilde{x}, \tilde{y}, \tilde{z})$. Let k be commodity of K. We know, by inequalities (5.1) (and by Menger's theorem) that the exists a path between the origin node of k o_k and its destination d_k. Inequalities (5.3) and (5.4) state that there exists a path in G_2 for each pair (e, w), $w \in W$, $e \in A_1$, such that w is installed on e. Moreover, every section of this path satisfies the disjunction constraints. As $(\tilde{x}, \tilde{y}, \tilde{z})$ is not feasible for OMBND problem, there is one arc say $\tilde{e}$ which have not receive enough subbands to carry the commodities using it. In other words, at least one subband $\tilde{w}$ is used without being installed, or its capacity is exceeded by the traffic flowing along $\tilde{e}$. Consequently, inequality (5.2) associated with $(\tilde{e}, \tilde{w})$ is violated and the result follows.

Similarly, we can show that any vector $(\overline{x}, \overline{y}, \overline{z})$ of $\{0,1\}^{(K+1+A_2)\times A_1\times W}$ that does not satisfy some inequality among (5.1)-(5.4) is not feasible for OMBND problem. □

Besides, we can easily check that with every solution of OMBND, we can associate a vector (x, y, z) that verifies inequalities (5.1)-(5.7). Thus, OMBND problem is equivalent to the following integer program

$$\min\{(x, y, z) \in \{0,1\}^{(K+1+A_2)\times A_1\times W} : (x, y, z) \text{ satisfies } (5.1) - (5.4)\} \tag{5.8}$$

Theorem 5.2 *The linear relaxation of* (5.8) *can be solved in polynomial time.*

Proof. Since inequalities (5.2) and (5.4) are in polynomial number, the complexity of the linear relaxation of (5.8) depends only on the complexity of separation problems related to inequalities (5.1) and (5.3) as well. Let us denote by $(\overline{x}, \overline{y}, \overline{z})$ a fractional solution to be cut off. Furthermore, the separation of inequalities (5.1) reduces to $|K|$ minimum $o_k d_k$-cuts in G_1, with weights $\overline{x}_k$, $k \in K$ on the pairs $(e, w) \in A_1 \times W$. And the separation of inequalities (5.4) reduces to compute $|A_1||W|$ minimum uv-cuts in G_2 with weights $\overline{z}_{ew}$, $e \in A_1$, $w \in W$ on the arcs of A_2. Both minimum cut computations can be done in polynomial time. □

Definition 5 *A solution S of OMBND problem is given by two subsets of arcs F_1, F_2 (with F_2 eventually empty), $|K|$ subsets of arcs $\mathcal{C}_1, \ldots, \mathcal{C}_k$, of A_1, a subset of subbands $\overline{W}$ of W installed on the arcs of $F_1 \cup F_2$, a subset of arcs Δ of A_2, and $|A_1| \times |W|$ subsets of arcs Δ_{ew}, $e \in A_1$, $w \in W$, in such a way that* *(i)* *at least one subband is installed on each arc of* $F_1 \cup F_2$,

(ii) $F_1 = \bigcup_{k \in K} \mathcal{C}_k$,

(iii) $\mathcal{C}_k$, $k \in K$, *contains a path between* o_k *and* d_k,

(iv) $\Delta = \bigcup_{e \in F_1 \cup F_2, w \in \overline{W}} \Delta_{ew}$,

(v) *with every arc* $e = (u, v) \in F_1 \cup F_2$ *and* $w \in \overline{W}$, *one can associate an arc subset* Δ_{ew} *(which may be empty), in such a way that if* w *is installed on* e, *then* Δ_{ew} *contains a path, say* $P_{ew} \subseteq \Delta_{ew}$ *between* u' *and* v',

(vi) *for every* $w \in \overline{W}$, *any arc of* Δ *belongs to at most one path* P_{ew}, *for* $e \in F_1 \cup F_2$.

We will denote by Γ the pairs (e, w) such that $e \in (F_1 \cup F_2)$ and $w \in \overline{W}$ such that w is installed on e. We then define the solution S by $S = (F_1, F_2, \Delta, \overline{W})$. The incidence vector of S, $(x^S, y^S, z^S) \in \mathbb{R}^{K \times A_1 \times W} \times \mathbb{R}^{A_1 \times W} \times \mathbb{R}^{A_1 \times W \times A_2}$, will be given by:

$$x^S_{kew} = \begin{cases} 1, & \text{if } e \in \mathcal{C}_k \text{ and } (e, w) \in \Gamma, \\ 0, & \text{otherwise.} \end{cases}$$

$$y^S_{ew} = \begin{cases} 1, & \text{if } w \in \overline{W}, e \in F_1 \cup F_2 \text{ and } (e, w) \in \Gamma, \\ 0, & \text{otherwise.} \end{cases}$$

$$z^S_{ew}(a) = \begin{cases} 1, & \text{if } a \in \Delta_{ew}, \\ 0, & \text{otherwise.} \end{cases}$$

5.3 Associated polytope

In this section, we introduce and discuss the OMBND polytope, that is the convex hull of the solutions of problem (5.8). Given an instance of OMBND, defined by the quadruplet (G_1, G_2, K, C), we denote by $P(G_1, G_2, K, C)$ this convex hull of incidence vectors $\mathcal{S}(G_1, G_2, K, C)$, that is

$$P(G_1, G_2, K, C) := conv\{(x, y, z) \in \mathbb{R}^{K \times A_1 \times W} \times \mathbb{R}^{A_1 \times W} \times \mathbb{R}^{A_1 \times W \times A_2} :$$

$$(x, y, z) \quad \text{satisfies } (5.1) - (5.4)\}$$

In what follows, we will characterize the dimension of polytope $P(G_1, G_2, K, C)$ and investigate the facial aspect of inequalities (5.1)-(5.7).

Theorem 5.3 *$P(G_1, G_2, K, C)$ is full dimensional.*

Proof. Assume that $P(G_1, G_2, K, C)$ is contained in the hyperplane defined by the linear equation

$$\alpha x + \beta y + \gamma z = \delta \tag{5.9}$$

where $\alpha = (\alpha^k_{ew}, k \in K, e \in A_1, w \in W) \in \mathbb{R}^{K \times A_1 \times W}$, $\beta = (\beta^{ew}, e \in A_1, w \in W) \in \mathbb{R}^{A_1 \times W}$, $\gamma = (\gamma^{ew}_a, e \in A_1, w \in W, a \in A_2) \in \mathbb{R}^{A_1 \times W \times A_2}$ and $\delta \in \mathbb{R}$. We will show that α=0, β=0, γ=0 and that $P(G_1, G_2, K, C)$ can not be included in the hyperplane (5.9), since it is not empty. To this end, let us first construct a solution $S^0 = (F^0_1, F^0_2, \Delta^0, W^0)$ of the problem.

For each commodity $k \in K$, we consider a path in G_1 between its origin and destination nodes, consisting of arc (o_k, d_k). This is possible since G_1 is complete. We install over this arc one subband. In other words, every subband is assigned at most to one commodity. Note that every arc (u, v) receives as much subbands as there are demands going from u to v. All the installed subbands are supposed to be different. After that, we associate with each subband, installed over (o_k, d_k), $k \in K$, a path in G_2 consisting in the arc (o'_k, d'_k). Again, this is possible since G_2 is also a complete graph.

Let $S^0 = (F^0_1, F^0_2, \Delta^0, W^0)$, be the solution given by $F^0_1 = \{(o_k, d_k), k \in K\}$, $F^0_2 = \emptyset$, $\Delta^0 = \{(o'_k, d'_k), k \in K\}$ and W^0 the subset of $|K|$ different subbands installed on the arcs of F^0_1.

Note that, as all the set up subbands are different, every considered path between o'_k and d'_k is associated with different subbands, and therefore, disjunction constraints (5.4) are satisfied. Moreover, since the capacities of the subbands are all greater than or equal to the commodity values, and a different subband is associated with each commodity, we have that capacity constraints (5.2) are also satisfied. Furthermore, by construction, the solution given above also satisfies the cut constraints (5.1) and (5.3). Thus the solution S^0 is feasible.

Consider a pair $(e, w) \in A_1 \times W$. Let $S^1 = (F^1_1, F^1_2, \Delta^1, W^1)$ be a solution obtained from S^0 by adding an arc $f \in A_2 \setminus \Delta^0$ to Δ^0_{ew}, while the other elements of S^0 remain the same. In other words, S^1 is such that $F_1^1 = F_1^0$, $F_2^1 = F_2^0$, $\Delta^1 = \Delta^0 \cup \{f\}$, and $W^1 = W^0$.

Obviously, S^1 is also feasible for the problem. As S^0 and S^1 are both feasible, their incidence vectors $(x^{S^0}, y^{S^0}, z^{S^0})$ and $(x^{S^1}, y^{S^1}, z^{S^1})$ both satisfy equality (5.9). Hence,

$$\alpha x^{S^0} + \beta y^{S^0} + \gamma z^{S^0} = \alpha x^{S^1} + \beta y^{S^1} + \gamma z^{S^1} = \alpha x^{S^0} + \beta y^{S^0} + \gamma z^{S^0} + \gamma_f^{ew}$$

This implies that $\gamma_f^{ew} = 0$. As f, e and w are chosen arbitrarily in $A_2 \setminus \Delta^0$, A_1 and W, respectively, we obtain that

$$\gamma_f^{ew} = 0, \text{ for all } f \text{ in } A_2 \setminus \Delta^0, e \in A_1 \text{ and } w \in W. \tag{5.10}$$

Now let $f = (u', v') \in \Delta^0$, $e = (u, v) \in A_1$ and $w \in W$. Suppose first that $f \in \Delta_{ew}^0$. Consider the solution $S^2 = (F_1^2, F_2^2, \Delta^2, W^2)$ such that $F_1^2 = F_2^2$, $F_2^2 = \emptyset$, $\Delta^2 = (\Delta^0 \cup \{f_1, f_2\}) \setminus \{f\}$, $W^2 = W^0$, where $f_1 = (u', s)$, $f_2 = (s, v')$ with $s \in V_2 \setminus \{u', v'\}$. In particular, $\Delta_{e'w'}^2 = \Delta_{e'w'}^0$ if $(e', w') \neq (e, w)$ and $\Delta_{ew}^2 = (\Delta_{ew}^0 \cup \{f_1, f_2\})\{f\}$. As both solutions S^0 and S^2 are feasible, their incidence vectors satisfy (5.9). It follows that $\gamma_f^{ew} = \gamma_{f_1}^{ew} + \gamma_{f_2}^{ew}$. As by 5.10, $\gamma_{f_1}^{ew} = \gamma_{f_2}^{ew} = 0$, we get $\gamma_f^{ew} = 0$.

If $f \notin \Delta_{ew}^0$, by considering the same solution S^0 where we add f to Δ_{ew}^0, we obtain that $\gamma_f^{ew} = 0$. We thus have, $\gamma_f^{ew} = 0$ for all $f \in \Delta^0$, $e \in A_1$ and $w \in W$. Hence,

$$\gamma_a^{ew} = 0, \text{ for all } a \in A_2, e \in A_1, \text{ and } w \in W. \tag{5.11}$$

Next, we will show that $\beta^{ew} = 0$, for all $(e, w) \in A_1 \times W$.

Consider an arc $g = (u, v) \in A_1 \setminus F_1^0$. Let us install a subband $\omega \in W$ over g. Let $S^3 = (F_1^3, F_2^3, \Delta^3, W^3)$, such that $F_1^3 = F_1^0$, $F_2^3 = F_2^0 \cup \{g\}$, $\Delta^3 = \Delta^0 \cup \{(u', v')\}$ and $W^3 = W^0 \cup \{\omega\}$. Solution S^3 is clearly feasible and its incidence vector satisfies (5.9). Therefore, we get

$$\beta^{g\omega} = 0, \text{ for all } g \in A_1 \setminus F_1^0 \text{ and } \omega \in W. \tag{5.12}$$

Now suppose that $g = (u, v) \in F_1^0$. Let w be a subband installed on g and k be a commodity of K using the pair (g, w). Let $S^4 = (F_1^4, F_2^4, \Delta^4, W^4)$ be a solution obtained from S^0 as follows. We consider two additional arcs $g_1 = (u, s)$ and $g_2 = (s, v)$ of $A_1 \setminus F_1^0$, where $s \in V_1 \setminus \{u, v\}$. And both g_1 and g_2 are added to the solution S^0 by receiving the subband w. In this solution, commodity k is moved from g to path $\{g_1, g_2\}$. In other words, the routing of k uses g_1, g_2 instead of g. Then, S^4 is such that $F_1^4 = F_1^0 \cup \{g_1, g_2\}$, $F_2^4 = F_2^0$, $\Delta^4 = \Delta^0 \cup \{(u', s'), (s', v')\}$, where $s' \in V_2 \setminus \{u', v'\}$. In addition, note that $W^4 = W^0$ and $\Gamma^4 = \Gamma^0 \cup \{(g_1, w), (g_2, w)\}$. $\mathcal{C}^4_k = (\mathcal{C}^0_k \setminus \{g\}) \cup \{g_1, g_2\}$, while the remaining elements of $\mathcal{C}^0$ do not change in $\mathcal{C}^4$. The solution S^4 is clearly feasible for OMBND problem.

Now we will introduce the solution S^5 which is obtained from S^4 by removing the pair (g, w) from Γ^4. Recall that, in S^4, (g, w) is not involved any more in the routing of k. In consequence, the removal of (g, w) does not affect the feasibility of this solution, which is actually ensured, since all the constraints of the problem are satisfied. Note that, in S^5, all the subsets are similar to those of S^4, except that $\Gamma^5 = \Gamma^4 \setminus \{(g, w)\}$. As both S^4 and S^5 are feasible, $(x^{S^4}, y^{S^4}, z^{S^4})$ and $(x^{S^5}, y^{S^5}, z^{S^5})$ verify (5.9). Hence, we get $\beta^{gw} = 0$. As g and w are arbitrary in F^0_1 and W, we obtain that

$$\beta^{ew} = 0, \text{ for all } e \in F^0_1 \text{ and for all } w \in W. \tag{5.13}$$

And, by (5.12) and (5.13), we have

$$\beta^{ew} = 0, \text{ for all } e \in A_1 \text{ and for all } w \in W. \tag{5.14}$$

Now let us show that $\alpha^k_{ew} = 0$, for all $k \in k$, $e \in A_1$, and $w \in W$.

Consider a commodity $\overline{k} \in K$, an arc $g = (u, v) \in A_1 \setminus F^0_1$, and a subband $\omega \in W$. We will install ω over g. Let $S^6 = (F^6_1, F^6_2, \Delta^6, W^6)$ be a solution defined as follows. $F^6_1 = F^0_1 \cup \{g\}$, $F^6_2 = F^0_2$, $\Delta^6 = \Delta^0 \cup \{(u', v')\}$ and $W^6 = W^0 \cup \{\omega\}$. In particular, $\Gamma^6 = \Gamma^0 \cup \{(g, \omega)\}$, and $\Delta^6_{g\omega} = \Delta^0_{g\omega} \cup \{(u', v')\}$. Moreover, $\mathcal{C}^6_k = \mathcal{C}^0_k$, for all $k \in K \setminus \{\overline{k}\}$ and $\mathcal{C}^6_{\overline{k}} = \mathcal{C}^0_{\overline{k}} \cup \{g\}$, while $\Delta^6_{ew} = \Delta^0_{ew}$, if $(e, w) \neq (g, \omega)$ and $\Delta^6_{ew} = \Delta^0_{ew} \cup \{(u', v')\}$ if $(e, w) = (g, \omega)$. S^6 is obviously a feasible solution. Hence, both incidence vectors of S^0 and S^6 verify (5.9), and consequently, we have,

$$\alpha^{\overline{k}}_{g\omega} + \beta^{g\omega} + \gamma^{g\omega}_{(u',v')} = 0,$$

As by (5.11) and (5.14), $\beta^{g\omega} = \gamma^{g\omega}_{(u',v')} = 0$, we get $\alpha^{\overline{k}}_{g\omega} = 0$. Since $g \in A_1 \setminus F^0_1$, $\omega \in W$ and $\overline{k} \in K$ are chosen arbitrarily and all the subbands play the same role, we obtain that

$$\alpha^k_{ew} = 0, \text{ for all } k \in K, e \in A_1 \setminus F^0_1 \text{ and } w \in W. \tag{5.15}$$

Suppose now that $g = (o_{\overline{k}}, d_{\overline{k}}) \in F^0_1$. Consider the subband $w_0 \in W^0$, such that (g, w_0) is involved in the routing of some commodity, say $\overline{k}$. Let S^7 be a solution obtained from S^0 as follows. We pick two arcs $g_1 = (o_{\overline{k}}, s)$ and $g_2 = (s, d_{\overline{k}})$ of $A_1 \setminus F^0_1$, with $s \in V_1 \setminus \{o_{\overline{k}}, d_{\overline{k}}\}$. We install w_0 on both g_1 and g_2, and we associate with pairs (g_1, w_0) and (g_2, w_0) paths $\{(o'_{\overline{k}}, s')\}$ and $\{(s', d'_{\overline{k}})\}$, respectively, with $s' \in V_2 \setminus \{o'_{\overline{k}}, d'_{\overline{k}}\}$. Then, $S^7 = (F^7_1, F^7_2, \Delta^7, W^7)$, where $F^7_1 = (F^0_1 \cup \{g_1, g_2\}) \setminus \{g\}$, $F^7_2 = F^0_2$, $\Delta^7 = \Delta^0 \cup \{(o'_{\overline{k}}, s'), (s', d'_{\overline{k}})\}$ and $W^7 = W^0$. Consider here $\mathcal{C}^7_k = \mathcal{C}^0_k$, for all $k \in K \setminus \{\overline{k}\}$ and

$\mathcal{C}^7_{\overline{k}} = (\mathcal{C}^0_{\overline{k}} \cup \{g_1, g_2\}) \setminus \{g\}$. Furthermore, $\Delta^7_{ew} = \Delta^0_{ew}$ if $(e,w) \notin \{(g_1,w_0),(g_2,w_0)\}$, while $\Delta^7_{g_1w_0} = \Delta^0_{g_1w_0} \cup \{(o'_{\overline{k}}, s')\}$ and $\Delta^7_{(s',d'_{\overline{k}})w_0} = \Delta^0_{g_2w_0} \cup \{(s', d'_{\overline{k}})\}$. Solution S^7 is also feasible, and its incidence vector as one of S^0 verifies equality (5.9). Thus we obtain that

$$\alpha^{\overline{k}}_{gw_0} + \beta^{gw_0} + \gamma^{gw_0}_{(o'_{\overline{k}},d'_{\overline{k}})} = \alpha^{\overline{k}}_{g_1w_0} + \alpha^{\overline{k}}_{g_2w_0} + \beta^{g_1w_0} + \beta^{g_2w_0} + \gamma^{g_1w_0}_{(o'_{\overline{k}},s')} + \gamma^{g_2w_0}_{(s',d'_{\overline{k}})},$$

By (5.11), $\gamma^{gw_0}_{(o'_{\overline{k}},d'_{\overline{k}})} = \gamma^{g_1w_0}_{(o'_{\overline{k}},s')} = \gamma^{g_2w_0}_{(s',d'_{\overline{k}})} = 0$. By (5.14) and (5.15) we also have $\beta^{gw_0} = \beta^{g_1w_0} = \beta^{g_2w_0} = 0$ and $\alpha^{\overline{k}}_{g_1w_0} = \alpha^{\overline{k}}_{g_2w_0} = 0$. This yields $\alpha^{\overline{k}}_{gw_0} = 0$. As $\overline{k}$, g and w_0 are chosen arbitrarily in K, F^0_1 and W, we get

$$\alpha^k_{ew} = 0, \text{ for all } k \in K, e \in F^0_1, \text{ and } w \in W. \tag{5.16}$$

Hence, by (5.15) and (5.16), we obtain

$$\alpha^k_{ew} = 0, \text{ for all } k \in K, e \in A_1, \text{ and } w \in W. \tag{5.17}$$

All together, and by (5.11), (5.13) and (5.16), $\alpha = \beta = \gamma = 0$. Moreover, since there exists at least one non-zero solution in polyhedron $P(G_1, G_2, K, C)$, it can not be included in hyperplane (5.9). Consequently, $P(G_1, G_2, K, C)$ is full dimensional. $\square$

5.3.1 Trivial inequalities

We can first remark that every inequality $y^{ew} \geq 0$, associated with a subband $w \in W$ and an arc $e \in A_1$ is dominated by the capacity constraint (5.2) associated with e and w. In what follows, we will focus on the inequalities $y^{ew} \leq 1$, for all $a \in A_1$ and for all $w \in W$.

Theorem 5.4 *For $\tilde{e} \in A_1$ and $\tilde{w} \in W$, inequality $y^{\tilde{e}\tilde{w}} \leq 1$ is facet defining for $P(G_1, G_2, K, C)$*

Proof. Let us denote by $\mathcal{F}^{\tilde{e}\tilde{w}}$ the face induced by inequality $y^{\tilde{e}\tilde{w}} \leq 1$, which is given by

$$\mathcal{F}^{\tilde{e}\tilde{w}} = \{(x,y,z) \in P(G_1, G_2, K, C) : y^{\tilde{e}\tilde{w}} = 1\}$$

We denote the inequality $y^{\tilde{e}\tilde{w}} \leq 1$ by $\alpha x + \beta y + \gamma z \leq \delta$. Let $\lambda x + \mu y + \nu z \leq \xi$ be a valid inequality that defines a facet $\mathcal{F}$ of $P(G_1, G_2, K, C)$. Suppose that $\mathcal{F}^{\tilde{e}\tilde{w}} \subseteq \mathcal{F}$. We show that there exists $\rho \in \mathbb{R}$ such that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$.

We will show that $\mu^{ew} = 0$, for all $(e, w) \in (A_1 \times W) \setminus \{(\tilde{e}, \tilde{w})\}$.

Consider the solution $S^0 = (F_1^0, F_2^0, \Delta^0, W^0)$ described in proof of Theorem 5.3. Suppose that $\tilde{e} \notin F_1^0 \cup F_2^0$. Then, let S^1 be a solution obtained from S^0, by adding $\tilde{e} = (u, v)$ to the solution and installing the subband $\tilde{w}$ on $\tilde{e}$. Suppose that $(\tilde{e}, \tilde{w})$ is associated with path $\{(u', v')\}$ in G_2 but is not involved in the routing of any commodity. In other words, $S^1 = (F_1^1, F_2^1, \Delta^1, W^1)$, where $F_1^1 = F_1^0$, $F_2^1 = F_2^0 \cup \{\tilde{e}\}$, $\Delta^1 = \Delta^0 \cup \{(u', v')\}$ and $W^1 = W^0 \cup \{\tilde{w}\}$. In particular, we have that $\Delta^1_{ew} = \Delta^0_{ew}$ if $(e, w) \neq (\tilde{e}, \tilde{w})$ and $\Delta^1_{\tilde{e}\tilde{w}} = \{(u', v')\}$. It is clear that S^1 is a feasible solution as it satisfies all the constraints of (5.1)-(5.7).

First, let us prove that $\nu_a^{ew} = 0$, for all $e \in A_1$, $w \in W$ and $a \in A_2$.

Consider an arc $a = (s, t)$ of $A_2 \setminus \Delta^1$. Let e and w be an arc of A_1 and a subband of W, respectively. Consider the solution S^2 that is obtained from S^1 by associating arc a with the pair (e, w) in addition to Δ^1_{ew}. In other words, $S^2 = (F_1^2, F_2^2, \Delta^2, W^2)$, where $F_1^2 = F_1^1$, $F_2^2 = F_2^1$, $\Delta^2 = \Delta^1 \cup \{a\}$, and $W^2 = W^1$. Note that $\Delta^2_{e_iw_i} = \Delta^1_{e_iw_i}$ if $(e_i, w_i) \neq (e, w)$ and $\Delta^2_{ew} = \Delta^1_{ew} \cup \{a\}$. S^2 is clearly feasible and both incidence vectors of S^2 and S^1 belong to $\mathcal{F}$ and thus to $\mathcal{F}^{\tilde{e}\tilde{w}}$. Hence, it follows that

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^2} + \mu y^{S^2} + \nu z^{S^2} = \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} + \nu_a^{ew},$$

Which implies that $\nu_a^{ew} = 0$. As a, e and w are chosen arbitrarily in $A_2 \setminus \Delta^1$, A_1 and W, we get

$$\nu_a^{ew} = 0, \quad \begin{array}{l} \text{for all } e \in A_1, \text{ for all } w \in W, \\ \text{and for all } a \in A_2 \setminus \Delta^1, \end{array} \tag{5.18}$$

Now assume that $a = (s, t)$ is in Δ^1. Let $a_1 = (s, r)$ and $a_2 = (r, t)$ be two arcs of $A_2 \setminus \Delta^1$, with $r \in V_2 \setminus \{s, t\}$. In particular $a \in \Delta^1_{ew}$ for some $e \in A_1$ and $w \in W$. Consider the solution S^3 obtained from S^1 by replacing the arc a with arcs a_1 and a_2 (see Figure 5.6). More formally, $S^3 = (F_1^3, F_2^3, \Delta^3, W^3)$, where $F_1^3 = F_1^1$, $F_2^3 = F_2^1$, $\Delta^3 = (\Delta^1 \setminus \{a\}) \cup \{a_1, a_2\}$, and $W^3 = W^1$.

Note that $\Delta^3_{e_iw_i} = \Delta^1_{e_iw_i}$ if $(e_i, w_i) \neq (e, w)$ while $\Delta^3_{ew} = (\Delta^1_{ew} \setminus \{a\}) \cup \{a_1, a_2\}$. It is easy to see that S^3 is a feasible solution. Moreover, both incidence vectors of S^3 and S^1 verify

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^3} + \mu y^{S^3} + \nu z^{S^3} = \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} - \nu_a^{ew} + \nu_{a_1}^{ew} + \nu_{a_2}^{ew},$$

Thus, we get

$$-\nu_a^{ew} + \nu_{a_1}^{ew} + \nu_{a_2}^{ew} = 0,$$

By (5.18), we have that $\nu_{a_1}^{ew} = \nu_{a_2}^{ew} = 0$. We thus obtain,

$$\nu_a^{ew} = 0, \quad \begin{array}{l} \text{for all } e \in A_1, \text{ for all } w \in W, \\ \text{and for all } a \in \Delta^1, \end{array} \tag{5.19}$$

Hence, by (5.18) and (5.19) we get

$$\nu_a^{ew} = 0, \quad \begin{array}{l} \text{for all } e \in A_1, \text{ for all } w \in W, \\ \text{and for all } a \in A^2, \end{array} \tag{5.20}$$

Let e be an arc of F_2^1 such that $e \notin \bigcup_{k\in K} \mathcal{C}_k^1$ and $w \in W^1$ is installed on e. Consider the solution S^4, obtained from S^1 by also considering the arc e for the routing of some commodity k. $S^4 = (F_1^4, F_2^4, \Delta^4, W^4)$, where $F_1^4 = F_1^1 \cup \{e\}$, $F_2^4 = F_2^1 \setminus \{e\}$, $\Delta^4 = \Delta^1$ and $W^4 = W^1$. Note that $\mathcal{C}_i^4 = \mathcal{C}_i^1$ if $i \neq k$, and $\mathcal{C}_k^4 = \mathcal{C}_k^1 \cup \{e\}$. One can easily check that S^4 is a feasible solution. Moreover, both incidence vectors of S^4 and S^1 are in $\mathcal{F}$ and in $\tilde{\mathcal{F}}$. Thus

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^4} + \mu y^{S^4} + \nu z^{S^4} = \lambda x^{S^1} + \lambda_{ew}^k + \mu y^{S^1} + \nu z^{S^1},$$

which implies that $\lambda_{ew}^k = 0$. As k, e and w are chosen arbitrarily in K, F_2^1 and W, we get

$$\lambda_{ew}^k = 0, \quad \begin{array}{l} \text{for all } k \in K, e \in F_2^1, \\ \text{for all } w \in W^1, \end{array} \tag{5.21}$$

Before showing that $\lambda_{ew}^k = 0$, for all $k \in K$, $e \in A_1 \setminus (F_1^1 \cup F_2^1)$, and for all $w \in W$, we need to prove that $\mu^{ew} = 0$, for all $e \in A_1 \setminus F_1^1 \cup F_2^1$ and for all $w \in W$.

Assume that $e = (s,t)$ is an arc of $A_1 \setminus (F_1^1 \cup F_2^1)$ and let w be a subband of W. Consider the solution S^5 obtained by S^1 as follows. We install the subband w on e and we associate with the pair (e,w) the path $\{(s',t')\}$ in G_2, with $(s',t') \in A_2 \setminus \Delta^1$. In this solution, we assume that $e \notin \bigcup_{k\in K} \mathcal{C}_k^5$. $S^5 = (F_1^5, F_2^5, \Delta^5, W^5)$, where $F_1^5 = F_1^1$, $F_2^5 = F_2^1 \cup \{e\}$, $\Delta^5 = \Delta^1 \cup \{(s',t')\}$ and $W^5 = W \cup \{w\}$. More precisely, we have $\Delta_{e_iw_i}^5 = \Delta_{e_iw_i}^1$ if $(e_i, w_i) \neq (e,w)$ while $\Delta_{ew}^5 = \Delta_{ew}^5 \cup \{(s',t')\}$. The solution S^5 is obviously feasible, and both incidence vectors of S^5 and S^1 are in $\mathcal{F}$ and $\mathcal{F}^{\tilde{e}\tilde{w}}$. Thus, we have

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^5} + \mu y^{S^5} + \nu z^{S^5} = \lambda x^{S^1} + \mu y^{S^1} + \mu^{ew} + \nu z^{S^1},$$

which implies that $\mu^{ew} = 0$. As e and w are chosen arbitrarily in $A_1 \setminus (F_2^1 \cup F_2^1)$ and W, respectively. It follows that,

$$\mu^{ew} = 0, \text{ for all } e \in A_1 \setminus (F_1^1 \cup F_2^1), w \in W, \tag{5.22}$$

Now consider an arc $e = (s,t)$ of $A_1 \setminus (F_1^1 \cup F_2^1)$. Let w be a subband of W and k any commodity of K. Let us introduce the solution S^6, obtained from S^1 as follows.

We install the subband w on the arc e, and we associate with the formed pair (e,w) path $\{(s',t')\}$ in G_2, where $(s',t') \in A_2 \setminus \Delta^1$. Then, we also associate e with the routing of commodity k in addition to its initial routing path. In other words, $S^6 = (F_1^6, F_2^6, \Delta^6, W^6)$, where $F_1^6 = F_2^1 \cup \{e\}$, $F_2^6 = F_2^1$, $\Delta^6 = \Delta^1 \cup \{(s',t')\}$, and $W^6 = W^1 \cup \{w\}$. Note that $\Delta^6_{e_iw_i} = \Delta^1_{e_iw_i}$ if $(e_i,w_i) \neq (e,w)$ and $\Delta^6_{ew} = \Delta^1_{ew} \cup \{(s',t')\}$. Moreover, $\mathcal{C}_i^6 = \mathcal{C}_i^1$, if $i \neq k$ while $\mathcal{C}_k^6 = \mathcal{C}_k^1 \cup \{e\}$. S^6 is also a feasible solution, and both incidence vectors of S^6 and S^1 verify

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^6} + \mu y^{S^6} + \nu z^{S^6} = \lambda x^{S^1} + \lambda^k_{ew} + \mu y^{S^1} + \mu^{ew} + \nu z^{S^1} + \nu^{ew}_{(s',t')},$$

which implies that

$$\lambda^k_{ew} + \mu^{ew} + \nu^{ew}_{(s',t')} = 0,$$

We have that $\nu^{ew}_{(s',t')} = 0$ by (5.20), and $\mu^{ew} = 0$ by (5.22). Thus, we get $\lambda^k_{ew} = 0$. As k, e and w are chosen arbitrarily in K, $A_1 \setminus (F_1^1 \cup F_2^1)$, and $w \in W$, we obtain

$$\lambda^k_{ew} = 0, \quad \begin{array}{l}\text{for all } k \in K, \text{ for all } e \in A_1 \setminus (F_1^1 \cup F_2^1), \\ \text{and for all } w \in W,\end{array} \tag{5.23}$$

Suppose now that $e = (u,v)$ is an arc of F_1^1 and let w be the subband of W^1 installed on e. We assume that $e \in \mathcal{C}_k^1$, for some commodity k. Let $e_1 = (u,r)$, $e_2 = (s,r)$ be two arcs of $A_1 \setminus (F_1^1 \cup F_2^1)$, with $s \in V_1 \setminus \{u,v\}$. Consider the solution S^7, obtained from S^1 by replacing the arc e with e_1 and e_2. We install the subband w on both e_1 and e_2, then we associate with pairs (e_1,w), (e_2,w) the paths $\{(u',r')\}$, $\{(r',v')\}$ in G_2, respectively, where (u',r'), $(r',v') \in A_2$. Figure 5.6 shows how a node r (respectively) may be inserted so as to replace any arc by a path between its end nodes.

$S^7 = (F_1^7, F_2^7, \Delta^7, W^7)$, where $F_1^7 = (F_2^1 \setminus \{e\}) \cup \{e_1,e_2\}$, $F_2^7 = F_2^1 \cup \{e\}$, $\Delta^7 = \Delta^1 \cup \{(u',s'),(s',v')\}$, and $W^7 = W^1$. Note that $\Delta^7_{e_iw_i} = \Delta^1_{e_iw_i}$ if $(e_i,w_i) \notin \{(e_1,w),(e_2,w)\}$ while $\Delta^7_{e_1w} = \Delta^1_{e_1w} \cup \{(u',s')\}$ and $\Delta^7_{e_2w} = \Delta^1_{e_2w} \cup \{(s',t')\}$. Finally, $\mathcal{C}_i^7 = \mathcal{C}_i^1$, if $i \neq k$ while $\mathcal{C}_k^7 = (\mathcal{C}_k^1 \setminus \{e\}) \cup \{e_1,e_2\}$. It is clear that S^7 is a feasible solution. Here, both incidence vectors of S^7 and S^1 are in $\mathcal{F}$. Thus, we have

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S1} = \lambda x^{S^7} + \mu y^{S^7} + \nu z^{S7} =$$

$$\lambda x^{S^1} + \lambda^k_{e_1w} + \lambda^k_{e_2w} - \lambda^k_{ew} + \mu y^{S^1} + \mu^{e_1w} + \mu^{e_2w} + \nu z^{S1} + \nu^{e_1w}_{(u',s')} + \nu^{e_2w}_{(s',v')},$$

which gives

$$\lambda^k_{e_1w} + \lambda^k_{e_2w} - \lambda^k_{ew} + \mu^{e_1w} + \mu^{e_2w} + \nu^{e_1w}_{(u',s')} + \nu^{e_2w}_{(s',v')} = 0,$$

We have that $\lambda^k_{e_1w} = \lambda^k_{e_2w} = 0$, by (5.23), $\mu^{e_1w} = \mu^{e_2w} = 0$ by (5.22), and $\nu^{e_1w}_{(u',s')} = \nu^{e_2w}_{(s',v')} = 0$ by (5.20). Thus, we get $\lambda^k_{ew} = 0$. As k and e are chosen arbitrarily in K and F_1^1, respectively, then we obtain

$$\lambda^k_{ew} = 0, \quad \begin{array}{l}\text{for all } k \in K, \text{ for all } e \in F_1^1, \\ \text{and for all } w \in W^1,\end{array} \tag{5.24}$$

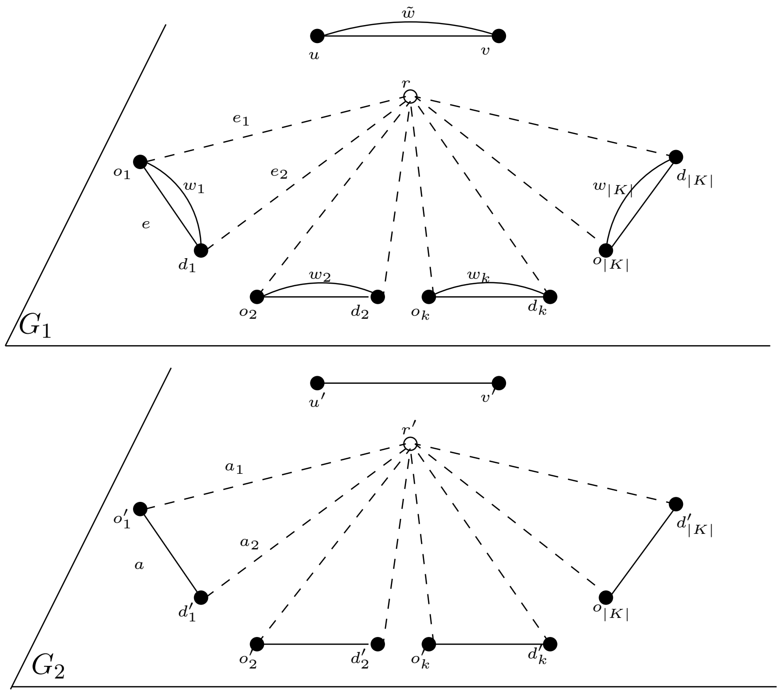


Figure 5.6: Getting further solutions by inserting a node

Consequently, by (5.21), (5.23) and (5.24), we conclude that

$$\lambda_{ew}^k = 0, \quad \begin{array}{l}\text{for all } k \in K, \text{ for all } e \in A_1, \\ \text{and for all } w \in W,\end{array} \tag{5.25}$$

Suppose now that $e = (u, v) \in (F_1^1 \cup F_2^1) \setminus \{\tilde{e}\}$, and let w be the subband of W^1 installed on e. Let $f = (u, r)$ and $g = (r, t)$ be two arcs of $A_1 \setminus (F_1^1 \cup F_2^1)$, with $s \in V_1 \setminus \{u, v\}$.

If $e \in F_1^1$ and $e \in \mathcal{C}_k^1$ for some commodity k, then we will consider the solution S^8 obtained from S^1 as follows. We replace e by f and g and we install the subband w on both f and g. We assign to the pairs (f, w), (g, w) the paths $\{(u', r')\}$ and $\{(r', v')\}$. Moreover, we consider that the routing of k uses f and g instead of e. More formally, $S^8 = (F_1^8, F_2^8, \Delta^8, W^8)$, where $F_1^8 = (F_1^1 \setminus \{e\}) \cup \{f, g\}$, $F_2^8 = F_2^1$, $\Delta^8 = (\Delta^1 \setminus \{(u', v')\}) \cup \{(u', r'), (r', v')\}$, $W^8 = W^1$. Note that $\Delta_{e_i w_i}^8 = \Delta_{e_i w_i}^1$ if $(e_i, w_i) \notin \{(e, w), (f, w), (g, w)\}$, $\Delta_{ew}^8 = \Delta_{ew}^1 \setminus \{(u', v')\}$, $\Delta_{fw}^8 = \Delta_{fw}^1 \cup \{(u', r')\}$ and $\Delta_{gw}^8 = \Delta_{gw}^1 \cup \{(r', v')\}$. Also note that $\mathcal{C}_i^8 = \mathcal{C}_i^1$, if $i \neq k$ while $\mathcal{C}_k^8 = (\mathcal{C}_k^1 \setminus \{e\}) \cup \{f, g\}$. S^8 is clearly feasible, and both incidence vectors of S^8 and S^1 are in $\mathcal{F}^{\tilde{e}\tilde{w}}$, and then in $\mathcal{F}$. Thus, we have

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^8} + \mu y^{S^8} + \nu z^{S^8} =$$

$$\lambda x^{S^1} + \lambda^k_{fw} + \lambda^k_{gw} - \lambda^k_{ew} + \mu y^{S^1} - \mu^{ew} + \mu^{fw} + \mu^{gw} + \nu z^{S^1} - \nu^{ew}_{(u',v')} + \nu^{fw}_{(u',s')} + \nu^{gw}_{(s',v')},$$

By (5.20), (5.25) and (5.22) We have that $\nu^{ew}_{(u',v')} = \nu^{fw}_{(u',r')} = \nu^{gw}_{(r',v')} = 0$, $\lambda^k_{fw} = \lambda^k_{gw} = \lambda^k_{ew} = 0$, and $\mu^{fw} = \mu^{gw} = 0$. Thus, we get $\mu^{ew} = 0$. As e is chosen arbitrarily in F_1^1, then

$$\mu^{ew} = 0, \text{ for all } e \in F_1^1, \text{ and for all } w \in W, \tag{5.26}$$

Let e be an arc of A_1 and w be a subband of W. Suppose that $e \in F_2^1$ and w is installed on e. Then we will construct the solution S^9 from S^1 by removing e as it is not used by any commodity. $S^9 = (F_1^1, F_2^1 \setminus \{e\}, \Delta^1, W^1)$, where the entries of S^9 are the same than those of S^1, except for subset F_2^1 who looses an element. Moreover, note that $\Gamma^9 = \Gamma^1 \setminus \{(e,w)\}$. It is clear that deleting e from F_2^1 does not impact on the feasibility of the solution. Hence, S^9 is feasible, and both $(x^{S^9}, y^{S^9}, z^{S^9})$ and $(x^{S^1}, y^{S^1}, z^{S^1})$ belong to $\mathcal{F}^{\tilde{e}\tilde{w}}$ and thus, to $\mathcal{F}$. Then, comparing S^9 and S^1 leads to

$$\lambda x^{S^1} + \mu y^{S^1} + \mu^{ew} + \nu z^{S^1} = \lambda x^{S^9} + \mu y^{S^9} + \nu z^{S^9},$$

We then have that $\mu^{ew} = 0$. As e was chosen arbitrarily in F_2^1 and the subbands of W are interchangeable, we get

$$\mu^{ew} = 0, \text{ for all } e \in F_2^1, \text{ and for all } w \in W, \tag{5.27}$$

Consequently, by (5.22), (5.26) and (5.27), we can then deduce that $\mu^{ew} = 0$ for all $(e,w) \in (A_1 \times W) \times \{(\tilde{e}, \tilde{w})\}$. Hence, $\mu^{\tilde{e}\tilde{w}} = \rho$, which ends the proof. ☐

Let us now study the facial structure of trivial constraints associated with x variables.

Theorem 5.5 *For $\tilde{k} \in K$, $\tilde{e} \in A_1$ and $\tilde{w} \in W$, inequality $x_{\tilde{k}\tilde{e}\tilde{w}} \leq 1$ is facet defining for $P(G_1, G_2, K, C)$*

Proof. Let us denote by $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$ the face induced by inequality $x_{\tilde{k}\tilde{e}\tilde{w}} \leq 1$, which is given by

$$\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}} = \{(x,y,z) \in P(G_1, G_2, K, C) : x_{\tilde{k}\tilde{e}\tilde{w}} = 1\},$$

We denote the inequality $x_{\tilde{k}\tilde{e}\tilde{w}} \leq 1$ by $\alpha x + \beta y + \gamma z \leq \delta$. Let $\lambda x + \mu y + \nu z \leq \xi$ be a valid inequality that defines a facet of $P(G_1, G_2, K, C)$. Suppose that $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}} \subseteq \mathcal{F}$. We show that there exists $\rho \in \mathbb{R}$ such that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$.

We will show that $\lambda^k_{ew} = 0$, for all $(k,e,w) \in (K \times A_1 \times W) \setminus \{(\tilde{k}, \tilde{e}, \tilde{w})\}$.

Consider the solution $S^0 = (F_1^0, F_2^0, \Delta^0, W^0)$ described in proof of Theorem 5.3. Suppose that $\tilde{e} = (u, v) \notin F_1^0 \cup F_2^0$. Let S^1 be a solution obtained from S^0 by installing the subband $\tilde{w}$ on $\tilde{e}$, and adding $\tilde{e}$ to the solution. In this solution, we associate with $(\tilde{e}, \tilde{w})$ the path in G_2 given by $\{(u', v')\}$, where $(u', v') \in A_2$. We will also consider the arc $\tilde{e}$ for the routing of the commodity $\tilde{k}$. In other words, $S^1 = (F_1^1, F_2^1, \Delta^1, W^1)$, where $F_1^1 = F_1^0 \cup \{\tilde{e}\}$, $F_2^1 = F_2^0$, $\Delta^1 = \Delta^0 \cup \{(u', v')\}$, and $W^1 = W^0 \cup \{\tilde{w}\}$. Note that $\mathcal{C}_i^1 = \mathcal{C}_i^0$ if $i \neq \tilde{k}$, and $\mathcal{C}_{\tilde{k}}^1 = \mathcal{C}_{\tilde{k}}^0 \cup \{\tilde{e}\}$. We also have $\Delta_{ew}^1 = \Delta_{ew}^0$ if $(e, w) \neq (\tilde{e}, \tilde{w})$ while $\Delta_{\tilde{e}\tilde{w}}^1 = \Delta_{\tilde{e}\tilde{w}}^0 \cup \{(u', v')\}$.

The solution S^1 is feasible and its incidence vector belongs to both $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$ and $\mathcal{F}$. In what follows, we will use S^1 as a reference solution. In other words, all the constructed solutions will be derived from S^1.

First, let us show that $\nu_a^{ew} = 0$, for all $e \in A_1$, and for all $w \in W$.

Let $a = (s', t')$ be an arc of A_2 that is not used in the solution S^1 ($a \notin \Delta^1$). Let e be an arc of A_1, and let w be a subband of W. We will construct the solution S^2, derived from S^1 by adding the arc a to the set Δ_{ew}^1. $S^2 = (F_1^2, F_2^2, \Delta^2, W^2)$ is then described as follows. $F_1^2 = F_1^1$, $F_2^2 = F_2^1$, $\Delta^2 = \Delta^1 \cup \{a\}$ and $W^2 = W^1$. Note that $\Delta_{e_i w_i}^2 = \Delta_{e_i w_i}^1$ if $(e_i, w_i) \neq (e, w)$ and $\Delta_{ew}^2 = \Delta_{ew}^1 \cup \{a\}$. One can easily check that S^2 is a feasible solution. Moreover, both incidence vectors of S^1 and S^2 belong to $\mathcal{F}$ and then, to $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$. Thus, we have

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^2} + \mu y^{S^2} + \nu z^{S^2} = \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} + \nu_a^{ew},$$

which implies that $\nu_a^{ew} = 0$. As a, e and w are chosen arbitrarily in $A_1 \setminus \Delta^1$, A_1 and W, we get

$$\nu_a^{ew} = 0, \quad \begin{array}{l} \text{for all } e \in A_1, \text{ for all } w \in W, \\ \text{and for all } a \in A_2 \setminus \Delta^1, \end{array} \tag{5.28}$$

Now suppose that $a = (s', t')$ is a part of the solution S^1. In other words, a is an arc of Δ^1 associated with some pair (e, w) of the solution ($(e, w) \in \Gamma^1$). Let $f = (s', r')$ and $g = (r', t')$ be two arcs of $A_2 \setminus \Delta^1$, with $r' \in V_2 \setminus \{s', t'\}$. Consider the solution S^3, obtained from S^1 by replacing the arc a by f and g. More formally, $S^3 = (F_1^3, F_2^3, \Delta^3, W^3)$, where $F_1^3 = F_1^1$, $F_2^3 = F_2^1$, $\Delta^3 = (\Delta^1 \setminus \{a\}) \cup \{f, g\}$, and $W^3 = W^1$. Note that $\Delta_{e_i w_i}^3 = \Delta_{e_i w_i}^1$ if $(e_i w_i) \neq (e, w)$ and $\Delta_{ew}^3 = (\Delta_{ew}^1 \setminus \{a\}) \cup \{f, g\}$. The solution S^3 is obviously feasible, and both incidence vectors of S^1 and S^3 verify

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^3} + \mu y^{S^3} + \nu z^{S^3} = \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} - \nu_a^{ew} + \nu_f^{ew} + \nu_g^{ew}, \tag{5.29}$$

which implies that $-\nu_a^{ew} + \nu_f^{ew} + \nu_g^{ew} = 0$. By (5.28) $\nu_f^{ew} = \nu_g^{ew} = 0$, we obtain $\nu_a^{ew} =$

0. As a is chosen arbitrarily in Δ^1, we conclude that

$$\nu_a^{ew} = 0, \quad \begin{array}{c}\text{for all } e \in A_1, \text{ for all } w \in W, \\ \text{and for all } a \in \Delta^1,\end{array} \tag{5.30}$$

Conequently, by (5.28) and (5.30), we conclude that

$$\nu_a^{ew} = 0, \quad \begin{array}{c}\text{for all } e \in A_1, \text{ for all } w \in W, \\ \text{and for all } a \in A_2,\end{array} \tag{5.31}$$

Now let us show that $\mu^{ew} = 0$, for all $e \in A_1$ and for all $w \in W$.

To do this, we will consider an arc $e \in A_1$ and a subband $w \in W$ that do not enter in the composition of S^1. In other words, $e = (u, v)$ is an arc of $A_1 \setminus (F_1^1 \cup F_2^1)$ and w is a subband of W. Let us construct the solution S^4, derived from S^1 as follows. We set up the subband w on the arc e, and we assign to the pair (e, w) the path $\{(u', v')\}$ in G_2. We assume that (e, w) is not associated with the routing of any commodity. In other words, $S^4 = (F_1^4, F_2^4, \Delta^4, W^4)$, where $F_1^4 = F_1^1$, $F_2^4 = F_2^1 \cup \{e\}$, $\Delta^4 = \Delta^1 \cup \{(u', v')\}$, and $W^4 = W^1 \cup \{w\}$. Note that $e \notin \bigcup_{k \in K} \mathcal{C}_k^4$, while $\Delta_{e_i w_i}^4 = \Delta_{e_i w_i}^1$, if $(e_i w_i) \neq (e, w)$ and $\Delta_{ew}^4 = \Delta_{ew}^1 \cup \{(u', v')\}$. It is clear that S^4 is a feasible solution. Moreover, both incidence vectors of S^1 and S^4 are in $\mathcal{F}$ and in $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$. Thus, we have

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^4} + \mu y^{S^4} + \nu z^{S^4} = \lambda x^{S^1} + \mu y^{S^1} + \mu^{ew} + \nu z^{S^1} + \nu_{(u',v')}^{ew},$$

and it follows that

$$\mu^{ew} + \nu_{(u',v')}^{ew} = 0,$$

We have that $\nu_{(u',v')}^{ew} = 0$ by (5.31). Hence, we get $\mu^{ew} = 0$. As e and w are chosen arbitrarily in $A_1 \setminus (F_1^1 \cup F_2^1)$ and W, respectively, we obtain

$$\mu^{ew} = 0, \text{ for all } e \in A_1 \setminus (F_1^1 \cup F_2^1), \text{ and for all } w \in W, \tag{5.32}$$

Assume now that $e = (u, v) \in F_2^1$, and w is the subband of W^1 that is installed on e. Let us consider a solution S^5 obtained from S^1 by removing the pair (e, w) from Γ^1. Clearly, this does not impact on feasibility of the solution and both incidence vectors $(x^{S^5}, y^{S^5}, z^{S^5})$ and $(x^{S^1}, y^{S^1}, z^{S^1})$ belong to $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$, and then, they also belong to $\mathcal{F}$. Hence, comparing S^1 and S^5 gives

$$\lambda x^{S^1} + \mu y^{S^1} + \mu^{ew} + \nu z^{S^1} = \lambda x^{S^5} + \mu y^{S^5} + \nu z^{S^5}$$

Thus, we get $\mu^{ew} = 0$. Since e and w are chosen arbitrarily in F_2^1 and W, we obtain

$$\mu^{ew} = 0, \text{for all } e \in F_2^1, w \in W, \tag{5.33}$$

Assume that $e=(u,v)$ is in F_1^1, and let w be the subband of W^1 that is installed on e. Suppose that $e \in \mathcal{C}_k^1$, where k is some commodity of K. We will introduce the solution S^6, obtained from S^1 by replacing e with arcs f and g. We install the subband w on both f and g, then we associate the pairs (f,w) and (g,w) with paths $\{(u',s')\}$ and $\{(s',v')\}$, in G_2, respectively. In this solution, we consider that the routing of the commodity k uses f and g instead of its initial routing that uses e. More formally, $S^6=(F_1^6,F_2^6,\Delta^6,W^6)$, where $F_1^6=(F_1^1\backslash\{e\})\cup\{f,g\}$, $\Delta^6=(\Delta^1\backslash\{(u',v')\})\cup\{(u',s'),(s',v')\}$, and the other subsets of S^1 do not change. In particular, we have $\mathcal{C}_i^6=\mathcal{C}_i^1$ if $i\neq k$, and $\mathcal{C}_k^6=(\mathcal{C}_k^1\backslash\{e\})\cup\{f,g\}$. Also note that $\Delta^6_{e_iw_i}=\Delta^1_{e_iw_i}$ if $(e_i,w_i)\notin\{(e,w),(f,w),(g,w)\}$, while $\Delta^6_{ew}=\Delta^1_{ew}\setminus\{(u',v')\}$, $\Delta^6_{fw}=\Delta^1_{fw}\cup\{(u',s')\}$ and $\Delta^6_{gw}=\Delta^1_{gw}\cup\{(s',v')\}$.

It is clear that S^6 is a feasible solution. Moreover, if we reintroduce the arc e to S^6, we obtain a solution S^7 which is also feasible. In S^7, we have $F_2^7=F_2^6\cup\{e\}$ and $\Delta^7_{ew}=\Delta^6_{ew}\cup\{(u',v')\}$. The other elements of S^6 remain the same. The incidence vectors of S^6 and S^7 are in $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$, and thus in $\mathcal{F}$. Hence, they verify

$$\lambda x^{S^6}+\mu y^{S^6}+\nu z^{S^6}=\lambda x^{S^7}+\mu y^{S^7}+\nu z^{S^7}=\lambda x^{S^6}+\mu y^{S^6}+\mu^{ew}+\nu z^{S^6}+\nu^{ew}_{(u',v')},$$

which gives

$$\mu^{ew}+\nu^{ew}_{(u',v')}=0,$$

We have by (5.31) that $\nu^{ew}_{(u',v')}=0$. Thus, we get $\mu^{ew}=0$. As the arc e is chosen arbitrarily in F_1^1, we obtain

$$\mu^{ew}=0,\ \text{for all } e\in F_1^1, w\in W, \tag{5.34}$$

Consequently, and by (5.32), (5.33) and (5.34), we conclude that

$$\mu^{ew}=0,\ \text{for all } e\in A_1 \text{ and } w\in W, \tag{5.35}$$

Next we will show that $\lambda^k_{ew}=0$, for all $(k,e,w)\in(K\times A_1\times W)\setminus\{(\tilde{k},\tilde{e},\tilde{w})\}$.

Suppose first that $e=(u,v)$ is an arc of $A_1\setminus(F_1^1\cup F_2^1)$ (e does not belong to the reference solution S^1). Let w be a subband of W and k some commodity of K. We will consider the solution S^8, obtained from S^1 as follows. We install the subband w on e and we associate with the pair (e,w) the path consisting in (u',v') of A_2. We will also consider the arc e for the routing of k. The solution $S^8=(F_1^8,F_2^8,\Delta^8,W^8)$, where $F_1^8=F_1^1\cup\{e\}$, $F_2^8=F_2^1$, $\Delta^8=\Delta^1\cup\{(u',v')\}$, and $W^8=W^1\cup\{w\}$. S^8 is clearly feasible and both incidence vectors of S^1 and S^8 satisfy

$$\lambda x^{S^1}+\mu y^{S^1}+\nu z^{S^1}=\lambda x^{S^8}+\mu y^{S^8}+\nu z^{S^8}=\lambda x^{S^1}+\lambda^k_{ew}+\mu y^{S^1}+\mu^{ew}+\nu z^{S^1}+\nu^{ew}_{(u',v')},$$

We have that $\mu y^{S^1} = 0$, and $\nu^{ew}_{(u',v')} = 0$, by (5.35) and (5.31) respectively. Thus, it follows that $\lambda^k_{ew} = 0$. As e and w are chosen arbitrarily in $A_1 \setminus (F^1_1 \cup F^1_2)$ and $w \in W$, respectively, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, e \in A_1 \setminus (F^1_1 \cup F^1_2), \text{ and } w \in W, \tag{5.36}$$

Now let us show that $\lambda^k_{ew} = 0$, for all $e \in (F^1_1 \cup F^1_2) \setminus \{\tilde{e}\}$, and $w \in W$.

If e is in $F^1_2 \setminus \{\tilde{e}\}$, then we can construct a solution say S^9, obtained from S^1 by also considering e for the routing of some commodity k. S^9 is such that $F^9_1 = F^1_1 \cup \{e\}$ and $F^9_2 = F^1_2 \setminus \{e\}$, the other subsets of S^1 remain the same in S^9. Note that $\mathcal{C}^9_i = \mathcal{C}^1_i$, if $i \neq k$ and $\mathcal{C}^9_k = \mathcal{C}^1_k \cup \{e\}$. It is easy to see that S^9 is feasible. Moreover, the incidence vectors of S^1 and S^9 belong to $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$, thus they satisfy

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} - (\lambda x^{S^9} + \mu y^{S^9} + \nu z^{S^9}) - \lambda^k_{ew} = 0$$

Since e, k and w are chosen arbitrarily in $F^1_2 \setminus \{\tilde{e}\}$, K and W, we obtain

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, e \in F^1_2 \setminus \{\tilde{e}\}, \text{ and } w \in W, \tag{5.37}$$

Now suppose that $e \in F^1_1 \setminus \{\tilde{e}\}$. In particular, suppose that $e \in \mathcal{C}^1_k$ for some k, and let w be the subband of W^1 installed on e. Recall that $f = (u,s)$ and $g = (s,v)$ denote two arcs of $A_1 \setminus (F^1_1 \cup F^1_2)$. We will construct a solution S^{10} obtained from S^1 by installing subband w on both f and g. The commodity k is then rerouted on f and g (instead of e). Let us assign to (f,w) the path $\{(u',s')\}$ with $(u',s') \in A_2$ while (g,w) is assigned path $\{(s',v')\}$, $(s',v') \in A_2$. The obtained solution is described as follows. $S^{10} = (F^{10}_1, F^{10}_2, \Delta^{10}, W^{10})$, where $F^{10}_1 = (F^1_1 \setminus \{e\}) \cup \{f,g\}$, and $\Delta^{10} = (\Delta^1 \setminus \{(u',v')\}) \cup \{(u',s'),(s',v')\}$. Note that $\mathcal{C}^{10}_k = (\mathcal{C}^1_k \setminus \{e\}) \cup \{f,g\}$ while $\Delta^{10}_{ew} = \Delta^1_{ew} \setminus \{(u',v')\}$, $\Delta^{10}_{fw} = \Delta^1_{fw} \cup \{(u',s')\}$ and $\Delta^{10}_{gw} = \Delta^1_{gw} \cup \{(s',v')\}$. All the other subsets of S^1 remain the same. S^{10} is obviously feasible. Moreover, incidence vectors of S^1 and S^{10} are in $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$, and then, in $\mathcal{F}$. Thus, we have

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^{10}} + \mu y^{S^{10}} + \nu z^{S^{10}}$$

$$\lambda x^{S^1} - \lambda^k_{ew} + \lambda^k_{fw} + \lambda^k_{gw} + \mu y^{S^1} - \mu^{ew} + \mu^{fw} + \mu^{gw} + \nu z^{S^1} - \nu^{ew}_{(u',v')} + \nu^{fw}_{(u',s')} + \nu^{gw}_{(s',v')},$$

By (5.31), (5.35) and (5.36), we have that $\nu^{ew}_{(u',v')} = \nu^{fw}_{(u',s')} = \nu^{gw}_{(s',v')} = \mu^{ew} = \mu^{fw} = \mu^{gw} = \lambda^k_{fw} = \lambda^k_{gw} = 0$. Thus, it remains that $\lambda^k_{ew} = 0$. As the arc e is chosen arbitrarily in $(F^1_1 \cup F^1_2) \setminus \{\tilde{e}\}$, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, e \in (F^1_1 \cup F^1_2) \setminus \{\tilde{e}\}, w \in W, \tag{5.38}$$

Let us show now that $\lambda^k_{\tilde{e}w} = 0$, for all $(k, w) \in (K \times W) \setminus \{(\tilde{k}, \tilde{w})\}$.

Let k be some commodity of K, w be a subband of $W \setminus \{\tilde{w}\}$. Let us consider the solution S^{11}, obtained from S^1 as follows. We set up the subband w on the arc $\tilde{e}$, and we associate the pair $(\tilde{e}, w)$ with the path $\{(u', v')\}$. We also consider $\tilde{e}$ for the routing of k, in addition to its initial routing path. S^{11} is such that $\Gamma^{11} = \Gamma^1 \cup \{(\tilde{e}, w)\}$, $\mathcal{C}^{11}_i = \mathcal{C}^1_i$ if $i \neq k$ and $\mathcal{C}^{11}_k = \mathcal{C}^1_k \cup \{\tilde{e}\}$. Note that $\Delta^{11}_{\tilde{e}w} = \Delta^1_{\tilde{e}w} \cup \{(u', v')\} = \{(u', v')\}$. The other subsets describing S^{11} remain the same as in S^1. Both incidence vectors of S^1 and S^{11} are in $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$, thus they satisfy

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^{11}} + \mu y^{S^{11}} + \nu z^{S^{11}} =$$

$$\lambda x^{S^1} + \lambda^k_{\tilde{e}w} + \mu y^{S^1} + \mu^{tildeew} + \nu z^{S^1} + \nu^{\tilde{e}w}_{(u',v')},$$

We have that $\nu^{\tilde{e}w}_{(u',v')} = \mu^{tildeew} = 0$, by (5.31) and (5.35). Thus, this implies that $\lambda^k_{\tilde{e}w} = 0$. As k and w are chosen arbitrarily in $(K \times W) \setminus \{(\tilde{k}, \tilde{w})\}$ respectively, we obtain

$$\lambda^k_{\tilde{e}w} = 0, \text{ for all } (k, w) \in (K \times W) \setminus \{(\tilde{k}, \tilde{w})\}, \tag{5.39}$$

Consequently, all together, we obtain that

$$\lambda^k_{ew} = \begin{cases} \rho, & \text{if } (k, e, w) = (\tilde{k}, \tilde{e}, \tilde{w}), \\ 0, & \text{otherwise.} \end{cases}$$

□

Theorem 5.6 *For $\tilde{k} \in K$, $\tilde{e} \in A_1$ and $\tilde{w} \in W$, inequality $x_{\tilde{k}\tilde{e}\tilde{w}} \geq 0$ is facet defining for $P(G_1, G_2, K, C)$*

Proof. Let us denote by $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$ the face induced by inequality $x_{\tilde{k}\tilde{e}\tilde{w}} \geq 0$, which is given by

$$\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}} = \{(x, y, z) \in P(G_1, G_2, K, C) : x_{\tilde{k}\tilde{e}\tilde{w}} = 0\},$$

We denote the inequality $x_{\tilde{k}\tilde{e}\tilde{w}} \geq 0$ by $\alpha x + \beta y + \gamma z \leq \delta$. Let $\lambda x + \mu y + \nu z \leq \xi$ be a valid inequality that defines a facet of $P(G_1, G_2, K, C)$. Suppose that $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}} \subseteq \mathcal{F}$. We show that there exists $\rho \in \mathbb{R}$ such that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$.

We will show that $\lambda^k_{ew} = 0$, for all $(k, e, w) \in (K \times A_1 \times W) \setminus \{(\tilde{k}, \tilde{e}, \tilde{w})\}$.

Consider the solution $S^0 = (F^0_1, F^0_2, \Delta^0, W^0)$ described in proof of Theorem 5.3. In what follows, we will suppose that $(\tilde{e}, \tilde{w})$ is in the solution S^0 and involved in the routing of $\tilde{k}$.

Let w be a subband of $W \setminus W^0$. Consider the solution S^1 obtained from S^0 as follows. We replace the subband $\tilde{w}$ installed on $\tilde{e}$ by any subband $w \in W \setminus \{\tilde{w}\}$ in the solution. The pair $(\tilde{e}, w)$ is assigned the path $\{(u', v')\}$ in G_2, where $(u', v') \in \Delta^0$. In this way, the commodity $\tilde{k}$ may use $(\tilde{e}, w)$ for its routing instead of $(\tilde{e}, \tilde{w})$. Note that this operation leads to $x^{S^1}_{\tilde{e}\tilde{w}} = 0$ while $x^{S^0}_{\tilde{e}\tilde{w}} = 1$. $S^1 = (F^1_1, F^1_2, \Delta^1, W^1)$, where $W^1 = W^0 \cup \{w\}$, and $\Delta^1_{\tilde{e}w} = \Delta^0_{\tilde{e}w} \cup \{(u', v')\}$, while the other subsets of S^1 remain the same as in S^0. The solution S^1 is clearly feasible and it will be considered as a reference solution in the rest of the proof.

First let us show that $\nu^{ew}_a = 0$, for all $e \in A_1$, $w \in W$ and $a \in A_2$.

Let $a = (s, t)$ be an arc of $A_2 \setminus \Delta^1$. Let e and w be an arc of A_1 and a subband of W, respectively. Consider the solution S^2, obtained from S^1 by adding the arc a. $S^2 = (F^2_1, F^2_2, \Delta^2, W^2)$ where $\Delta^2_{ew} = \Delta^1_{ew} \cup \{a\}$, and all the remaining subsets are the same as in S^1. It is easy to see that S^2 is a feasible solution. Moreover, both incidence vectors of S^1 and S^2 are in $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$ (and in $\mathcal{F}$). Thus,

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^2} + \mu y^{S^2} + \nu z^{S^2} = \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} + \nu^{ew}_a,$$

which implies that $\nu^{ew}_a = 0$. As the arc a is chosen arbitrarily in $A_2 \setminus \Delta^1$, we get

$$\nu^{ew}_a = 0, \text{ for all } e \in A_1, w \in W, a \in A_1 \setminus \Delta^1, \tag{5.40}$$

Suppose now that a belongs to the solution S^1. In other words, a is an arc of Δ^1_{ew}, where $e \in A_1$ and w is in W. Let $f = (s', r')$ and $g = (r', t')$ be two arcs of $A_2 \setminus \Delta^1$, with $r' \in V_2 \setminus \{s, t\}$. Let us introduce the solution S^3, obtained from S^1 by replacing a by the arcs f and g. The solution $S^3 = (F^3_1, F^3_2, \Delta^3, W^3)$ is described as follows. Δ^3_{ew} $= (\Delta^1_{ew} \setminus \{a\}) \cup \{f, g\}$, and all the other subsets of S^3 are the same as in S^1. S^3 is still a feasible solution, and both incidence vectors of S^1 and S^3 satisfy

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^3} + \mu y^{S^3} + \nu z^{S^3} = \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} - \nu^{ew}_a + \nu^{ew}_f + \nu^{ew}_g,$$

since they belong to $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$ and $\mathcal{F}$ as well. This implies that - $\nu^{ew}_a + \nu^{ew}_f + \nu^{ew}_g = 0$. We have $\nu^{ew}_f = \nu^{ew}_g = 0$, by (5.40), we get $\nu^{ew}_a = 0$. As one can chose arbitrarily a in Δ^1, we obtain

$$\nu^{ew}_a = 0, \text{ for all } e \in (A_1, w \in W, a \in \Delta^1, \tag{5.41}$$

Hence, by (5.40) and (5.41), we conclude that

$$\nu^{ew}_a = 0, \text{ for all } e \in A_1, w \in W, a \in A_2, \tag{5.42}$$

Next, we will show that $\mu^{ew} = 0$, for all $e \in A_1$ and $w \in W$.

Given an arc $e = (s,t) \in A_1$ which does not appear in the solution S^1. Let w be a subband of W. Let us construct a solution S^4 by adding the arc e to the solution S^1. We set up the subband w on the arc e, and we assign the arc (s',t') of A_2 to the pair (e,w) as a routing path. $S^4 = (F_1^4, F_2^4, \Delta^4, W^4)$, where $F_1^4 = F_1^1$, $F_2^4 = F_2^1 \cup \{e\}$, $\Delta^4 = \Delta^1 \cup \{(s',t')\}$ and $W^4 = W^1 \cup \{w\}$. Here, we have $\Delta^4_{ew} = \Delta^1_{ew} \cup \{(s',t')\}$ while the other subsets of Δ^1 do not change. In addition, we assume that e is not involved in the routing of any commodity. The solution S^4 is feasible and its incidence vector as one of S^1 are in $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$ (and in $\mathcal{F}$), so they verify

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^4} + \mu y^{S^4} + \nu z^{S^4} = \lambda x^{S^1} + \mu y^{S^1} + \mu^{ew} + \nu z^{S^1} + \nu^{ew}_{(s',t')},$$

which implies that $\mu^{ew} + \nu^{ew}_{(s',t')} = 0$. We have $\nu^{ew}_{(s',t')} = 0$ by (5.42). Then, it follows that $\mu^{ew} = 0$. As e was chosen arbitrarily in $A_1 \setminus (F_1^1 \cup F_2^1)$, we get that

$$\mu^{ew} = 0, \text{ for all } e \in A_1 \setminus (F_1^1 \cup F_2^1), w \in W, \tag{5.43}$$

Now assume that e is in $F_1^1 \cup F_2^1$. Let $f = (s,r)$ and $g = (r,t)$ be two arcs of $A_1 \setminus (F_1^1 \cup F_2^1)$, with $r \in V_1 \setminus \{s,t\}$. We will construct three solutions S^5 and S^6 and S^7 in order to show that $\mu^{ew} = 0$, for $e \in F_1^1 \cup F_2^1$, $w \in W$.

First, suppose that $e \in F_2^1$. Consider the solution S^5, that is obtained from S^1 by replacing the arc e by f and g. We assume that the subband w, initially installed on e is reused for both f and g. The pairs (f,w) and (g,w) are then assigned the arcs (s',r') and (r',t') of A_2 for their routing in G_2, respectively. This solution is feasible, and both incidence vectors of S^1 and S^5 satisfy

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^5} + \mu y^{S^5} + \nu z^{S^5} =$$

$$\lambda x^{S^1} + \mu y^{S^1} - \mu^{ew} + \mu^{fw} + \mu^{gw} + \nu z^{S^1} - \nu^{ew}_{(s',t')} + \nu^{fw}_{(s',r')} + \nu^{gw}_{(r',t')},$$

since they belong to $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$ and $\mathcal{F}$. We have that $\nu^{ew}_{(s',t')} = \nu^{fw}_{(s',r')} = \nu^{gw}_{(r',t')} = 0$, by (5.42), while $\mu^{fw} = \mu^{gw} = 0$ by (5.43). We then get $\mu^{ew} = 0$. As e is chosen arbitrarily in F_2^1, we obtain

$$\mu^{ew} = 0, \text{ for all } e \in F_2^1, w \in W, \tag{5.44}$$

If $e \in F_1^1$, then e is considered in the routing of some commodity, say k ($e \in \mathcal{C}_k^1$). Let us construct the solution S^6 based on S^1, and where the arc e is replaced by f and g. Again, we consider that the subband w is reused for the arcs f and g. We assume that (f,w) and (g,w) are assigned the arcs (s',r') and (r',t'), respectively. In this solution, the commodity k is rerouted in f and g (instead of e). More formally, S^6 is described as follows. $S^6 = (F_1^1, F_2^1, \Delta^6, W^6)$, where $F_1^6 = (F_1^1 \setminus \{e\}) \cup \{f,g\}$

and $\Delta^6 = (\Delta^1 \setminus \{(s',t')\}) \cup \{(s',r'),(r',t')\}$. In particular F_1^6 and Δ^6 are such that, $\mathcal{C}_k^6 = (\mathcal{C}_k^1 \setminus \{e\}) \cup \{f,g\}$, $\Delta_{ew}^6 = \Delta_{ew}^1 \setminus \{(s',t')\}$, $\Delta_{fw}^6 = \Delta_{fw}^1 \cup \{(s',r')\}$ and $\Delta_{gw}^6 = \Delta_{gw}^1 \cup \{(r',t')\}$.

Now consider the solution S^7 which is obtained from S^6 by adding the arc e to the solution. We set up the subband w on the arc e and we assign the arc (s',t') to the pair (e,w). $S^7 = (F_1^7, F_2^7, \Delta^7, W^7)$, where $F_2^7 = F_2^6 \cup \{e\}$, $\Delta_{ew}^7 = \Delta_{ew}^6 \cup \{(s',t')\}$ and the other subsets are still the same as in S^6. Both solutions are feasible and their incidence vectors are in $\mathcal{F}^{\tilde{e}\tilde{e}\tilde{w}}$ and $\mathcal{F}$. Thus, we have

$$\lambda x^{S^6} + \mu y^{S^6} + \nu z^{S^6} = \lambda x^{S^7} + \mu y^{S^7} + \nu z^{S^7} = \lambda x^{S^6} + \mu y^{S^6} + \mu^{ew} + \nu z^{S^6} + \nu_{(s',t')}^{ew},$$

which gives $\mu^{ew} + \nu_{(s',t')}^{ew} = 0$. By (5.42), we have that $\nu_{(s',t')}^{ew} = 0$, and thus $\mu^{ew} = 0$. As the arc e was chosen arbitrarily in $F_1^1 \setminus \{\tilde{e}\}$, we get

$$\mu^{ew} = 0, \text{ for all } e \in F_1^1, w \in W, \tag{5.45}$$

Hence, by (5.43), (5.44) and (5.43), we obtain

$$\mu^{ew} = 0, \text{ for all } e \in A_1, w \in W, \tag{5.46}$$

Finally, let us show that $\lambda_{ew}^k = 0$, for all $(k,e,w) \in (K \times A_1 \times W) \setminus \{(\tilde{k},\tilde{e},\tilde{w})\}$.

Suppose that $e = (u,v)$ is in $A_1 \setminus (F_1^1 \cup F_2^1)$ and let k be some commodity of K. Consider the solution S^8, obtained from S^1 by also considering the arc e for the routing of k. In other words, e is added to the solution, and receives a subband $w \in W$. The pair (e,w) is then assigned the path $\{(u',v')\}$ in G_2. $S^8 = (F_1^8, F_2^8, \Delta^8, W^8)$ where $F_1^8 = F_1^1 \cup \{e\}$, $F_2^8 = F_2^1$, $\Delta^8 = \Delta^1 \cup \{(u',v')\}$ and $W^8 = W^1 \cup \{w\}$. In particular, we have that $\mathcal{C}_k^8 = \mathcal{C}_k^1 \cup \{e\}$ and $\Delta_{ew}^8 = \Delta_{ew}^1 \cup \{(u',v')\}$, while the remaining subsets still the same as in S^1. S^8 is a feasible solution, and both incidence vectors of S^1 and S^8 verify

$$\begin{aligned} \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} &= \lambda x^{S^8} + \mu y^{S^8} + \nu z^{S^8} \\ \lambda x^{S^1} + \lambda_{ew}^k + \mu y^{S^1} + \mu^{ew} + \nu z^{S^1} + \nu_{(u',v')}^{ew}, \end{aligned}$$

which gives that $\lambda_{ew}^k + \mu^{ew} + \nu_{(u',v')}^{ew} = 0$. We know by (5.42) and (5.46), that $\mu^{ew} = \nu_{(u',v')}^{ew} = 0$. Thus, we get $\lambda_{ew}^k = 0$. As e and k are chosen arbitrarily in $A_1 \setminus (F_1^1 \cup F_2^1)$ and K, respectively, we obtain

$$\lambda_{ew}^k = 0, \text{ for all } k \in K, e \in A_1 \setminus (F_1^1 \cup F_2^1), w \in W, \tag{5.47}$$

Now consider $e = (u,v) \in (F_1^1 \cup F_2^1) \setminus \{\tilde{e}\}$, and let w be the subband installed on e. Suppose that $e \in \mathcal{C}_k^1$ for some commodity k. Let S^9 be a solution, obtained from S^1

by replacing e with two arcs $f = (u, s)$ and $g = (s, v)$ of $A_1 \setminus (F_1^1 \cup F_2^1)$. Both f and g receive the subband w, and we associate with the pairs (f, w) and (g, w) the arcs (u', s'), (s', v') of A_2, respectively. We also consider the arcs f and g for the routing of k (instead of e). $S^9 = (F_1^9, F_2^9, \Delta^9, W^9)$, where $F_1^9 = (F_1^1 \setminus \{e\}) \cup \{f, g\}$, $F_2^9 = F_2^1$, $\Delta^9 = \Delta^1 \cup \{(u', s'), (s', v')\}$ and $W^9 = W^1$. In particular, F_1^9 and Δ^9 are such that $\mathcal{C}_k^9 = (\mathcal{C}_k^1 \setminus \{e\}) \cup \{f, g\}$, $\Delta_{fw}^9 = \Delta_{fw}^1 \cup \{(u', s')\}$ and $\Delta_{gw}^9 = \Delta_{gw}^1 \cup \{(s', v')\}$.

Let us introduce the solution S^{10}, obtained by reinserting the arc e in the solution S^9. In this solution, we consider the arc e for the routing of the commodity k, instead of f and g. More formally, the solution $S^{10} = (F_1^{10}, F_2^{10}, \Delta^{10}, W^{10})$ is described as follows. $F_1^{10} = F_1^9 \cup \{e\}$ while the remaining subsets are still the same as in S^9. The solutions S^9 and S^{10} are both feasible, and as their incidence vectors belong to $\mathcal{F}^{\tilde{k}\tilde{e}\tilde{w}}$ (and to $\mathcal{F}$), they verify

$$\lambda x^{S^9} + \mu y^{S^9} + \nu z^{S^9} = \lambda x^{S^{10}} + \mu y^{S^{10}} + \nu z^{S^{10}} =$$

$$\lambda x^{S^9} + \lambda_{ew}^k - \lambda_{fw}^k - \lambda_{gw}^k + \mu y^{S^9} + \mu^{ew} + \nu z^{S^9}$$

which gives that $\lambda_{ew}^k - \lambda_{fw}^k - \lambda_{gw}^k + \mu^{ew} = 0$. By (5.46) and (5.47), we have that $\mu^{ew} = 0$ and $\lambda_{fw}^k = \lambda_{gw}^k = 0$. Thus, we have that $\lambda_{ew}^k = 0$. As, e was chosen arbitrarily in $(F_1^1 \cup F_2^1) \setminus \{\tilde{e}\}$, we obtain

$$\lambda_{ew}^k = 0, \text{ for all } k \in K, e \in (F_1^1 \cup F_2^1) \setminus \{\tilde{e}\}, w \in W, \tag{5.48}$$

Now suppose that $e = \tilde{e}$, and let $w \in W \setminus \{\tilde{w}\}$ be the subband installed on $\tilde{e}$. Recall that, by construction of S^1, we have $\tilde{e} \in \mathcal{C}_{\tilde{k}}^1$. Let w' be a subband of $W \setminus \{\tilde{w}, w\}$. We will construct a solution S^{11}, based on S^1, where we replace the subband w installed on $\tilde{e}$, by the subband w'. The pair $(\tilde{e}, w')$ is then assigned the path $\{(u', v')\}$ in G_2. $S^{11} = (F_1^{11}, F_2^{11}, \Delta^{11}, W^{11})$, where, $\Delta_{\tilde{e}w'}^{11} = \Delta_{\tilde{e}w'}^1 \cup \{(u', v')\}$, and $W^{11} = W^1 \cup \{w'\}$, the other subsets of S^1 remain unchanged. S^{11} is clearly feasible, and both inicidence vectors of S^1 and S^{11} satisfy

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^{11}} + \mu y^{S^{11}} + \nu z^{S^{11}} = \lambda x^{S^1} + \lambda_{\tilde{e}w'}^{\tilde{k}} + \mu y^{S^1} + \mu^{\tilde{e}w'} + \nu z^{S^1} + \nu_{(u',v')}^{\tilde{e}w'},$$

We have by (5.42) and (5.46) that $\mu^{\tilde{e}w'} = \nu_{(u',v')}^{\tilde{e}w'} = 0$. Thus, it follows that $\lambda_{\tilde{e}w'}^{\tilde{k}} = 0$. As w' was chosen arbitrarily in $W \setminus \{\tilde{w}\}$, we obtain that

$$\lambda_{\tilde{e}w}^{\tilde{k}}, \text{ for all } w \in W \setminus \{\tilde{w}\}, \tag{5.49}$$

Thus, we conclude that

$$\lambda_{ew}^k = 0, \text{ for all }, (k, e, w) \in (K \times A_1 \times W) \setminus \{(\tilde{k}, \tilde{e}, \tilde{w})\}, \tag{5.50}$$

Consequently, we conclude by (5.42), (5.46) and (5.50), we can deduce that $\lambda_{ew}^k = 0$, for all $(k, e, w) \in (K \times A_1 \times W) \setminus \{(\tilde{k}, \tilde{e}, \tilde{w})\}$, while $\lambda_{\tilde{e}\tilde{w}}^{\tilde{k}} = \rho$, which ends the proof. $\square$

Next, we will investigate the facial structure of trivial constraints related to z variables. Note that each inequality $z_{ewa} \leq 1$, associated with a subband $w \in W$, an arc $e \in A_1$ and an arc $a \in A_2$, is dominated by a disjunction constraint (5.4), associated with w and a. Thus, we will only study inequalities $z_{ewa} \geq 0$, for all $e \in A_1$, $w \in W$ and $a \in A_2$.

Theorem 5.7 *For $\tilde{e} \in A_1$, $\tilde{w} \in W$ and $\tilde{a} \in A_2$, inequality $z_{\tilde{e}\tilde{w}\tilde{a}} \geq 0$ is facet defining for $P(G_1, G_2, K, C)$.*

Proof. Let us denote by $\mathcal{F}_{\tilde{a}}^{\tilde{e}\tilde{w}}$ the face induced by inequality $z_{\tilde{e}\tilde{w}\tilde{a}} \geq 0$, which is given by

$$\mathcal{F}_{\tilde{a}}^{\tilde{e}\tilde{w}} = \{(x, y, z) \in P(G_1, G_2, K, C) : z_{\tilde{e}\tilde{w}\tilde{a}} = 0\}$$

We denote the inequality $z_{\tilde{e}\tilde{w}\tilde{a}} \geq 0$ by $\alpha x + \beta y + \gamma z \leq \delta$. Let $\lambda x + \mu y + \nu z \leq \xi$ be a valid inequality that defines a facet $\mathcal{F}$ of $P(G_1, G_2, K, C)$. Suppose that $\mathcal{F}_{\tilde{a}}^{\tilde{e}\tilde{w}} \subseteq \mathcal{F}$. We show that both inequalities are equal up to a scalar $\rho \in \mathbb{R}^*$.

First, let us show that $\nu_a^{ew} = 0$, for all $(e, w, a) \in (A_1 \times W \times A_2) \setminus \{(\tilde{e}, \tilde{w}, \tilde{a})\}$.

Consider the solution $S^0 = (F_1^0, F_2^0, \Delta^0, W^0)$ described in proof of Theorem 5.3. We will assume without loss of generality that $(\tilde{e}, \tilde{w}, \tilde{a})$ does not belong to the solution S^0. In other words, the subband $\tilde{w}$ is not installed on the arc $\tilde{e}$, and the pair $(\tilde{e}, \tilde{w})$ is not associated with the arc $\tilde{a}$ for its routing in G_2. Let $\overline{a}$ be an arc of $A_2 \setminus \Delta^0$ such that $\overline{a} \neq \tilde{a}$ and (e, w) be some pair of $A_1 \times W$. We will introduce the solution S^1 obtained from S^0 by adding $\overline{a}$ to Δ_{ew}^0. The solution given by $S^1 = (F_1^0, F_2^0, \Delta^0 \cup \{\overline{a}\}, W^0)$ is feasible for the problem, and its incidence vector belongs to $\mathcal{F}_{\tilde{a}}^{\tilde{e}\tilde{w}}$ and $\mathcal{F}$ as well. Thus, we have

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} + \nu_{\overline{a}}^{ew},$$

which implies that $\nu_{\overline{a}}^{ew} = 0$. As the arcs $\overline{a}$ and e, and the subband w are chosen arbitrarily in the subsets $A_2 \setminus \Delta^0$, A_1 and W, we get

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in A_2 \setminus (\Delta^0 \cup \{\tilde{a}\}), \tag{5.51}$$

Now if $\overline{a} = (s', t') \in \Delta^0$, in particular $\overline{a} \in \Delta_{ew}^0$, with $e \in A_1$, $w \in W$, then consider two arcs of $A_2 \setminus (\Delta^0 \cup \{\tilde{a}\})$, denoted (s', r') and (r', t'). Consider the solution S'^1 obtained from S^0 by replacing the arc $\overline{a}$ by (s', r') and (r', t') in Δ_{ew}^0. In other words, $S'^1 = (F_1^0, F_2^0, (\Delta^0 \setminus \{\overline{a}\}) \cup \{(s', r'), (r', t')\}, W^0)$ where $\Delta_{ew}'^1 = (\Delta_{ew}^0 \setminus \{\overline{a}\}) \cup \{(s', r'), (r', t')\}$ and $\Delta_{e_i w_i}'^1 = \Delta_{e_i w_i}^0$ if $(e_i, w_i) \neq (e, w)$. S'^1 remains clearly feasible, and its incidence vector verifies

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S'^1} + \mu y^{S'^1} + \nu z^{S'^1} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} - \nu_{\overline{a}}^{ew} + \nu_{(s',r')}^{ew} + \nu_{(r',t')}^{ew},$$

and it follows that $-\nu^{ew}_{\overline{a}} + \nu^{ew}_{(s',r')} + \nu^{ew}_{(r',t')} = 0$. As by (5.51), $\nu^{ew}_{(s',r')} = \nu^{ew}_{(r',t')} = 0$, we have that $\nu^{ew}_{\overline{a}} = 0$. Since $\overline{a}$ is chosen arbitrarily in Δ^0, we get

$$\nu^{ew}_a = 0, \text{ for all } e \in A_1, w \in W, a \in \Delta^0, \tag{5.52}$$

Thus, and by (5.51) and (5.52), we obtain

$$\nu^{ew}_a = 0, \text{ for all } e \in A_1, w \in W, a \in A_2, (e, w, a) \neq (\tilde{e}, \tilde{w}, \tilde{a}), \tag{5.53}$$

In what follows, we will show that $\mu^{ew} = 0$, for all $e \in A_1$, $w \in W$.

Suppose that $\overline{e} = (u, v)$ is an arc of $A_1 \setminus (F^0_1 \cup F^0_2)$ and $\overline{w}$ a subband of W. Consider the solution S^2 obtained from S^0 by installing the subband $\overline{w}$ on the arc $\overline{e}$, then adding $(\overline{e}, \overline{w})$ to the solution. We associate the arc (u', v') of $A_2 \setminus \Delta^0$ with the routing of $(\overline{e}, \overline{w})$. We will assume that no commodity uses this pair for its routing. S^2 is then given by $(F^0_1, F^0_2 \cup \{\overline{e}\}, \Delta^0 \cup \{(u', v')\}, W^0 \cup \{\overline{w}\})$, where $\Delta^2_{ew} = \Delta^0_{ew}$ if $(e, w) \neq (\overline{e}, \overline{w})$ and $\Delta^2_{\overline{ew}} = \Delta^0_{\overline{ew}} \cup \{(u', v')\}\}$. It is not hard to see that S^2 is a feasible solution. Hence, it satisfies

$$\begin{aligned} \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} &= \lambda x^{S^2} + \mu y^{S^2} + \nu z^{S^2} \\ &= \lambda x^{S^0} + \mu y^{S^0} + \mu^{\overline{ew}} + \nu z^{S^0} + \nu^{\overline{ew}}_{(u',v')}, \end{aligned}$$

that is to say that $\mu^{\overline{ew}} + \nu^{\overline{ew}}_{(u',v')} = 0$. We have that $\nu^{\overline{ew}}_{(u',v')} = 0$ by 5.53. Thus, $\mu^{\overline{ew}} = 0$. As $\overline{e}$ and $\overline{w}$ are selected out of the solution, we get

$$\mu^{ew} = 0, \text{ for all } e \in A_1 \setminus (F^0_1 \cup F^0_2), w \in W, \tag{5.54}$$

Assume that $\overline{e}$ and $\overline{w}$ are used in the solution S^0. In other words, $\overline{e} = (u, v) \in (F^0_1 \cup F^0_2)$ and $\overline{w}$ is installed on $\overline{e}$. Then, $\overline{e} \in F^0_1$, as F^0_2 is empty. In particular, let k be a commodity such that $\overline{e} \in \mathcal{C}^0_k$. Let $f = (u, r)$ and $g = (r, v)$ be two arcs of $A_1 \setminus (F^0_1 \cup F^0_2) \cup \{\tilde{e}\}$. Consider the solutions S'^2 and S''^2 which are obtained from S^0 as follows. S'^2 is constructed by adding the arcs f and g to the solution S^0. Both arcs receive the subband $\overline{w}$, and the pairs $(f, \overline{w})$ and $(g, \overline{w})$ are assigned the arcs (u', r') and (r', v') of $A_2 \setminus \Delta^0$. We assume that the commodity k uses f and g instead of $\overline{e}$. The solution S'^2 is then described as follows $((F^0_1 \setminus \{\overline{e}\}) \cup \{f, g\}, F^0_2 \cup \{\overline{e}\}, \Delta^0 \cup \{(u', r'), (r', v')\}, W^0)$, where $\mathcal{C}'^2_k = (\mathcal{C}^0_k \setminus \{\overline{e}\}) \cup \{f, g\}$, while $\Delta'^2_{f\overline{w}} = \Delta^0_{f\overline{w}} \cup \{(u', r')\}$ and $\Delta'^2_{g\overline{w}} = \Delta^0_{g\overline{w}} \cup \{(r', v')\}$. S''^2 is obtained by simply removing the arc $\overline{e}$ from the solution S'^2. In other words, $S''^2 = (F'^2_1, F'^2_2 \setminus \{\overline{e}\}, \Delta'^0, W'^2)$. Both solutions S'^2 and S''^2 are feasible for the problem, and their incidence vectors belong to $\mathcal{F}^{\tilde{e}\tilde{w}}_{\tilde{a}}$ and $\mathcal{F}$. Thus, they verify

$$\lambda x^{S'^2} + \mu y^{S'^2} + \nu z^{S'^2} = \lambda x^{S''^2} + \mu y^{S''^2} + \nu z^{S''^2} = \lambda x^{S'^2} + \mu y^{S'^2} - \mu^{\overline{ew}} + \nu z^{S'^2},$$

which implies that $\mu^{\overline{ew}} = 0$. As $\overline{e}$ is selected arbitrarily in the solution, we get

$$\mu^{ew} = 0, \text{ for all } e \in F_1^0 \cup F_2^0, w \in W, \tag{5.55}$$

By (5.54) and (5.55), we have that

$$\mu^{ew} = 0, \text{ for all } e \in A_1, w \in W, \tag{5.56}$$

Next, we will show that $\lambda^k_{ew} = 0$, for all $k \in K$, $e \in A_1$ and $w \in W$.

Let $\overline{k}$ be a commodity of K, and let (e, w) be some pair of $A_1 \times W$. Two cases may hold here.

Case 1.

Suppose that (e, w), $e = (u, v)$ does not appear in the solution S^0. We will consider a solution S^3 obtained by adding (e, w) to Γ^0, that is to install the subband w on the arc e. We associate to (e, w) the path in G_2 composed by arc (u', v'), where $(u', v') \in A_2 \setminus \{\tilde{a}\}$, and we consider the arc e for the routing of $\overline{k}$, in addition to its initial routing. More formally, $S^3 = (F_1^3, F_2^3, \Delta^3, W^3)$, where $F_1^3 = F_1^0 \cup \{e\}$, $F_2^3 = F_2^0$, $\Delta^3 = \Delta^0 \cup \{(u', v')\}$ and $W^3 = W^0 \cup \{w\}$. In particular, $\Gamma^3 = \Gamma^0 \cup \{(e, w)\}$, $\Delta^3_{ew} = \Delta^0_{ew} \cup \{(u', v')\}$ and $\mathcal{C}^3_{\overline{k}} = \mathcal{C}^0_{\overline{k}} \cup \{e\}$. It is clear that S^3 induces a feasible solution for OMBND problem, and its incidence vector belongs to $\mathcal{F}^{\tilde{e}\tilde{w}}_{\tilde{a}}$, and thus, it also belong to $\mathcal{F}$. Comparing $(x^{S^3}, y^{S^3}, z^{S^3})$ and $(x^{S^0}, y^{S^0}, z^{S^0})$ yields

$$\lambda^{\overline{k}}_{ew} + \mu^{ew} + \nu^{ew}_{(u',v')} = 0,$$

As by (5.53) and (5.56), we have $\lambda^{\overline{k}}_{ew} = 0$, we can conclude that

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, (e, w) \in (A_1 \times W) \setminus \Gamma^0, \tag{5.57}$$

Case 2.

Now assume that $(e, w) \in \Gamma^0$. Note that the case where $e \in F_2^0$ is rather easy, so we will assume that $e \in \mathcal{C}^0_k$ for some commodity k of K. Suppose without loss of generality that (e, w) are not involved in the routing of commodity $\overline{k}$. Let w' be a subband of W different from w. We will install w' on the arc e and associate (e, w') with the routing of k. In other words, we set the entry $x^{S^0}_{kew'}$ to 1. The pair (e, w') is associated with path $\{(u', v')\}$ in G_2. Note that in this solution, we just move the commodity k from the subband w to the subband w' on the same arc e. This means that k still use arc

e for its routing, but is carried by subband w' instead of w. Let us denote by S^4 the solution described above. It is clear that S^4 is feasible as all the constraints of (5.8) are satisfied. We will derive an other solution, based on S^4, that consists in associating the arc e with the commodity $\overline{k}$, in addition to its initial routing. Then, we can set the entry $x^{S^4}_{\overline{k}ew}$ to 1, and induce a feasible solution. Note that this is possible, since operations done in S^4 allow to free up the capacity of w, which can now be used for $\overline{k}$. The solution S^5 is obviously feasible, and both incidence vectors of S^4 and S^5 belong to $\mathcal{F}^{\tilde{e}\tilde{w}}_{\tilde{a}}$, and thus, to $\mathcal{F}$. Hence, we obtain that $\lambda^{\overline{k}}_{ew} = 0$. Since $\overline{k}$, e and w are arbitrary and interchangeable in K, A_1 and W, we obtain

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, (e, w) \in \Gamma^0, \tag{5.58}$$

Consequently, and by (5.57) and (5.58), we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, (e, w) \in A_1 \times W, \tag{5.59}$$

All together, we obtain that all the coefficients are equal to zero except $\nu^{\tilde{e}\tilde{w}}_{\tilde{a}}$ which is equal to some $\rho \in \mathbb{R}$. □

5.3.2 Disjunction constraints

In this section, we study the facial structure of disjunction constraints. Let $\tilde{a} = (u', v')$ and $\tilde{w}$ be an arc of A_2 and a subband of W, respectively. We denote by $\mathcal{F}^{\tilde{w}}_{\tilde{a}}$, the face induced by the inequality (5.4). In other words,

$$\mathcal{F}^{\tilde{w}}_{\tilde{a}} = \{(x, y, z) \in P(G_1, G_2, K, C) : \sum_{e \in A_1} z_{e\tilde{w}\tilde{a}} = 1\}.$$

In what follows, we show that (5.4) are facet defining.

Theorem 5.8 *For $\tilde{w} \in W$ and $\tilde{a} \in A_2$, the inequality $\sum_{e \in A_1} z_{e\tilde{w}\tilde{a}} \leq 1$ defines a facet of $P(G_1, G_2, K, C)$.*

Proof. Let $\alpha x + \beta y + \gamma z \leq \delta$ be the disjunction constraint (5.4) related to the arc $\tilde{a}$ and the subband $\tilde{w}$. Consider the valid inequality, denoted $\lambda x + \mu y + \nu z \leq \xi$, that defines a facet $\mathcal{F}$ for $P(G_1, G_2, K, C)$. Suppose that $\mathcal{F}^{\tilde{w}}_{\tilde{a}}$. We will show that there exists $\rho \in \mathbb{R}$ such that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$.
Consider the solution $S^0 = (F^0_1, F^0_2, \Delta^0, W^0)$ described in proof of Theorem 5.3. Suppose that $\tilde{w} \notin W^0$, and $\tilde{a} \notin \Delta^0$. We will introduce a new solution S^1, obtained from

S^0 by adding the subband $\tilde{w}$ to W^0. We assume that $\tilde{w}$ is installed on some arc, say $e = (o_1, d_1)$, but the pair $(e, \tilde{w})$ is not involved in the routing of any commodity. $S^1 = (F_1^1, F_2^1, \Delta^1, W^1)$, is then defined as follows. $F_1^1 = F_1^0$, $F_2^1 = F_2^0$, $\Delta^1 = \Delta^0$ and $W^1 = W \cup \{\tilde{w}\}$. In particular, note that $\Delta^1_{ew} = \Delta^0_{ew}$ if $w \neq \tilde{w}$, and $\Delta^1_{e\tilde{w}} = \{(o_1', d_1')\} \cup \{\tilde{a}\}$. In other words, a new subband $\tilde{w}$ is added to the arc $e = (o_1, d_1)$, and the pair $(e, \tilde{w})$ is assigned two arcs in G_2, (o_1', d_1') and $\tilde{a}$.

The solution S^1 is clearly feasible, and its incidence vector belongs to both $\mathcal{F}_{\tilde{a}}^{\tilde{w}}$ and $\mathcal{F}$. Moreover, S^1 will be considered as a reference solution in the rest of the proof.

First, let us show that $\nu_a^{ew} = 0$, for all $e \in A_1$ and for all $(w, a) \in (A_2 \times W) \setminus \{(\tilde{w}, \tilde{a})\}$.

Let e be an arc of A_1, w a subband of W and a an arc of $A_2 \setminus \Delta^1$. Let us introduce the solution S^2, obtained from S^1, by adding a to Δ^1_{ew}. In other words, the pair (e, w) is assigned the arc a. $S^2 = (F_1^2, F_2^2, \Delta^2, W^2)$, where $\Delta^2_{ew} = \Delta^1_{ew} \cup \{a\} = \{a\}$, and the other elements of S^2 remain the same as in S^1.

We can easily see that the solution S^2 is feasible, and its incidence vector belongs to $\mathcal{F}_{\tilde{a}}^{\tilde{w}}$ and $\mathcal{F}$. Thus, we have

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^2} + \mu y^{S^2} + \nu z^{S^2} = \lambda x^{S^1} + \mu y^{S^1} + \mu^{ew} + \nu z^{S^1} + \nu_a^{ew},$$

which gives $\nu_a^{ew} = 0$. As e, w, and a were chosen arbitrarily in A_1, W and $A_2 \setminus \Delta^1$, we obtain

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, \text{ and } a \in A_2 \setminus \Delta^1, \tag{5.60}$$

Suppose now that we select an arc $a = (s', t')$ in the subset $\Delta^1 \setminus \{\tilde{a}\}$. Let e and w be an arc of A_1 and a subband of W, respectively, such that $a \in \Delta^1_{ew}$. Let $a_1 = (s', r')$ and $a_2 = (r', t')$ be two arcs of $A_2 \setminus \Delta^1$, with $r' \in V_2 \setminus \{s', t'\}$. Consider the solution S^3, which is obtained from S^1, by replacing a in S^2 by a_1 and a_2. $S^3 = (F_1^3, F_2^3, \Delta^3, W^3)$, where $\Delta^3_{ew} = (\Delta^1_{ew} \setminus \{a\}) \cup \{a_1, a_2\}$. The solution S^3 is feasible and both incidence vectors of S^1 and S^3 verify

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^3} + \mu y^{S^3} + \nu z^{S^3} = \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} - \nu_a^{ew} + \nu_{a_1}^{ew} + \nu_{a_2}^{ew},$$

Thus, we have that $-\nu_a^{ew} + \nu_{a_1}^{ew} + \nu_{a_2}^{ew} = 0$. By (5.60), we know that $\nu_{a_1}^{ew} = \nu_{a_2}^{ew} = 0$. We then obtain $\nu_a^{ew} = 0$. As e, w and a were chosen arbitrarily in A_1, W, and $\Delta^1 \setminus \{\tilde{a}\}$, respectively, we get that

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, \text{ and } a \in \Delta^1 \setminus \{\tilde{a}\}, \tag{5.61}$$

We can conclude, by (5.60) and (5.61) that

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, \text{ and } a \in A_2, (w, a) \neq (\tilde{w}, \tilde{a}), \tag{5.62}$$

Next, we will show that $\mu^{ew} = 0$, for all $e \in A_1$ and for all $w \in W$.

Let $e = (s,t)$ and w be an arc of $A_1 \setminus (F_1^1 \cup F_2^1)$ and $W \setminus W^1$, respectively. Consider the solution S^4, defined as follows. We install the subband w over the arc e, and we assign to the pair (e,w) the arc (s',t') in G_2. In other words, $S^4 = (F_1^4, F_2^4, \Delta^4, W^4)$, where $F_2^4 = F_2^1 \cup \{e\}$, $\Delta_{ew}^4 = \Delta_{ew}^1 \cup \{(s',t')\}$ and $W^4 = W^1 \cup \{w\}$. All the other subsets defining S^4 remain the same as in S^1. S^4 is clearly feasible, and both incidence vectors of S^1 and S^4 belong to $\mathcal{F}_{\tilde{a}}^{\tilde{w}}$ and $\mathcal{F}$, thus

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^4} + \mu y^{S^4} + \nu z^{S^4} = \lambda x^{S^1} + \mu y^{S^1} + \mu^{ew} + \nu z^{S^1} + \nu_{(s',t')}^{ew},$$

which implies $\mu^{ew} + \nu_{(s',t')}^{ew} = 0$. As by (5.62), $\nu_{(s',t')}^{ew} = 0$, we get $\mu^{ew} = 0$. The arc e and the subband w were chosen arbitrarily in $A_1 \setminus (F_1^1 \cup F_2^1)$ and $W \cup W^1$, hence, we obtain

$$\mu^{ew} = 0, \text{ for all } e \in A_1 \setminus (F_1^1 \cup F_2^1), w \in W \setminus W^1, \tag{5.63}$$

Assume now that e and w belong to the solution S^1. In other words, e is an arc of $(F_1^1 \cup F_2^1)$, and $w \in W^1$. We will use three solutions S^5, S^6 and S^7 in order to show that $\mu^{ew} = 0$, for all $e \in (F_1^1 \cup F_2^1)$, and for all $w \in W^1$.

First, suppose that $e = (s,t) \in F_2^1$ is not involved in the routing of any commodity. Let $w \in W^1$, be the subband installed on e. Consider the arcs $f = (s,r)$ and $g = (r,t)$, with $r \in V_1 \setminus \{s,t\}$ that do not appear in S^1. Consider the solution S^5, obtained from S^1 as follows. The arc e is replaced by f and g, and the subband w, initially installed on e is reused for both f and g. Moreover, the pairs (f,w) and (g,w) are assigned the arcs $f' = (s',r')$ and $g' = (r',t')$, respectively. f' and g' are not considered in the solution S^1. More formally, S^5 is such that $F_2^5 = (F_2^1 \setminus \{e\}) \cup \{f,g\}$, $\Delta_{ew}^5 = \Delta_{ew}^1 \setminus \{(s',t')\}$, $\Delta_{fw}^5 = \Delta_{fw}^1 \cup \{(s',r')\}$ and $\Delta_{gw}^5 = \Delta_{gw}^1 \cup \{(r',t')\}$. The other subsets of S^1 remain unchanged.

It is easy to see that S^5 is a feasible solution. Moreover, its incidence vector, as one of S^1, satisfy

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^5} + \mu y^{S^5} + \nu z^{S^5}$$
$$= \lambda x^{S^1} + \mu y^{S^1} - \mu^{ew} + \mu^{fw} + \mu^{gw} + \nu z^{S^1} + \nu z^{S^1} + \nu_{(s',r')}^{fw} + \nu_{(r',t')}^{gw},$$

which gives

$$-\mu^{ew} + \mu^{fw} + \mu^{gw} + \nu_{(s',r')}^{fw} + \nu_{(r',t')}^{gw} = 0,$$

We have by (5.62) that $\nu_{(s',r')}^{fw} + \nu_{(r',t')}^{gw} = 0$. It follows then that $-\mu^{ew} + \mu^{fw} + \mu^{gw} = 0$. As by (5.63), $\mu^{fw} = \mu^{gw} = 0$, we get $\mu^{ew} = 0$. The arc e is chosen arbitrarily in F_2^1. Thus we get

$$\mu^{ew} = 0, \text{ for all } e \in F_2^1, w \in W, \tag{5.64}$$

Assume now that $e = (s,t)$ is in F_1^1 and let w be the subband installed on e. Suppose that $e \in \mathcal{C}_k^1$, where k is some commodity of K. Consider the solution S^6, obtained from S^1 by replacing e by arcs f and g. The subband w is reused for both f and g, while the pairs (f,w) and (g,w) are assigned the arcs (s',r') and (r',t') of $A_2 \setminus \Delta^1$, respectively. The commodity k is supposed to use the arcs f and g for its routing, instead of e. In other words, S^6 is such that $F_1^6 = (F_1^1 \setminus \{e\}) \cup \{f,g\}$, $\mathcal{C}_k^6 = (\mathcal{C}_k^1 \setminus \{e\}) \cup \{f,g\}$, while $\Delta_{fw}^6 = \Delta_{fw}^1 \cup \{(s',r')\}$ and $\Delta_{gw}^6 = \Delta_{gw}^1 \cup \{(r',t')\}$.

We introduce the solution S^7, obtained from S^6 by reintroducing the arc e to the solution S^6. e receives the subband w, and (e,w) is assigned again the arc (s',t'). $S^7 = (F_1^7, F_2^7, \Delta^7, W^7)$, where $F_1^7 = F_1^6 \cup \{e\}$ and the other entries remain the same as in S^6.

Both solutions S^6 and S^7 are feasible, and their incidence vectors belong to $\mathcal{F}_{\tilde{a}}^{\tilde{w}}$ and $\mathcal{F}$. Thus, they satisfy

$$\lambda x^{S^6} + \mu y^{S^6} + \nu z^{S^6} = \lambda x^{S^7} + \mu y^{S^7} + \nu z^{S^7}$$

$$= \lambda x^{S^6} + \mu y^{S^6} + \mu^{ew} + \nu z^{S^6},$$

we thus obtain $\mu^{ew} = 0$. The arc e was selected arbitrarily in the subset F_1^1, we conclude that,

$$\mu^{ew} = 0, \text{ for all } e \in F_1^1, w \in W, \tag{5.65}$$

We can conclude by (5.63), (5.64) and (5.65) that

$$\mu^{ew} = 0, \text{ for all } e \in A_1, w \in W. \tag{5.66}$$

Next, we will show that $\lambda_{ew}^k = 0$, for all $k \in K$, for all $e \in A_1$ and for all $w \in W$.

Suppose that $e = (s,t)$ is an arc of $A_1 \setminus (F_1^1 \cup F_2^1)$, and w is a subband of $W \setminus W^1$. Consider the solution S^8 obtained from S^1 as follows. Let us install w on the arc e, and assign to the pair (e,w) the arc (s',t'). We associate (e,w) with the routing of some commodity, say k. S^8 is such that $\mathcal{C}_k^8 = \mathcal{C}_k^1 \cup \{e\}$, $\Delta_{ew}^8 = \Delta_{ew}^1 \cup \{(s',t')\}$ and $W^8 = W^1 \cup \{w\}$. One can easily check that S^8 is a feasible solution. In addition, both incidence vectors of S^1 and S^8 verify

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^8} + \mu y^{S^8} + \nu z^{S^8}$$

$$= \lambda x^{S^1} + \lambda_{ew}^k + \mu y^{S^1} + \mu^{ew} + \nu z^{S^1} + \nu_{(s',t')}^{ew},$$

which implies that

$$\lambda_{ew}^k + \mu^{ew} + \nu_a^{ew} = 0,$$

We have that $\mu^{ew} = \nu_a^{ew} = 0$, by (5.66) and (5.62). Thus, we get $\lambda_{ew}^k = 0$. As e, w and k were chosen arbitrarily in $A_1 \setminus (F_1^1 \cup F_2^1)$, $W \setminus W^1$ and K, we obtain

$$\lambda_{ew}^k = 0, \text{ for all } k \in K, e \in A_1 \setminus (F_1^1 \cup F_2^1), w \in W \setminus W^1, \tag{5.67}$$

Suppose now that $e = (s,t)$ is an arc of the solution. If $e \in F_2^1$, we associate e with the routing of the commodity $k \in K$, and define a solution S^9, where $\mathcal{C}_k^9 = \mathcal{C}_k^1 \cup \{e\}$. S^9 is a feasible solution, and both incidence vectors of S^1 and S^9 allow to state that

$$\lambda_{ew}^k = 0, \text{ for all } k \in K, e \in F_2^1, w \in W, \tag{5.68}$$

Finally, if $e = (s,t) \in F_1^1$, and w is the subband installed on e. Suppose that e is involved in the routing of the commodity k. We consider the solution S^{10}, obtained from S^1 as follows. We replace e by two arcs $f = (s,r)$ and $g = (r,t)$ of $A_1 \setminus (F_1^1 \cup F_2^1)$, and we install w on both f and g. The commodity k is associated the arcs f and g instead of e for its routing. S^{10} is such that $F_1^{10} = (F_1^1 \setminus \{e\}) \cup \{f,g\}$, $\Delta_{fw}^{10} = \{(s',r')\}$ and $\Delta_{gw}^{10} = \{(r',t')\}$. In particular, we have $\mathcal{C}_k^{10} = (\mathcal{C}_k^1 \setminus \{e\}) \cup \{f,g\}$. The solution S^{10} is feasible, and both incidence vectors of S^1 and S^{10} are in $\mathcal{F}$. So they verify

$$\lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^{10}} + \mu y^{S^{10}} + \nu z^{S^{10}}$$

$$= \lambda x^{S^1} - \lambda_{ew}^k + \lambda_{fw}^k + \lambda_{gw}^k + \mu y^{S^1} - \mu^{ew} + \mu^{fw} + \mu^{gw} + \nu z^{S^1},$$

We have by (5.66) and (5.68) that $\lambda_{ew}^k = 0$. As we selected e and k arbitrarily in F_1^1 and K, we obtain

$$\lambda_{ew}^k = 0, \text{ for all } k \in K, e \in F_1^1, w \in W, \tag{5.69}$$

Hence, (5.67), (5.68) and (5.69) allow to conclude that

$$\lambda_{ew}^k = 0, \text{ for all } k \in K, e \in A_1, w \in W, \tag{5.70}$$

Now we will show that all the coefficient ν related to $(\tilde{a}, \tilde{w})$ are equal.

Recall that in the solution S^0, the arc $\tilde{a}$ belongs to a subset $\Delta_{e\tilde{w}}^0$, where $e \in A_1$. We will introduce a solution S^{11}. To this end, consider an arc $\tilde{e} \in A_1$, $\tilde{e} = (s,t)$, and a subband $w \in W \setminus \{\tilde{w}\}$. We will install w on the arc e, then move $\tilde{w}$ from e to $\tilde{e}$. In other words, $y_{e\tilde{w}}^{S^{11}} = 0$, $y_{ew}^{S^{11}} = 1$, and $y_{\tilde{e}\tilde{w}}^{S^{11}} = 1$ (see Figure 5.7).

In this solution, the pairs (e,w) and $(\tilde{e}, \tilde{w})$ are associated path in G that are $\{(u',v')\}$ and $\{(s',t')\}$, where $(s',t') \in A_2$. However, we will associate $\tilde{a}$ with the routing of both pairs (e,w) and $(\tilde{e},\tilde{w})$. More formally, S^{11} is such that $S^{11} = (F_1^0, F_2^0 \cup \{e\}, \Delta^0 \cup$

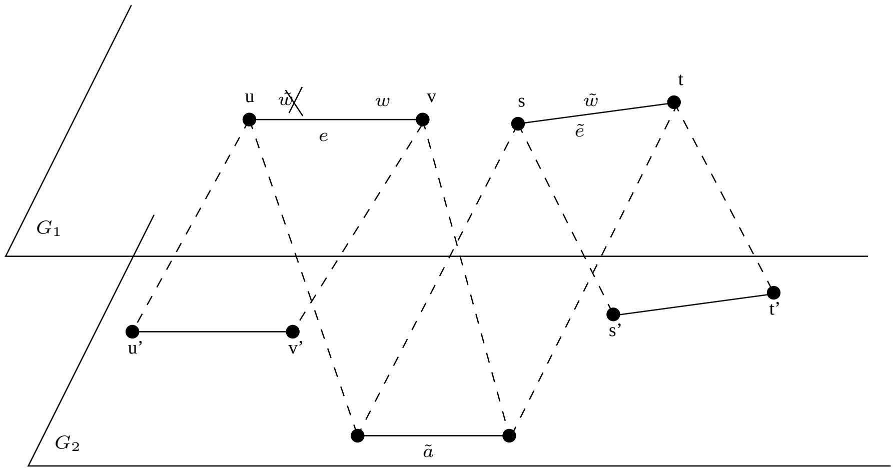


Figure 5.7: Obtaining S^{11} from S^0

$\{(s',t')\}, W^0 \cup \{w\})$, where $\Gamma^{11} = (\Gamma^0 \setminus \{(e,\tilde{w})\}) \cup \{(e,w),(\tilde{e},\tilde{w})\}$. Moreover, $\Delta^{11}_{ew} = \Delta^0_{ew} \cup \{(u',v'),\tilde{a}\}$ and $\Delta^{11}_{\tilde{e}\tilde{w}} = \Delta^0_{\tilde{e}\tilde{w}} \cup \{(s',t'),\tilde{a}\}$. The solution S^{11} is clearly feasible, as routing path holding enough capacity are still available for the commodities of K and the installed subbands, so all the constraints of (5.8) are satisfied. Hence, $(x^{S^{11}}, y^{S^{11}}, z^{S^{11}})$ belongs to $\mathcal{F}$. Thus, comparing S^{11} and S^0 yields

$$\nu^{e\tilde{w}}_{\tilde{a}} = \nu^{ew}_{\tilde{a}} + \nu^{\tilde{e}\tilde{w}}_{\tilde{a}},$$

As by (5.62) $\nu^{ew}_{\tilde{a}} = 0$, we get $\nu^{e\tilde{w}}_{\tilde{a}} = \nu^{\tilde{e}\tilde{w}}_{\tilde{a}}$. Since, the arcs e, $\tilde{e}$ are arbitrary in A_2, we conclude that there exists a scalar $\rho \in \mathbb{R}$, such that

$$\nu^{e\tilde{w}}_{\tilde{a}} = \rho, \text{ for all } e \in A_1, \tag{5.71}$$

In consequence, and by (5.62), (5.66) and (5.70), we get

$$\nu^{e\tilde{w}}_{\tilde{a}} = \begin{cases} \rho, & \text{for all } e \in A_1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $(\alpha,\beta,\gamma) = \rho(\lambda,\mu,\nu)$ with $\rho \in \mathbb{R}$, and the results follows. □

5.3.3 Cut inequalities

In what follows, we will investigate the facial structure of cut inequalities (5.1).

Theorem 5.9 *For $\tilde{k} \in K$, every cut inequality defines a facet of $P(G_1, G_2, K, C)$.*

Proof. Let $T \subseteq V_1$ such that $T = \{o_{\tilde{k}}\}$ and $\overline{T} = V_1 \setminus \{o_{\tilde{k}}\}$. Observe that the arc $(o_{\tilde{k}}, d_{\tilde{k}}) \in \delta^+_{G_1}(T)$, as G_1 is a complete graph. Let us denote inequality

$$\sum_{w \in W} \sum_{e \in \delta^+_{G_1}(T)} x^{\tilde{k}}_{ew} \geq 1$$

by $\alpha x + \beta y + \gamma z \geq \delta$, and let $\lambda x + \mu y + \nu z \geq \xi$ be a facet defining inequality of $P(G_1, G_2, K, C)$, such that

$$\mathcal{F}^{\tilde{k}} = \{(x, y, z) \in P(G_1, G_2, K, C) : \alpha x + \beta y + \gamma z = \delta\}$$

$$\subseteq \mathcal{F} = \{(x, y, z) \in P(G_1, G_2, K, C) : \lambda x + \mu y + \nu z = \xi\}$$

We will show that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$ for some $\rho \in \mathbb{R}$.

First, let us show that coefficients $\nu^{ew}_a = 0$, for all $e \in A_1$, $w \in W$ and $a \in A_2$.

Consider the solution $S^0 = (F^0_1, F^0_2, \Delta^0, W^0)$ described in proof of Theorem 5.3. S^0 is a feasible solution and its incidence vector belongs to $\mathcal{F}^{\tilde{k}}$ and $\mathcal{F}$. Let a be an arc of A_2. If $a \in A_2 \setminus \Delta^0$, then it is clear that $(F^0_1, F^0_2, \Delta^0 \cup \{a\}, W^0)$ still induces a feasible solution for the problem. In particular, a can be added to any subset Δ^0_{ew}, with $e \in A_1$ and $w \in W$. Let this solution be denoted by S'^0. Since

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S'^0} + \mu y^{S'^0} + \nu z^{S'^0} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} + \nu^{ew}_a,$$

which implies that $\nu^{ew}_a = 0$. As one can select any a in $A_2 \setminus \Delta^0$, we can state that

$$\nu^{ew}_a = 0, \text{ for all } e \in A_1, w \in W, a \in A_2 \setminus \Delta^0, \tag{5.72}$$

Suppose that $a = (s', t')$ is an arc of the subset Δ^0 (a is used in S^0). In particular $a \in \Delta^0_{ew}$, for some $e \in A_1$, $w \in W$. Let (s', r') and (r', t') be two arcs of $A_2 \setminus \Delta^0$, with $r' \in V_2 \setminus \{s', t'\}$. Consider the solution $S''^0 = (F''^0_1, F''^0_2, \Delta''^0, W''^0)$, where $\Delta''^0 = (\Delta^0 \setminus \{a\}) \cup \{(s', r'), (r', t')\}$. S''^0 is a feasible solution, and its incidence vector belong to $\mathcal{F}^{\tilde{k}}$ and $\mathcal{F}$, so

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S''^0} + \mu y^{S''^0} + \nu z^{S''^0} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} - \nu^{ew}_a + \nu^{ew}_{(s',r')} + \nu^{ew}_{(r',t')},$$

and it follows that $-\nu^{ew}_a + \nu^{ew}_{(s',r')} + \nu^{ew}_{(r',t')} = 0$. We have by (5.72) that $\nu^{ew}_{(s',r')} = \nu^{ew}_{(r',t')} = 0$, then we get $\nu^{ew}_a = 0$. As a was chosen arbitrarily in Δ^0, we obtain

$$\nu^{ew}_a = 0, \text{ for all } e \in A_1, w \in W, a \in \Delta^0, \tag{5.73}$$

Consequently, by (5.72) and (5.73), we conclude that

$$\nu^{ew}_a = 0, \text{ for all } e \in A_1, w \in W, a \in A_2, \tag{5.74}$$

Next, we will show that $\mu^{ew} = 0$, for all $e \in A_1$ and $w \in W$.

Let e be an arc of A_1. Assume first that $e = (u,v) \in A_1 \setminus (F_1^0 \cup F_2^0)$ (e is not used in the solution S^0), and let w be a subband of $W \setminus W^0$. One can easily see that the subsets $(F_1^0, F_2^0 \cup \{e\}, \Delta^0 \cup \{(u',v')\}, W^0 \cup \{w\})$ induces a feasible solution. Let us denote this solution by S^1. Since $(x^{S^1}, y^{S^1}, z^{S^1}) \in \mathcal{F}$, we have

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^1} + \mu x^{S^1} + \nu z^{S^1} = \lambda x^{S^0} + \mu y^{S^0} + \mu^{ew} + \nu z^{S^0} + \nu^{ew}_{(u',v')},$$

which implies $\mu^{ew} + \nu^{ew}_{(u',v')} = 0$. By (5.74), we have that $\nu^{ew}_{(u',v')} = 0$. Thus, $\mu^{ew} = 0$. As we selected e and w arbitrarily in $A_1 \setminus (F_1^0 \cup F_2^0)$ and $W \setminus W^0$, we get

$$\mu^{ew} = 0, \text{ for all } e \in A_1 \setminus (F_1^0 \cup F_2^0), w \in W \setminus W^0, \tag{5.75}$$

Now, suppose that $e = (u,v) \in (F_1^0 \cup F_2^0)$. Since the subset F_2^0 is empty, by construction of S^0, then $e \in F_1^0$. In particular, assume that $e \in \mathcal{C}_k^0$, for $k \in K$, and w is the subband installed on e. Let $f = (u,r)$ and $g = (r,v)$ be two arcs of $A_1 \setminus (F_1^0 \cup F_2^0)$, with $r \in V_1 \setminus \{u,v\}$. Consider the solutions S^2 and S'^2 defined as follows. S^2 is obtained from S^0 by installing the subband w on both f and g, and assigning with the couples (f,w) and (g,w) the arcs (u',r') and (r',v') of A_2, respectively. In addition, the commodity k uses f and g instead of e for its routing. More formally, $S^2 = (F_1^2, F_2^2, \Delta^2, W^2)$, where $F_1^2 = F_1^0 \cup \{f,g\}$, $\mathcal{C}_k^2 = (\mathcal{C}_k^0 \setminus \{e\}) \cup \{f,g\}$, $F_2^2 = F_2^0 \cup \{e\}$, $\Delta^2_{fw} = \Delta^0_{fw} \cup \{(u',r')\}$ and $\Delta^2_{gw} = \Delta^0_{gw} \cup \{(r',v')\}$.

S^2 is a feasible solution for the problem, and its incidence vector verifies

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^2} + \mu y^{S^2} + \nu z^{S^2}$$

$$\lambda x^{S^0} + -\lambda^k_{ew} + \lambda^k_{fw} + \lambda^k_{gw} + \mu y^{S^0} + \mu^{fw} + \mu^{gw} + \nu z^{S^0} + \nu^{fw}_{(u',r')} + \nu^{gw}_{(r',v')},$$

As by (5.74), we have $\nu^{fw}_{(u',r')} = \nu^{gw}_{(r',v')} = 0$, it follows that

$$-\lambda^k_{ew} + \lambda^k_{fw} + \lambda^k_{gw} + \mu^{fw} + \mu^{gw} = 0, \tag{5.76}$$

The solution S'^2 results from the removal of the arc e of S^0. As for solution S^2, e is replaced by the arcs f and g for the routing of the commodity k. Since S'^2 is feasible, and both incidence vectors of S^0 and S'^2 are in $\mathcal{F}$, we have

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S'^2} + \mu y^{S'^2} + \nu z^{S'^2}$$

$$\lambda x^{S^0} + -\lambda^k_{ew} + \lambda^k_{fw} + \lambda^k_{gw} + \mu y^{S^0} + -\mu^{ew} + \mu^{fw} + \mu^{gw} + \nu z^{S^0} + \nu^{fw}_{(u',r')} + \nu^{gw}_{(r',v')},$$

Again, we have $\nu^{fw}_{(u',r')} = \nu^{gw}_{(r',v')} = 0$, by (5.74), and thus

$$-\lambda^k_{ew} + \lambda^k_{fw} + \lambda^k_{gw} + -\mu^{ew} + \mu^{fw} + \mu^{gw} = 0, \tag{5.77}$$

By (5.76) and (5.77), we have $-\mu^{ew} + \mu^{fw} + \mu^{gw} = 0$. We yet know that $\mu^{fw} = \mu^{gw} = 0$ by (5.75), which yields $\mu^{ew} = 0$. As e was selected arbitrarily in F^0_1, we get

$$\mu^{ew} = 0, \text{ for all } e \in F^0_1 \cup F^0_1, w \in W, \tag{5.78}$$

We conclude, by (5.75) and (5.78) that

$$\mu^{ew} = 0, \text{ for all } e \in A_1, w \in W, \tag{5.79}$$

Finally, we will show that $\lambda^k_{ew} = 0$, for all $e \in A_1 \setminus \delta^+_{G_1}(T)$, $w \in W$.

First, suppose that $k \in K \setminus \{tildek\}$. Let $e = (u, v)$ and w be an arc of A_1 such that $e \notin (F^0_1 \cup F^0_2)$ and a subband of $W \setminus W^0$, respectively. The subsets $(F^0_1 \cup \{e\}, F^0_2, \Delta^0 \cup \{(u', v')\}, W^0 \cup \{w\})$, with $F^0_1 \cup \{e\} = (\bigcup_{i \in K\setminus\{\tilde{k}\}} \mathcal{C}^0_i) \cup (\mathcal{C}^0_k \cup \{e\})$ clearly induces a feasible solution of the problem. We will denote by S^3 this solution. Since $(x^{S^3}, y^{S^3}, z^{S^3}) \in \mathcal{F}$, we have

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^3} + \mu y^{S^3} + \nu z^{S^3} = \lambda x^{S^0} + \lambda^k_{ew} + \mu y^{S^0} + \mu^{ew} + \nu z^{S^0} + \nu^{ew}_{(u',v')},$$

which gives $\lambda^k_{ew} + \mu^{ew} + \nu^{ew}_{(u',v')} = 0$. We have by (5.74) and (5.79) that $\mu^{ew} = \nu^{ew}_{(u',v')} = 0$. Thus, $\lambda^k_{ew} = 0$. We selected e and w arbitrarily in $A_1 \setminus (F^0_1 \cup F^0_2)$ and $W \setminus W^0$, respectively. Hence, we obtaine

$$\lambda^k_{ew} = 0, \text{ for all } k \in K \setminus \{\tilde{k}\}, e \in A_1 \setminus (F^0_1 \cup F^0_2), w \in W \setminus W^0, \tag{5.80}$$

Now, if $e \in \mathcal{C}^0_k \subseteq F^0_1$ and w is the subband used for e in the solution S^0, then the subsets $(F^0_1, F^0_2, \Delta^0, (W^0 \setminus \{w\}) \cup \{\tilde{w}\})$, where $\tilde{w} \in W \setminus W^0$, induces a feasible solution of the problem. This solution will be referred to as S'^3. Note that $\Delta'^3_{e\tilde{w}} = \Delta^0_{e\tilde{w}} \cup \{(u', v')\}$. The incidence vector of S'^3 satisfies

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S'^3} + \mu y^{S'^3} + \nu z^{S'^3} =$$

$$\lambda x^{S^0} - \lambda^k_{ew} + \lambda^k_{e\tilde{w}} + \mu y^{S^0} - \mu^{ew} + \mu^{e\tilde{w}} + \nu z^{S^0} + \nu^{e\tilde{w}}_{(u',v')},$$

which leads to $-\lambda^k_{ew} + \lambda^k_{e\tilde{w}} - \mu^{ew} + \mu^{e\tilde{w}} + \nu^{e\tilde{w}}_{(u',v')} = 0$. As by (5.74), (5.79), and (5.80), we have $\lambda^k_{e\tilde{w}} = \mu^{ew} = \mu^{e\tilde{w}} = \nu^{e\tilde{w}}_{(u',v')} = 0$. The remaining term, that is λ^k_{ew}, is also equal to zero. Since the couple (e,w) was chosen arbitrarily in the solution S^0, we conclude that

$$\lambda^k_{ew} = 0, \text{ for all } k \in K \setminus \{\tilde{k}\}, e \in F^0_1 \cup F^0_2, w \in W^0, \tag{5.81}$$

We still have to show that $\lambda^{\tilde{k}}_{ew} = 0$, for all $e \in A_1 \setminus \delta^+_{G_1}(T)$, $w \in W$.

To do this, we will consider two cases : e is used in the solution and the subband w is installed on e, e does not appear in the solution.

Case 1.

Suppose that $e = (u,v) \notin F^0_1$, and let w be a subband that was not used before. Then, $S^4 = (F^0_1 \cup \{e\}, F^0_2, \Delta^0 \cup \{(u',v')\}, W^0 \cup \{w\})$, with $(u',v') \in A_2$ defines a feasible solution of the problem. Notice that $F^0_1 \cup \{e\} = \bigcup_{i \in K\setminus\{\tilde{k}\}} \mathcal{C}^0_i \cup (\mathcal{C}^0_{\tilde{k}} \cup \{e\})$. Since

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^4} + \mu y^{S^4} + \nu z^{S^4} =$$

$$\lambda x^{S^0} + \lambda^{\tilde{k}}_{ew} + \mu y^{S^0} + \mu^{ew} + \nu z^{S^0} + \nu^{ew}_{(u',v')},$$

and it follows that $\lambda^{\tilde{k}}_{ew} + \mu^{ew} + \nu^{ew}_{(u',v')} = 0$. By (5.74) and (5.79), we have $\lambda^{\tilde{k}}_{ew} = 0$. As e and w were chosen arbitrarily in $A_1 \setminus (\delta^+_{G_1}(T) \cup (F^0_1))$ and $W \setminus W^0$, respectively, we obtain that

$$\lambda^{\tilde{k}}_{ew} = 0, \text{ for all } e \in A_1 \setminus (F^0_1), w \in W \setminus W^0, \tag{5.82}$$

Case 2.

Now, if $e \in F^0_1$, $e \notin \delta^+_{G_1}(T)$, let w be the subband installed on e. Observe that, as e appears in the solution, but as it does not belong to $\delta^+_{G_1}(T)$, it can not be involved in the routing of $\tilde{k}$. In other words, $e \in \mathcal{C}^0_i$, with $i \in K \setminus \{\tilde{k}\}$. In this case, we install an additionnal subband on e, say $\tilde{w}$. The couple $(e,\tilde{w})$ is associated with the routing of $\tilde{k}$.

Consider the solution $S'^4 = (F^0_1, F^0_2, \Delta^0, W^0 \cup \{\tilde{w}\})$, is clearly feasible. Note that $F'^4_0 = F^0_1 = \bigcup_{i \in K\setminus\{\tilde{k}\}} \mathcal{C}^0_i \cup (\mathcal{C}^0_{\tilde{k}} \cup \{e\})$, and $\Delta'^4_{e\tilde{w}} = \Delta^0_{e\tilde{w}} \cup \{(u',v')\}$. As the incidence vector of S'^4 belongs to $\mathcal{F}$, we have

$$\lambda^{\tilde{k}}_{e\tilde{w}} + \mu^{e\tilde{w}} + \nu^{e\tilde{w}}_{e\tilde{w}} = 0,$$

Since $\mu^{e\tilde{w}} = \nu^{e\tilde{w}}_{e\tilde{w}} = 0$, we obtain $\lambda^{\tilde{k}}_{e\tilde{w}} = 0$. As e was chosen arbitrarily in $F^0_1 \setminus \delta^+_{G_1}(T)$, it follows that

$$\lambda^{\tilde{k}}_{ew} = 0, \text{ for all } e \in F^0_1, w \in W \setminus W^0, \tag{5.83}$$

In consequence, and by (5.80), (5.81), (5.82) and (5.83), we obtain that

$$\lambda_{ew}^{\tilde{k}} = 0, \text{ for all } e \in A_1 \setminus \delta_{G_1}^{+}(T), w \in W, \tag{5.84}$$

and

$$\lambda_{ew}^{k} = 0, \text{ for all } k \in K \setminus \{\tilde{k}\}, e \in A_1, w \in W, \tag{5.85}$$

Now let us show that all the coefficient λ related to $\tilde{k}$ and the arcs of $\delta_{G_1}^{+}(T)$ are equal.

Recall that commodity $\tilde{k}$ is associated with path $\{(o_{\tilde{k}}, d_{\tilde{k}})\}$ in solution S^0. Let us denote by w the subband installed on $(o_{\tilde{k}}, d_{\tilde{k}})$ in S^0. Consider the solution S^5, where we introduce three additional arcs $(o_{\tilde{k}}, u)$, (u, v) and $(v, d_{\tilde{k}})$ in the subset F_1^0. We will shift the subband w from $(o_{\tilde{k}}, d_{\tilde{k}})$ to the arcs $(o_{\tilde{k}}, u)$, (u, v) and $(v, d_{\tilde{k}})$, and associate the path $\{(o_{\tilde{k}}, u), (u, v), (v, d_{\tilde{k}})\}$ with the routing of $\tilde{k}$ instead of its initial routing path (see Figure 5.8).

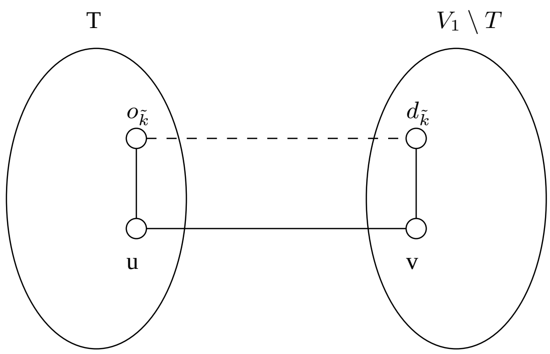


Figure 5.8: Obtaining the solution S^5

Furthermore, we assign a path in G_2 to each pair (e, w) such that w is installed on e. Indeed, the pair $((o_{\tilde{k}}, u), w)$, $((e, v), w)$ and $((v, d_{\tilde{k}}), w)$ are associated with paths $\{(o'_{\tilde{k}}, u')\}$, $\{(u', v')\}$ and $\{(v', d'_{\tilde{k}})\}$, respectively, with $(o'_{\tilde{k}}, u')$, (u', v'), $(v', d'_{\tilde{k}}) \in A_2$. It is clear that the solution S^5 is clearly feasible and differs from S^0 in what $\tilde{k}$ crosses the cut using the arc (u, v) instead of $(o_{\tilde{k}}, d_{\tilde{k}})$. Thus, we can set $x_{\tilde{k}(u,v)w}^{S^0}$ to 1 while $x_{\tilde{k}(o_{\tilde{k}},d_{\tilde{k}})w}^{S^0}$ is set to 0. Comparing both incidence vectors of S^6 and S^0 gives us

$$\lambda_{(o_{\tilde{k}},d_{\tilde{k}})w}^{\tilde{k}} = \lambda_{(o_{\tilde{k}},u)w}^{\tilde{k}} + \lambda_{(u,v)w}^{\tilde{k}} + \lambda_{(v,d_{\tilde{k}})w}^{\tilde{k}},$$

By (5.84), we get that $\lambda_{(o_{\tilde{k}},d_{\tilde{k}})w}^{\tilde{k}} = \lambda_{(u,v)w}^{\tilde{k}}$. Since (u, v) is arbitrary in $\delta_{G_1}^{+}(T)$, we conclude that

$$\lambda_{ew}^{\tilde{k}} = \rho, \text{ for all } e \in \delta_{G_1}^{+}(T), w \in W, \tag{5.86}$$

Hence, all together, we obtain that

$$\lambda^k_{ew} = \begin{cases} \rho, & \text{if } k = \tilde{k}, \text{ and for all } e \in \delta^+_{G_1}(T), \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$, and the results follows. $\square$

In this section, we will show that inequalities (5.3) define facets for $P(G_1, G_2, K, C)$.

Theorem 5.10 *For $\tilde{e} = (\tilde{u}, \tilde{v}) \in A_1$, $\tilde{w} \in W$, every cut inequality* (5.3) *defines a facet of $P(G_1, G_2, K, C)$.*

Proof. Consider a subset of nodes T of V_2 such that $\tilde{u}' \in T$ and $\tilde{v}' \in \overline{T} = V_2 \setminus T$. Let us denote inequality $z^{\tilde{e}\tilde{w}}(\delta^+_{G_2}(T)) \geq y^{\tilde{e}\tilde{w}}$ by $\alpha x + \beta y + \gamma z \geq \delta$, and let $\lambda x + \mu y + \nu z \geq \xi$ be a valid inequality that defines a facet $\mathcal{F}$ of $P(G_1, G_2, K, C)$, such that

$$\mathcal{F}^{\tilde{e}\tilde{w}} = \{(x, y, z) \in P(G_1, G_2, K, C) : z^{\tilde{e}\tilde{w}}(\delta^+_{G_2}(T)) - y^{\tilde{e}\tilde{w}} = 0\} \subseteq \mathcal{F}.$$

We show that there exists a scalar $\rho \in \mathbb{R}$ such that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$.

To do this, let us first show that $\nu^{ew}_a = 0$, for all $a \in A_2$, $(e, w) \in (A_1 \times W) \setminus \{(\tilde{e}, \tilde{w})\}$.

Consider the solution $S^0 = (F^0_1, F^0_2, \Delta^0, W^0)$ defined in proof of Theorem 5.3. Observe that in both cases, whether $(\tilde{e}, \tilde{w}) \in \Gamma^0$ or not $(x^{S^0}, y^{S^0}, z^{S^0}) \in \mathcal{F}^{\tilde{e}\tilde{w}}$. In fact, if $(\tilde{e}, \tilde{w}) \in \Gamma^0$, then $\tilde{e} = (o_k, d_k)$ for some commodity k, and hence $y^{S^0}_{\tilde{e}\tilde{w}} = 1$, $z^{S^0}_{\tilde{e}\tilde{w}}((o'_k, d'_k)) = 1$, and $z^{S^0}_{\tilde{e}\tilde{w}}(a) = 0$ for all $a \in A_2 \setminus (o'_k, d'_k)$. Thus, $z^{S^0}_{\tilde{e}\tilde{w}}(\delta^+_{G_2}(T)) = 1$.
If $(\tilde{e}, \tilde{w}) \notin \Gamma^0$, then $y^{S^0}_{\tilde{e}\tilde{w}} = 0$, and in consequence, $z^{S^0}_{\tilde{e}\tilde{w}}(a) = 0$, for all $a \in A_2$, and trivially $(x^{S^0}, y^{S^0}, z^{S^0}) \in \mathcal{F}^{\tilde{e}\tilde{w}}$.
Let $a \in A_2 \setminus \Delta^0$ and $(e, w) \in (A_1 \times W) \setminus (\tilde{e}, \tilde{w})$. We define the solution S^1 which is obtained from S^0 by adding arc a to Δ^0_{ew}. $S^1 = (F^0_1, F^0_2, \Delta^1, W^0)$, where $\Delta^1 = \Delta^0 \cup \{a\}$, with $\Delta^1_{e_i w_i} = \Delta^0_{e_i w_i}$ if $(e_i, w_i) \neq (e, w)$ and $\Delta^1_{ew} = \Delta^0_{ew} \cup \{a\}$. Solution S^1 is feasible for the problem and its incidence vector is in $\mathcal{F}^{\tilde{e}\tilde{w}}$. Hence, $(x^{S^1}, y^{S^1}, z^{S^1})$ and also $(x^{S^0}, y^{S^0}, z^{S^0})$ belong to $\mathcal{F}$, we get

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} + \nu^{ew}_a$$

which implies that $\nu^{ew}_a = 0$. Since the elements a, e and w were chosen arbitrarily in the sets $A_2 \setminus \Delta^0$, $A_1 \setminus \{\tilde{e}\}$, $W \setminus \{\tilde{w}\}$, respectively, we obtain

$$\nu^{ew}_a = 0, \text{ for all } (e, w) \in (A_1 \times W) \setminus \{(\tilde{e}, \tilde{w})\}, a \in A_2 \setminus \Delta^0 \tag{5.87}$$

Now, consider an arc $a = (s', t') \in A_2$ that is used in the solution S^0 ($a \in \Delta^0$). Assume that $a \in \Delta^0_{ew}$, where $(e, w) \in \Gamma^0$. Let S^2 be a solution, obtained from S^0 by replacing the arc a by $f = (s', r')$ and $g = (r', t')$, with $r' \in V_2 \setminus \{s', t'\}$ in the subset Δ^0_{ew}. Clearly, the solution S^2 is feasible, and its incidence vector belongs to $\mathcal{F}^{\tilde{e}\tilde{w}}$. Thus, we have

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^2} + \mu y^{S^2} + \nu z^{S^2} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} - \nu^{ew}_a + \nu^{ew}_f + \nu^{ew}_g,$$

that gives $-\nu^{ew}_a + \nu^{ew}_f + \nu^{ew}_g = 0$. As by (5.87), $\nu^{ew}_f = \nu^{ew}_g = 0$, we obtain $\nu^{ew}_a = 0$. The arc a we selected arbitrarily in the subset Δ^0, we then conclude that

$$\nu^{ew}_a = 0, \text{ for all } (e, w) \in \Gamma^0, a \in Delta^0, \tag{5.88}$$

Thus, by (5.87) and (5.88), we get

$$\nu^{ew}_a = 0, \text{ for all } (e, w) \in (A_1 \times W) \setminus \{(\tilde{e}, \tilde{w})\}, a \in A_2, \tag{5.89}$$

Consider again solution S^0. For the rest of the proof, we will suppose without loss of generality that $\tilde{e} = (o_k, d_k)$ for some $k \in K$, and $(\tilde{e}, \tilde{w}) \in \Gamma^0$. Let a be an arc of $A_2 \setminus \delta^+_{G_2}(T)$. Consider the solution S^3 obtained from S^0 by adding a to $\Delta^0_{\tilde{e}\tilde{w}}$. As both $(x^{S^0}, y^{S^0}, z^{S^0})$ and $(x^{S^3}, y^{S^3}, z^{S^3})$ are in $\tilde{\mathcal{F}}$ and hence in $\mathcal{F}$, it follows that $\nu^{\tilde{e}\tilde{w}}_a = 0$. As a is arbitrary in $A_2 \setminus \Delta^+_{G_2}(T)$, we have that

$$\nu^{\tilde{e}\tilde{w}}_a = 0, \text{ for all } a \in A_2 \setminus \delta^+_{G_2}(T), \tag{5.90}$$

Now, we will show that all the coefficients $\nu^{\tilde{e}\tilde{w}}_a$ are the same for the arcs of the cut $\delta^+_{G_2}(T)$. Indeed, let $a = (u', v')$ be an arc of $\delta^+_{G_2}(T)$ different from (o'_k, d'_k). Consider the solution S^4 obtained from S^0 by replacing in Δ^0 the arc (o'_k, d'_k) by the path (o'_k, u'), (u', v'), (v', d'_k). Remark that the nodes o'_k and u' (respectively d'_k and v') may be the same. We have that $(x^{S^4}, y^{S^4}, z^{S^4})$ belongs to $\mathcal{F}^{\tilde{e}\tilde{w}}$ and also to $\mathcal{F}$. In consequence,

$$\nu^{\tilde{e}\tilde{w}}_{(o'_k,d'_k)} = \nu^{\tilde{e}\tilde{w}}_{(o'_k,u')} + \nu^{\tilde{e}\tilde{w}}_{(u',v')} + \nu^{\tilde{e}\tilde{w}}_{(v',d'_k)},$$

By (5.90), it follows that $\nu^{\tilde{e}\tilde{w}}_{(o'_k,d'_k)} = \nu^{\tilde{e}\tilde{w}}_{(u',v')}$. This implies that

$$\nu^{\tilde{e}\tilde{w}}_a = \begin{cases} \rho, & \text{for some } \rho \in \mathbb{R}, \text{ for all } a \in \delta^+_{G_2}(T), \\ 0, & \text{otherwise.} \end{cases} \tag{5.91}$$

Next, we will show that $\mu^{ew} = 0$, for all $(e, w) \in (A_1 \times W)$.

Let $e = (u, v)$ be an arc of $A_1 \setminus (F^0_1 \cup F^0_2)$ and $w \in W \setminus W^0$ such that $(e, w) \neq (\tilde{e}, \tilde{w})$. Consider the solution S^5, constructed from S^0 by adding (e, w) to Γ^0. S^5 is then defined as follows $(F_1^0, F_2^0 \cup \{e\}, \Delta^0 \cup \{a = (u', v')\}, W^0 \cup \{w\})$. The solution S^5 is clearly feasible, and $(x^{S^5}, y^{S^5}, z^{S^5}) \in \mathcal{F}^{\tilde{e}\tilde{w}}$, thus we have

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^5} + \mu y^{S^5} + \nu z^{S^5} = \lambda x^{S^0} + \mu y^{S^0} + \mu^{ew} + \nu z^{S^0} + \nu_a^{ew},$$

and it follows that $\mu^{ew} + \nu_a^{ew} = 0$. As $\nu_a^{ew} = 0$, by (5.89), we obtain $\mu^{ew} = 0$. Since the arc e and the subband w were selected arbitrarily out of the solution, we get

$$\mu^{ew} = 0, \text{ for all } e \in A_1 \setminus (F_1^0 \cup F_2^0), w \in W \setminus W^0, \tag{5.92}$$

Now, if $e = (u, v)$ is selected among the arcs used in the solution S^0, then $e \in F_1^0 \setminus \{\tilde{e}\}$. Let us denote by w the subband installed on e. In other words, $(e, w) \in \Gamma^0$ and $e \in \mathcal{C}_k^0$, for some commodity k. Here, we need to introduce two solutions S^6 and S^7. Let $f = (u, r)$ and $g = (r, v)$ be two arcs of $A_1 \setminus (F_1^0 \cup F_2^0)$, with $r \in V_1 \setminus \{u, v\}$. First, consider the solution S^6 which is obtained from S^0 by adding (f, w) and (g, w) to Γ^0. In particular, (f, w) and (g, w) are added to $\mathcal{C}_k^0$. $S^6 = (F_1^6, F_2^6, \Delta^6, W^0)$, where $F_1^6 = (F_1^0 \setminus \{e\}) \cup \{f, g\}$, $F_2^6 = F_2^0 \cup \{e\}$ and $\Delta^6 = \Delta^0 \cup \{(u', r'), (r', v')\}$ with $\Delta_{fw}^6 = \Delta_{fw}^0 \cup \{(u', r')\}$ and $\Delta_{gw}^6 = \Delta_{gw}^0 \cup \{(r', v')\}$.

Consider now the solution S^7 that is obtained by removing e from the solution S^6. $S^7 = (F_1^6, F_1^7, \Delta^6, W^6)$, where $F_2^7 = F_2^6 \setminus \{e\}$. Both solutions S^6 and S^7 are clearly feasible, and their incidence vectors are in $\mathcal{F}^{\tilde{e}\tilde{w}}$, thus $(x^{S^6}, y^{S^6}, z^{S^6})$ and $(x^{S^7}, y^{S^7}, z^{S^7})$ are in $\mathcal{F}$. Hence, we get

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^6} + \mu y^{S^6} + \nu z^{S^6} =$$

$$\lambda x^{S^7} + \mu y^{S^7} + \nu z^{S^7} = \lambda x^{S^6} + \mu y^{S^6} - \mu^{ew} + \nu z^{S^6},$$

And it follows that $\mu^{ew} = 0$. Since, e is chosen arbitrarily in $F_1^0 \cup F_2^0$, we obtain

$$\mu^{ew} = 0, \text{ for all } (e, w) \in \Gamma^0, \tag{5.93}$$

Therefore, by (5.92) and (5.93), we get that

$$\mu^{ew} = 0, \text{ for all } e \in A_1, w \in W, (e, w) \neq (\tilde{e}, \tilde{w}), \tag{5.94}$$

Finally, we will show that $\lambda_{ew}^k = 0$, for all $k \in K$, $e \in A_1$ and $w \in W$.

Consider the solution S^8, obtained from S^0 as follows. Let $e = (u, v)$ be an arc of $\in A_1 \setminus (F_1^0 \cup F_2^0)$ and w a subband of $W \setminus W^0$. We add (e, w) to Γ^0 and e to $\mathcal{C}_k^0$, where k is a commodity of K. Then, the solution S^8 is such that $S^8 = (F_2^8, F_2^0, \Delta^8, W^8)$, where $F_1^8 = F_1^0 \cup \{e\}$, $\Delta^8 = \Delta^0 \cup \{(u', v')\}$ with $\Delta_{ew}^8 = \Delta_{ew}^0 \cup \{(u', v')\}$ and $\Delta_{e_i w_i}^8 = \Delta_{e_i w_i}^0$

if $(e_i, w_i) \neq (e, w)$, and $W^8 = W^0 \cup \{w\}$. The solution S^8 is clearly feasible, and its incidence vector verifies

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^8} + \mu y^{S^8} + \nu z^{S^8}$$

$$= \lambda x^{S^0} + \lambda^k_{ew} + \mu y^{S^0} + \mu^{ew} + \nu z^{S^0} + \nu^{ew}_{(u',v')},$$

as it belongs to $\mathcal{F}^{\tilde{e}\tilde{w}}$ and $\mathcal{F}$. Hence, it follows that

$$\lambda^k_{ew} + \mu^{ew} + \nu^{ew}_{(u',v')} = 0,$$

As $\nu^{ew}_{(u',v')} = 0$, by (5.89), and $\mu^{ew} = 0$ by (5.94), we get $\lambda^k_{ew} = 0$. Since the arc e and the subband w are chosen arbitrarily in $A_1 \setminus (F_1^0 \cup F_2^0)$ and $W \setminus W^0$, respectively (out of the solution S^0), it implies that

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, e \in A_1 \setminus (F_1^0 \cup F_2^0), w \in W \setminus W^0, \tag{5.95}$$

Now suppose that $e = (u, v) \in A_1$ and $w \in W$ are such that $(e, w) \in \Gamma^0$. Assume that $e \in \mathcal{C}^0_k$, where k is a commodity of K. Let $f = (u, r)$ and $g = (r, v)$ be two arcs of $A_1 \setminus (F_1^0 \cup F_2^0)$. Consider the solution S^9 that is obtained from S^0 by replacing the arc e by f and g in $\mathcal{C}^0_k$. Here we reuse the subband w for both f and g. In other words, (f, w) and (g, w) are added to Γ^0. The solution S^9 is then defined as follows. $S^9 = (F_1^9, F_2^9, \Delta^9, W^0)$, where $F_1^9 = (F_1^0 \setminus \{e\}) \cup \{f, g\}$, $F_2^9 = F_2^0 \cup \{e\}$, $\Delta^9 = \Delta^0 \cup \{(u', r'), (r', v')\}$. Notice that $\Delta^9_{fw} = \Delta^0 \cup \{(u', r')\} = \{(u', r')\}$ while $\Delta^9_{gw} = \Delta^0_{gw} \cup \{(r', v')\}$. Also remark that $\mathcal{C}^9_k = (\mathcal{C}^0_k \setminus \{e\}) \cup \{f, g\}$, while the other subsets of F_1^9 remain the same. It is clear that the solution S^9 is feasible for the problem, and its incidence vector belongs to $\mathcal{F}^{\tilde{e}\tilde{w}}$ and $\mathcal{F}$. Thus,

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^9} + \mu y^{S^9} + \nu z^{S^9}$$

$$= \lambda x^{S^0} - \lambda^k_{ew} + \lambda^k_{fw} + \lambda^k_{gw} + \mu y^{S^0} + \mu^{fw} + \mu^{gw} + \nu z^{S^0} + \nu^{fw}_{(u',r')} + \nu^{gw}_{(r',v')},$$

which implies that

$$-\lambda^k_{ew} + \lambda^k_{fw} + \lambda^k_{gw} + \mu^{fw} + \mu^{gw} + \nu^{fw}_{(u',r')} + \nu^{gw}_{(r',v')} = 0,$$

By (5.89), (5.94) and (5.95), we get $\lambda^k_{ew} = 0$. Since the arc e is chosen arbitrarily in the subset F_1^0, we obtain

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, e \in F_1^0 \cup F_2^0, w \in W, \tag{5.96}$$

Hence, by (5.95) and (5.96), we conclude that

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, e \in A_1, w \in W, \tag{5.97}$$

Consider now the solution $\overline{S}^0$ obtained from S^0 by replacing in F_1^0 $\tilde{e} = (o_k, d_k)$ by (o_k, r), (r, d_k), and assigning to both arcs (o_k, r) and (r, d_k) the subband $\tilde{w}$. Hence, the new set $\overline{\Delta}^0$ is given by $(\Delta^0 \setminus (o'_k, d'_k)) \cup \{(o'_k, r'), (r', d'_k)\}$. We have $y_{\tilde{e}\tilde{w}}^{\overline{S}^0} = 0$, and $z_{\tilde{e}\tilde{w}}^{\overline{S}^0}(a) = 0$, for all $a \in A_2$. As $(x^{\overline{S}^0}, y^{\overline{S}^0}, z^{\overline{S}^0}) \in \mathcal{F}^{\tilde{e}\tilde{w}}$, $(x^{\overline{S}^0}, y^{\overline{S}^0}, z^{\overline{S}^0}) \in \mathcal{F}$. This implies that $\xi = 0$.

By considering the solution S^0, we have

$$\mu^{\tilde{e}\tilde{w}} + \nu_{(o'_k, d'_k)}^{\tilde{e}\tilde{w}} = 0,$$

From (5.91), it follows that $\mu^{\tilde{e}\tilde{w}} = -\rho$.

All together, we have obtained

$$\lambda_{ew}^k = 0, \text{ for all } k \in K, e \in A_2, w \in W,$$

$$\nu_a^{\tilde{e}\tilde{w}} = \begin{cases} \rho, & \text{for some } \rho \in \mathbb{R}, \text{ for all } a \in \delta_{G_2}^+(T), \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu^{ew} = \begin{cases} -\rho, & \text{for } (e, w) = (\tilde{e}, \tilde{w}), \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$, and the proof is complete. □

5.3.4 Capacity inequalities

We will focus on the facial structure of the capacity constraints given by inequalities (5.2).

Theorem 5.11 *For $\tilde{e} \in A_1$ and $\tilde{w} \in W$, inequality* (5.2) *defines facet for $P(G_1, G_2, K, C)$ only if the following holds*

(i) for all $k' \in K$, there exists a subset $R_{k'} \subset K$ such that $\sum_{k \in R_{k'}} D^k = C$,

(ii) for each k', $k'' \in K$, there exists $R_{k'}$, $R_{k''}$, such that $R_{k'} \subset K \setminus \{k''\}$, $k' \in R_{k'}$, $R_{k''} \subset K \setminus \{k''\}$, $k'' \in R_{k''}$, such that

$$\sum_{k \in R_{k'}} D^k = \sum_{k \in R_{k''}} D^k = C.$$

Proof. (i) Suppose that the first condition is not verified. Then, the face induced by inequality $\sum_{k\in K} D^k x_{k\tilde{e}\tilde{w}} \leq C y_{\tilde{e}\tilde{w}}$ is contained in the face induced by $x_{\tilde{k}\tilde{e}\tilde{w}} = 0$. In fact, since there is no subset $R_{\tilde{k}}$ verifying $\sum_{k\in R_{\tilde{k}}} D^k = C$, we can not find a solution such that $x_{\tilde{k}\tilde{e}\tilde{w}} = 1$ that satisfies inequality (5.2) with equality.

(ii) Now assume that condition (ii) is not verified. Then, every solution of the of the face induced by $\sum_{k\in K} D^k x_{k\tilde{e}\tilde{w}} \leq C y_{\tilde{e}\tilde{w}}$ either does not use the pair $(\tilde{e}, \tilde{w})$, or both k' and k'' use $(\tilde{e}, \tilde{w})$. Consequently, each solution of the face also verifies

$$x_{k'\tilde{e}\tilde{w}} = x_{k''\tilde{e}\tilde{w}},$$

but this inequality can not be a multiple of $\sum_{k\in K} D^k x_{k\tilde{e}\tilde{w}} \leq C y_{\tilde{e}\tilde{w}}$, which is a contradiction.

□

Theorem 5.12 *For $\tilde{e} \in A_1$ and $\tilde{w} \in W$, inequality (5.2) defines facet for $P(G_1, G_2, K, C)$ if the following holds*

(i) conditions (i), (ii) of Theorem 5.11 are satisfied,

(ii) $D^k = q$, for all $k \in K$, where $q \in \mathbb{R}_+$ (the commodities are equivalent in size).

Proof. Suppose that there exists a subset of commodities $\tilde{K}$ such that $\sum_{k\in\tilde{K}} D^k = C$. This is possible because of condition (i) of Theorem 5.11. Let us denote by $\alpha x + \beta y + \gamma z \leq \delta$ the capacity inequality (5.2) induced by the arc $\tilde{e}$ and the subband $\tilde{w}$, and let

$$\mathcal{F}^{\tilde{e}\tilde{w}} = \{(x, y, z) \in P(G_1, G_2, K, C) : \sum_{k\in K} D^k x_{k\tilde{e}\tilde{w}} - C y_{\tilde{e}\tilde{w}} = 0\},$$

Let $\lambda x + \mu y + \nu z \leq \xi$ be a valid inequality that defines a facet of $P(G_1, G_2, K, C)$. Suppose that $\mathcal{F}^{\tilde{k}}_{\tilde{e}\tilde{w}} \subseteq \mathcal{F}$. We show that there exists $\rho \in \mathbb{R}$ such that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$. We will construct a feasible solution S^0 that satisfies (5.2) with equality.

For each commodity $k \in K \setminus \tilde{K}$, we consider a path in G_1 between its origin and destination nodes, consisting of arc (o_k, d_k). We install over this arc a subband w_k. In other words, every subband is assigned at most one commodity. We install the subband $\tilde{w}$ on the arc $\tilde{e} = (u, v)$ ($\tilde{e}$ may be equal to some arc (o_k, d_k), $k \in K \setminus \tilde{K}$). We will assume without loss of generality that $\tilde{e} \notin \{(o_k, d_k), k \in K\}$. Then, we install a subband w_i on each arc (o_i, u), $u \neq d_i$ and a subband w'_i on each arc (v, d_i), $v \neq o_i$, where $i \in \tilde{K}$. Every couple (e, w) such that w is installed on $e = (s, t)$ is associated the arc (s', t') in A_2. This is possible since G_2 is a complete graph, and the installed subbands are all different. Observe that, in this solution, each commodity $k \in K \setminus \tilde{K}$ uses the couple $((o_k, d_k), w_k)$ for its routing, while the commodities of $\tilde{K}$ have a path of length at most three $\{(o_i, u), (u, v), (v, d_i)\}$, $i \in \tilde{K}$. Moreover, note that a subband is associated to each commodity of $K \setminus \tilde{K}$, while the commodities of $\tilde{K}$ use different subbands on the arcs (o_i, u), (v, d_i), $i \in \tilde{K}$ and share the same subband $\tilde{w}$ on the arc $\tilde{e}$. More formally, the solution S^0 is such that $S^0 = (F_1^0, F_2^0, \Delta^0, W^0)$, where $F_1^0 = \{(\bigcup_{k \in K \setminus \tilde{K}} (o_k, d_k)) \cup (\bigcup_{i \in \tilde{K}} \{(o_i, u), (v, d_i)\}) \cup \{\tilde{e}\}\}$, $F_2^0 = \emptyset$, $\Delta^0 = \{(o'_k, d'_k), k \in K \setminus \tilde{K}\} \cup \{(o'_i, u'), (v', d'_i), i \in \tilde{K}\} \cup \{(u', v')\}$ and W^0 contains all the used subbands.

Observe that all the subbands installed here are different, thus, disjunction constraints (5.4) are satisfied. Moreover, since the capacities of the subbands are greater than or equal to the commodity values, and a different subband is associated with each commodity of $K \setminus \tilde{K}$, we have the capacity constraints (5.2), that are also satisfied. Note that the unique subband that may be shared by all the commodities of $\tilde{K}$, is $\tilde{w}$, and this is possible since $\sum_{k \in \tilde{K}} D^k = C$, by hypothesis. Therefore, the capacity constraints (5.2) are again satisfied. Furthermore, by construction, the solution given above also satisfies the connectivity constraints (5.1) and (5.3). Consequently, the solution S^0 is feasible for the problem.

Now, let us show that us show that $\nu_a^{ew} = 0$, for all $e \in A_1$, $w \in W$ and $a \in A_2$.

To do this, we will introduce the solution S^1 that is obtained from S^0 by simply adding to Δ^0, an arc a of $A_2 \setminus \Delta^0$. Assume that a is added to the subset Δ^0_{ew} where $e \in A_1$ and $w \in W$. Then, $S^1 = (F_1^0, F_2^0, \Delta^0 \cup \{a\}, W^0)$, with $\Delta^1_{ew} = \Delta^0_{ew} \cup \{a\}$ and $\Delta^1_{e_i w_i} = \Delta^0_{e_i w_i}$ if $(e_i, w_i) \neq (e, w)$. It is not hard to see that S^1 is still feasible. Moreover, both incidence vectors of S^0 and S^1 belong to $\mathcal{F}^{\tilde{e}\tilde{w}}$ and $\mathcal{F}$. Thus, they satisfy

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} + \nu_a^{ew},$$

which implies that $\nu_a^{ew} = 0$. Since a is chosen arbitrarily out of the solution, and so as for e in A_1 and $w \in W$, we get

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in A_1 \setminus \Delta^0, \tag{5.98}$$

Suppose now that the arc $a = (s', t') \in A_2$ belongs to the solution, more precisely $a \in \Delta^0_{ew}$ where $e \in A_1$ and $w \in W$. And let, $f = (s', r')$ and $g = (r', t')$ be two arcs of $A_2 \setminus \Delta^0$. Consider the solution S'^1 obtained from S^0 by replacing the arc a by f and g in Δ^0_{ew}. S'^1 is then equal to $(F_1^0, F_2^0, (\Delta^0 \setminus \{a\}) \cup \{f, g\}, W^0)$. S'^1 is clearly feasible, and its incidence vector verifies

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S'^1} + \mu y^{S'^1} + \nu z^{S'^1} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} - \nu_a^{ew} + \nu_f^{ew} + \nu_g^{ew},$$

which gives $-\nu_a^{ew} + \nu_f^{ew} + \nu_g^{ew} = 0$. As by (5.98), $\nu_f^{ew} = \nu_g^{ew} = 0$, we get ν_a^{ew} which is also equal to 0. Since the arc a is chosen arbitrarily within the solution, it follows that

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in \Delta^0, \tag{5.99}$$

In consequence, we have

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in A_1, \tag{5.100}$$

Next, we will show that $\mu^{ew} = 0$, for all $(e, w) \in (A_1 \times W) \setminus \{(\tilde{e}, \tilde{w})\}$.

Consider the solution S^2, obtained from S^0 by adding an arc $e = (s, t) \in A_1 \setminus (F_1^0 \cup F_2^0)$ to the solution. The arc e receives a subband w that is not used in S^0. The couple (e, w) is then assigned the arc (s', t') of $A_2 \setminus \Delta^0$ and is not involved in the routing of any commodity. In other words, $S^2 = (F_1^0, F_2^0 \cup \{e\}, \Delta^0 \cup \{(s', t')\}, W^0 \cup \{w\})$. Note that $\Delta^2_{ew} = \Delta^0_{ew} \cup \{(s', t')\}$ while $\Delta^2_{e_i w_i} = \Delta^0_{e_i w_i}$ if $(e_i, w_i) \neq (e, w)$. It is easy to check that the solution S^2 is feasible. In addition, its incidence vector belongs to $\mathcal{F}^{\tilde{e}\tilde{w}}$ and $\mathcal{F}$. Thus, we have

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^2} + \mu y^{S^2} + \nu z^{S^2} = \lambda x^{S^0} + \mu y^{S^0} + \mu^{ew} + \nu z^{S^0} + \nu^{ew}_{(s', t')},$$

and it follows that $\mu^{ew} + \nu^{ew}_{(s', t')} = 0$. Since $\nu^{ew}_{(s', t')} = 0$, by (5.100), we have $\mu^{ew} = 0$. As e and w are selected arbitrarily in the sets $A_1 \setminus (F_1^0 \cup F_2^0)$ and $W \setminus W^0$, we get

$$\mu^{ew} = 0, \text{ for all } e \in A_1 \setminus (F_1^0 \cup F_2^0), w \in W \setminus W^0, \tag{5.101}$$

Assume now that $e = (s, t)$ is an arc of the solution, and let w be the subband installed on e. As $F_2^0 = \emptyset$, this means that e is in $F_1^0 \setminus \{\tilde{e}\}$. In particular, $e \in \mathcal{C}_k^0$, where k is some commodity of K. Then, let $f = (s, r)$ and $g = (r, t)$ be two arcs of $A_1 \setminus (F_1^0 \cup F_2^0)$. We will define a new solution S'^2 that is obtained from S^0 as follows. The arcs f and g are added to the solution and receive the subband w. The couples (f, w) and (g, w) are assigned the arcs (s', r') and (r', t'), respectively (their corresponding arcs in A_2). In this solution, (f, w) and (g, w) are supposed to be involved in the routing of k. In other words, k uses f and g instead of e. More formally, $S'^2 = ((F_1^0 \setminus \{e\}) \cup \{f, g\}, F_2^0 \cup$

$\{e\}, \Delta^0 \cup \{(s',r'),(r',t')\}, W^0)$, where $\mathcal{C}_k'^2 = (\mathcal{C}_k^0 \setminus \{e\}) \cup \{f,g\}$, $\Delta_{fw}'^2 = \Delta_{ew}^0 \cup \{(s',r')\}$, $\Delta_{gw}'^2 = \Delta_{ew}^0 \cup \{(r',t')\}$, and the remaining subsets still the same. The solution S'^2 is obviously feasible. Consider the solution S''^2, that is obtained by removing the arc e from S'^2. $S''^2 = (F_1'^2, F_2'^2 \setminus \{e\}, \Delta'^2, W'^2)$ is also feasible for the problem. In addition, both incidence vectors of S'^2 and S''^2 belong to $\mathcal{F}^{\tilde{e}\tilde{w}}$ and $\mathcal{F}$. Hence,

$$\lambda x^{S'^2} + \mu y^{S'^2} + \nu z^{S'^2} = \lambda x^{S''^2} + \mu y^{S''^2} + \nu z^{S''^2} = \lambda x^{S'^2} + \mu y^{S'^2} - \mu^{ew} + \nu z^{S'^2},$$

which implies that $\mu^{ew} = 0$. Since the arc e is chosen arbitrarily in $F_1^0 \setminus \{\tilde{e}\}$, we obtain

$$\mu^{ew} = 0, \text{ for all } e \in F_1^0 \setminus \{\tilde{e}\}, w \in W^0, \tag{5.102}$$

and it follows that

$$\mu^{ew} = 0, \text{ for all } (e,w) \in (A_1 \times W) \setminus \{(\tilde{e},\tilde{w})\}, \tag{5.103}$$

It remains to show that $\lambda_{ew}^k = 0$, for all $k \in K$, $(e,w) \in (A_1 \times W) \setminus \{(\tilde{e},\tilde{w})\}$.

Consider the solution S^3 obtained from S^0 by including to the solution an arc $e = (s,t)$ of $A_1 \setminus (F_1^0 \cup F_2^0)$. The arc e receives a subband $w \in W$ and (e,w) is assigned the arc (s',t') of $A_2 \setminus \Delta^0$. Moreover, we assume in this solution that (e,w) is involved in the routing of some commodity, say k. The solution S^3 is then equal to $(F_1^0 \cup \{e\}, F_2^0, \Delta^0 \cup \{(s',t')\}, W^0 \cup \{w\})$, where $\mathcal{C}_k^3 = \mathcal{C}_k^0 \cup \{e\}$ and $\Delta_{ew}^3 = \Delta_{ew}^0 \cup \{(s',t')\}$. It is clear that the solution S^3 is feasible for the problem. Furthermore, its incidence vector satisfies

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^3} + \mu y^{S^3} + \nu z^{S^3}$$
$$= \lambda x^{S^0} + \lambda_{ew}^k + \mu y^{S^0} + \mu^{ew} + \nu z^{S^0} + \nu_{(s',t')}^{ew},$$

which implies that $\lambda_{ew}^k + \mu^{ew} + \nu_{(s',t')}^{ew} = 0$. As by (5.100) and (5.103), we have $\mu^{ew} = \nu_{(s',t')}^{ew} = 0$, it follows that $\lambda_{ew}^k = 0$. Since e, w and k are chosen arbitrarily in the subsets $A_1 \setminus (F_1^0 \cup F_2^0)$, W and K, we get

$$\lambda_{ew}^k = 0, \text{ for all } k \in K, e \in A_1 \setminus (F_1^0 \cup F_2^0), w \in W, \tag{5.104}$$

Now if $e = (s,t)$ is an arc of $(F_1^0 \cup F_2^0) \setminus \{\tilde{e}\} = F_1^0 \setminus \{\tilde{e}\}$ and w is a subband installed on e. Assume that $e \in \mathcal{C}_k^0$ where k is a commodity of K. Let $f = (s,r)$ and $g = (r,t)$ be two arcs of $A_1 \setminus (F_1^0 \cup F_2^0)$. Then, consider the solution $S'^3 = ((F_1^0 \setminus \{e\}) \cup \{f,g\}, F_2^0, \Delta^0 \cup \{(s',r'),(r',t')\}, W^0)$, where $\mathcal{C}_k'^3 = (\mathcal{C}_k^0 \setminus \{e\}) \cup \{f,g\}$ while $\Delta_{fw}'^3 = \Delta_{fw}^0 \cup \{(s',r')\}$ and $\Delta_{gw}'^3 = \Delta_{fw}^0 \cup \{(r',t')\}$. S'^3 is also feasible for the problem, and its incidence vector belongs to $\mathcal{F}^{\tilde{e}\tilde{w}}$. Therefore, we have

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S'^3} + \mu y^{S'^3} + \nu z^{S'^3}$$

$$= \lambda x^{S^0} - \lambda^k_{ew} + \lambda^k_{fw} + \lambda^k_{gw} + \mu y^{S^0} + \mu^{fw} + \mu^{gw} + \nu z^{S^0} + \nu^{fw}_{(s',r')} + \nu^{gw}_{(r',t')},$$

and it follows that $-\lambda^k_{ew} + \lambda^k_{fw} + \lambda^k_{gw} + \mu^{fw} + \mu^{gw} + \nu^{fw}_{(s',r')} + \nu^{gw}_{(r',t')} = 0$. As by (5.100) we have $\nu^{fw}_{(s',r')} = \nu^{gw}_{(r',t')} = 0$, by (5.103) we have $\mu^{fw} = \mu^{gw} = 0$, and $\lambda^k_{fw} = \lambda^k_{gw} = 0$ by (5.104), we obtain $\lambda^k_{ew} = 0$. Since the arc e is chosen arbitrarily in $(F^0_1 \cup F^0_2) \setminus \{\tilde{e}\}$, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, e \in (F^0_1 \cup F^0_2) \setminus \{\tilde{e}\}, w \in W, \tag{5.105}$$

Hence, by (5.104) and (5.105), we have

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, e \in A_1 \setminus \{\tilde{e}\}, w \in W, \tag{5.106}$$

Now let us turn ourselves to the coefficients μ and λ related to $(\tilde{e}, \tilde{w})$.

Consider again the solution S^0. Let k', k'' be two commodities such that $k' \in \tilde{K}$ and $k'' \in K \setminus \tilde{K}$. Note that k' and k'' are interchangeable in solution S^0 since $D^{k'} = D^{k''}$, by condition (ii). Then, let us introduce the solution S^4 obtained from S^0 by replacing k' in $(\tilde{e}, \tilde{w})$ by k''. In other words, we associate $(\tilde{e}, \tilde{w})$ with the routing of k'' while k' uses an other path. Comparing both solutions yields $\lambda^{k'}_{\tilde{e}\tilde{w}} = \lambda^{k''}_{\tilde{e}\tilde{w}}$. Since k' and k'' are arbitrary, we get

$$\lambda^k_{\tilde{e}\tilde{w}} = \rho = D^k, \text{ for all } k \in K,$$

Furthermore, by replacing S^0 in $\lambda x + \mu y + \nu z \leq \xi$, we obtain $\sum_{k \in \tilde{K}} D^k \leq \mu^{\tilde{e}\tilde{w}}$. Since S^0 belongs to $\mathcal{F}$, it verifies $\sum_{k \in \tilde{K}} D^k \rho = \mu^{\tilde{e}\tilde{w}}$. In addition, by supposition we have $\sum_{k \in \tilde{K}} D^k =$ C. Consequently, we get $\mu^{\tilde{e}\tilde{w}} = C$. Thus, $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$, and the results follows. □

5.4 Valid inequalities and facets

In what follows, we present several families of inequalities that are valid for OMBND problem. We give the necessary conditions and sufficient conditions for some of them to define facets.

5.4.1 Capacitated Cutset Inequalities

In this section, we will present a first class of valid inequalities arising directly from the capacity requirement of the problem. Similar inequalities have been introduced by Magnanti[78], then studied by Barahona [13] and Bienstock et al. [30] for different variants of *capacitated network loading problem.*

Consider the graphs of Figure 5.9 which hold four nodes denoted 1 to 4 for graphs (a) and (b), and 1' to 4' for graph (c). The instance contains three commodities, denoted k_1, k_2 and k_3, all going from node 1 to node 3, with values $D^{k_1} = D^{k_2} = D^{k_3} = 6$. The capacity of a subband is $C = 10$.

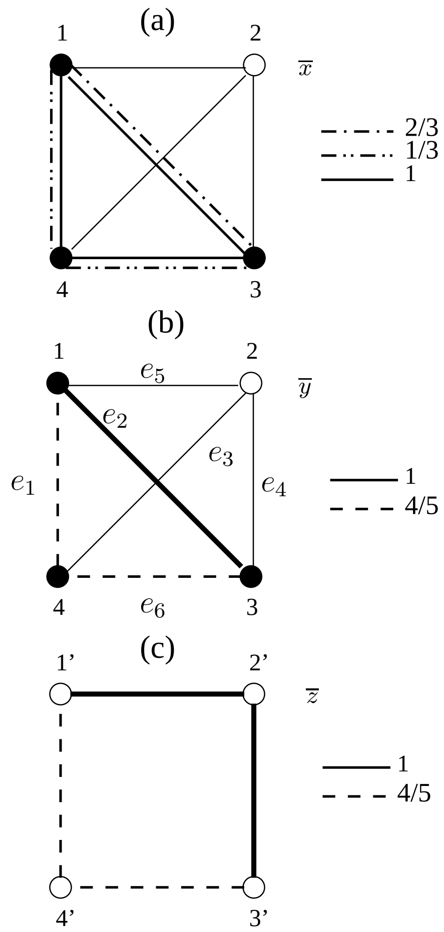


Figure 5.9: First fractional solution

Figure 5.9 shows a fractional solution $(\overline{x}, \overline{y}, \overline{z})$ for this instance, whose representation is subdivided into three graphs, each one associated with a family of variables. Graphs (a) and (b) are associated with G_1. They are related with variables $\overline{x}$ and $\overline{y}$, respectively. Values of variables $\overline{z}$ are reported in graph (c), which is associated with G_2. In this solution, the same subband, denoted w is installed over arcs e_1, e_2 and e_3 (see Figure 5.9 (b)). The paths associated with k_1, k_2, k_3, and with the pairs (e_i, w), $i = 1$, 2 3, as well, can be found in graphs (a) and (c). The solution for the design variables

$\overline{y}$ is particularly given by $\overline{y}_{e_1w_1} = \overline{y}_{e_6w_1} = \frac{4}{5}$, $\overline{y}_{e_2w_1} = 1$, and 0 for the other entries (see Figure 5.9 (b)).

It is clear that $(\overline{x}, \overline{y}, \overline{z})$ satisfies all the constraints of the linear relaxation of (5.8). However, $\overline{y}$ violates the inequality

$$y_{e_1w} + y_{e_2w} + y_{e_5w} \geq 2,$$

which is valid for OMBND problem.

In what follows, we give a generalization of this inequality for $P(G_1, G_2, K, C)$, that will be referred to as *capacitated cutset inequalities.*

Given a partition of G_1 nodes in two subsets T and $\overline{T} = V_1 \setminus T$. We denote by $K(T)$ (respectively $K(\overline{T})$) the commodities of K having their origin and destination nodes in the subset T (respectively in $\overline{T}$), while the remaining subset of K will be denoted by P^+ and P^-. Note that P^+ (respectively P^-) is the subset of commodities having their origin node in T (respectively in $\overline{T}$) and their destination node in $\overline{T}$ (respectively in T). We will also denote by $D(P^+)$ the total traffic amount of the commodities of K having their origin in T and their destination in $\overline{T}$. In other words, $D(P^+) = \sum_{k \in P^+} D^k$, and $D(P^-) = \sum_{k \in P^-} D^k$. Moreover, recall that $BP(P^+)$ is the smallest number of subbands needed to carry the commodities of P^+.

Proposition 5.13 *Let $T \subseteq V_1$, $\emptyset \neq T \neq V_1$, then the following cut-set inequality*

$$\sum_{e \in \delta^+_{G_1}(T)} \sum_{w \in W} y_{ew} \geq \lceil \frac{D(P^+)}{C} \rceil \tag{5.107}$$

is valid for $P(G_1, G_2, K, C)$.

Proof. The total capacity of the subbands installed over the cut must be greater than or equal to the traffic amount of the commodities going from T to $\overline{T} = V_1 \backslash T$ and using the arcs of that cut. Then, inequality

$$C \sum_{e \in \delta^+_{G_1}(T)} \sum_{w \in W} y_{ew} \geq D(P^+)$$

Is clearly valid. One can divide this inequality by C, round up the right-hand side and thus, obtain the inequality

$$\sum_{e \in \delta^+_{G_1}(T)} \sum_{w \in W} y_{ew} \geq \lceil \frac{D(P^+)}{C} \rceil$$

which is valid for $P(G_1, G_2, K, C)$. $\square$

In this section, we will provide the conditions under which cut-set inequalities (5.107) define facets for $P(G_1, G_2, K, C)$. Let $\tilde{T}$ be a subset of V_1, and P^+ (respectively P^-) a subset of commodities having their origin node is in $\tilde{T}$ (respectively in $V_1 \setminus \tilde{T}$) and their destination node in $V_1 \setminus \tilde{T}$ (respectively in $\tilde{T}$). We also denote by $K(u, v)$ the set of commodities such that $K(u, v) = \{k \in K : o_k = u, d_k = v, uv \in A_1\}$.

Theorem 5.14 *The cutset inequality* (5.107) *induced by* $\tilde{T}$ *and* P^+ *defines a facet of* $P(G_1, G_2, K, C)$*, only if* $\lceil \frac{D(P^+)}{C} \rceil = BP(P^+)$.

Proof. Let $(\overline{x}, \overline{y}, \overline{z})$ a fractional solution satisfying all constraint of linear relaxation of (5.8) but such that (5.107) is violated. Let P^+ the set of commodities crossing the cut inducing a violated inequality. Suppose that $\lceil \frac{D(P^+)}{C} \rceil < BP(P^+)$ for those commodities. In this case, (5.107) can not be tight, since the commodities of P^+ might not fit in $\lceil \frac{D(P^+)}{C} \rceil$ subbands, and thus (5.107) does not induce a proper face. $\square$

Theorem 5.15 *The cutset inequality* (5.107) *induced by* $\tilde{T}$ *and* P^+ *defines a facet of* $P(G_1, G_2, K, C)$ *if*

(i) $\lceil \frac{D(P^+)}{C} \rceil = BP(P^+)$,

(ii) $BP(P^+ \cup \{k\}) = BP(P^+)$*, for all* $k \in K \setminus P^+$,

(iii) $\forall\ k' \in P^+$, $\exists\ k'' \in P^+$ *such that* $D^{k'} + D^{k''} \leq C$,

(iv) $\forall k \in P^+$, $BP(P^+ \setminus \{k\}) = BP(P^+)$ *-1.*

Proof. Suppose that conditions (i) to (iv) of Theorem 5.15 are fulfilled. Let us denote by $\alpha x + \beta y + \gamma z \geq \delta$ the capacitated cutset inequality induced by $\tilde{T}$, and let

$$\tilde{\mathfrak{F}} = \{(x, y, z) \in P(G_1, G_2, K, C) : \sum_{e \in \delta^+_{G_1}(\tilde{T})} \sum_{w \in W} y_{ew} = BP(P^+)\}.$$

We first show that $\tilde{\mathfrak{F}}$ is a proper face of $P(G_1, G_2, K, C)$. To do this, let us construct a feasible solution S^0 that satisfies (5.107) with equality.

For each pair of nodes $(s, t) \in \tilde{T}$ (respectively in $V_1 \setminus \tilde{T}$), we install $BP(K(s, t))$ different subbands on the arc (s, t) of A_1. Notice that if there is no commodity $k \in K$ such that $o_k = s$ and $d_k = t$ we do not use (s, t) in the solution. Moreover, each commodity k in $K(\tilde{T})$ (respectively in $K(V_1 \setminus \tilde{T})$) is associated with path $\{(s, t)\} =$

$\{(o_k, d_k)\}$ and the subband w_k. Note that, in this solution, a subband w_k may be associated with more than one commodity (see Figure 5.10).

Now, we choose a node $u \in \tilde{T}$ and a node $v \in V_1 \backslash \tilde{T}$. Observe that $(u, v) \in \delta^+_{G_1}(\tilde{T})$. We then install on the arcs (o_k, u) (respectively (v, d_k)) of A_1, $BP(K(o_k, u))$ (respectively $BP(v, d_k)$) new subbands of W, while (u, v) receives $BP(P^+)$ new subbands. Note that (u, v) is the only arc of the cut $\delta^+_{G_1}(\tilde{T})$ that is used in this solution. We do the same operation for the commodities of P^-. Furthermore, we associate with each pair (e, w) such that w is installed on $e = (i, j)$ the arc (i', j') of A_2. This is possible since G_2 is a complete graph. Notice that, in this solution, each commodity $k \in K(\tilde{T})$ (respectively in $K(V_1 \setminus \tilde{T})$) uses the subband w_k on path $\{e_k\}$, $e_k = (o_k, d_k)$ for its routing, while the commodities of P^+ have a path of length at most three $\{(o_i, u), (u, v), (v, d_i)\}$, $i \in P^+$. The node u (respectively v) can obviously be equal to some o_i (respectively d_i), $i \in P^+$. Moreover, in this solution, every commodity of K uses at least one subband for its routing, and we assume that all the set up subbands are different so that the disjunction constraints (5.4) are satisfied. Also note that many commodities may share the same subband, however, as $BP(K(s, t))$ subbands are installed on each pair of nodes $s, t \in \tilde{T}$ and $V_1 \setminus \tilde{T}$, we ensure that the capacity constraints (5.2) are satisfied. In this solution, a path in G_1 is assigned to each commodity of K. Moreover, a path is also associated with every pair (e, w) such that w is installed on e. Furthermore, both capacity constraints (5.2) and disjunction constraints (5.4), are satisfied, as enough different subbands are installed on each arc used in the solution. It is not hard to see that S^0 induces a feasible solution of $P(G_1, G_2, K, C)$ whose incidence vector satisfies $\alpha x + \beta y + \gamma z \geq \delta$ with equality. Hence, $\tilde{\mathfrak{F}} \neq \emptyset$ and, therefore, is a proper face of $P(G_1, G_2, K, C)$.

Now suppose that there exists a facet defining inequality $\lambda x + \mu y + \nu z \geq \xi$ such that

$$\tilde{\mathfrak{F}} \subseteq \mathfrak{F} = \{(x, y, z) \in P(G_1, G_2, K, C) : \lambda x + \mu y + \nu z = \xi\}.$$

We will show that there exists a scalar $\rho \in \mathbb{R}$ such that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$.

Let us first show that $\nu^{ew}_a = 0$, for all $e \in A_1$, $w \in W$ and $a \in A_2$.

Consider an arc $a \in A_2 \setminus \Delta^0$, and a pair $(e^*, w^*) \in A_1 \times W$. Clearly, the solution $S^1 = (F^0_1, F^0_2, \Delta^1, W^0)$, where $\Delta^1_{e_i w_i} = \Delta^0_{e_i w_i}$ and $\Delta^1_{e^* w^*} = \Delta^0_{e^* w^*} \cup \{a\}$ is a solution of the $P(G_1, G_2, K, C)$ and its incidence vector satisfies $\alpha x + \beta y + \gamma z \geq \delta$ with equality. It then follows that

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} + \nu^{e^* w^*}_a,$$

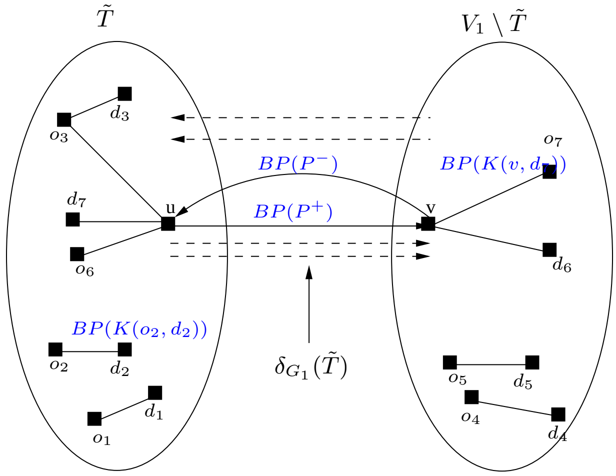


Figure 5.10: Solution S^0

which implies that $\nu_a^{e^*w^*} = 0$. Since, a, e^* and w^* are arbitrary in $A_2 \setminus \Delta^0$, A_1 and W, we obtain

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in A_2 \setminus \Delta^0, \tag{5.108}$$

Now let $a = (s', t') \in \Delta^0$, such that $a \in \Delta^0_{e^*w^*}$ for some $(e^*, w^*) \in \Gamma^0$. Then, consider the solution S^2 obtained from S^0 by replacing a by (s', r') and (r', t') in $\Delta^0_{e^*w^*}$, with (s', r'), $(r', t') \in A_2 \setminus \Delta^0$ and $r' \in V_2 \setminus \{s', t'\}$. $S^2 = (F_1^0, F_2^0, \Delta^2, W^0)$, with $\Delta^2_{ew} = \Delta^0_{ew}$ and $\Delta^2_{e^*w^*} = (\Delta^0_{e^*w^*} \setminus \{a\}) \cup \{(s', r'), (r', t')\}$ is obviously feasible for $P(G_1, G_2, K, C)$. As its incidence vector belongs to $\tilde{\mathcal{F}}$ and thus, to $\mathcal{F}$, we have

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^2} + \mu y^{S^2} + \nu z^{S^2} =$$

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} - \nu_a^{e^*w^*} + \nu_{(s',r')}^{e^*w^*} + \nu_{(r',t')}^{e^*w^*},$$

which gives that $\nu_a^{e^*w^*} = \nu_{(s',r')}^{e^*w^*} + \nu_{(r',t')}^{e^*w^*}$. As by (5.108), $\nu_{(s',r')}^{e^*w^*} = \nu_{(r',t')}^{e^*w^*} = 0$, $\nu_a^{e^*w^*}$ is also equal to zero which yields

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in \Delta^0, \tag{5.109}$$

Thus, by (5.108) and (5.109), we obtain

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in A_2. \tag{5.110}$$

Next, we will show that $\mu^{ew} = 0$, for all $e \in A_1 \setminus \delta^+_{G_1}(\tilde{T})$ and $w \in W$.

Let $e^* = (s, t) \in A_1 \setminus (F_1^0 \cup F_2^0)$ such that $e \notin \delta^+_{G_1}(T)$. Let w^* be a subband of W. We introduce the solution S^3, obtained from S^0 by adding e^* to the subset F_2^0.

Thus, $S^3 = (F_1^0, F_2^0 \cup \{e^*\}, \Delta^0 \cup \{(s',t')\}, W^0 \cup \{w^*\})$, where $(s',t') \in A_2 \setminus \Delta^0$, induces a feasible solution of $P(G_1, G_2, K, C)$. Note that $\Gamma^3 = \Gamma^0 \cup (e^*, w^*)$. In addition, $(x^{S^3}, y^{S^3}, z^{S^3}) \in \tilde{\mathcal{F}}$, and then, $(x^{S^3}, y^{S^3}, z^{S^3}) \in \mathcal{F}$. Hence we have

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^3} + \mu y^{S^3} + \nu z^{S^3} =$$

$$\lambda x^{S^0} + \mu y^{S^0} + \mu^{e^*w^*} + \nu z^{S^0} + \nu^{e^*w^*}_{(s',t')},$$

and it follows that $\mu^{e^*w^*} = -\nu^{e^*w^*}_{(s',t')}$. Consequently, and by (5.110), we obtain $\mu^{e^*w^*} = 0$. Since e^* and w^* are arbitrarily selected in $A_1 \setminus (F_1^0 \cup F_2^0)$ and W, we get

$$\mu^{ew} = 0, \text{ for all } e \in A_1 \setminus (F_1^0 \cup F_2^0), e \notin \delta^+_{G_1}(\tilde{T}), w \in W, \tag{5.111}$$

Now consider an arc $e^* = (s,t) \in F_1^0 \cup F_2^0 = F_1^0$ such that $e^* \notin \delta^+_{G_1}(\tilde{T})$. Let w^* be a subband installed on e^* and assume that $e^* \in \mathcal{C}^0_{k^*}$ for some commodity $k^* \in K$. Consider the solution S^4 defined as follows. $S^4 = (F_1^0 \cup \{(s,r),(r,t)\}, F_2^0, \Delta^0 \cup \{(s',r'),(r',t')\}, W^0)$, with $(s,r),(r,t) \in A_1 \setminus F_1^0 \cup F_2^0$, $r \in V_1 \setminus \{s,t\}$ and $(s',r'),(r',t') \in A_2 \setminus \Delta^0$, $r' \in V_2 \setminus \{s',t'\}$. Notice that $\Gamma^4 = \Gamma^0 \cup \{((s,r),w^*),((r,t),w^*)\}$. Moreover, $\mathcal{C}^4_k = \mathcal{C}^0_k$ if $k \neq k^*$ and $\mathcal{C}^4_{k^*} = (\mathcal{C}^0_{k^*} \setminus \{e^*\}) \cup \{(s,r),(r,t)\}$, while $\Delta^4_{(s,r)w^*} = \Delta^0_{(s,r)w^*} \cup \{(s',r')\}$ and $\Delta^4_{(r,t)w^*} = \Delta^0_{(r,t)w^*} \cup \{(r',t')\}$. We will construct a further solution S^5, obtained from S^4 by removing the pair (e^*, w^*) from Γ^4. More formally, S^5 is such that $S^5 = (F_1^4, F_2^4, \Delta^4, W^4)$, and $\Gamma^5 = \Gamma^4 \setminus \{(e^*, w^*)\}$. Both solutions S^4 and S^5 are feasible for $P(G_1, G_2, K, C)$ and their incidence vectors belong to $\tilde{\mathcal{F}}$ and then, to $\mathcal{F}$. In consequence, it follows that

$$\lambda x^{S^4} + \mu y^{S^4} + \nu z^{S^4} = \lambda x^{S^5} + \mu y^{S^5} + \nu z^{S^5} = \lambda x^{S^4} + \mu y^{S^4} - \mu^{e^*w^*} + \nu z^{S^4},$$

Hence, we get that $\mu^{e^*w^*} = 0$. Since e^* is arbitrary in $(F_1^0 \cup F_2^0) \setminus \delta^+_{G_1}(\tilde{T})$, we conclude that

$$\mu^{ew} = 0, \text{ for all } e \in (F_1^0 \cup F_2^0) \setminus \delta^+_{G_1}(\tilde{T}), (e,w) \in \Gamma^0, \tag{5.112}$$

By (5.111) and (5.112) we obtain

$$\mu^{ew} = 0, \text{ for all } e \in A_1 \setminus \delta^+_{G_1}(\tilde{T}), w \in W. \tag{5.113}$$

In what follows, we will show that $\lambda^k_{ew} = 0$, for all $k \in K$, $e \in A_1$ and $w \in W$.

Let $e^* = (s,t)$ be an arc $A_1 \setminus (F_1^0 \cup F_2^0)$ that does not belong to $\delta^+_{G_1}(\tilde{T})$ and let w^* be a subband of W. Consider the solution S^6, obtained from S^0 by installing w^* on e^*, and adding e^* to any subset $\mathcal{C}^0_{k^*}$, $k^* \in K$. This means setting $x^{S^0}_{k^*e^*w^*}$ to 1. Then $S^6 = (F_1^0 \cup \{e^*\}, F_2^0, \Delta^0 \cup \{(s',t')\}, W^0 \cup \{w^*\})$, where $(s',t') \in A_2 \setminus \Delta^0$. Observe that $\Gamma^6 = \Gamma^0 \cup \{(e^*, w^*)\}$. In addition, note that $\mathcal{C}^6_k = \mathcal{C}^0_k$ if $k \neq k^*$ and $\mathcal{C}^6_{k^*} = \mathcal{C}^0_{k^*} \cup \{e^*\}$, while

$\Delta^6_{e^*w^*} = \Delta^0_{e^*w^*} \cup \{(s',t')\}$ and $\Delta^6_{ew} = \Delta^0_{ew}$ if $(e,w) \neq (e^*,w^*)$. It is easy to see that S^6 induces a feasible solution of $P(G_1,G_2,K,C)$ whose incidence vector verifies $\lambda x + \mu y + \nu z \geq \xi$ with equality. Hence, we have that

$$\lambda^{k^*}_{e^*w^*} + \mu^{e^*w^*} + \nu^{e^*w^*}_{(s',t')} = 0,$$

Since $\mu^{e^*w^*} = \nu^{e^*w^*}_{(s',t')} = 0$, by (5.110) and (5.113), we obtain that $\lambda^{k^*}_{e^*w^*} = 0$. As e^*, w^* and k^* are arbitrary, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, e \in A_2 \setminus (F^0_1 \cup F^0_2), e \notin \delta^+_{G_1}(\tilde{T}), w \in W, \tag{5.114}$$

Now consider an arc $e^* = (s,t)$ of $(F^0_1 \cup F^0_2)$ and let w^* be a subband of W such that $(e^*,w^*) \in \Gamma^0$. Assume without loss of generality that e^* is different from (u,v), and that the pair (e^*,w^*) is associated with the routing of some commodity, say k^*. Let us introduce the solution S^7, obtained from S^0 by replacing e^* in $\mathcal{C}^0_{k^*}$ by two arcs (s,r) and (r,t) of $A_1 \setminus (F^0_1 \cup F^0_2)$. Then, $S^7 = (F^0_1 \cup \{(s,r),(r,t)\}, F^0_2, \Delta^0 \cup \{(s',r'),(r',t')\}, W^0)$, where (s',r'), $(r',t') \in A_2 \setminus \Delta^0$, $\Gamma^7 = \Gamma^0 \cup \{((s,r),w^*),((r,t),w^*)\}$, and $\mathcal{C}^7_{k^*} = (\mathcal{C}^0_{k^*} \setminus \{e^*\}) \cup \{(s,r),(r,t)\}$. Also remark that $\Delta^7_{(s,r)w^*} = \Delta^0_{(s,r)w^*} \cup \{(s',r')\}$ while $\Delta^7_{(r,t)w^*} = \Delta^0_{(r,t)w^*} \cup \{(r',t')\}$. It is clear that S^7 is a feasible solution whose incidence vector is in $\tilde{\mathcal{F}}$ and $\mathcal{F}$. Hence, we have

$$\lambda^{k^*}_{e^*w^*} = \lambda^{k^*}_{(s,r)w^*} + \lambda^{k^*}_{(r,t)w^*},$$

which implies that $\lambda^{k^*}_{e^*w^*} = 0$, as $\lambda^{k^*}_{(s,r)w^*} = \lambda^{k^*}_{(r,t)w^*} = 0$ by (5.114). Furthermore, since (e^*,w^*) is arbitrary in Γ^0, $e^* \notin \delta^+_{G_1}(\tilde{T})$, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, (e,w) \in \Gamma^0, e \notin \delta^+_{G_1}(\tilde{T}), \tag{5.115}$$

Now consider a commodity $k^* \in K$. We will show that coefficient λ related to commodities of K and arcs of $\delta^+_{G_1}(\tilde{T})$ are equal to zero. Two cases may hold here.

Case 1.

Suppose that $k^* \in K \setminus P^+$. Consider an arc e^* of $\delta^+_{G_1}(\tilde{T})$ and a subband w^* of W. We will assume that $e^* = (u,v)$, since the arcs of the cut $\delta^+_{G_1}(\tilde{T})$ are interchangeable. Also suppose that w^* is installed on e^*. Consider the solution S^8, obtained from S^0 by associating e^* to k^* in addition to its routing. In other words, $S^8 = (F^0_1, F^0_2, \Delta^0, W^0)$ and $\mathcal{C}^8_{k^*} = \mathcal{C}^0_8 \cup \{e^*\}$. Condition (ii) makes the solution feasible for the problem, as it allows capacity constraints to be satisfied. Thus, S^8 as well as S^0 belong to $\mathcal{F}$, and consequently to $\mathcal{F}$. Hence, both incidence vectors $(x^{S^0}, y^{S^0}, z^{S^0})$ and $(x^{S^0}, y^{S^0}, z^{S^0})$ satisfy the following

$$\lambda x^{S^8} + \mu y^{S^8} + \nu z^{S^8} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} + \lambda^{k^*}_{e^*w^*},$$

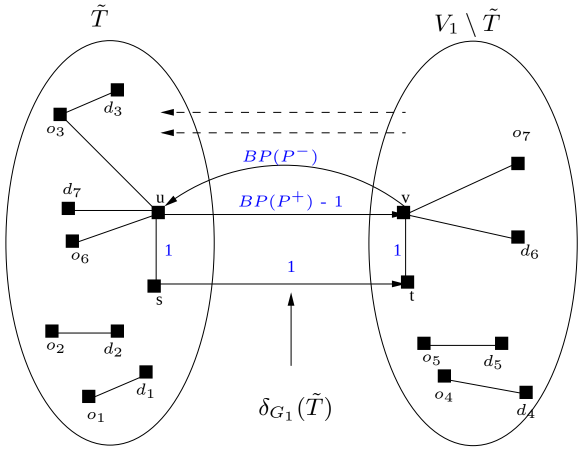


Figure 5.11: Obtaining the solution S^9

which yields $\lambda^{k^*}_{e^*w^*} = 0$. Since, k^*, e^* and w^* are arbitrary in $K \setminus P^+$, $\delta^+_{G_1}(\tilde{T})$ and W, we obtain that

$$\lambda^k_{ew} = 0, \text{ for all } k \in K \setminus P^+, e \in \delta^+_{G_1}(\tilde{T}), w \in W, \tag{5.116}$$

Case 2.

Now consider the case where $k^* \in P^+$, and k be a commodity of P^+ such that $D^{k^*} + D^k \leq C$. Such a commodity exists because of condition (iii). Let $e^* = (s,t)$ be an arc of $\delta^+_{G_1}(\tilde{T})$ and let w^* be one of the commodities installed on (u,v). We will construct a solution S^9 obtained from S^0 by moving w^* from arc (u,v) to arc (s,t), and associating with $((s,t),w^*)$ the path $\{(s',t')\}$ in G_2. In this solution, we will also replace (u,v) in the routing path of k^* by $\{(u,s),e^*,(t,v)\}$, where (u,s) and (t,v) are two arcs of $A_1 \setminus \delta^+_{G_1}(\tilde{T})$. (u,s) and (t,v) also receive the subband w^* and are assigned the paths $\{(u',s')\}$ and $\{(t',v')\}$, in G_2, respectively. S^9 is feasible, since we know, by condition (iv) that capacity constraints (5.2) are satisfied. Now let us derive a solution S^{10} which slightly differs from S^9 in that we associate (s,t) to k in addition to its routing. Again, this is possible thanks to condition (iii). This variation in the solution induces $x^{S^{10}}_{k(s,t)w^*} = 1$ while $x^{S^9}_{k(s,t)w^*} = 0$. Solution S^{10} is clearly feasible, and both incidence vectors $(x^{S^9}, y^{S^9}, z^{S^9})$ and $(x^{S^{10}}, y^{S^{10}}, z^{S^{10}})$ are in $\tilde{\mathfrak{F}}$, and then, also in $\mathfrak{F}$. Thus, we obtain that $\lambda^k_{(s,t)w^*} = 0$. By the interchangeability argument on the elements of P^+, $\delta^+_{G_1}(\tilde{T})$ and W, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in P^+, e \in \delta^+_{G_1}(\tilde{T}), w \in W, \tag{5.117}$$

And, by (5.114), (5.115), (5.116) and (5.117), we finally obtain

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, e \in A_1, w \in W, \tag{5.118}$$

We still have to show that all the coefficients μ^{ew} are the same for the arcs of the cut $\delta^+_{G_1}(\tilde{T})$.

Indeed, let $e^* = (s,t)$ be an arc of $\delta^+_{G_1}(\tilde{T})$, different from (u,v). Recall that $BP(P^+)$ different subbands are installed over the arc (u,v). Let $\overline{w}$ be one of these subbands. Consider the solution S^9 where we replace the pair $((u,v),\overline{w})$ in Γ^0 by $((u,s),\overline{w})$, $((s,t),\overline{w})$ and $((t,v),\overline{w})$, with (u,s), $(t,v) \in A_1 \setminus (F_1^0 \cup F_2^0)$ (Figure 5.11). Comparing solutions S^0 and S^9 give

$$\mu^{(u,v)\overline{w}} = \mu^{(u,s)\overline{w}} + \mu^{(s,t)\overline{w}} + \mu^{(t,v)\overline{w}} + \nu^{(u,s)\overline{w}}_{(u',s')} + \nu^{(s,t)\overline{w}}_{(s',t')} + \nu^{(t,v)\overline{w}}_{(t',v')},$$

By (5.110) and (5.113), we obtain that

$$\mu^{(u,v)\overline{w}} = \mu^{(s,t)\overline{w}},$$

Since (s,t) is arbitrary in $\delta^+_{G_1}(\tilde{T})$, we get

$$\mu^{ew} = \begin{cases} \rho, & \text{for some } \rho \in \mathbb{R}^*, \text{ for all } e \in \delta^+_{G_1}(\tilde{T}), w \in W, \\ 0, & \text{otherwise.} \end{cases} \tag{5.119}$$

Hence, all together, and when replacing $(x^{S^0}, y^{S^0}, z^{S^0})$ in hyperplane $\lambda x + \mu y + \nu z \geq \xi$ we obtain

$$\rho \sum_{w \in W} \sum_{e \in \delta^+_{G_1}(\tilde{T})} y_{ew} = \xi,$$

Note that $\rho \neq 0$, since $\mathcal{F} \neq \emptyset$. Consequently, $\sum_{w \in W} \sum_{e \in \delta^+_{G_1}(\tilde{T})} y_{ew} = \xi/\rho = \lceil \frac{D(P^+)}{C} \rceil$. Thus, $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$, and the proof is complete. $\square$

5.4.2 Flow-Cutset Inequalities

In what follows, we will describe a set of strong valid inequalities for $P(G_1, G_2, K, C)$ that are a generalization of cutset inequalities presented below. Similar inequalities have been introduced by Chopra et al. [31] and were discussed in [8], [21] and [93] for network design problems where discrete modular capacities are installed on the arcs of the graph.

Consider a fixed non empty subset of nodes $T \subseteq V_1$ and a partition F, $\overline{F}$ of the cut $\delta^+_{G_1}(T)$ induced by T (figure 5.12). We denote by P^+ the set of commodities having their origin node in T and their destination node in $\overline{T}$. In other words, $P^+ = K(T, \overline{T})$, and $D(P^+) = \sum_{k \in P^+} D^k$.

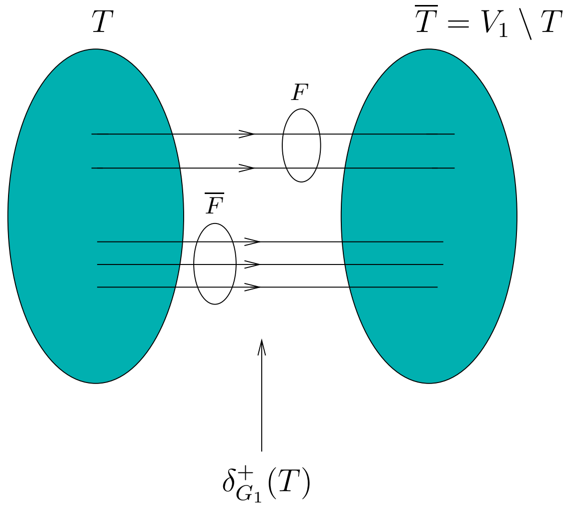


Figure 5.12: Flow-cutset inequality configuration

Proposition 5.16 *The following flow-cutset inequalities are valid for* $P(G_1, G_2, K, C)$

$$\sum_{w\in W}\sum_{e\in F} y_{ew} + \sum_{w\in W}\sum_{e\in\overline{F}}\sum_{k\in P^+} x_{kew} \geq \lceil \frac{D(P^+)}{C} \rceil. \tag{5.120}$$

Proof. It is clear that the following inequalities are valid for $P(G_1, G_2, K, C)$, as they express the connectivity constraints for the commodities of P

$$\sum_{w\in W}\sum_{e\in\delta^+_{G_1}(T)} x_{kew} \geq 1, \quad \text{for all } k \in P,$$

Multiplying by D^k and summing over the commodities of P allows to obtain

$$\sum_{w\in W}\sum_{e\in\delta^+_{G_1}(T)}\sum_{k\in P} D^k x_{kew} \geq D(P), \tag{5.121}$$

that is also valid for $P(G_1, G_2, K, C)$. In addition, we have from the capacity constraints (5.2), restricted to the commodities of P and the arcs of F, that

$$\sum_{k\in P} D^k x_{kew} - C y_{ew} \leq 0, \quad \text{for all } e \in F, w \in W,$$

is valid and leads to the following inequality, when summing over F and W

$$\sum_{w\in W}\sum_{e\in F} C y_{ew} - \sum_{w\in W}\sum_{e\in F}\sum_{k\in P} D^k x_{kew} \geq 0, \tag{5.122}$$

Recall that $\delta^+_{G_1}(T) = F \cup \overline{F}$. Consequently, by doing (5.121) + (5.122), and dividing the resulting inequality by C, we get

$$\sum_{w\in W}\sum_{e\in F} y_{ew} + \sum_{w\in W}\sum_{e\in\overline{F}}\sum_{k\in P}\frac{D^k}{C}x_{kew} \geq \frac{D(P)}{C}, \tag{5.123}$$

Moreover, the trivial constraints $x_{kew} \geq 0$, for all $k \in P$, $e \in \overline{F}$, $w \in W$, can be multiplied by $(1 - \frac{D^k}{C})$ for all $k \in P$, and by summing over P, $\overline{F}$ and W, we obtain

$$\sum_{w\in W}\sum_{e\in\overline{F}}\sum_{k\in P}(1 - \frac{D^K}{C})x_{kew} \geq 0, \tag{5.124}$$

Notice that this inequality is valid for $P(G_1, G_2, K, C)$, since $D^k \leq C$, for all $k \in K$ and $(1 - \frac{D^K}{C}) \geq 0$, for all $k \in K$. Now by doing (5.123) + (5.124) we get

$$\sum_{w\in W}\sum_{e\in F} y_{ew} + \sum_{w\in W}\sum_{e\in\overline{F}}\sum_{k\in P} x_{kew} \geq \frac{D(P)}{C}, \tag{5.125}$$

(5.120) is then obtained from inequality (5.125) by rounding up its right hand side. □

In what follows we will investigate the facial structure of flow-cutset inequalities and provide necessary conditions and sufficient conditions under which the constraints define facets of $P(G_1, G_2, K, C)$.

Theorem 5.17 *A flow-cutset inequality* (5.120) *defines a facet of* $P(G_1, G_2, K, C)$, *different from* (5.107) *only if the following conditions hold*

(i) $F \neq \emptyset \neq \overline{F}$,

(ii) $D(P^+) > C$,

(iii) $D(P^+)$ *is not a multiple of* C,

(iv) $\lceil \frac{D(P^+)}{C} \rceil = BP(P^+)$,

(v) $BP(P^+) < |P^+|$,

(vi) $\exists\ q \subsetneq P^+$ *such that* $BP(P^+ \setminus q) \leq BP(P^+) - |q|$,

(vii) $\forall k \in K \setminus P^+$ *such that* $BP(P^+ \cup \{k\}) \leq BP(P^+)$.

Proof. Let T be a subset of nodes of V_1 and $\overline{T} = V_1 \setminus T$. Consider the cut $\delta^+_{G_1}(T)$ induced by T, and let F, $\overline{F}$ be a partition of $\delta^+_{G_1}(T)$. Now consider the flow-cutset inequality induced by T and F

$$\sum_{e\in F}\sum_{w\in W} y_{ew} + \sum_{e\in\overline{F}}\sum_{w\in W}\sum_{k\in P} x_{kew} \geq \lceil \frac{D(P)}{C} \rceil.$$

(i) Assume that $F = \delta^+_{G_1}(T)$ ($\overline{F} = \emptyset$). Then, (5.120) is equivalent to

$$\sum_{w\in W}\sum_{e\in F} y_{ew} \geq \lceil \frac{D(P^+)}{C} \rceil,$$

which reduces to the cutset inequality (5.107) when $\lceil \frac{D(P^+)}{C} \rceil = BP(P^+)$. Hence, (5.120) cannot be a facet of $P(G_1, G_2, K, C)$ different from (5.107). If $F = \emptyset$, then $\overline{F} = \delta^+_{G_1}(T)$ and (5.120) is equivalent to

$$\sum_{e\in\delta^+_{G_1}(T)}\sum_{w\in W}\sum_{k\in P^+} x_{kew} \geq \lceil \frac{D(P^+)}{C} \rceil, \tag{5.126}$$

which implies that the number of commodities allowed to use the cut $\delta^+_{G_1}(T)$ is greater than or equal to $\lceil \frac{D(P^+)}{C} \rceil$. Note that $\lceil \frac{D(P^+)}{C} \rceil \leq |P^+|$, as $D^k \leq C$, for all $k \in P^+$. Thus, inequality (5.126) is dominated by inequality

$$\sum_{k\in P^+}\sum_{e\in\delta^+_{G_1}(T)}\sum_{w\in W} x_{kew} \geq |P^+|,$$

that is the sum of the connectivity constraints (5.1) over the commodities of P^+. Thus, (5.120) can not define a facet for $P(G_1, G_2, K, C)$.

(ii) Now if $D(P^+) < C$. Then, (5.120) is equivalent to

$$\sum_{w\in W}\sum_{e\in F} y_{ew} + \sum_{w\in W}\sum_{e\in\overline{F}}\sum_{k\in P^+} x_{kew} \geq 0,$$

which is nothing but a linear combination of trivial inequalities $y_{ew} \geq 0$, and $x_{kew} \geq 0$, summed up over the subsets F, W and P^+, $\overline{F}$, W, respectively.

(iii) If $D(P^+)/C$ is integer, then (5.120) can be obtained from inequalities (5.1), (5.2) and the trivial constraints $x_{kew} \geq 0$.

(iv) Suppose that $\lceil \frac{D(P^+)}{C} \rceil < BP(P^+)$. Then inequality (5.120) does not induce a proper face, as it may be empty.

(v) Now assume that $BP(P^+) = |P^+|$. Then, inequality (5.120) is equivalent to the following expression

$$\sum_{e \in F} \sum_{w \in W} y_{ew} + \sum_{k \in P^+} \sum_{e \in \overline{F}} \sum_{w \in W} x_{kew} \geq BP(P^+) = |P^+|,$$

which is a linear combination of inequalities (5.126) and trivial constraints y_{ew} summed up over F and W. Hence, (5.120) can not define a facet.

(vi) Suppose that condition (vi) is not verified, that is to say $BP(P^+ \backslash q) \geq BP(P^+) - q + 1$ for all $q \subsetneq P^+$. Then we can find no solution with $x_{kew} = 1$, for some commodity $k \in P^+$, $e \in \overline{F}$ and $w \in W$. In other words, the face induced by (5.120) is included in

$$\overline{\mathfrak{F}} = \{(x, y, z) \in P(G_1, G_2, K, C) : x_{kew} = 0, for k \in q, e \in \overline{F}, w \in W\},$$

and then, (5.120) can not define a facet.

(vii) Now if there exists a commodity k in $K \backslash P^+$ such that $BP(P^+ \backslash \{k\}) \geq BP(P^+) + 1$. Then it is not possible to identify a solution of the problem with $x_{kew} = 1$, for $e \in F \cup \overline{F}$, $w \in W$. In this case also, the face induced by (5.120) is included in

$$\overline{\mathfrak{F}} = \{(x, y, z) \in P(G_1, G_2, K, C) : x_{kew} = 0, for k \in q, e \in \overline{F}, w \in W\},$$

and then, (5.120) can not define a facet.

$\square$

Theorem 5.18 *A flow-cutset inequality* (5.120) *defines a facet of* $P(G_1, G_2, K, C)$*, different from* (5.107) *if the following conditions are satisfied*

(i) conditions (i) *to* (vii) *of Theorem 5.18,*

(ii) if $|F| = 1$*, for each* $k \in P^+$, $BP(P^+ \setminus \{k\}) \leq BP(P^+)$ *- 1,*

(iii) if $|F| = 1$*, for each* $k \in P^+$, $\exists k' \in P^+ : D^k + D^{k'} \leq C$.

Proof. Suppose that conditions (i) to (iii) are satisfied. Let $\alpha x + \beta y + \gamma z \geq \delta$ denote the flow-cutset inequality produced by T and F, and let

$$\tilde{\mathfrak{F}} = \{(x, y, z) \in P(G_1, G_2, K, C) : \sum_{e \in F} \sum_{w \in W} y_{ew} + \sum_{k \in P^+} \sum_{e \in \overline{F}} \sum_{w \in W} x_{kew} = \lceil \frac{D(P^+)}{C} \rceil\},$$

Let us first show that $\tilde{\mathfrak{F}} \neq \emptyset$. To this end, we will construct a solution S^0 whose incidence vector belongs to $\tilde{\mathfrak{F}}$.

We install, for each $k \in K(T)$ (resp. $k \in K(\overline{T})$), a subband w_k on the arc (o_k, d_k). This is to associate with every commodity of $K(T)$ (resp. $K(\overline{T})$) a path linking o_k and d_k composed by one arc, and entirely contained in T (resp. in $\overline{T}$). Recall that for the commodities of $K(T)$ (resp. $K(\overline{T})$), both nodes o_k and d_k are in T (resp. $\overline{T}$)). This solution is such that each arc (i,j) of $A_1(T)$ (resp. $A_1(\overline{T})$), receives as many subbands as there exist commodities with $(o_k, d_k) = (i,j)$, $k \in K(T)$ (resp. $k \in K(\overline{T})$). In other words, every commodity k of $K(T) \cup K(\overline{T})$ is associated with the pair (e_k, w_k) for its routing, where $e_k = (o_k, d_k)$ (see Figure 5.13).

Recall that P^+ (resp. P^-) commodities of K having their origin in T (resp. $\overline{T}$) and their destination in $\overline{T}$ (resp. T). Consider two nodes u, s in T and two nodes v, t in $\overline{T}$. Note that u, s (resp. v, t) may be the same. Notice that both arcs (u,v) and (s,t) belong to the directed cut $\delta^+_{G_1}(T)$. And, we can assume without loss of generality that $(u,v) \in F$ and $(s,t) \in \overline{F}$. Now, for every commodity $k \in P^+$ (resp. P^-), we install a subband w_k over the arc (o_k, u) (resp. (o_k, v)). Similarly, we install a subband w_k over (v, d_k) (resp. (u, d_k)), for every $k \in P^+$ (resp. P^-). We then set up $\lceil \frac{D(P^+)}{C} \rceil$ different subbands on the arc (u,v), so as all the commodities of P^+ may be routed across the cut $\delta^+_{G_1}(T)$ by using (u,v). Note that we exactly need $BP(P^+)$ subbands to route commodities of P^+. The same is done on the arc (v,u), so as the commodities of P^- may be routed as well from their origins in $\overline{T}$, to their destinations in T using the cut $\delta^-_{G_1}(T)$. Remark that nodes o_k and u (resp. d_k and v) may coincide. Observe that, (u,v) is the unique arc of $\delta^+_{G_1}(T)$ used in this solution. In addition, we assign to each pair (e,w) such that $w \in W$ is installed on $e = (i,j) \in A_1$, the path $\{(i',j')\}$ in G_2. This is possible since G_2 is a complete graph, and no two established subbands are associated with the same path. So both constraints given by (5.3) and (5.4) are satisfied. Remark that, in this solution, every installed subband is associated with at most one commodity, except for the subbands set up on (u,v). Indeed, several commodities may share the same subband on this arc, and the solution is feasible, as we do not need more than $BP(P^+)$ subbands to pack all the commodities of P^+. Thus, it is clear that the remaining constraints of the problem are satisfied, as a feasible path is ensured for each commodity of K. The solution S^0 is clearly feasible for OMBND problem. Moreover, its incidence vector is such that

$$\sum_{e \in F} \sum_{w \in W} y^{S^0}_{ew} = \sum_{w \in W} y^{S^0}_{(u,v)w} = \lceil \frac{D(P^+)}{C} \rceil,$$

$$\sum_{k \in P^+} \sum_{e \in \overline{F}} \sum_{w \in W} x^{S^0}_{kew} = 0.$$

Thus, by condition (iv), the solution S^0 satisfies $\alpha x + \beta y + \gamma z \geq \delta$ with equality. And hence, $(x^{S^0}, y^{S^0}, z^{S^0})$ belongs to $\tilde{\mathcal{F}}$, and then to $\mathcal{F}$. Hence, $\tilde{\mathcal{F}} \neq \emptyset$ is a proper face of $P(G_1, G_2, K, C)$. In what follows, we give a more formal definition of S^0.

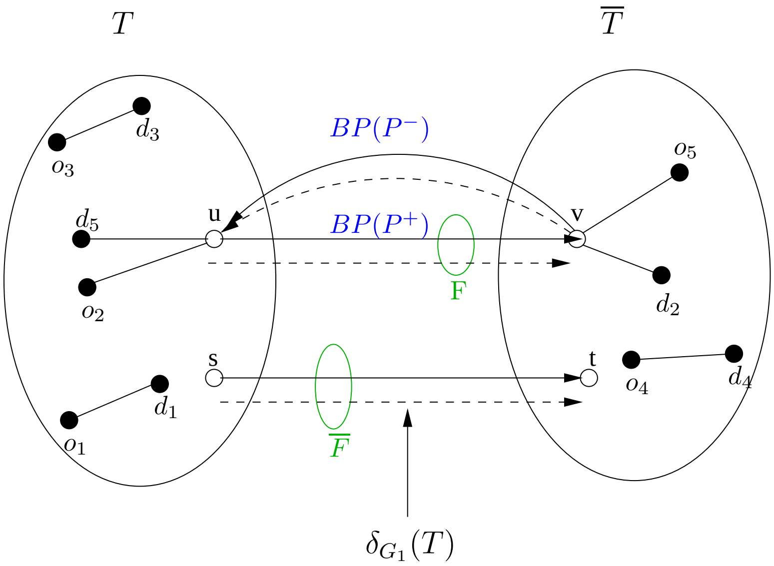


Figure 5.13: Solution S^0

$S^0 = (F_1^0, F_2^0, \Delta^0, W^0)$, where F_1^0 is the set of all arcs of A_1 used by the traffic in the solution described above. $F_2^0 = \emptyset$. Δ^0 is the set of paths assigned to the installed subbands and W^0 is the set of subbands used in S^0. Observe that, in this solution, a path is assigned to each commodity of K. Indeed, $\mathcal{C}_k^0 = \{(o_k, d_k)\}$ if $k \in K(T) \cup K(\overline{T})$, $\mathcal{C}_k^0 = \{(o_k, u), (u, v), (v, d_k)\}$ if $k \in P^+$, and $\mathcal{C}_k^0 = \{(o_k, v), (v, u), (u, d_k)\}$ if $k \in P^-$. Moreover, for each pair (e, w), such that w is installed on $e = (i, j)$, $\Delta_{ew}^0 = \{(i', j')\}$, with $(i', j') \in A_2$.

Let $\lambda x + \mu y + \nu z \geq \xi$ be a constraint that defines a facet of $P(G_1, G_2, K, C)$ and such that

$$\tilde{\mathcal{F}} \subseteq \mathcal{F} = \{(x, y, z) \in P(G_1, G_2, K, C) : \lambda x + \mu y + \nu z = \xi\}.$$

We will show that there exists a scalar $\rho \in \mathbb{R}$ such that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$. First, let us prove that $\nu_a^{ew} = 0$, for all $e \in A_1$, $w \in W$ and $a \in A_2$.

Consider an arc $a \in A_2 \setminus \Delta^0$ and a pair $(e^*, w^*) \in A_1 \times W$. It is not hard to see that the solution $S^1 = (F_1^0, F_2^0, \Delta^1, W^0)$, where $\Delta_{e^*w^*}^1 = \Delta_{e^*w^*}^0 \cup \{a\}$ and $\Delta_{ew}^1 = \Delta_{ew}^0$ for $(e, w) \neq (e^*, w^*)$ is feasible for $P(G_1, G_2, K, C)$. Moreover, $(x^{S^1}, y^{S^1}, z^{S^1}) \in \tilde{\mathcal{F}} \subseteq \mathcal{F}$.

Thus, we have the following

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} + \nu_a^{e^*w^*},$$

which implies that $\nu_a^{e^*w^*} = 0$. Since, a, e^* and w^* are arbitrary in $A_2 \setminus \Delta^0$, A_1 and W, we obtain

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in A_2 \setminus \Delta^0, \tag{5.127}$$

Now let $a = (i', j')$ be an arc of $\Delta^0_{e^*w^*}$, where (e^*, w^*) is some pair of Γ^0. Consider then the solution S^2 obtained from S^0 by replacing a by arcs (i', r'), (r', j') in $\Delta^0_{e^*w^*}$. The arcs (i', r') and (r', j') are in $A_2 \setminus \Delta^0$, with $r' \in V_2 \setminus \{i', j'\}$. $S^2 = (F_1^0, F_2^0, \Delta^2, W^0)$, where $\Delta^2_{e^*w^*} = (\Delta^0_{e^*w^*} \setminus \{a\}) \cup \{(i', r'), (r', j')\}$ and $\Delta^2_{ew} = \Delta^0_{ew}$ for $(e, w) \neq (e^*, w^*)$. S^2 is obviously feasible, and its incidence vector belongs to $\tilde{\mathcal{F}}$ and then to $\mathcal{F}$. Thus, we have

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^2} + \mu y^{S^2} + \nu z^{S^2} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} - \nu_a^{e^*w^*} + \nu_{(i',r')}^{e^*w^*} + \nu_{(r',j')}^{e^*w^*},$$

which leads to $-\nu_a^{e^*w^*} + \nu_{(i',r')}^{e^*w^*} + \nu_{(r',j')}^{e^*w^*} = 0$. Since by (5.127), $\nu_{(i',r')}^{e^*w^*} = \nu_{(r',j')}^{e^*w^*} = 0$, we get $\nu_a^{e^*w^*} = 0$. As a is selected arbitrarily in Δ^0, we obtain

$$\nu_a^{ew} = 0, \text{ for all } (e, w) \in \Gamma^0, a \in \Delta^0, \tag{5.128}$$

Hence, by (5.127) and (5.128), we have

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in A_2, \tag{5.129}$$

Next, we will show that $\mu^{ew} = 0$, for all $(e, w) \in (A_1 \setminus F) \times W$.

Consider first an arc $e^* = (i, j)$ that does not belong to the solution S^0. That is to say $e^* \in A_1 \setminus (F \cup F_1^0 \cup F_2^0)$. Then one may install any subband, say w^*, over e^* and form a new solution $S^3 = (F_1^3, F_2^3, \Delta^3, W^3)$, where $F_1^3 = F_1^0$, $F_2^3 = F_2^0 \cup \{e^*\}$, $\Delta^3 = \Delta^0 \cup \{(i', j')\}$ and $W^3 = W^0 \cup \{w^3\}$. In particular, $\Gamma^3 = \Gamma^0 \cup \{(e^*, w^*)\}$, $\Delta^3_{e^*w^*} = \{(i', j')\}$ while $\Delta^3_{ew} = \Delta^0_{ew}$ for $(e, w) \neq (e^*, w^*)$. It is clear that S^3 is a feasible solution whose incidence vector is in $\mathcal{F}$ and hence, in$\tilde{\mathcal{F}}$. Thus,

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^3} + \mu y^{S^3} + \nu z^{S^3} = \lambda x^{S^0} + \mu y^{S^0} + \mu^{e^*w^*} + \nu z^{S^0} + \nu_a^{e^*w^*},$$

which yields $\mu^{e^*w^*} + \nu_a^{e^*w^*} = 0$. And by (5.129), we get $\mu^{e^*w^*} = 0$. As e^* and w^* are chosen arbitrarily in $A_1 \setminus (F \cup F_1^0 \cup F_2^0)$ and W respectively, we obtain

$$\mu^{ew} = 0, \text{ for all } e \in A_1 \setminus (F \cup F_1^0 \cup F_2^0), w \in W, \tag{5.130}$$

Now suppose that $e^* \in F_1^0$ (recall that F_2^0 is empty). Suppose without loss of generality, that $e^* \notin F$, and let w^* be a subband such that $(e^*, w^*) \in \Gamma^0$. Suppose that the pair (e^*, w^*) is involved in the routing of a commodity k^*. Then, consider two arcs $f = (i, r)$ and $g = (r, j)$ of $A_1 \setminus (F \cup F_1^0 \cup F_2^0)$, where $r \in V_1 \setminus \{i, j\}$. Let us introduce the solution S^4 which is obtained from S^0 by adding f and g to F_1^0. Both f and g receive the subband w^* and we associate them to the routing of k^* (instead of e^*). $S^4 = (F_1^0 \cup \{f, g\}, F_2^0, \Delta^0 \cup \{(i', r'), (r', j'), W^0\})$. Note that here we have $\Gamma^4 = \Gamma^0 \cup \{(f, w^*), (g, w^*)\}$, $\Delta^4_{fw^*} = \{(i', r')\}$ and $\Delta^4_{gw^*} = \{(r', j')\}$. Also note that $\mathcal{C}^4_{k^*} = (\mathcal{C}^0_{k^*} \setminus \{e^*\}) \cup \{f, g\}$. In addition, we will consider the solution S^5, obtained by removing the pair (e^*, w^*) from Γ^4. Remark that this is not equivalent to removing e^* from the solution S^4, as e^* may be supporting further subbands. Obviously, both solutions S^4 and S^5 are feasible, and their incidence vectors belong to $\mathcal{F}$ and hence, to $\tilde{\mathcal{F}}$. Moreover, every component of $(x^{S^4}, y^{S^4}, z^{S^4})$ equals the corresponding component in $(x^{S^4}, y^{S^4}, z^{S^4})$ except for $y^{S^4}_{e^*w^*}$ whose value is 1 while $y^{S^5}_{e^*w^*}$ is set to 0. Hence, the corresponding coefficient $\mu^{e^*w^*}$ equals to 0. As e^* is arbitrary in $(F_1^0 \cup F_2^0)$, $e^* \notin F$, we obtain

$$\mu^{ew} = 0, \text{ for all } e \in F_1^0 \cup F_2^0, e \notin F, w \in W, \tag{5.131}$$

In consequence, we have by (5.130) and (5.131) that

$$\mu^{ew} = 0, \text{ for all } (e, w) \in (A_1 \setminus F) \times W, \tag{5.132}$$

The case where $e^* \in F$ will be treated further in the proof.

In what follows, we will examine the λ coefficients related to commodities not in P^+.

Consider a commodity k^* of $K \backslash P^+$. We will show that $\lambda^{k^*}_{ew} = 0$ for all $(e, w) \in A_1 \times W$. Let $e^* = (i, j)$ and w^* be an arc of A_1 and a subband of W, respectively, such that w^* is not already installed on e^*. First, assume that $e^* \notin F$. Consider the solution S^6 obtained from S^0 as follows. We install w^* on e^* and we associate e^* to the commodity k^* in addition to its initial routing. In other words, the component $x^{S^6}_{k^*e^*w^*} = 1$ while $x^{S^0}_{k^*e^*w^*} = 0$. Furthermore, we assign to (e^*, w^*) a path $\{(i', r'), (r', j')\}$, where r' is some node of $V_2 \setminus \{i', j'\}$, that is not used in S^0. Clearly, the solution S^6 is feasible for the problem and $(x^{S^6}, y^{S^6}, z^{S^6}) \in \tilde{\mathcal{F}} \subseteq \mathcal{F}$. Hence, we have

$$\lambda x^{S^6} + \mu y^{S^6} + \nu z^{S^6} = \lambda x^{S^0} + \lambda^{k^*}_{e^*w^*} + \mu y^{S^0} + \mu^{e^*w^*} + \nu z^{S^0} + \nu^{e^*w^*}_{(i',r')} + \nu^{e^*w^*}_{(r',j')},$$

which implies, by (5.129) and (5.132) that $\lambda^{k^*}_{e^*w^*} = 0$. As k^*, e^*, and w^* are arbitrary elements in $K \setminus P^+$, $A_1 \setminus F$ and W, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in K \setminus P^+, e \in A_1 \setminus F, w \in W.$$

Suppose now that e^* is an arc of F. Note that if $|F| = 1$, then $e^* = (u, v)$. Let w^* be some subband installed on e^*, such that w^* still has enough residual capacity to carry k^*. Because of condition (vii), we know that such subband exists. Consider the solution S^7, obtained from S^0 by associating e^* to the commodity k^*. Here the routing of k^* does not change, we only set the variable $x^{S^7}_{k^*e^*w^*}$ to 1 while $x^{S^0}_{k^*e^*w^*} = 0$. More formally, $S^7 = (F^7_1, F^7_2, \Delta^7, W^7)$ where $\mathcal{C}^7_{k^*} = \mathcal{C}^0_{k^*} \cup \{e^*\}$ and the other elements of S^0 do not change. Clearly S^7 is feasible and both $(x^{S^0}, y^{S^0}, z^{S^0})$ and $(x^{S^7}, y^{S^7}, z^{S^7})$ belong to $\tilde{\mathcal{F}}$ and hence to $\mathcal{F}$. Thus, we have that $\lambda^{k^*}_{e^*w^*} = 0$. As k^*, e^* and w^* are arbitrary in $K \setminus P^+$, F and W, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in K \setminus P^+, e \in F, w \in W.$$

Now, let us look at the λ coefficients related to the commodities of P^+.

Let k^* be a commodity of P^+ and $e^* = (i,j)$ an arc of A_1 such that $e^* \notin (F \cup \overline{F})$. Consider a subband $w^* \in W$ such that $(e^*, w^*) \notin \Gamma^0$, and (i', r'), (r', j') two arcs of $A_2 \setminus \Delta^0$, where $r' \in V_2 \setminus \{i', j'\}$. Let S^8 be a solution obtained from S^0 by adding (e^*, w^*) to Γ^0 and e^* to $\mathcal{C}^0_{k^*}$. In other words, w^* is installed on e^*, and k^* is assigned arc e^* in addition to its initial routing path contained in $\mathcal{C}^0_{k^+}$. In addition, (e^*, w^*) is associated with the path $\{(i', r'), (r', j')\}$, that is to say $\Delta^8_{e^*w^*} = \Delta^0_{e^*w^*} \cup \{(i', r'), (r', j')\}$. Clearly, S^8 forms a feasible solution for OMBND problem, and its incidence vector as well as one of S^0 verify

$$\lambda x^{S^8} + \mu y^{S^8} + \nu z^{S^8} = \lambda x^{S^0} + \lambda^{k^*}_{e^*w^*} + \mu y^{S^0} + \mu^{e^*w^*} + \nu z^{S^0} + \nu^{e^*w^*}_{(i',r')} + \nu^{e^*w^*}_{(r',j')},$$

By (5.129) and (5.132), this yields $\lambda^{k^*}_{e^*w^*} = 0$. As k^* and e^* are arbitrary in P^+ and $A_1 \setminus (F \cup \overline{F})$, respectively, we obtain

$$\lambda^k_{ew} = 0, \text{ for all } k \in P^+, e \in A_1 \setminus (F \cup \overline{F}), w \in W,$$

In what follows, we will turn ourselves to arcs of F, and show that $\lambda^k_{ew} = 0$, for $k \in P^+$, $e \in F$ and $w \in W$.

First, if $|F| = 1$, that is to say $F = \{(u,v)\}$. Let k^* be a commodity of P^+ and let w^* be some subband of W that is installed on (u,v) in the solution S^0. Consider the solution S^{11} that is obtained from S^0 as follows. The subband w^* is involved in the routing of k^* while the remaining $BP(P^+)$ - 1 subbands are re-assigned for the routing of the left $P^+ \setminus \{k^*\}$ commodities using (u,v). Condition (ii) ensures that this induces a feasible solution (see Figure 5.14). Now let k' be a commodity of $P^+ \setminus \{k^*\}$ and such that $D^{k^*} + D^{k'} \leq C$. This is possible since condition (iii) guarantees that such a commodity exists. Consider the solution S^{12}, which slightly differs from S^{11} in what follows. We associate with k' the subband w^* in addition to its initial routing.

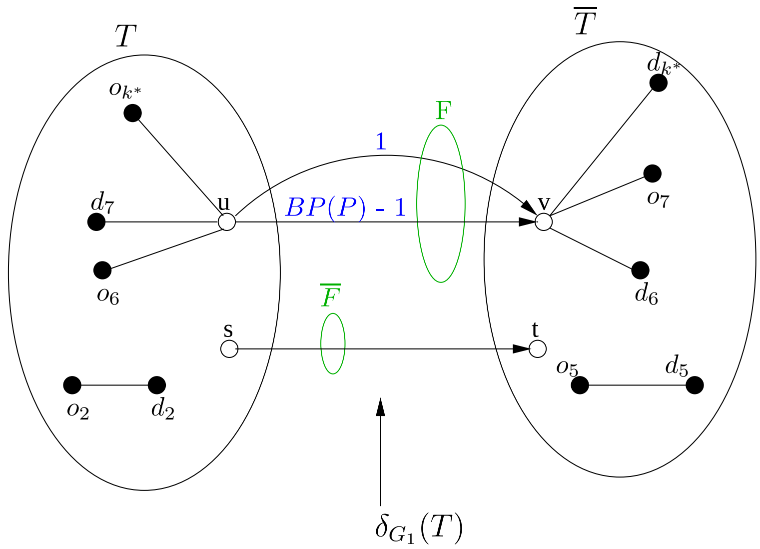


Figure 5.14: Getting the solution S^{11}

In other words, $x^{S^{12}}_{k'(u,v)w^*}$ is set to 1, while $x^{S^{11}}_{k'(u,v)w^*} = 0$. Clearly, S^{12} is feasible for the problem, and both incidence vectors of S^{11} and S^{12} belong to $\tilde{\mathcal{F}}$, and then to $\mathcal{F}$. Hence, we obtain that $\lambda^{k'}_{(u,v)w^*} = 0$. As k^* is arbitrary in P^+, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in P^+, e \in F, w \in W,$$

Furthermore, we will show that λ related to commodities of P^+ on arcs of $\overline{F}$ and μ coefficients for F are equal. Let k^* be some commodity of P^+ and let w^* the subband used for its routing along the arc (u,v). Consider the solution S^{13} obtained by S^0 as follows. We move the subband w^* from (u,v) to (s,t) and we install two subbands w' and w'' on the arcs (o_{k^*}, s) and (t, d_{k^*}). We then replace the routing of k^* by $\{(o_{k^*}, s), (s,t), (t, d_{k^*})\}$ (the initial routing is $\{(o_{k^*}, u), (u,v), (v, d_{k^*})\}$). This solution is feasible as condition (ii) ensures that enough capacity is available on (u,v) to carry the commodities of $P^+ \setminus \{k^*\}$. The solution S^{13} is obviously feasible for the problem, and comparing $(x^{S^{13}}, y^{S^{13}}, z^{S^{13}})$ and $(x^{S^0}, y^{S^0}, z^{S^0})$ gives

$$\lambda^{k^*}_{(u,v)w^*} + \mu^{(u,v)w^*} = \lambda^{k^*}_{(s,t)w^*} + \mu^{(s,t)w^*},$$

Together with (5.132), () implies that $\mu^{(u,v)w^*} = \lambda^{k^*}_{(s,t)w^*}$. As k^*, w^* and (s,t) are arbitrary in P^+, W and $\overline{F}$, we obtain that those coefficients are equal to a positive scalar ρ

$$\mu^{(u,v)w} = \lambda^k_{(s,t)w} = \rho, \text{ for all } k \in P^+, (s,t) \in \overline{F}, w \in W,$$

Suppose now that $|F| \geq 2$. Let $e^* = (i,j)$ be an arc of F different from (u,v). Consider two commodities k', k'' of P^+, such that $D^{k'} + D^{k''} \leq C$. Condition (iii)

guarantees that such commodities exists. Recall that in S^0, all the commodities of P^+ are routed along (u, v). Let w^* be the subband installed on (u, v) and involved in the routing of k'. We will consider a new solution S^{14} obtained from S^0 as follows : we install w^* on (i, j) and replace the routing path $\{(o_{k'}, u), (u, v), (v, d_{k'})\}$ of k' by $\{(o_{k'}, i), (i, j), (j, d_{k'})\}$ (see Figure 5.15). By condition (ii), the remaining traffic can be routed along (u, v) using the left $BP(P) - 1$ subbands. It is clear that S^{14} is a feasible solution for the problem. Now consider the solution S^{15}, obtained from S^{14} as follows : associate with the commodity k'' one more arc, namely (i, j). Note that k'' is still routed through (u, v). Arc (i, j) is just added to the solution. As $D^{k'} + D^{k''} \leq C$, the capacity constraint (5.2) related to $((i, j), w^*)$ is satisfied. Hence, S^{15} is feasible. Moreover, as the incidence vectors of S^{14} and S^{15} both belong to $\tilde{\mathcal{F}}$ and hence to $\mathcal{F}$, we have that $\lambda^{k''}_{(i,j)w^*} = 0$.

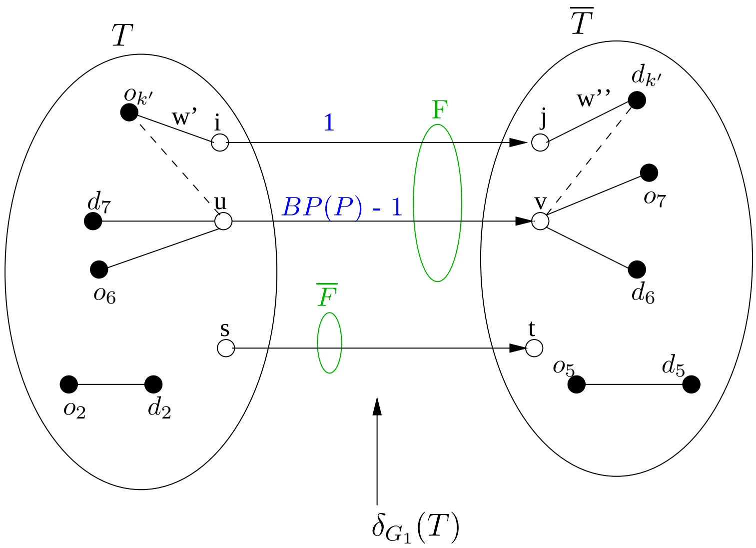


Figure 5.15: Obtaining the solution S^{14}

Now let us show that all the coefficients λ related to the commodities of P^+ and the arcs of F are equal.

Let k^* be a commodity different from k' (commodity whose routing is changed in S^{14}). Consider an arc $(i, j) \in F$ different from (u, v) and w^* a subband installed on (u, v) and involved in the routing of k^*. We will construct a solution S^{16} similar to S^{14}, that is obtained from S^0 as follows. We shift w^* from (u, v) to (i, j) and we replace the routing path of k^* by $\{(o_{k^*}, i), (i, j), (j, d_{k^*})\}$. The remaining operations are all similar to solution S^{14}. Obviously, S^{16} is feasible for the problem, and incidence vectors of S^{14} and S^{16} both belong to $\tilde{\mathcal{F}}$ and then to $\mathcal{F}$. Thus, comparing the components of

$(x^{S^{14}}, y^{S^{14}}, z^{S^{14}})$ and $(x^{S^{16}}, y^{S^{16}}, z^{S^{16}})$ yields $\lambda^{k'}_{(i,j)w^*} = \lambda^{k^*}_{(u,v)w^*}$. As commodities k' and k^* are arbitrary in P^+, we obtain that all the coefficients λ^k_{ew}, $k \in P^+$, $e \in F$, $w \in W$ are equal and, in consequence

$$\lambda^k_{ew} = 0, \text{ for all } k \in P^+, e \in F, w \in W,$$

We will go over the coefficients related to the demands in P^+ and the arcs of $\overline{F}$ at the end of the proof. Let us first get back to the coefficients μ for the arcs of F.

Simply compare solutions S^{14} and S^0, together with (5.129), (5.132) and (), allows to conclude that

$$\mu^{(i,j)w^*} = \mu^{(u,v)w^*},$$

Since the arc (i, j) is arbitrary in F, we get the equality of coefficients μ for the arcs of F. Hence, we conclude that there exists a positive scalar $\rho \in \mathbb{R}$, such that

$$\mu^{ew} = \rho, \text{ for all } e \in F, w \in W,$$

The last case of our proof concerns the coefficients of commodities of P^+ related to arcs of $\overline{F}$.

Consider the commodity $k^* \in P^+$, and let w^* be a subband installed on (u, v), such that the pair $((u, v), w^*)$ is involved the routing of k^*. Assume that w^* is moved from (u, v) to the arc (s, t) (see Figure 5.16). This allows to introduce the later arc in the solution S^0. Let us install subbands w' and w'' on arcs (o_{k^*}, s) and (t, d_{k^*}), respectively. In this way, k^* is assigned the path $\{(o_{k^*}, s), (s, t), (t, d_{k^*})\}$ instead of the initial routing path $\{(o_{k^*}, u), (u, v), (v, d_{k^*})\}$. And the sections of this path are themselves assigned the paths $\{(o'_{k^*}, s')\}$, $\{(s', t')\}$ and $\{(t', d'_{k^*})\}$ in G_2, respectively.

Let us denote by S^{17} the solution described above, and give in what follows its different subsets. $S^{17} = (F_1^0 \cup \{(o_{k^*}, s), (s, t), (t, d_{k^*})\}, F_2^0, \Delta^0 \cup \{(o'_{k^*}, s'), (s', t'), (t', d'_{k^*})\}, W^0 \cup \{w', w''\})$. S^{17} is obviously feasible, and $(x^{S^{17}}, y^{S^{17}}, z^{S^{17}})$ together with $(x^{S^0}, y^{S^0}, z^{S^0})$ belong to $\tilde{\mathcal{F}}$ and then to $\mathcal{F}$. In addition, S^{17} is such that

$$\sum_{e \in F} \sum_{w \in W} y^{S^{17}}_{ew} = \sum_{w \in W} y^{S^{17}}_{(u,v)w} = BP(P^+) - 1,$$
$$\sum_{e \in \overline{F}} \sum_{w \in W} y^{S^{17}}_{ew} = y^{S^{17}}_{(s,t)w^*} = 1,$$
$$\sum_{k \in P^+} \sum_{e \in \overline{F}} \sum_{w \in W} x^{S^{17}}_{kew} = x^{S^{17}}_{k^*(s,t)w^*} = 1.$$

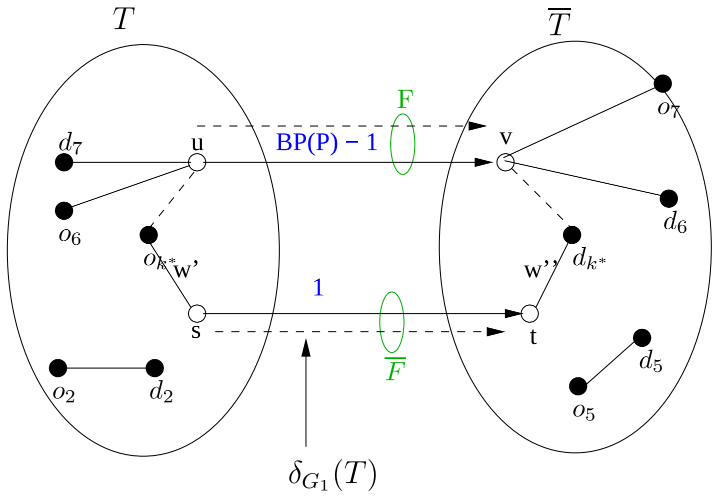


Figure 5.16: Obtaining the solution S^{17}

Comparing both incidence vectors $(x^{S^{17}}, y^{S^{17}}, z^{S^{17}})$ and $(x^{S^0}, y^{S^0}, z^{S^0})$ induces the following

$$\lambda x^{S^{17}} + \mu y^{S^{17}} + \nu z^{S^{17}} = \lambda x^{S^0} - \lambda^{k^*}_{(u,v)w^*} + \lambda^{k^*}_{(o_{k^*},s)w'} + \lambda^{k^*}_{(s,t)w^*} + \lambda^{k^*}_{(t,d_{k^*})w''}$$
$$+\mu y^{S^0} - \mu^{(u,v)w^*} + \mu^{(o_{k^*},s)w'} + \mu^{(s,t)w^*} + \mu^{(t,d_{k^*})w''} + \nu z^{S^0} + \nu^{(o_{k^*},s)w'}_{(o'_{k^*},s')} + \nu^{(s,t)w^*}_{(s',t')} + \nu^{(t,d_{k^*})w''}_{(t',d'_{k^*})},$$

By (5.129), (5.132), () and (), it remains from the previous equality that $\lambda^{k^*}_{(s,t)w^*} = \mu^{(u,v)w^*}$. As k^* is arbitrary in P^+, we conclude that all the coefficients λ of P^+ and $\overline{F}$ are equal up to the scalar ρ.

$$\lambda^k_{ew} = \rho, \text{ for } k \in P^+, e \in \overline{F}, w \in W,$$

To summarize, all together, we finally get

$$\nu^{ew}_a = 0, \text{ for all } e \in A_1, w \in W, a \in A_2,$$
$$\mu^{ew} = \begin{cases} \rho, & \text{for some scalar } \rho \in \mathbb{R}^*_+, \text{ for all } e \in F, w \in W, \\ 0, & \text{otherwise.} \end{cases}$$
$$\lambda^k_{ew} = \begin{cases} \rho, & \text{for } k \in P^+, e \in \overline{F}, w \in W, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\rho \neq 0$, since $\mathcal{F} \neq \emptyset$. Thus, replacing the values of our coefficients in $\lambda x + \mu y + \nu z \geq \xi$, yields

$$\sum_{e \in F} \sum_{w \in W} \rho y^{ew} + \sum_{k \in P^+} \sum_{e \in \overline{F}} \sum_{w \in W} \rho x^k_{ew} \geq \xi$$

And, as $(x^{S^0}, y^{S^0}, z^{S^0}) \in \mathcal{F}$, it follows that $\rho BP(P) = \xi$ and hence $\frac{\xi}{\rho} = BP(P^+)$, which completes the proof. $\square$

5.4.3 Clique-based Inequalities

In what follows, we will study an additional class of inequalities that are valid for $P(G_1, G_2, K, C)$. These inequalities are based on the so-called *clique inequalities* introduced by Manfred Padberg in the context of *stable set polytope* investigation [89]. They have also been studied in [14] for the *Balanced Induced Subgraph problem*, where authors provide necessary conditions for these inequalities to define facets. More generally, clique inequalities arise in problems where *conflicts* may occur between objects (see [60, 25]). In order to identify these facet-defining inequalities, we will introduce the definition of a *conflict graph* for an instance of $OMBND$ problem.

Definition 6 *A conflict graph H is composed by a set of nodes N and a set of edges E. Each node $n \in N$ is a commodity of K and two commodities u, v are connected by an edge $(u, v) \in E$ if and only if u and v cannot be packed in a subbband together. In other words, there exists an edge (u, v) in E is and only if $D^u + D^v > C$.*

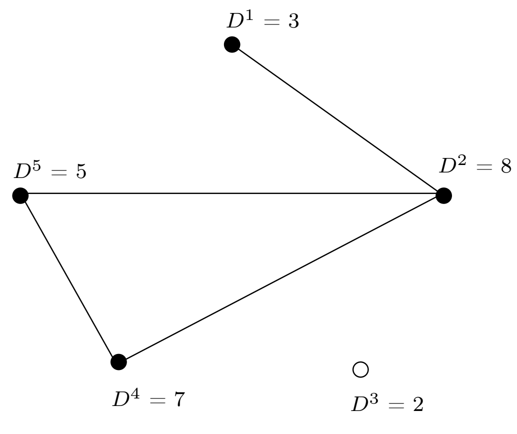


Figure 5.17: The conflict graph associated with 5 commodities

A *clique* $\mathcal{C} \subseteq N$ in the conflict graph H is a set a set of nodes such that an edge is associated with each pair (u, v), $u, v \in \mathcal{C}$. In other words, $\mathcal{C}$ is a set of nodes that induces a complete subgraph of H. Consequently, two nodes u and v of a clique $\mathcal{C}$ cannot be included together in a subband, that is to say u and v cannot be associated with the same pair (e, w), $e \in A_1$ and $w \in W$. A clique $\mathcal{C}$ is said to be *maximal* if it cannot be extended by including one more node that is connected to the other nodes.

Two commodities k' and k'' are said to be *compatible* if the corresponding nodes in H are not adjacent.

The Figure 5.17 represents the conflict graph associated with an instance of the OMBND problem with five commodities. In other words, $|K| = |N| = 5$. The available subbands have a capacity $C = 10$. In this example, two cliques are represented $\mathcal{C}_1 = \{2, 4, 5\}$ and $\mathcal{C}_2 = \{1, 2\}$. The maximal clique in H is $\mathcal{C}_1$.

Figure 5.18 shows a partial description of a fractional solution denoted $(\overline{x}, \overline{y}, \overline{z})$ obtained by solving the linear relaxation of OMBND for the following instance. Consider a graph of six nodes, denoted 1 to 6 (see Figure 5.18), and a set of three installable subbands, namely w_1, w_2 and w_3. The capacity of each subband is $C = 10$. The instance includes six commodities, denoted k_1, to k_6 with the traffic amounts $D^{k_1} = D^{k_2} = D^{k_3} = D^{k_4} = 6$, and $D^{k_5} = D^{k_6} = 4$. The values of design variables $\overline{y}$ are such that $\overline{y}_{e_1w_1} = \overline{y}_{e_6w_3} = 0.6$, $\overline{y}_{e_2w_1} = \overline{y}_{e_3w_1} = 1$, $\overline{y}_{e_4w_1} = 0.4$, $\overline{y}_{e_5w_1} = 0.2$ and $\overline{y}_{e_7w_2} = 0.33$. We can remark that e_1, e_2, e_3, e_4 and e_5 receive the subband w_1, while e_6 receives the subband w_3 and two subbands, namely, w_1 and w_2 are installed on e_7.

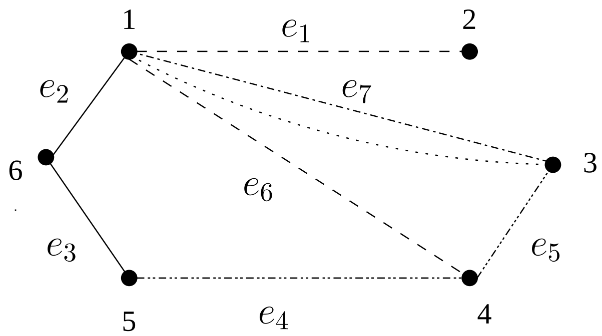


Figure 5.18: Second fractional solution

Let us focus on pair (e_7, w_1) whose corresponding entry in $\overline{y}$ is 0.66. Consider the conflict graph related to the commodities of this instances (see Figure 5.19), that will be called H. Figure 5.19 shows a graph of six nodes, denoted k_1 to k_6, each one corresponding to a commodity of the instance described above. We can see that there exists an edge between each pair of nodes such that the associated commodities are not compatible. A weight $w(k_i)$ is associated with each node k_i, $i = 1, \ldots, 6$, which is given by the value of $\overline{x}_{k_ie_7w_1}$. In other words, nodes whose weight is different from zero induce commodities that uses e_7 and particularly w_1 for their routing.

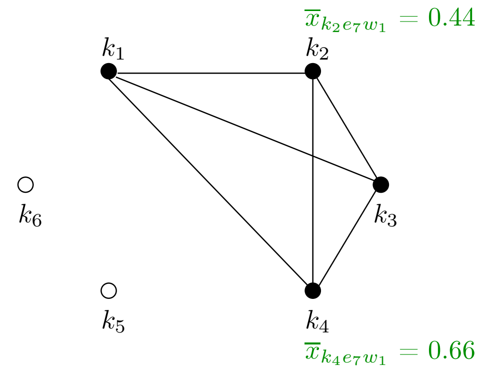


Figure 5.19: The associated conflict graph H

Although fractional solution $(\overline{x}, \overline{y}, \overline{z})$ satisfies all constraints of linear relaxation of (5.8), it violates the following inequality

$$x_{k_2e_7w_1} + x_{k_4e_7w_1} \leq y_{e_7w_1}, \tag{5.133}$$

which is valid for $P(G_1, G_2, K, C)$ polytope.

Observe that commodities k_1, k_2, k_3 and k_4 form a clique in the conflict graph H, as no two commodities among them can fit in a subband. Hence, (5.133) can be strengthened to give the following inequality

$$x_{k_1e_7w_1} + x_{k_2e_7w_1} + x_{k_3e_7w_1} + x_{k_4e_7w_1} \leq y_{e_7w_1}, \tag{5.134}$$

which is also valid for $P(G_1, G_2, K, C)$. In what follows, we prove that these inequalities belong to a more general class of valid inequalities for $P(G_1, G_2, K, C)$ polytope, that we refer to as *clique-based inequalities.*

Proposition 5.19 *Let $\mathcal{C} \subseteq K$ be a clique in the conflict graph, and $(\tilde{e}, \tilde{w}) \in A_1 \times W$, then the following clique-based inequality*

$$\sum_{k \in \mathcal{C}} x_{k\tilde{e}\tilde{w}} - y_{\tilde{e}\tilde{w}} \leq 0, \tag{5.135}$$

is valid for $P(G_1, G_2, K, C)$.

Proof. The proof is quite trivial. Indeed, two commodities belong to the clique $\mathcal{C}$ if they can not be packed together in one subband on a given arc. So they can not be associated with the same pair $(\tilde{e}, \tilde{w})$. In other words, each edge (u, v) of the clique $\mathcal{C}$ represents an infeasible packing of the commodities u and v in the subband $\tilde{w}$. $\square$

Theorem 5.20 *Let $\tilde{\mathcal{C}} \subseteq N$ be a clique in the conflict graph H. Let $\tilde{e} = (u, v)$ and $\tilde{w}$ be an arc of A_1 and a subband of W, respectively. The clique-based inequality* (5.135) *induced by $\tilde{\mathcal{C}}$ and $(\tilde{e}, \tilde{w})$ define a facet of $P(G_1, G_2, K, C)$, if and only if the $\tilde{\mathcal{C}}$ is maximal.*

Proof. We will denote by $\alpha x + \beta y + \gamma z \leq \delta$ the inequality (5.135) produced by $\tilde{\mathcal{C}}$ and $(\tilde{e}, \tilde{w})$, and let $\tilde{\mathcal{F}}$ be the face induced by this inequality. We will first show that $\tilde{\mathcal{F}}$ is a proper face of $P(G_1, G_2, K, C)$. To this end, we will construct a feasible solution S^0 whose incidence vector belongs to $\tilde{\mathcal{F}}$.

Consider the solution S^0 defined in the proof of Theorem 5.3. Suppose without loss of generality that $\tilde{e}$ and $\tilde{w}$ are not used in the solution S^0 ($(\tilde{e}, \tilde{w}) \notin \Gamma^0$). Let us introduce the solution S^1, obtained from S^0 by adding the pair $(\tilde{e}, \tilde{w})$ to Γ^0. We assign to $(\tilde{e}, \tilde{w})$ the path $\{(u', v')\}$ in G_2. The pair $(\tilde{e}, \tilde{w})$ is then associated with some commodity of the clique $\tilde{\mathcal{C}}$, say $\tilde{k}$. Every remaining commodity of $\tilde{\mathcal{C}}$ is associated with the path $\{(o_k, d_k)\}$, and uses the subband w_k, as described in the construction of S^0. More formally, this solution is equivalent to $S^1 = (F_1^0 \cup \{\tilde{e}\}, F_2^0, \Delta^0 \cup \{(u', v')\}, W^0 \cup \{\tilde{w}\})$. It is easy to see that S^1 is a feasible solution for the problem. In addition, $y^{S^1}_{\tilde{e}\tilde{w}} = 1$, while $\sum_{k \in \tilde{\mathcal{C}}} x^{S^1}_{k\tilde{e}\tilde{w}} = x^{S^1}_{\tilde{k}\tilde{e}\tilde{w}} = 1$. Hence, $(x^{S^1}, y^{S^1}, z^{S^1})$ belongs to $\tilde{\mathcal{F}}$, and $\tilde{\mathcal{F}} \neq \emptyset$ is a proper face of $P(G_1, G_2, K, C)$.

Consider a facet-defining inequality denoted by $\lambda x + \mu y + \nu z \leq 0$ and let $\mathcal{F}$ be the face induced by this inequality, and such that

$$\tilde{\mathcal{F}} \subseteq \mathcal{F} = \{(x, y, z) \in P(G_1, G_2, K, C) : \lambda x + \mu y + \nu z = \xi\},$$

We will show that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$. Let us first show that $\nu_a^{ew} = 0$, for all $e \in A_1$, $w \in W$ and $a \in A_2$.

Let a^* be an arc of $A_2 \setminus \Delta^0$. Consider the solution S^2, obtained from S^1 by adding the arc a^* to some pair (e^*, w^*) of $A_1 \times W$. The solution $S^2 = (F_1^1, F_2^1, \Delta^2, W^1)$, with $\Delta^2_{e^*w^*} = \Delta^1_{e^*w^*} \cup \{a^*\}$ and $\Delta^2_{ew} = \Delta^1_{ew}$ for $(e, w) \in (A_1 \times W) \setminus \{(e^*, w^*)\}$, is clearly feasible and its incidence vector belongs to $\tilde{\mathcal{F}}$ and then, to $\mathcal{F}$. Hence, we have that $\nu_{a^*}^{e^*w^*} = 0$. As a^*, e^* and w^* are arbitrary in $A_2 \setminus \Delta^1$, A_1 and W, it follows that

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in A_2 \setminus \Delta^1, \tag{5.136}$$

Now assume that $a^* = (s', t')$ is used in the solution S^1, that is to say $a^* \in \Delta^1_{e^*w^*}$ for some pair $(e^*, w^*) \in A_1 \times W$. Then, the solution $S^3 = (F_1^1, F_2^1, (\Delta^1 \setminus \{a^*\}) \cup \{(s', r'), (r', t')\}, W^1)$, with $(s', r'), (r', t') \in A_2 \setminus \Delta^1$ and $r' \in V_2 \setminus \{s', t'\}$, is also feasible.

Moreover, $(x^{S^3}, y^{S^3}, z^{S^3}) \in \tilde{\mathcal{F}}$ and hence, $(x^{S^3}, y^{S^3}, z^{S^3}) \in \mathcal{F}$. Thus, comparing solutions S^3 and S^1 give

$$\nu_{a^*}^{e^*w^*} = \nu_{(s',r')}^{e^*w^*} + \nu_{(r',t')}^{e^*w^*},$$

which implies, by (5.136), that $\nu_{a^*}^{e^*w^*} = 0$. Since a^* is chosen arbitrarily in Δ^1, we get

$$\nu^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in \Delta^1, \tag{5.137}$$

we then obtain by (5.136) and (5.137)

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in A_2, \tag{5.138}$$

Now we will show that coefficient μ, related to the pairs (e, w) of $A_1 \times W \setminus \{(\tilde{e}, \tilde{w})\}$ are equal to zero.

Let $e^* = (s, t)$ and w^* be an arc of A_1 and a subband of W, respectively, such that $(e^*, w^*) \in A_1 \times W \setminus \Gamma^1$. Consider the solution S^4, obtained from S^1 by adding e^* to F_2^1 and (e^*, w^*) to Γ^1. We assign to (e^*, w^*) the path $\{(s', t')\}$ where $(s', t') \in A_2 \setminus \Delta^1$. The solution S^4 is then defined as follows. $S^4 = (F_1^1, F_2^1 \cup \{e^*\}, \Delta^1 \cup \{(s', t')\}, W^1 \cup \{w^*\})$ where $\Gamma^4 = \Gamma^1 \cup \{(e^*, w^*)\}$, $\Delta^4_{e^*w^*} = \Delta^1_{e^*w^*} \cup \{(s', t')\}$ and $\Delta^4_{ew} = \Delta^1_{ew}$, for $(e, w) \neq (e^*, w^*)$. The solution S^4 is obviously feasible, and its incidence vector $(x^{S^4}, y^{S^4}, z^{S^4})$ belongs to $\tilde{\mathcal{F}}$ and, consequently, to $\mathcal{F}$. Thus, we have $\mu^{e^*w^*} + \nu_{(s',t')}^{e^*w^*} = 0$, which implies that $\mu^{e^*w^*} = 0$, by (5.138). Since e^* and w^* are arbitrary in $A_1 \setminus (F_1^1 \cup F_2^1)$ and $W \setminus W^1$, we get

$$\mu^{ew} = 0, \text{ for all } (e, w) \in (A_1 \times W) \setminus \Gamma^1, \tag{5.139}$$

Assume now that $(e^*, w^*) \in \Gamma^1 \setminus \{(\tilde{e}, \tilde{w})\}$, and let k^* be a commodity of K such that $e^* = (s, t) \in \mathcal{C}_{k^*}^1$. Recall that $\tilde{k}$ is the only commodity of the clique $\tilde{\mathcal{C}}$ that uses $(\tilde{e}, \tilde{w})$, and suppose that $k^* \neq \tilde{k}$. Let $f = (s, r)$, $g = (r, t)$ be two arcs of $A_1 \setminus (F_1^1 \cup F_2^1)$, and (s', r'), (r', t') be the corresponding arcs in A_2. Consider the solution S^5 that is obtained from S^1 by adding the pairs (f, w^*), (g, w^*) to Γ^1 and assigning to them the paths $\{(s', r')\}$, $\{(r', t')\}$, respectively. Moreover, f and g are added to $\mathcal{C}_{k^*}^1$. In other words, k^* uses the path $\{f, g\}$ and the associated subbands, instead of the original routing path $\{e^*\}$. These operations lead to a feasible solution $S^5 = (F_1^1 \cup \{f, g\}, F_2^1, \Delta^1 \cup \{(s', r'), (r', t')\}, W^1)$ whose incidence vector belongs to $\tilde{\mathcal{F}}$ and then, to $\mathcal{F}$. Now, consider a new feasible solution S^6, obtained from S^5 by removing the pair (e^*, w^*) from the subset Γ^5. We then obtain a further feasible solution S^6, different from S^5 in what $y_{e^*w^*}^{S^5} = 1$, while $y_{e^*w^*}^{S^6} = 0$. It is clear that $(x^{S^6}, y^{S^6}, z^{S^6}) \in \tilde{\mathcal{F}} \subseteq \mathcal{F}$, we then have

$$\lambda x^{S^6} + \mu y^{S^6} + \nu z^{S^6} = \lambda x^{S^5} + \mu y^{S^5} + \mu^{e^*w^*} + \nu z^{S^5},$$

and it follows directly that $\mu^{e^*w^*} = 0$, which yields

$$\mu^{ew} = 0, \text{ for all } (e, w) \in \Gamma^1 \setminus (\tilde{e}, \tilde{w}), \tag{5.140}$$

Since (e^*, w^*) is arbitrarily selected in Γ^1. 5.139 and 5.140 together give

$$\mu^{ew} = 0, \text{ for all } (e, w) \in (A_1 \times W) \setminus (\tilde{e}, \tilde{w}), \tag{5.141}$$

Next, we will show that $\lambda^k_{ew} = 0$, for all $(k, e, w) \in (K \times A_1 \times W \times) \setminus \tilde{\mathcal{C}} \times \{(\tilde{e}, \tilde{w})\}$.

Let (e^*, w^*) be any pair of $(A_1 \times W) \setminus \{(\tilde{e}, \tilde{w})\}$, and let k^* be some commodity of K. Suppose first that (e^*, w^*) is not a part from solution S^1. Then let us consider the solution S^7, obtained from S^1 by adding (e^*, w^*) to Γ^1. In other words, $y^{S^7}_{e^*w^*} = 1$ while $y^{S^1}_{e^*w^*} = 0$. In particular we also add e^* to the set $\mathcal{C}^1_{k^*}$, that is to set the element $x^{S^7}_{k^*e^*w^*}$ to 1, while $x^{S^1}_{k^*e^*w^*} = 0$. Furthermore, we add the arc $(s', t') \in A_2$ to $\Delta^1_{e^*w^*}$, which means to associate the path $\{(s', t')\}$ to (e^*, w^*) ($z^{S^7}_{e^*w^*(s',t')} = 1$ whereas $z^{S^1}_{e^*w^*(s',t')} = 0$). The solution constructed above is given by $S^7 = (F^1_1 \cup \{e^*\}, F^1_2, \Delta^1 \cup \{(s', t')\}, W^1 \cup \{w^*\})$ and is clearly feasible for the OMBND problem. In addition, its incidence vector satisfies $\lambda x + \mu y + \nu z \leq \xi$ with equality, and it belongs to $\tilde{\mathcal{F}}$, and consequently to $\mathcal{F}$. Hence, we get

$$\lambda^{k^*}_{e^*w^*} + \mu^{e^*w^*} + \nu^{e^*w^*}_{(s',t')} = 0,$$

and, by (5.138) and (5.141), we consequently obtain $\lambda^{k^*}_{e^*w^*} = 0$. As the pair (e^*, w^*) is arbitrarily chosen in $(A_1 \times W) \setminus \Gamma^1$, and so as for k^*, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, (e, w) \in (A_1 \times W) \setminus \Gamma^1, \tag{5.142}$$

Now if $(e^*, w^*) \in \Gamma^1 \setminus \{(\tilde{e}, \tilde{w})\}$ where $e^* = (s, t)$, then consider a commodity of K, say k^* such that $e^* \in \mathcal{C}^1_{k^*}$. Let $f = (s, r)$, $g = (r, t)$ be two arcs of $A_1 \setminus (F^1_1 \cup F^1_2)$ and $f' = (s', r')$, $g' = (r', t')$ be two arcs of $A_2 \setminus \Delta^1$. Consider the solution S^8, obtained from S^1 as follows. We relocate commodity k^* in the path formed by f and g instead of its original routing path, then we remove the pair (e^*, w^*) from Γ^1, as it becomes no more used. The subband w^* is then reused for both f and g. More formally, $S^8 = (F^8_1, F^8_2, \Delta^8, W^8)$, where $F^8_1 = F^1_1 \cup \{(f, g)\}$, $F^8_2 = F^1_2$, $\Delta^8 = \Delta^1 \cup \{f', g'\}$ and $W^8 = W^1$. In particular, note that $\Gamma^8 = \Gamma^1 \cup \{(f, w^*), (g, w^*)\}$, $\mathcal{C}^8_{k^*} = (\mathcal{C}^1_{k^*} \setminus \{e^*\}) \cup \{f, g\}$, $\Delta^8_{fw^*} = \Delta^1_{fw^*} \cup \{f'\}$ while $\Delta^8_{gw^*} = \Delta^1_{gw^*} \cup \{g'\}$. It is straightforward to see that S^8 induces a feasible solution, and comparing both incidence vectors of S^8 and S^1 allows to get

$$\lambda^{k^*}_{fw^*} + \lambda^{k^*}_{gw^*} + \mu^{fw^*} + \mu^{gw^*} + \nu^{fw^*}_{f'} + \nu^{gw^*}_{g'} = \lambda^{k^*}_{e^*w^*} + \mu^{e^*w^*},$$

which implies by (5.138), (5.141) and (5.142) that $\lambda^{k^*}_{e^*w^*} = 0$. Since (e^*, w^*) is arbitrary, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, (e, w) \in \Gamma^1 \setminus \{(\tilde{e}, \tilde{w})\}, \tag{5.143}$$

Thus, by 5.142 and 5.143 we obtain

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, (e, w) \in (A_1 \times W) \setminus \{(\tilde{e}, \tilde{w})\}, \tag{5.144}$$

Now let us turn ourselves to coefficients λ related to commodities of $K \setminus \tilde{\mathcal{C}}$ and $(\tilde{e}, \tilde{w})$.

Consider a commodity k^* of $K \setminus \tilde{\mathcal{C}}$. It is clear that k^* is compatible with at least one commodity of the clique $\tilde{\mathcal{C}}$, otherwise $\tilde{\mathcal{C}}$ should not be maximal. Let $\tilde{k}$ be a commodity of $\tilde{\mathcal{C}}$ such that k^* and $\tilde{k}$ are compatible, and $(\tilde{e}, \tilde{w})$ are involved in the routing of $\tilde{k}$. Then, consider the solution S^9, obtained from S^1 by associating the pair $(\tilde{e}, \tilde{w})$ with the commodity k^*, that is to set element $x^{S^1}_{k^*\tilde{e}\tilde{w}}$ to 1. In other words, $\mathcal{C}^9_{k^*} = \mathcal{C}^1_{k^*} \cup \{\tilde{e}\}$. Note that, here, $(\tilde{e}, \tilde{w})$ is involved in the routing of two commodities that are compatible. So, no capacity constraint is violated and the solution verifies all the remaining constraints. S^9 still then obviously feasible for the problem, and its incidence vector satisfies $\lambda x + \mu y + \nu z \leq \xi$ with equality. In consequence, we have that $\lambda^{k^*}_{\tilde{e}\tilde{w}} = 0$. And one can state that

$$\lambda^k_{\tilde{e}\tilde{w}} = 0, \text{ for all } k \in K \setminus \tilde{\mathcal{C}}, \tag{5.145}$$

since k^* is arbitrary in $K \setminus \tilde{\mathcal{C}}$.

Now let us show that all the coefficient λ related to commodities of $\tilde{\mathcal{C}}$ and $(\tilde{e}, \tilde{w})$ are equal, which is to show that $\lambda^k_{\tilde{e}\tilde{w}} = \rho$, for all $k \in \tilde{\mathcal{C}}$, $\rho \in \mathbb{R}^+$.

Recall that $\tilde{k}$ denote the commodity using $(\tilde{e}, \tilde{w})$ in the solution S^1 and let k^* be a commodity of $\tilde{\mathcal{C}} \setminus \{\tilde{k}\}$. Consider the solution S^{10}, obtained from S^1 by switching roles of $\tilde{k}$ and k^* in the use of $(\tilde{e}, \tilde{w})$. More precisely, we move $\tilde{e}$ from $\mathcal{C}^1_{\tilde{k}}$ to $\mathcal{C}^1_{k^*}$. The pair $(\tilde{e}, \tilde{w})$ is then associated with the routing of k^* instead of one of $\tilde{k}$. This modification does not impact on feasibility of the solution, and $y^{S^{10}}_{\tilde{e}\tilde{w}} = 1$ while $\sum_{k \in \tilde{\mathcal{C}}} x^{S^{10}}_{k\tilde{e}\tilde{w}} = x^{S^{10}}_{k^*\tilde{e}\tilde{w}} = 1$. Then, $(x^{S^{10}}, y^{S^{10}}, z^{S^{10}})$ belongs to $\tilde{\mathcal{F}}$ and, consequently, it also belongs to $\mathcal{F}$. Hence, the following is true

$$\lambda x^{S^{10}} + \mu y^{S^{10}} + \nu z^{S^{10}} = \lambda x^{S^1} - \lambda^{\tilde{k}}_{\tilde{e}\tilde{w}} + \lambda^{k^*}_{\tilde{e}\tilde{w}} + \mu y^{S^1} + \nu z^{S^1},$$

which implies that $\lambda^{\tilde{k}}_{\tilde{e}\tilde{w}} = \lambda^{k^*}_{\tilde{e}\tilde{w}}$. Since the commodities k^* and $\tilde{k}$ are arbitrary and interchangeable in $\tilde{\mathcal{C}}$, we conclude that there exists a positive scalar $\rho \in \mathbb{R}$ such that

$$\lambda^k_{\tilde{e}\tilde{w}} = \rho, \text{ for all } k \in \tilde{\mathcal{C}}, \tag{5.146}$$

The last part of the proof is to show that $\lambda^k_{\tilde{e}\tilde{w}} =$ - $\mu^{\tilde{e}\tilde{w}}$, for every commodity k of $\tilde{\mathcal{C}}$.

Recall that the solution S^0 is such that $(\tilde{e}, \tilde{w}) \notin \Gamma^0$. In other words, $(\tilde{e}, \tilde{w})$ is not used in S^0 and no commodity of $\tilde{\mathcal{C}}$ is associated with $(\tilde{e}, \tilde{w})$. In consequence, $y^{S^0}_{\tilde{e}\tilde{w}} = 0$, and $\sum_{k\in\tilde{\mathcal{C}}} x^{S^0}_{k\tilde{e}\tilde{w}} = 0$. Moreover, $(x^{S^0}, y^{S^0}, z^{S^0}) \in \tilde{\mathcal{F}} \subseteq \mathcal{F}$. Thus, replacing $(x^{S^0}, y^{S^0}, z^{S^0})$ in the hyperplane inducing $\mathcal{F}$ gives us $\xi = 0$. Furthermore, by replacing $(x^{S^1}, y^{S^1}, z^{S^1})$ in the same hyperplane, we get

$$\lambda^{\tilde{k}}_{\tilde{e}\tilde{w}} + \mu^{\tilde{e}\tilde{w}} = 0,$$

and it follows by (5.146) that

$$\mu^{\tilde{e}\tilde{w}} = -\rho, \tag{5.147}$$

All together, we get

$$\nu^{ew}_a = 0, \text{ for all } e \in A_1, w \in W, a \in A_2,$$

$$\mu^{ew} = \begin{cases} -\rho, & \text{if } (e,w) = (\tilde{e},\tilde{w}), \\ 0, & \text{otherwise.} \end{cases}$$

$$\lambda^k_{ew} = 0, \text{ for all } (k,e,w) \in (K \times A_1 \times W) \setminus (\tilde{\mathcal{C}} \times \{(\tilde{e},\tilde{w})\}),$$

$$\lambda^k_{\tilde{e}\tilde{w}} = \begin{cases} \rho, & \text{if } k \in \tilde{\mathcal{C}}, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$, and the proof is complete. □

5.4.4 Cover Inequalities

Cover inequalities Cover inequalities have been introduced independently by Balas [12], Hammer *et al.* [59] and Wolsey [108] for the knapsack problem. They have also been used more recently for problems where knapsack appears as an embedded structure, like the *Generalised Assignment Problem* [56, 29] and the capacitated newtork design problems [49, 32]. The reader is referred to [69, 9] for detailed surveys on strong valid inequalities related to knapsack structures.

Definition 7 *A cover $\mathcal{I} \subseteq K$ is a subset of commodities such that $\sum_{k\in\mathcal{I}} D^k > C$. A cover is said to be minimal if it does not contain any cover as a subset.*

In other words, $\mathcal{I}$ is a set of commodity that can not be packed together in a subband, as their total traffic amount exceeds the subband capacity.

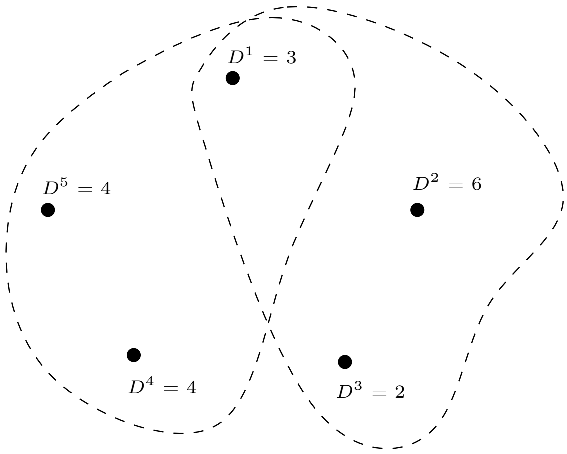


Figure 5.20: Examples of covers in an instance with $C = 10$

Example Suppose that K includes five commodities k_1 to k_5, with the following traffic amounts 3, 6, 2, 4 and 4. Then, capacity constraints (5.2) would be $3x_{k_1ew} + 6x_{k_2ew} + 2x_{k_3ew} + 4x_{k_4ew} + 4x_{k_5ew} \leq 10$, for all $(e, w) \in A_1 \times W$. The sets $\{k_1, k_2, k_3\}$ and $\{k_1, k_4, k_5\}$ form covers, as $D^{k_1} + D^{k_2} + D^{k_3} = 3 + 6 + 2 > 10$, and $D^{k_1} + D^{k_4} + D^{k_5} = 3 + 4 + 4 > 10$ (see Figure 5.20). The cover inequalities induced by these subsets are then given as follows:

$$x_{k_1ew} + x_{k_2ew} + x_{k_3ew} \leq 2y_{ew}, \quad \forall (e, w) \in A_1 \times W,$$
$$x_{k_1ew} + x_{k_4ew} + x_{k_5ew} \leq 2y_{ew}, \quad \forall (e, w) \in A_1 \times W.$$

Proposition 5.21 *Consider an arc $\tilde{e} \in A_1$, a subband $\tilde{w} \in W$ and a subset of commodities $\tilde{\mathfrak{I}} \subseteq K$ defining a cover. Then, the following inequality*

$$\sum_{k \in \tilde{\mathfrak{I}}} x_{k\tilde{e}\tilde{w}} \leq (|\tilde{\mathfrak{I}}| - 1) y_{\tilde{e}\tilde{w}} \tag{5.148}$$

is valid for $P(G_1, G_2, G_2, K, C)$.

Proof. If $y_{\tilde{e}\tilde{w}} = 0$, then it is clear that no commodity can use $\tilde{e}$ and $\tilde{w}$, that is to say $x_{k\tilde{e}\tilde{w}} = 0$, for all $k \in K$, in particular for all $k \in \tilde{\mathfrak{I}}$. Now suppose that $y_{\tilde{e}\tilde{w}} = 1$, and $\sum_{k \in \tilde{\mathfrak{I}}} x_{k\tilde{e}\tilde{w}} \geq (|\tilde{\mathfrak{I}}| - 1) y_{\tilde{e}\tilde{w}} + 1 = |\tilde{\mathfrak{I}}|$. This means that all the commodities of $\tilde{\mathfrak{I}}$ use $(\tilde{e}\tilde{w})$. In other words, $x_{k\tilde{e}\tilde{w}} = 1$, for all $k \in \tilde{\mathfrak{I}}$, which violates the capacity constraint (5.2) induced by $(\tilde{e}, \tilde{w})$. Contradiction. $\square$

Cover inequalities define facets under some known conditions (see [86, 107]). They should also define facets for $P(G_1, G_2, K, C)$ polytope with appropriate additional conditions. Furthermore, note that facets based on covers and extensions of covers may be derived by using procedure as *sequentiel lifting* (see [55, 90]).

5.5 Conclusion

In this chapter, we have proposed a cut-based integer linear programming formulation. We studied the basic properties of the associated polytope, and performed a facial investigation of the basic inequalities. We have also introduced further valid inequalities and discussed necessary conditions and sufficient conditions for these inequalities to define facets. The next chapter will be dedicated to the description of the Branch-and-Cut algorithm to solve the OMBND problem, and to give an insight of the efficiency of theoretical results provided within this chapter.

Chapter 6

Branch-and-Cut Algorithm for OMBND problem

Contents

In this chapter we present a branch-and-cut algorithm that we have devised and implemented to solve the optical multi-band network design problem. This algorithm is based on the polyhedral results introduced in the previous chapter. The purpose of this chapter is to substantiate the efficiency of the valid inequalities described in the polyhedral study, and provide exact solutions for realistic instances of networks.

6.1 Branch-and-Cut algorithm for Cut formulation

6.1.1 Overview

We describe the framework of our algorithm. Consider given two graphs $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$, that instantiate the virtual layer and the physical layer of the network, respectively. Also suppose given a set of commodities K where each commodity k is characterized by a pair $(o_k, d_k) \in V_1 \times V_1$ and a traffic value D^k. We consider a set W of available subbands having a capacity C. A cost vector $c \in R_+^{W \times A_1}$, is given as well.

To start the optimization, we set up the restricted linear program given by the degree cuts associated with the origin and destination nodes of the commodities of K, the capacity constraints (5.2) and the disjunction constraints (5.4), together with the trivial constraints. Inequalities (5.3) are not included in this restricted linear program. We will denote this formulation by $LP_{initial}$

$$
\begin{aligned}
& Min \sum_{e \in A_1} \sum_{w \in W} c(w) y^{ew} + \sum_{e \in A_1} \sum_{w \in W} \sum_{a \in A_2} z_a^{ew} \\
& s.t: \\
& \sum_{w \in W} \sum_{e \in \delta^+_{G_1}(s)} x^k_{ew} \geq 1, && \forall k \in K, s \in \{o_k, d_k\}, \\
& \sum_{k \in K} D^k x^k_{ew} \leq C y^{ew}, && \forall e \in A_1, \forall w \in W, \\
& \sum_{e \in A_1} z_a^{ew} \leq 1, && \forall w \in W, \forall a \in A_2, \\
& 0 \leq x^k_{ew} \leq 1, && \forall k \in K, e \in A_1, \\
& 0 \leq y^{ew} \leq 1, && \forall w \in W, e \in A_1, \\
& 0 \leq z_a^{ew} \leq 1, && \forall e \in A_1, \forall w \in W, \forall a \in A_2.
\end{aligned}
$$

We denote by $(\overline{x}, \overline{y}, \overline{z})$, $\overline{x} \in \mathbb{R}^{K \times W \times A_1}$, $\overline{y} \in \mathbb{R}^{W \times A_1}$, $\overline{z} \in \mathbb{R}^{W \times A_1 \times A_2}$, the optimal solution of the restricted linear relaxation of OMBND problem. This solution is feasible for the problem if $(\overline{x}, \overline{y}, \overline{z})$ is an integer vector that satisfies all the cut constraints of type (5.1) and (5.3). In most of the cases, the solution obtained by this way is not feasible for OMBND problem. We then manage to identify, at each iteration of the algorithm, valid inequalities that are violated by the solution of the current restricted linear program. This is referred to as the *separation problem.* Namely, given a class of valid inequalities, the separation problem is to check whether if the solution $(\overline{x}, \overline{y}, \overline{z})$ meets all the inequalities of this class, and, if this is not the case, to find an inequality that is violated by $(\overline{x}, \overline{y}, \overline{z})$. The detected inequalities are then added to the current linear program, and such procedure is reiterated until no violated inequality can be identified. The algorithm use then to branch over the fractional variables. The algorithm 6 summarizes the principal steps of the branch-and-cut algorithm.

Algorithm 6: Branch-and-cut algorithm

Data : two graphs $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$, a set of commodities K, a set of available subbands W, and a cost vector $c \in I\!R^{W \times A_1}$.
Output : optimal solution of OMBND problem, or best feasible upper bound.

1: LP $\leftarrow$ LP$_{initial}$
2: solve the linear program LP.
  let $(\overline{x}, \overline{y}, \overline{z})$ be the optimal solution of LP.
3: **If** $(\overline{x}, \overline{y}, \overline{z})$ is feasible for OMBND **then**
  $(\overline{x}, \overline{y}, \overline{z})$ is an optimal solution. STOP
4: **If** constraints (cut, capacitated cutset) violated by $(\overline{x}, \overline{y}, \overline{z})$ are found **then**
  add them to LP.
  go to 2.
5: **else**
  create two sub-problems by branching on a fractional variable.
6: return the best solution for all the sub-problems.

The branch-and-cut algorithm includes the inequalities described in the previous chapter, and their separations are accomplished in the following order

1. cut inequalities
2. min set I inequalities
3. capacitated cutset inequalities
4. flow-cutset inequalities
5. clique-based inequalities
6. cover inequalities
7. min set II inequalities

Observe that all the inequalities are global (i.e., valid for the whole Branch-and-Cut tree), and several inequalities may be added at each iteration. Furthermore, we move to the next class only if no violated inequalities of the current class is identified. Our strategy is to try to detect violated inequalities at each node of the Branch-and-Cut tree, in order to obtain the best possible lower bound by strengthening the linear relaxation, and thus limit the number of generated nodes.

In the sequel, we describe the separation procedures embedded in our algorithm. We use exact and heuristic algorithms as well, depending on the class of inequalities. Except for cut inequalities (5.4), all the separation routines are applied on the graph G_1. In fact, weighs, given by the current solution $(\overline{x}, \overline{y}, \overline{z})$, are distributed on the arcs of G_1. We present beforehand our feasibility test.

6.1.2 Feasibility test

Since OMBND cut formulation holds an exponential number of cut constraints, they can not be enumerated and added explicitly to the initial linear programming formulation. Thus, an optimal solution of the initial linear program is not needfully feasible, even if it is integer. Actually, this solution must satisfy all the cut constraints. To deal with this, we have added a feasibility test that checks whether if a given solution is feasible or not. This test is based on an implementation of the so-called *push-relabel* algorithm of Goldberg and Tarjan [54] for computing the maximum flow/minimum cut in a graph.

6.1.3 Separation of Cut constraints

6.1.3.1 Connectivity constraints

The separation problem consists, given a vector $(\overline{x}, \overline{y}, \overline{z})$, in deciding whether this solution meets all the inequalities (5.1), and if not, to identify an inequality of this class, violated by $(\overline{x}, \overline{y}, \overline{z})$ and add it to the current linear program. Such problem may be solved by using the Goldberg and Tarjan's preflow push-relabel algorithm [54] on the graph G_1, by considering for each commodity the cost $\overline{x}$ associated with the pairs (e, w), $e \in A_1$, $w \in W$. Recall that each commodity is assigned a path in G_1 using the subbands installed on the arcs of A_1. In addition, a subband set up over an arc is considered as a copy of that arc. Hence, for every commodity k, the pairs (e, w) are assigned a weigh $c(e,w) = \overline{x}^k_{ew}$. This algorithm produces for each commodity k, the minimum cut separating o_k and d_k, using the previously defined weigh function.

Due to maximum flow - minimum cut theorem of Ford and Fulkerson [47], it is possible to solve the problem of finding a minimum cut in polynomial time. Actually, the algorithm of Goldberg and Tarjan for maximum flow is one of the fastest known maximum flow algorithms. This algorithm is also the most commonly used, as it is the case in LEMON GRAPH [3] which is a C++ library. This algorithm has a worst case complexity of $\mathcal{O}(n_1^2\sqrt{m_1})$ where n_1 and m_1 are the number of nodes and arcs of G_1, respectively. Furthermore, the algorithm requires for each commodity $k \in K$ a minimum cut computation. Then, the separation of cut constraints (5.1) for $k \in K$ has a worst-case complexity of $\mathcal{O}(n^2\sqrt{m})$. Therefore, the separation algorithm for cut constraints (5.1) for all $k \in K$ is exact and runs in $\mathcal{O}(n^2 t\sqrt{m})$, where $t = |K|$.

6.1.3.2 Subband connectivity constraints

For the cut constraints (5.3), we have to solve the separation problem that consists in deciding, each pair $(e,w) \in A_1 \times W$, such that subband w is installed on the arc $e = (u,v)$, whether if there exists a cut constraint (5.3) violated by the solution $(\overline{x}, \overline{y}, \overline{z})$. One has to identify, for each pair $(e,w) \in A_1 \times W$, such that subband w is installed on the arc $e = (u,v)$, the minimum cut in the graph G_2 separating u' and v', $u', v' \in V_2$. Recall that theses inequalities ensure that a path is associated with each (e,w) whenever w is installed on e. In other words, (e,w) may be viewed as a commodity for the physical layer. Furthermore, for every pair (e,w), the weighs of the arcs in G_2 are given by the value of $\overline{z}^{ew}_a$, $a \in A_2$. By the same way as the previous cut constraint, we use the Goldberg and Tarjan maximum flow algorithm. For each pair (e,w) This algorithm has a worst case complexity of $O(n^2\sqrt{m})$. Hence, the separation algorithm has a complexity of $O(n^2 mq\sqrt{m})$, where $q = |W|$.

6.1.4 Separation of Capacitated Cut inequalities

Given a solution $(\overline{x}, \overline{y}, \overline{z})$, the separation problem associated with the capacitated cut-set inequalities is to identify an inequality of this class, violated by $(\overline{x}, \overline{y}, \overline{z})$, if such inequality exists. The separation problem associated with cutset inequalities has been proven NP-hard in general [30]. In our case, the separation problem related to capacitated cut-set inequalities (5.107) is also NP-hard. Therefore, we have developed two heuristics to separate the capacitated cutset inequalities. The former is based on the so-called n-cut heuristic, proposed by Bienstock et al. in [30] for the minimum cost capacity installation for multicommodity network flows. We adapt this heuristic in order to make it suitable with our problem.

This heuristic works as follows. For any commodity $k \in K$, we check whether if there is a path in G_1 connecting nodes o_k and d_k, and using only pair (e,w), $e \in A_1$, $w \in W$ with $\overline{y}^{ew} > 0$. Since this can be performed by using any path finding algorithm, we use Dijkstra's algorithm. If such path does not exist, then it is clear that a capacitated cutset inequality is violated. This inequality is induced by a subset of nodes T such that $o_k \in T$ and $d_k \notin T$. If a path between o_k and d_k is identified in G_1 for each commodity k, then we randomly pick a subset of nodes, say $T \subseteq V_1$, $0 \neq T \neq V_1$, and we identify the subset of commodity P^+ having their origin node in T and their destination in $V_1 \setminus T$. After that, we compute the right-hand side, and we check if the constraint thus constructed is violated or not. Since we check the existence of a path for each commodity between its origin and its destination, the worst-case complexity of this procedure is $\mathcal{O}(|K|(m_1|W| + n_1 log(n_1)))$, where $n_1 = |V_1|$ and $m_1 = |A_1|$.

In the second separation heuristic, we use Goldberg-Tarjan max-flow algorithm to find violated capacitated cut-set inequalities (5.107). We attribute to each pair $(e,w) \in A_1 \times W$ the capacity $\overline{y}^{ew}$, and determine for each $k \in K$ a minimum $o^k d^k$-dicut in G_1, say $\delta^+_{G_1}(T^*)$, with $T^* \subseteq V_1$. We then identify the subset of commodities $P^+ \subseteq K$ passing through this directed cut. We finally add inequality

$$\sum_{e \in \delta^+_{G_1}(T^*)} \sum_{w \in W} y^{ew} \geq \lceil \frac{D(P^+)}{C} \rceil,$$

in case it is violated. This procedure is based on max-flow computations, thus the worst case complexity is $\mathcal{O}(n_1^2 t \sqrt{m_1})$.

6.1.5 Separation of Flow-Cutset inequalities

Now we discuss our separation procedure for the flow-cutset inequalities (5.120). Atamtürk shows in [8] that the separation problem associated with of a more general form of flow-cutset inequalities is NP-hard even for one commodity. In case of a multiple commodity set, the complexity of simultaneously determining P^+ and F is not known [93]. As we do not know an efficient procedure to separate flow-cutset inequalities in general, we use here a simple heuristic based on Goldberg-Tarjan max-flow algorithm. The main idea is to identify, for each commodity the minimum cut separating its origin and its destination, then to consider the subset of commodities whose origin and destination nodes are separated by the current cut. In other words, for each $k \in K$, we assign the capacity $\overline{y}^{ew} + \overline{x}^k_{ew}$ to each pair $(e, w) \in A_1 \times W$, and we compute the minimum cut separating o^k from d^k in the graph G_1. Let $\delta^+_{G_1}(T^*)$, $T^* \subseteq V_1$, denote this cut. We then pick an arbitrary subset of arcs, say F^* of $\delta^+_{G_1}(T^*)$, such that $\emptyset \neq F^* \neq \delta^+_{G_1}(T^*)$. We then determine the subset of commodities $P^+ \subseteq K$ using $\delta^+_{G_1}(T^*)$. If $D(P^+)/C$ is not integer, we add the succeeding flow-cutset inequality

$$\sum_{e \in F^*} \sum_{w \in W} y^{ew} + \sum_{k \in P^+} \sum_{e \in \overline{F}^*} \sum_{w \in W} x^k_{ew} \geq \lceil \frac{D(P^+)}{C} \rceil,$$

if it is violated by the current fractional solution $(\overline{x}, \overline{y}, \overline{z})$.

6.1.6 Separation of Clique-based and Cover inequalities

Given a fractional solution $(\overline{x}, \overline{y}, \overline{z})$, and a pair $(\tilde{e}, \tilde{w}) \in A_1 \times W$. The separation problem associated with the clique-based inequalities (5.135) consists in identifying a clique $\mathcal{C}^*$ in the conflict graph H, such that

$$\sum_{k \in \mathcal{C}^*} \overline{x}^k_{ew} > \overline{y}^{ew},$$

If there is some. To do so, we use a greedy algorithm introduced by Nemhauser and Sigismondi [85] for the independant set problem. This heuristic works as follows. We first construct the conflict graph $H = (V, E)$ where each node $v \in V$ corresponds to a commodity in K and an edge $e \in E$ exists between two nodes u, $v \in V$ if $D^u + D^v > C$. For each pair $(e, w) \in A_1 \times W$, we assign a weight to each node v of V that is $\overline{x}^v_{ew}$, then we choose a node, say u, having the largest weight and we set $\mathcal{C}^* = \{u\}$. We then iteratively add to $\mathcal{C}^*$ the maximum weighted node of $V \setminus \mathcal{C}^*$ whenever it is neighbouring all the nodes of the current clique $\mathcal{C}^*$. We add the clique-based inequality induced by $\mathcal{C}^*$ if it is violated.

We use a similar approach to identify violated cover inequalities (5.148) if any. Indeed, we put the largest weighted node u in $\mathcal{N}^*$, then we repeat the following operation

Let v be the maximum weighted node of $V \setminus \mathcal{N}^*$, then we simply insert v to $\mathcal{N}^*$ if $\mathcal{N}^* \cup \{v\}$ does not form a clique

until a cover is obtained ($\sum_{v \in \mathcal{N}^*} D^v > C$). Every node $v \in \mathcal{N}^*$ such that $\sum_{i \in \mathcal{N}^* \setminus \{v\}} D^i > C$ is deleted from the subset $\mathcal{N}^*$. Finally, we add the inequality

$$\sum_{k \in \mathcal{N}^*} x^k_{ew} \leq (|\mathcal{N}^*| - 1) y^{ew},$$

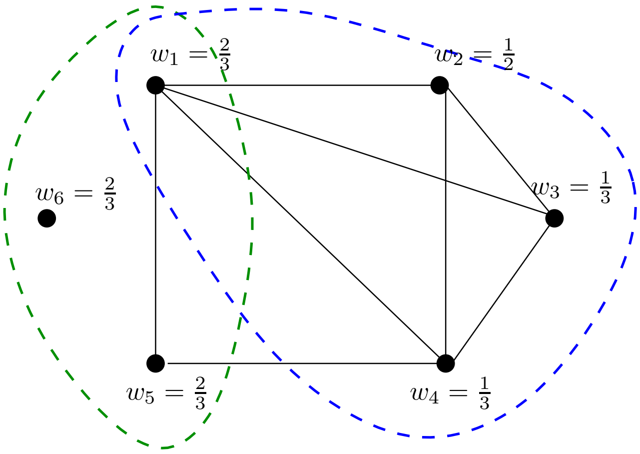


Figure 6.1: Clique and cover configurations in the conflict graph

if it is violated. Note that there exists plenty more sophisticated algorithms to solve the separation problem associated with cover inequalities (see for example [36, 50, 9, 69, 68] and references therein for separation of cover inequalities and [58, 69] for lifted cover inequalities), but our first idea was to take advantage from the separation performed for the clique-based inequalities and try to find subsets of commodities that form covers, if the heuristic fails to identify a clique. Besides, we consider only violated clique (respectively cover) inequalities where $|\mathcal{C}^*| \geq 3$ (respectively $|\mathcal{N}^*| \geq 3$) in our branch-and-cut algorithm.

We show in figure 6.1 an example of fractional point where $\overline{y}^{ew} = \frac{2}{3}$ for some pair (e, w) and we have six commodities with the values $D^1 = 7$, $D^2 = 6$, $D^3 = 5$, $D^4 = 7$, $D^5 = 4$, $D^6 = 3$ and the facilities have a capacity $C = 10$. We have assigned to each node a weigh w_i, $i = 1, \ldots, 6$ that is the value of $\overline{x}^i_{ew}$. Then, we can see that the subset of nodes surrounded by the blue dashed lines induces the violated clique inequality $x^1_{ew} + x^2_{ew} + x^3_{ew} + x^4_{ew} \leq y^{ew}$, while the subset the green dashed lines subset induces the following cover inequality $x^1_{ew} + x^5_{ew} + x^6_{ew} \leq 2y^{ew}$ which is also violated by the current fractional solution.

6.2 Computational results

Based on the polyhedral results presented in the former sections, we devised a branch-and-cut algorithm to solve OMBND problem. Similarly to implementation features described in Chapter 4, the Branch-and-Cut algorithm for OMBND problem has been implemented in C++, using Cplex 12.5 callable library [2]. Also recall that we used the LEMON GRAPH C++ library for the Goldberg-Tarjan max-flow algorithm. It was tested on a processor Intel Core i5-3210M CPU 2.50GHz $\times$ 4 with 3.7 Gb RAM, running under ubuntu 12.10 platform. We fixed the maximum CPU time to 5 hours.

6.2.1 Instances description

The results show in this chapter have been obtained by solving instances coming from real networks as well as realistic topologies. For all the instances, the graph G_1 representing the virtual (subbands) layer is supposed to be complete. The cost induced by installing each subband is given by

$$c(w) = (1 + w)c,$$

where w is the subband index and c is a fixed cost associated with the ROADM generating the subband. This cost is justified by our wish to install the subbands progressively. In other words, a subband w_i is not used before w_{i-1} is filled. We also take into account the length of routing path in G_2 associated with each installed subband. This length is given in terms of number of sections in the path. Note that we use the same objective function for both classes of instances.

The realistic instances come from SNDlib [1] library. The graph G_1 is obtained by considering an edge between each pair of nodes. Moreover, if the topology corresponds to a non directed graph, we replace each edge by two anti-parallel arcs in both G_2 and G_1. The number of available subbands per arc is set to $|W| = 5$ for all the instances. Based on these topologies, we have considered two sub-classes of instances. The first one is obtained by using SNDlib topologies with randomly generated traffic commodities. We have tested 3 examples of each instance size and we give the average of the results for these examples. The second sub-class uses SNDlib topologies and traffic matrices. We pick the K most important commodities for each topology and traffic matrix. We have considered the topologies *pdh*, *polska*, *nobel_ us*, *atlanta*, *newyork*, *nobel_ germany*, *geant*, *ta1*, *france*, and *india35*.

The real instances are derived from real network topologies provided by Orange Labs. Three topologies of real instances are considered here, all related to Bretagne area backhaul network. The traffic commodities, as well as the subbands capacities are also given by Orange Labs. For each topology, we have considered three subband capacities $C = 10$ Gbit/s, $C = 12.5$ Gbit/s and $C = 25$ Gbit/s, so as to compare the performances of each type of OFDM multi-band solution.

Our experimental results are reported in tables of following sections. The entries of the columns in these tables are:

$\|V_2\|$	:	number of nodes in G_2,
$\|A_2\|$	:	number of arcs,
$\|K\|$	:	number of commodities,
NcI	:	number of generated connectivity constraints,
NcII	:	number of generated subband connectivity constraints,
NMSI	:	number of min set I inequalities generated,
NCCS	:	number of capacitated cutset inequalities generated,
NFCS	:	number of flow-cutset inequalities generated,
NC	:	number of clique inequalities generated,
NCo	:	number of cover inequalities generated,
NMSII	:	number of min set II inequalities generated,
nodes	:	number of nodes in the Branch-and-Cut tree,
o/p	:	number of problem solved to optimality over number of tested instances (only for instances with randomly generated traffic),
Gap	:	the relative error between the best upper bound (optimal solution if the problem has been solved to optimality) and the lower bound obtained at the root node of the Branch-and-Cut tree (before branching),
TT	:	total CPU time in h:m:s,
TTsep	:	CPU time spent in performing the constraints separation, in seconds.

6.2.2 Effectiveness of the constraints

Before giving the results of our experiments for the instances described above, we first propose to evaluate the impact of the valid inequalities that we use within the Branch-and-Cut algorithm. To this end, we show some numerical results obtained by considering, on one hand the basic cut formulation (5.1)-(5.7), and adding the valid inequalities on the other hand. We have tested our approach on a subset of instances whose topologies are pdh, polska, nobel_us, newyork and geant. We rely here on three criteria to make our comparison: the gap, the number of nodes in the Branch-andt-Cut tree, and the CPU time computation. The results reported in Table 6.1.

Table 6.1 shows results obtained for graphs having up to 22 nodes, and 72 arcs. The number of commodities ranges from 2 to 14. It appears clearly from this table that

				Basic B&C			B&C with valid inequalities		
Instance	$\lvert V_2\rvert$	$\lvert A_2\rvert$	$\lvert K\rvert$	Gap	Nodes	TT	Gap	Nodes	TT
pdh	10	68	2	0.00	1	13	0.00	1	5
pdh	10	68	4	22.22	115	254	0.00	1	16
pdh	10	68	6	11.17	71	140	3.00	3	541
pdh	10	68	8	20.17	10205	7071	6.11	10	1514
pdh	10	68	10	*25.64*	*34709*	18000	22.54	6	1593
pdh	10	68	12	17.54	2454	3573	4.63	104	4833
pdh	10	68	14	1.39	31	560	0.00	1	133
polska	12	36	2	0.00	1	5	0.00	1	4
polska	12	36	4	23.18	462	726	0.00	1	49
polska	12	36	6	0.00	1	51	0.00	1	44
polska	12	36	8	*29.92*	40201	18000	10.68	144	2299
polska	12	36	10	*13.40*	34896	18000	3.95	56	2059
polska	12	36	12	*36.75*	30954	18000	*13.19*	3114	18000
polska	12	36	14	*34.48*	14983	18000	8.27	1115	13768
nobel_us	14	42	2	0.00	1	21	0.00	1	20
nobel_us	14	42	4	31.33	58	308	0.00	1	140
nobel_us	14	42	6	0.00	1	62	0.00	1	59
nobel_us	14	42	8	*34.93*	17921	18000	2.72	3	978
nobel_us	14	42	10	*36.89*	15682	18000	7.74	205	12653
nobel_us	14	42	12	*41.20*	5479	18000	7.77	322	12016
nobel_us	14	42	14	*42.95*	10937	18000	*28.33*	513	18000
newyork	16	98	2	0.00	1	25	0.00	1	51
newyork	16	98	4	33.09	2421	5514	0.00	1	273
newyork	16	98	6	20.47	380	1459	0.00	1	270
newyork	16	98	8	*35.52*	19739	18000	0.00	1	634
newyork	16	98	10	13.87	44	306	3.22	11	6102
newyork	16	98	12	*33.09*	18179	18000	*11.65*	88	18000
newyork	16	98	14	14.97	7769	9942	0.00	1	1064
geant	22	72	2	0.00	1	17	0.00	1	38
geant	22	72	4	13.33	40	436	0.00	1	264
geant	22	72	6	27.65	4126	4773	0.00	1	305
geant	22	72	8	*42.76*	25860	18000	0.22	3	7577
geant	22	72	10	*15.18*	24635	18000	3.57	17	8716
geant	22	72	12	*47.27*	20686	18000	*5.75*	2	18000
geant	22	72	14	*41.46*	17057	18000	*6.23*	14	18000

Table 6.1: The impact of adding valid inequalities

the formulation with valid inequalities performs much more better than the basic formulation on all the instances. In fact, we first notice from Table 6.1 that using valid inequalities enables solving some instances that are not solved to optimality when considering the basic formulation. See for example instance nobel_us with 8 commodities, that is not solved to optimality within 5 hours when using the basic formulation. Introducing valid inequalities allows to solve this instances in less than one hour. Also we can see that both gap value and CPU time are smaller when adding the valid inequalities, for all the considered instances. In fact, 17 among the considered instances are solved to optimality at the root node by using valid inequalities, while only 7 instances are solved at the root node without adding cuts. Furthermore, observe that the number nodes in the Branch-and-Cut tree decreases drastically when introducing valid inequalities. For example, see instance geant_8, where the Branch-and-Cut algorithm for basic formulation explores no less than 25860 nodes, while this number drops to 3 nodes, by adding valid inequalities.

All these observations lead us to conclude that using valid inequalities to strengthen linear relaxation of (5.1)-(5.7) is a key issue to solve efficiently OMBND problem. As we could see, this enabled to improve significantly the gap value, number of Branch-and-Cut tree as well as the time for computation.

Table 6.2 shows more accurately the gap evolution when adding the valid inequalities progressively. In fact, the column Gap(0) contains the gap values for basic formulation and Gap(6) contains the gap value when considering all the cuts. The remaining columns are intermediate gap values obtained by considering an additional family of valid inequalities. The constraints are separated in the order given in section 6.1. It appears from Table 6.2 that the gap value decreases when adding valid inequalities. However, it seems that some inequalities are more efficient than other in strengthening the linear relaxation. In fact, the most significant improvement is observed when adding Min Set I inequalities (see columns Gap(0) and Gap(1)). Adding capacitated cutset and flow-cutset inequalities also allows to improve the gap value, while only a slight gain is notified when adding the remaining families of valid inequalities. In practice, their interest lies in the number of nodes in the Branch-and-Cut tree, which gets smaller as further families of valid inequalities are being separated.

In what follows, we will get benefit from these valid inequalities to solve realistic and real instances.

6.2.3 Realistic instances

Our first series of experiments concerns the SNDlib topologies with randomly generated traffic commodities. The instances considered here have graphs with 10 up to 24 nodes and the graphs vary from sparse (like for polska) to highly meshed (like for ta1) topology. The number of commodities for each size of graph ranges from 2 to 18 with values generated randomly in the interval $]\epsilon C, C]$, with $\epsilon = 0.2$ for these instances. For

Table 6.2: Effectiveness of the cuts - Gap evolution

Instance	V	A	K	Gap(0)	Gap(1)	Gap(2)	Gap(3)	Gap(4)	Gap(5)	Gap(6)
pdh	10	68	2	0.00	0.00	0.00	0.00	0.00	0.00	0.00
pdh	10	68	4	22.22	0.00	0.00	0.00	0.00	0.00	0.00
pdh	10	68	6	11.17	3.00	3.00	3.00	3.00	3.00	3.00
pdh	10	68	8	20.17	6.11	6.11	6.11	6.11	6.11	6.11
pdh	10	68	10	25.64	22.67	22.54	22.54	22.54	22.54	22.54
pdh	10	68	12	17.54	6.63	4.76	4.63	4.63	4.63	4.63
pdh	10	68	14	1.39	0.29	0.00	0.00	0.00	0.00	0.00
polska	12	36	2	0.00	0.00	0.00	0.00	0.00	0.00	0.00
polska	12	36	4	23.18	3.64	0.00	0.00	0.00	0.00	0.00
polska	12	36	6	0.00	0.00	0.00	0.00	0.00	0.00	0.00
polska	12	36	8	29.92	12.38	12.36	10.68	10.68	10.68	10.68
polska	12	36	10	13.40	4.03	3.95	3.95	3.95	3.95	3.95
polska	12	36	12	36.75	13.50	13.50	13.47	13.19	13.19	13.19
polska	12	36	14	34.48	15.63	9.60	8.27	8.27	8.27	8.27
nobel_us	14	42	2	0.00	0.00	0.00	0.00	0.00	0.00	0.00
nobel_us	14	42	4	31.33	0.00	0.00	0.00	0.00	0.00	0.00
nobel_us	14	42	6	0.00	0.00	0.00	0.00	0.00	0.00	0.00
nobel_us	14	42	8	34.93	3.00	2.72	2.72	2.72	2.72	2.72
nobel_us	14	42	10	36.89	16.00	8.84	8.16	7.74	7.74	7.74
nobel_us	14	42	12	41.20	41.13	41.13	8.10	7.77	7.77	7.77
nobel_us	14	42	14	42.95	33.82	32.41	28.33	28.33	28.33	28.33
newyork	16	98	2	0.00	0.00	0.00	0.00	0.00	0.00	0.00
newyork	16	98	4	33.09	0.00	0.00	0.00	0.00	0.00	0.00
newyork	16	98	6	20.47	0.00	0.00	0.00	0.00	0.00	0.00
newyork	16	98	8	35.52	0.00	0.00	0.00	0.00	0.00	0.00
newyork	16	98	10	13.87	4.27	3.27	3.27	3.27	3.22	3.22
newyork	16	98	12	33.09	24.08	11.97	11.65	11.65	11.65	11.65
newyork	16	98	14	14.97	1.35	0.00	0.00	0.00	0.00	0.00
geant	22	72	2	0.00	0.00	0.00	0.00	0.00	0.00	0.00
geant	22	72	4	13.33	0.00	0.00	0.00	0.00	0.00	0.00
geant	22	72	6	27.65	0.00	0.00	0.00	0.00	0.00	0.00
geant	22	72	8	42.76	3.11	1.11	0.22	0.22	0.22	0.22
geant	22	72	10	15.18	5.54	4.14	4.14	3.57	3.57	3.57
geant	22	72	12	47.27	16.96	8.71	8.46	8.46	5.85	5.75
geant	22	72	14	41.46	17.60	14.37	13.42	13.42	6.23	6.23

each instance size, we have generated 3 examples. The results are reported in Table 6.3.

It appears from Table 6.3 that 20 over 45 groups of instances have been solved to optimality within the fixed time limit. Besides, only 6 groups of instances among the

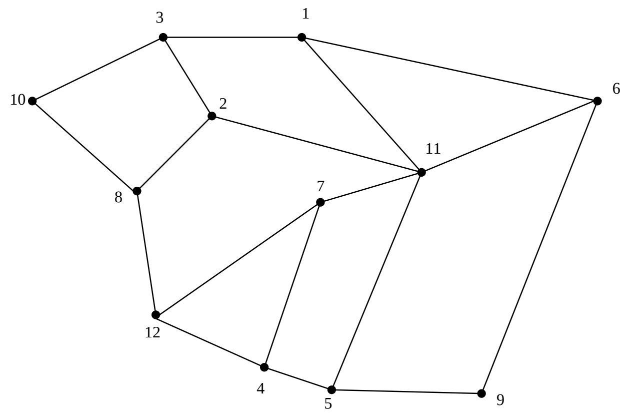


Figure 6.2: Polska network

remaining groups could not obtain any optimal solution over the three tested instances, within 5 hours. Observe that no more than 4 groups of instances among those solved to optimality have a gap value greater than 20%. For the remaining groups of instances, apart from newyork with |K| = 6, the gap does not exceed 30%.

Table 6.3 also shows that the difficulty of solving an instance is not only related to its size, but also to the nature of the commodities. For example, instances polska with 12 commodities are solved to optimality within the time limit, while 2 over 3 instances polska with 10 commodities are solved to optimality. Even though the second group of instances are larger in size, they are solved more easily. In fact, OMBND problem presents the same behaviour as CSLND problem (see Chapter 4) in terms of difficulty of instance. Moreover, it should be emphasized again that parallel arcs of G_1 are considered as additional commodities. Indeed, since two levels of routing must be performed, there are $|K| + |W|(n_1(n_1 - 1))$ commodities, where $n_1 = |V_1|$, $|K|$ being the traffic demands and $|W|(n_1(n_1 - 1))$ the number of subbands that can be installed in G_1.

Remark also that an important number of min set I, capacitated cutset and flows-cutset inequalities are being generated along the Branch-and-Cut tree, which means that they are helpful for solving the problem. However, the number of clique and cover inequalities separated is less high. This can be explained by the fact that each arc of G_1 potentially induces the same cliques (respectively cover subsets), since it depends on the commodities size. Thus, if all the commodities are "small" regarding to the capacity of a subband, then clique and eventually cover inequalities are unlikely to appear.

Figure 6.2 shows the topology of a realistic instance having 12 nodes and 36 arcs,

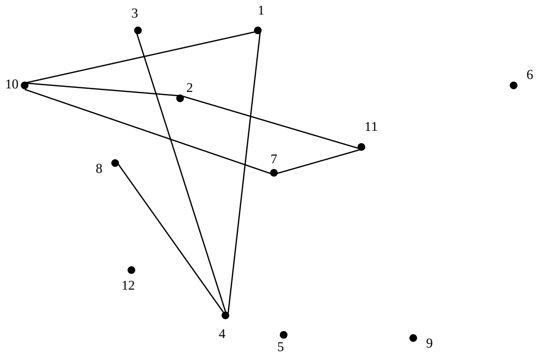


Figure 6.3: Design solution in G_1

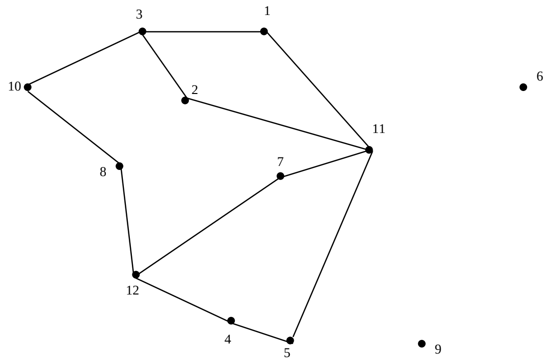


Figure 6.4: Routing in G_2

namely polska. Figures 6.3 and 6.4 depict a partial description of the solution obtained for polska graph with $|K| = 10$ and $|W| = 5$. In particular, Figure 6.3 shows the design solution in terms of number of subbands installed in G_1. In fact, a link is represented in this graph for each installed subband. This solution requires only one subband per link. Note that the commodities use these links for their routing. We can see in Figure 6.4 the solution in term of routing for the subbands. It is easy to check that a path in G_2 is associated with each link supporting a subband in G_1.

The second series of experiments that we have conducted concerns SNDlib instances with realistic traffic commodities. We have considered instances with graphs having 10 to 35 nodes while the commodities number varies from 2 to 20 commodities, 2 to 10 commodities for larger instances. A total of 70 instances have been tested. Among them, 38 instances have been solved to optimality within the time limit. The remaining instances, often having more than 18 commodities could not reach the optimal solution after 5 hours of computation. Also we can see that, for the smaller instances that could be solved to optimality, the gap value does not exceed 30% are the number of nodes in the Branch-and-Cut tree remains reasonable. 35 among the instances for which the algorithm provided an optimal solution have been solved in less than 3 hours.

Similarly to random instances, some instances may be more difficult to solve than other instances, even larger in size. In fact, we could previously see that the proportion occupied by a traffic commodity in a subband capacity was a key factor in the difficulty of an instance. Yet this does not totally explain the algorithm behaviour for some instances. For example, instances atlanta seem to be harder to solve than newyork which are larger in size. In fact, the algorithm could not reach the optimal solution from 8 commodities for atlanta instances. This behaviour is in reality caused by some conflict that may arise in the subband routing, because of the disjunction constraints. Actually, the topology of atlanta instances corresponds to a graph that is not so dense,

Instance	$\|V_2\|$	$\|A_2\|$	|K|	Opt	NcI	NcII	NmsI	NCCS	NFCS	NC	NCo	NmsII	Nodes	Gap	TT
pdh	10	68	2	3/3	24.00	156.33	7.33	1.00	0.00	0.00	0.00	0.00	13.33	9.87	29.70
pdh	10	68	4	3/3	114.33	1133.33	57.67	9.33	8.33	0.00	0.67	0.00	151.00	28.51	393.80
pdh	10	68	6	3/3	134.66	643.33	40.67	19.33	31.67	31.67	1.33	0	71.33	12.33	262.48
pdh	10	68	8	3/3	426.67	2405.33	92.00	60.00	800.00	4.00	18.00	1.00	730.33	24.44	3145.93
pdh	10	68	10	3/3	247.00	832.67	74.33	33.00	83.67	2.33	0.67	0.00	40.67	7.19	350.93
pdh	10	68	12	3/3	625.67	1504.00	150.67	72.33	1389.67	31.67	6.33	1.33	194.67	10.29	2244.68
pdh	10	68	14	3/3	450.33	1030.00	113.00	57.33	1031.33	11.00	6.33	0.00	63.33	5.95	1259.59
pdh	10	68	16	2/3	478.67	1047.33	133.00	23.00	7021.67	17.33	3.67	0.67	191.67	2.83	6061.61
pdh	10	68	18	0/3	3602.67	1703.33	124.67	40.67	11683.67	12.00	815.00	0.00	493.33	6.54	18000.00
polska	12	36	2	3/3	38.67	292.67	22.00	1.67	2.67	0.00	0.00	0.00	8.33	7.05	26.54
polska	12	36	4	3/3	122.00	1187.00	55.67	8.67	15.33	0.00	1.67	0.00	70.33	16.90	264.58
polska	12	36	6	3/3	164.33	1508.00	35.00	21.33	57.00	0.33	0.00	0.00	168.67	11.62	471.90
polska	12	36	8	3/3	435.67	1817.67	89.00	59.00	490.00	4.00	4.33	0.00	161.00	12.06	872.21
polska	12	36	10	2/3	536.33	2782.33	185.33	61.33	6473.33	12.00	1.67	0.00	1263.67	9.08	6142.68
polska	12	36	12	3/3	1071.67	3042.00	197.33	81.33	5191.00	57.00	29.67	0.33	549.33	6.30	6224.30
polska	12	36	14	2/3	1019.67	3896.33	259.67	73.67	7919.00	31.33	42.67	0.00	918.33	11.53	9440.76
polska	12	36	16	2/3	955.33	3484.33	166.33	62.33	7991.67	30.67	15.33	0.33	852.33	13.81	10248.26
polska	12	36	18	2/3	1234.67	3881.33	286.00	46.00	7527.67	78.33	14.67	0.33	1098.00	8.77	12166.49
nobel_us	14	42	2	3/3	46.33	476.00	3.67	0.67	0.00	0.00	0.00	0.00	20.67	13.78	127.57
nobel_us	14	42	4	3/3	174.00	1812.33	74.67	13.00	4.33	0.00	0.00	0.00	127.33	15.72	807.94
nobel_us	14	42	6	3/3	228.33	1241.00	34.33	19.00	149.33	1.67	0.67	0.00	116.33	9.22	1001.01
nobel_us	14	42	8	3/3	715.00	2873.33	195.33	92.33	566.33	13.33	5.67	0.33	366.00	12.20	3429.44
nobel_us	14	42	10	2/3	1504.67	5064.67	214.00	120.67	4303.33	30.67	34.67	0.67	1388.67	19.05	14031.90
nobel_us	14	42	12	2/3	1529.00	4793.00	294.33	93.67	4377.67	76.00	1.67	2.33	1350.67	14.83	15391.00
nobel_us	14	42	14	0/3	1543.33	4569.33	256.67	116.33	8392.33	59.00	19.33	1.00	877.33	26.14	18000.00
nobel_us	14	42	16	1/3	1755.33	5120.33	301.67	105.33	4693.33	114.33	0.67	0.00	828.33	16.35	17840.40
nobel_us	14	42	18	1/3	1440.67	3590.33	318.33	55.00	8551.00	78.67	8.33	0.00	211.33	13.53	15614.03
newyork	16	98	2	3/3	23.33	205.67	15.00	0.00	0.00	0.00	0.00	0.00	7.33	5.73	78.60
newyork	16	98	4	3/3	327.67	3683.00	57.33	32.33	60.00	0.00	2.33	0.00	1185.00	30.14	5032.24
newyork	16	98	6	2/3	686.00	5251.33	95.00	50.33	334.33	0.67	7.33	0.00	1033.33	38.88	10273.94
newyork	16	98	8	3/3	720.67	2982.67	115.00	56.00	651.00	10.33	6.00	0.33	238.00	21.77	6384.00
newyork	16	98	10	2/3	664.00	2763.67	186.33	43.67	1554.67	9.67	3.33	0.67	315.00	14.74	8385.19
newyork	16	98	12	1/3	1102.33	4453.67	217.00	104.00	915.33	52.00	8.00	0.00	672.33	24.31	13484.55
newyork	16	98	14	2/3	1721.00	4043.33	245.33	109.00	1822.67	38.67	4.33	0.00	308.67	15.01	17578.47
newyork	16	98	16	2/3	911.67	3468.00	170.67	107.67	1484.33	16.67	8.33	0.00	291.00	10.37	14519.00
newyork	16	98	18	1/3	1428.33	3492.00	264.33	105.00	1924.00	48.00	6.33	0.33	330.00	6.35	16097.23
ta1	24	102	2	3/3	27.00	258.33	30.67	0.33	0.00	0.00	0.00	0.00	129.00	8.27	430.00
ta1	24	102	4	1/3	377.67	4749.33	131.67	14.00	16.67	0.00	2.33	0.00	75.33	16.26	653.67
ta1	24	102	6	1/3	464.67	4377.33	255.33	39.67	19.00	0.33	0.33	0.00	133.00	9.98	15096.67
ta1	24	102	8	0/3	937.00	6481.33	426.00	115.67	53.67	2.33	0.00	0.00	60.00	13.42	18000.00
ta1	24	102	10	2/3	169.00	3551.67	19.33	32.67	82.33	0.33	0.00	0.00	2.33	7.10	810.45
ta1	24	102	12	2/3	2.67	287.67	5.33	0.00	0.00	0.00	0.00	0.00	15.00	20.26	357.88
ta1	24	102	14	0/3	779.89	6955.22	148.11	166.00	199.00	19.00	4.67	0.00	146.00	14.74	18000.00
ta1	24	102	16	0/3	249.33	3414.00	23.33	33.00	93.00	4.00	0.33	0.00	515.00	21.16	18000.00
ta1	24	102	18	0/3	996.33	10244.33	157.67	150.33	1187.33	45.33	1.67	0.00	185.67	15.51	18000.00

Table 6.3: Branch-and-Cut results for SNDlib instances with randomly generated traffic

which makes more challenging to find a workable routing for the subbands that does not yield a too large cost. Besides, it should be pointed out that CPU time spent by the algorithm in performing separation of valid inequalities can be important. In fact, we noticed that this time could reach more than 50% of the tota CPU time of computation (see for example instances newyork with 10 and 12 commodities). More precisely, we noticed that the separation procedure for generating flow-cutset inequalities is the most time consuming routine.

In what follows, we intend to propose an alternative approach to get full advantage of our valid inequalities in solving real instances of networks provided by Orange Labs.

6.2.4 Real instances

Results given in previous section for SNDlib instances with both random and realistic commodities have shown that, even though valid inequalities added are very helpful, it still difficult to tackle real instances by using an approach fully oriented on cuts. Actually, since CPU time dedicated to identify violated valid inequalities may constitute an important part of the total time, we propose a second Branch-and-Cut algorithm using a flow based formulation for the problem. This allows to save the time dedicated to separate basic cut constraints, since they are replaced by flow conservation constraints in the compact formulation. This formulation is given in Chapter 7, and holds a polynomial number of constraints, while the variables are the same as in formulation (5.1)-(5.7). The separation routines as well as the order for inserting valid inequalities remains the same as in previous Branch-and-Cut algorithm.

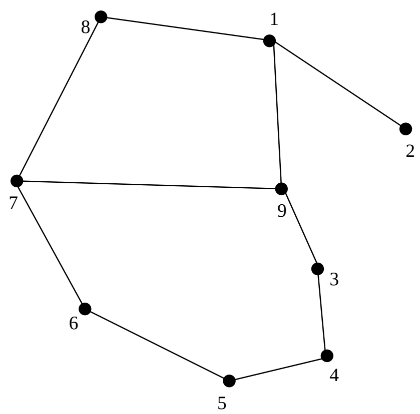


Figure 6.5: A real instance with 9 nodes

Instance	$\lvert V_2\rvert$	$\lvert A_2\rvert$	$\lvert K\rvert$	NcI	NcII	NmsI	NCCS	NFCS	NC	NCo	NmsII	Nodes	Gap	TT	TTsep
pdh	10	68	2	6	44	0	0	0	0	0	0	1	0.00	8	0
pdh	10	68	4	40	158	119	0	0	0	0	0	1	0.00	23	0
pdh	10	68	6	123	363	204	6	30	0	0	0	1	0.00	74	10
pdh	10	68	8	2435	1808	2000	7	64	0	1	0	245	6.00	5191	42
pdh	10	68	10	5861	1082	2000	8	217	0	3	0	2	13.40	4500	340
pdh	10	68	12	1498	1454	540	15	2004	0	0	0	112	10.05	4311	1167
pdh	10	68	14	1494	1738	647	24	2004	0	0	0	177	2.50	4939	1312
pdh	10	68	16	3141	1241	2000	12	2002	0	1	0	88	*8.88*	18000	1433
pdh	10	68	18	4317	2298	2001	10	2002	0	0	0	869	*16.68*	18000	1178
pdh	10	68	20	3641	1840	2000	6	164	0	1	0	733	*7.40*	18000	122
nobel_us	14	42	2	17	86	0	0	0	0	0	0	1	0.00	6	0
nobel_us	14	42	4	135	1385	51	12	4	0	0	0	1	12.22	455	285
nobel_us	14	42	6	52	254	0	0	0	0	0	0	1	0.00	21	2
nobel_us	14	42	8	446	1945	198	60	47	12	0	0	33	18.84	502	258
nobel_us	14	42	10	1759	5245	180	147	8795	41	56	0	67	17.74	1239	654
nobel_us	14	42	12	1221	4807	319	76	4465	56	3	5	111	11.18	1427	680
nobel_us	14	42	14	1706	4514	298	109	7611	123	0	1	210	5.67	1443	740
nobel_us	14	42	16	189	10240	167	16	1129	7	0	0	254	14.33	1533	590
nobel_us	14	42	18	540	10078	141	60	6813	78	6	0	826	17.00	5438	2845
nobel_us	14	42	20	1528	9023	263	66	9378	57	2	0	126	19.37	12343	4528
atlanta	15	44	2	165	2682	0	12	2000	0	0	0	2493	21.63	4366	1899
atlanta	15	44	4	1519	5718	363	26	2000	0	0	0	2972	25.71	9247	2792
atlanta	15	44	6	2951	5227	2001	22	61	0	0	0	315	3.94	3580	78
atlanta	15	44	8	2277	9661	786	31	2001	1	0	0	3065	*27.20*	18000	8791
atlanta	15	44	10	3680	9823	2005	10	0	0	0	0	4363	*40.41*	18000	4
atlanta	15	44	12	4975	8260	2001	22	0	0	0	0	11823	*41.96*	18000	13
atlanta	15	44	14	3655	9214	2004	10	0	0	0	0	4379	*48.72*	18000	4
atlanta	15	44	16	4148	8943	2001	10	0	0	0	0	2531	*34.14*	18000	3
atlanta	15	44	18	2354	9801	467	9	2004	0	0	0	1827	*33.81*	18000	6760.46
atlanta	15	44	20	5708	9915	2001	20	0	0	0	0	2681	*34.07*	18000	6
newyork	16	98	2	13	36	0	0	0	0	0	0	1	0.00	9	0
newyork	16	98	4	28	95	0	0	0	0	0	0	1	0.00	21	1
newyork	16	98	6	33	148	0	0	0	0	0	0	1	0.00	28	3
newyork	16	98	8	118	687	9	0	0	0	0	0	35	10.53	1275	974
newyork	16	98	10	658	3041	108	60	368	7	0	0	281	9.58	9081	6686
newyork	16	98	12	645	2203	117	41	711	1	0	0	94	20.29	9122	5384
newyork	16	98	14	1047	2708	233	96	871	7	0	1	226	*14.70*	18000	12322
newyork	16	98	16	551	10100	149	63	4564	2	0	0	359	25.86	10082	2319
newyork	16	98	18	1569	10153	312	175	11271	5	0	0	512	*25.88*	18000	4768
newyork	16	98	20	1328	10245	261	103	11389	5	0	0	264	*27.94*	18000	3171
nobel_germany	17	52	2	0	24	0	0	0	0	0	0	1	0.00	4	0
nobel_germany	17	52	4	19	1684	1	0	0	0	0	0	862	0.00	697	69
nobel_germany	17	52	6	183	6353	14	33	39	0	0	0	1599	41.20	2674	377
nobel_germany	17	52	8	210	6872	21	48	129	0	0	0	945	41.60	4460	765
nobel_germany	17	52	10	244	7634	113	66	234	8	0	0	1124	40.86	5700	829
france	25	90	2	33	101	0	0	0	0	0	0	1	0.00	43	1
france	25	90	4	172	1074	0	19	2	0	0	0	17	25.00	938	505
france	25	90	6	1037	4912	0	95	105	5	0	0	92	37.50	5952	2875
france	25	90	8	1934	6277	0	151	176	12	0	0	128	18.94	10230	5184
france	25	90	10	1118	4079	0	119	16	3	0	0	139	*11.24*	18000	5507
india	35	160	2	0	42	0	0	0	0	0	0	4	30.00	295	2
india	35	160	4	68	2029	10	0	0	0	0	0	61	36.75	17230	540
india	35	160	6	38	2451	15	6	219	0	0	0	2	*47.00*	18000	680
india	35	160	8	1146	5074	0	143	23	3	0	0	2	*42.34*	18000	5386

Table 6.4: Branch-and-Cut results for SNDlib instances with realistic traffic

Instance	$\lvert V_2\rvert$	$\lvert A_2\rvert$	|K|	NMSI	NCCS	NFCS	NC	NCo	NMSII	Nodes	Gap	TT
Bretagne_10	9	20	15	36	39	597	0	2	0	42	29.49	107
Bretagne_10	9	20	20	36	38	353	8	6	0	24	35.34	104
Bretagne_10	9	20	25	31	30	1344	0	0	0	8	*40.26*	18000
Bretagne_10	9	20	30	30	23	2341	0	0	0	4	*33.77*	18000
Bretagne_10	9	20	35	22	16	960	0	0	0	2	*48.48*	18000
Bretagne_10	9	20	42	39	20	590	2	0	0	12	*38.42*	18000
Bretagne_12	9	20	5	6	1	43	0	5	0	368	32.17	37
Bretagne_12	9	20	10	126	31	165	0	103	0	4483	43.28	3044
Bretagne_12	9	20	15	24	24	951	0	4	1	22	44.53	76
Bretagne_12	9	20	20	42	41	98	0	11	0	22	48.63	46
Bretagne_12	9	20	30	24	35	443	0	2	0	28	*47.68*	18000
Bretagne_12	9	20	35	33	32	256	0	0	0	25	*38.98*	18000
Bretagne_12	9	20	42	118	23	122	0	0	0	787	*24.50*	18000
Bretagne_25	9	20	5	28	14	373	0	3	0	1888	26.00	150
Bretagne_25	9	20	10	34	24	399	0	0	0	2101	24.20	340
Bretagne_25	9	20	15	49	74	652	0	11	0	249	28.33	1036
Bretagne_25	9	20	20	73	44	821	5	6	0	327	33.33	1100
Bretagne_25	9	20	30	112	81	789	11	4	0	23509	*39.93*	18000
Bretagne_25	9	20	42	139	66	1203	0	4	0	16704	*24.56*	18000
Bretagne_10	22	52	5	3	3	0	0	0	0	1	0.00	22
Bretagne_10	22	52	10	28	8	164	4	0	0	32	34.00	302
Bretagne_10	22	52	15	21	8	347	0	0	0	38	41.30	1242
Bretagne_10	22	52	20	21	4	5	0	0	0	14	36.96	247
Bretagne_10	22	52	30	28	28	1076	1	0	0	16	*47.71*	18000
Bretagne_12	22	52	5	31	17	376	0	24	0	5776	38.71	12138
Bretagne_12	22	52	10	67	49	6192	4	47	1	2082	*44.77*	18000
Bretagne_12	22	52	15	6	0	3	0	5	0	38	43.76	209
Bretagne_12	22	52	20	36	17	3825	1	13	2	60	44.78	*18000*
Bretagne_12	22	52	20	26	33	1483	0	0	0	16	37.96	*18000*
Bretagne_25	22	52	5	511	419	4465	21	14	3	7149	*31.00*	18000
Bretagne_25	22	52	10	9	22	3308	0	0	4	273	*49.00*	18000
Bretagne_25	22	52	15	38	12	9825	0	0	21	876	*53.00*	18000

Table 6.5: Branch-and-Cut results for real instances

The tested instances have graphs with 9 to 45 nodes and a number of commodities that varies between 5 and 42 for the smaller instances. Figure 6.5 shows the topology of the first group of instances. In particular, we have considered $|W| = 4$ for all the instances, and three possible subband capacities, namely $C = 10$ Gbit/s, 12.5 Gbit/s and 25 Gbit/s. Table 6.5 shows the results obtained for two over the three families of instances considered. We further give and example of solution obtained when solving an instance with 45 nodes and 10 commodities.

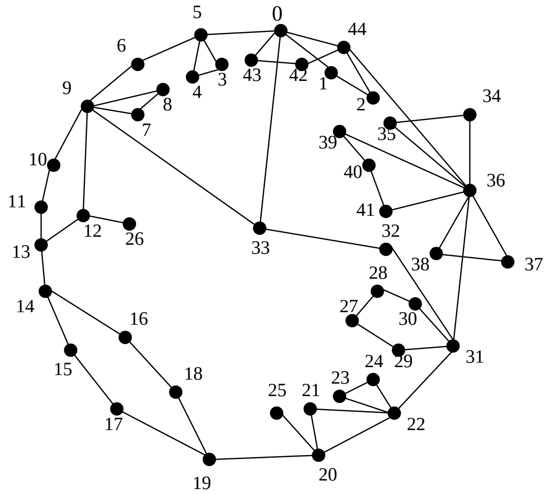


Figure 6.6: A real instance with 45 nodes and |K| = 10

It appears from Table 6.5 that 16 instances among the 32 tested instances were solved to optimality within the CPU time limit. Except for Bretagne_12 with 5 commodities, an optimal solution could be obtained within one hour for all the solved instances. Several observations can be maid based on these results. First concerning instances Bretagne with 9 nodes, we can see that we get better results when using a larger subband capacity C. This is due to the topology of these instances which is quite sparse (see Figure 6.5). Basically, finding a feasible routing for the commodities by using less subbands is a challenging task because of the graph topology. Indeed, the disjunction constraints make difficult to reuse the same paths in G_2 for the installed subbands. Besides, when $C = 25$ Gbit/s, commodities are more likely to be packed in the same subbands, which makes easier to find a good solution within the fixed time limit.

We noticed from Table 6.5 that results for instances with 22 nodes gets better when $C = 10$ Gbit/s. In fact, since the graph holds more nodes and arcs, it offers more possible paths, and hence more routing alternatives for both commodities and subbands. Finally, we notice that an important number of cover inequalities are generated for these instances. In fact, the traffic commodities here are relatively small and tends to have the same size. Cover inequalities are then more expected to appear than clique based inequalities.

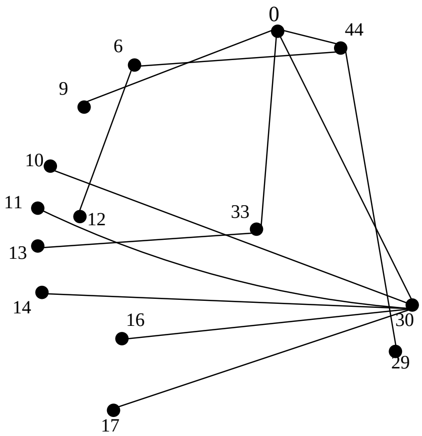


Figure 6.7: Design solution in G_1

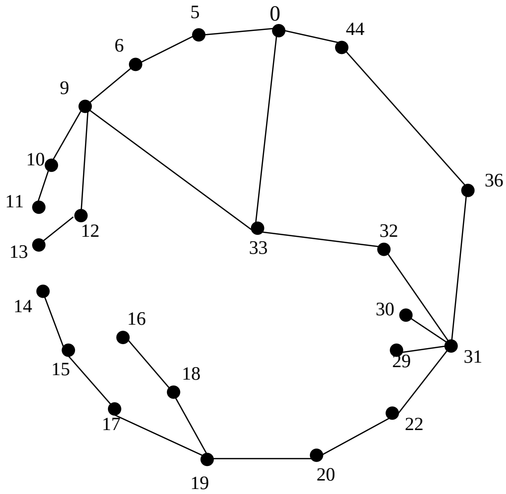


Figure 6.8: Routing in G_2

Figure 6.6 shows a real instance related to the backhaul network of Bretagne area. This instance consists of 45 nodes and 10 commodities that must be routed. The number of available subbands is $|W| = 4$ and the capacity of each subband is set to $C = 25$ Gbits/s. The instance have been solved by the Branch-and-Cut algorithm within 3 hours. The optimal solution obtained for this instance is depicted in Figure 6.7 and Figure 6.8.

6.3 Concluding remarks

In this chapter we have described a Branch-and-Cut algorithm to solve efficiently OMBND problem. This algorithm is based on the polyhedral results introduced in Chapter 5. We have first presented an overview of the main steps in the algorithm, then we discussed the separation problems associated with valid inequalities introduced in the previous chapter. We have tested our approach on SNDlib instances with realistic and randomly generated traffic commodities. We could show the gain provided by the separated valid inequalities regarding to the basic cut formulation. In particular, Min Set I, capacitated cutset and flow-cutset inequalities reduce the integrality gap at the root node, and solve OMBND problem more effectively. The remaining classes of valid inequalities improve the Branch-and-Cut algorithm but not significantly. However, it seems that more sophisticated separation routines are necessary to get full advantage of these valid inequalities without paying too much in CPU time.

Alternatively, these valid inequalities are further used within a Branch-and-Cut framework, to strengthen the flow-based formulation given in Chapter 7. This approach enabled to tackle real instances provided by Orange Labs, and to get good solutions for the problem within few hours. Also it could be of great interest to use a primal heuristic to get quickly good feasible solutions, and being able to handle larger instances.

In the subsequent, we discuss further modelling approaches for OMBND problem and present new algorithms for the problem using paths. We study the underlying column generation procedures and embed them within a Branch-and-Price framework.

Chapter 7

Optical Multi-Band Network Design using paths

Contents

In this chapter, we present a column generation approach to tackle the OMBND problem. First we propose a compact formulation for the problem that is used to deduce a *path formulation, obtained by a Dantzig-Wolfe decomposition. This path formulation holds a polynomial number of constraints, and two families of path variables that may be exponential. We then devise a Branch-and-Price algorithm using a double column generation procedure to solve the path formulation. A further "aggregated" path formulation is presented for the problem. We manage to solve this formulation using a second Branch-and-Price algorithm based on a two-stage column generation. Both approaches are then compared empirically, and some experiments are conducted on random and realistic instances to show their efficiency.*

7.1 Path formulation

In this section, we give two integer linear programming formulations based on path variables. For this purpose, we first introduce a compact (node-arc) formulation for the OMBND problem, that will be the starting point of a Dantzig-Wolfe decomposition to get path formulations.

7.1.1 Compact formulation

Let us first introduce some necessary notations. In this formulation, we use the families of variables introduced in formulation (5.1)-(5.7) (see Chapter 5). Recall that $y \in \{0,1\}^{|A_1||W|}$ are referred to as design variables, and are such that for each $e \in A_1$ and for each $w \in W$

$$y_{ew} = \begin{cases} 1, & \text{if } w \text{ is installed on } e, \\ 0, & \text{otherwise.} \end{cases}$$

Also, let $z \in \mathbb{R}^{A_1 \times W \times A_2}$ be such that for each arc $e \in A_1$, for each subband $w \in W$ and for each arc $a \in A_2$

$$z_a^{ew} = \begin{cases} 1, & \text{if } a \text{ belongs to a path in } G_2 \text{ associated with pair } (e,w), \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, let $x \in \mathbb{R}^{K \times A_1 \times W}$ such that for each commodity $k \in K$, for each arc $e \in A_1$ and for each subband $w \in W$

$$x_{ew}^k = \begin{cases} 1, & \text{if } k \text{ uses } (e,w) \textit{ for its routing}, \\ 0, & \text{otherwise.} \end{cases}$$

We will denote by m_1 and m_2 the number of arcs of G_1 and G_2, respectively. That is to say, $m_1 = |A_1|$ and $m_2 = |A_2|$. Furthermore, for each node s in V_1, we denote by $\delta^+(s)$ (resp. $\delta^-(s)$) the set of arcs in A_1 outgoing (resp. incoming) from s. Similarly, we denote by $\delta^+(s')$ (resp. $\delta^-(s')$) the set of arcs in A_2 outgoing (resp. incoming) from s', for each node s' in V_2.

Consider then the following integer programming formulation:

$$\min \sum_{e \in A_1} \sum_{w \in W} c(w) y_{ew}$$

$$\sum_{e \in \delta^-(s)} \sum_{w \in W} x^k_{ew} - \sum_{e \in \delta^+(s)} \sum_{w \in W} x^k_{ew} = \begin{cases} 1, & \text{if } s = d_k, \\ -1, & \text{if } s = o_k, \\ 0, & \text{otherwise}, \end{cases} \qquad \begin{array}{l} \forall k \in K, \\ \forall s \in V_1, \end{array} \tag{7.1}$$

$$\sum_{k \in K} D^k x^k_{ew} \leq C y_{ew}, \qquad \forall e \in A_1, w \in W, \tag{7.2}$$

$$\sum_{a \in \delta^-(s')} z^{ew}_a - \sum_{a \in \delta^+(s')} z^{ew}_a = \begin{cases} y_{ew}, & \text{if } s' = v', \\ -y_{ew}, & \text{if } s' = u', \\ 0, & \text{otherwise}, \end{cases} \qquad \begin{array}{l} \forall e = (u,v) \in A_1, \\ \forall w \in W, \\ \forall s' \in V_2, \end{array} \tag{7.3}$$

$$\sum_{e \in A_1} z^{ew}_a \leq 1, \qquad \forall w \in W, \forall a \in A_2, \tag{7.4}$$

$$0 \leq x^k_{ew} \leq 1, x^k_{ew} \in \{0,1\}, \qquad \forall k \in K, e \in A_1, w \in W, \tag{7.5}$$

$$0 \leq y_{ew} \leq 1, y_{ew} \in \{0,1\}, \qquad \forall e \in A_1, w \in W, \tag{7.6}$$

$$0 \leq z^{ew}_a \leq 1, z^{ew}_a \in \{0,1\}, \qquad \forall e \in A_1, w \in W, a \in A_2. \tag{7.7}$$

In this formulation, there are $m_1|W|$ binary design variables, $|K|m_1|W|$ flow variables for the routing of commodities in G_1, and $m_1|W|m_2$ flow variables for the routing of installed subbands in G_2. The objective is to minimize the total cost of the design, which is the overall cost driven by the subbands installation.

Equalities (7.1) are the flow conservation constraints for commodities of K. They ensure that a path is associated with each $k \in K$, between its origin node and its destination node, by using arcs of A_1 and subbands installed therein. They will be referred to as *commodities routing constraints*. Inequalities (7.2) are *capacity constraints* for the subbands. They guarantees that the flow using a certain arc does not exceed the capacity of any subband carried by that arc. Moreover, such a constraint, as we could see in previous chapters, ensures that a feasible solution can be obtained by installing enough subbands on G_1. Equalities (7.3) are the flow conservation constraints for the routing of installed subbands. They ensure that a path in G_2 is associated with each pair $(e, w) \in A_1 \times W$, between nodes corresponding to the extremities of e. Also recall that inequalities (7.4) express the *disjunction constraints* for the subbands of W. Finally, (7.5) to (7.7) are the *trivial* and *integrity constraints* associated with the variables of the formulation.

Note that the linear relaxation of this formulation is obtained by considering inequalities

$$0 \leq x_{ew}^k \leq 1, \quad \forall k \in K, e \in A_1, w \in W, \tag{7.8}$$
$$0 \leq y_{ew} \leq 1, \quad \forall e \in A_1, w \in W, \tag{7.9}$$
$$0 \leq z_a^{ew} \leq 1, \quad \forall e \in A_1, w \in W, a \in A_2. \tag{7.10}$$

instead of inequalities (7.5)-(7.7).

It is straightforward to see that integer linear programming formulation (7.1)-(7.7) is equivalent to OMBND problem. Formulation (7.1)-(7.7) will be referred to as *compact formulation* since the variables of the model as well as the constraints, are in polynomial number.

This model, as well as compact formulation of CSLND problem (see Chapter 3), suffers from many symmetries due to the large number of possible subbands location, and routing alternatives for both commodities and subbands. Thus, it is unlikely that handling the compact formulation by using a Branch-and-Bound approach allows to solve the problem efficiently, for realistic instances.

Besides, it is quite intuitive and natural to reformulate this model using path variables. In fact, the compact formulation suggests that underlying structures embedded in the problem, would benefit from being exploited. Furthermore, as we could see in Chapter 5, a solution to OMBND problem is essentially given by a set of paths in both graphs G_1 and G_2 (corresponding to virtual and physical layer respectively).

In what follows, we will apply a Dantzig-Wolfe decomposition to the compact formulation (7.1)-(7.7) in order to obtain a first path formulation.

7.1.2 Dantzig-Wolfe decomposition

The Dantzig-Wolfe decomposition was originally introduced by Dantzig and Wolfe, in 1960, for solving large scale integer linear programming problems [103]. This technique becomes now widely used for providing reformulations of ILP problems having specific structure, and tighter linear relaxation bounds (see [103, 105] and references therein for more details on this approach).

We propose here a Dantzig-Wolfe decomposition on the compact formulation (7.1)-(7.7). However, let us first introduce some necessary notations.

Recall that the subbands installed on arcs of G_1 are used independently by the commodities for their routing. In other words, every subband set up on an arc is considered as a copy of that arc. Consequently, G_1 is such that there exists $|W|$ parallel arcs between each pair of nodes $u, v \in V_1 \times V_1$. We will re-use the notation $(e, w) \in A_1 \times W$ to designate a pair such that w may be installed on e. (e, w) also denotes the copy having index w, of arc e. It In what follows, we will consider a path in G_2 between two nodes $u', v' \in V_2$ as a sequence of arcs $\{a_1, a_2, \dots, a_r\}$, such that $a_1 = (u', i'), i' \in V_2 \setminus \{u'\}$ and $a_r = (j', v')$, $j' \in V_2 \setminus \{v'\}$. Similarly, we define a path in G_1 between nodes u and v as a sequence of pairs $\{(e_1, w_1), (e_2, w_2), \dots, (e_r, w_r)\}$, where $e_1 = (u, i)$, $i \in V_1 \setminus \{u\}$, $e_r = (j, v)$, $j \in V_1$, and $w_1, w_2, \dots, w_r$ are the copies of $e_1, e_2, \dots, e_r$ used (see Figure 7.1).

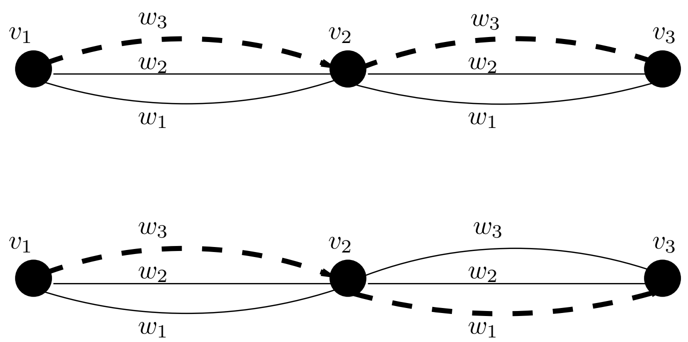


Figure 7.1: Two non equivalent paths in G_1

We then let Π_k be the set of paths routing k, computed in graph G_1, and using pairs $(e, w) \in A_1 \times W$. By the same way, we denote by P_{ew} the set of paths associated with (e, w), computed in G_2 and using arcs of A_2. We define the coefficients $a_k^{ew}(\pi)$, $e \in A_1$, $w \in W$, $k \in K$, $\pi \in \Pi_k$, that indicates whether if a pair $(e, w) \in A_1 \times W$ belongs to a path π that may be selected to route k, and 0 otherwise. We also introduce a coefficient $b_{ew}^a(p)$, for $a \in A_2$, $e \in A_1$, $w \in W$, $p \in P_{ew}$, that takes the value 1 if arc a is involved in the path associated with (e, w), and 0 otherwise.

For each path $\pi \in \Pi_k$, we define the variable $x^k(\pi)$, that takes the value 1 if π is used for the routing of k, and 0 otherwise. x^k will be referred to as *commodity path variables.* Also, for each path $p \in P_{ew}$, we define the binary variable $z^{ew}(p)$ that takes the value 1 if p is selected to be assigned to (e,w), and 0 otherwise. z^{ew} will be referred to as *subband path variables*. Both families of path variables are linked with the original "arc" variables. This relationship is given by

$$x_{ew}^{k} = \sum_{\pi\in\Pi_k} a_k^{ew}(\pi)x^k(\pi), \qquad \text{for all } k\in K, (e,w)\in A_1\times W, \tag{7.11}$$

$$z_a^{ew} = \sum_{p\in P_{ew}} b_{ew}^{a}(p)z^{ew}(p), \qquad \text{for all } (e,w)\in A_1\times W, a\in A_2. \tag{7.12}$$

Replacing the right hand-side of equalities (7.11) and (7.12) in formulation (7.1)-(7.7), yields a new formulation, given in what follows

$$\min \sum_{e\in A_1}\sum_{w\in W} c(w)y_{ew}$$

$$\sum_{\pi\in\Pi_k} x^k(\pi) \geq 1, \qquad \forall k\in K, \tag{7.13}$$

$$\sum_{k\in K}\sum_{\pi\in\Pi_k} a_k^{ew}(\pi)D^k x^k(\pi) \leq C y_{ew}, \qquad \forall (e,w)\in A_1\times W, \tag{7.14}$$

$$\sum_{p\in P_{ew}} z^{ew}(p) \geq y_{ew}, \qquad \forall (e,w)\in A_1\times W, \tag{7.15}$$

$$\sum_{e\in A_1}\sum_{p\in P_{ew}} b_{ew}^{a}(p)z^{ew}(p) \leq 1, \qquad \forall a\in A_2, w\in W, \tag{7.16}$$

$$0\leq x^k(\pi)\leq 1, x^k(\pi)\in\{0,1\}, \qquad \forall k\in K, \pi\in\Pi_k, \tag{7.17}$$

$$0\leq y_{ew}\leq 1, y_{ew}\in\{0,1\}, \qquad \forall (e,w)\in A_1\times W, \tag{7.18}$$

$$0\leq z^{ew}(p)\leq 1, z^{ew}(p)\in\{0,1\}, \qquad \begin{array}{l}\forall (e,w)\in A_1\times W,\\ p\in P_{ew}\end{array}. \tag{7.19}$$

By a commonly admitted result in network flow theory, inequalities (7.13) and (7.15) are equivalent to inequalities (7.1) and (7.3), respectively (see [5]). The remaining constraints are clearly the same as in (7.1)-(7.7). Inequalities (7.13)-(7.19) constitutes a path formulation for OMBND problem. Replacing constraints (7.17)-(7.19) by the following constraints

$$0\leq x^k(\pi)\leq 1, \qquad \forall k\in K, \pi\in\Pi_k, \tag{7.20}$$

$$0\leq y_{ew}\leq 1, \qquad \forall (e,w)\in A_1\times W, \tag{7.21}$$

$$0\leq z^{ew}(p)\leq 1, \quad \forall (e,w)\in A_1\times W, p\in P_{ew}. \tag{7.22}$$

gives the linear relaxation of path formulation.

This formulation holds a polynomial number of constraints with the same structure as in formulation (7.1)-(7.7). However, the number of variables may be exponential. Indeed, there is a huge number of candidates paths in both graphs G_1 and G_2. The *column generation* is a method that suits well to this kind of formulations.

In what follows, we describe such procedure and apply it how it can be applied to solve the linear relaxation of (7.13)-(7.19).

7.1.3 Double column generation

Column generation is a technique for solving linear programming formulations having a huge (exponential) number of variables. This approach consists in solving iteratively the problem with a subset of columns (path variables). We start the process by solving the linear program restricted to a subset of variables. Then at each iteration, an auxiliary (pricing) problem identifies the variables that should enter the current basis. If the auxiliary problem fails to identify additional variables, then the current solution is optimal for the linear program with all the variables.

In our case, formulation (7.13)-(7.19) holds two families of path variables, too large to appear explicitly in the formulation. Those families of variables correspond to paths computed in two different graphs, by considering different costs on the arcs. Therefore, we use two pricing problems, each one providing a subset of paths belonging to one of the families. In what follows, we describe the procedure that is used to generate the subset of variables that will appear in the initial linear program.

7.1.3.1 Initial solution

We use a heuristic procedure based on an idea presented in [20] to construct a feasible solution for OMBND problem. This procedure mainly consists in following steps.

Let $H = (V_H, A_H)$ be a graph corresponding to the solution in terms of design variables. In other words, H is a sub-graph such that $V_H = V_1$, and A_H containing arcs of G_1 where at least one subband is installed.

1) We start with $A_H = \emptyset$,

2) Then, for each commodity $k \in K$, we try to identify a path in H using the pre-installed subbands,

3) If such path exists, we associate it with k. Otherwise, we add $a_k = (o_k, d_k)$ to A_H and set up a subband, say w_k, over this arc.

4) We associate a path in G_2 with the pair (a_k, w_k) such that none of its sections has been assigned to subband w_k before.

5) If such path does not exists, we replace w_k by a subband that has not been used in previously, and we go back to step 2.

We assume that the set of available subbands W is large enough, so that a feasible solution, even expensive, can be identified. Moreover, it is clear that paths computed in H correspond to paths in G_1.

Let us denote by P_1 and P_2, the set of paths identified in H and G_2, respectively. We then start the column generation procedure with a subset of variables corresponding to paths of $P_1 \cup P_2$. The linear programming formulation (7.13)-(7.16)-(7.20)-(7.22) restricted to the design variables together with a subset of variables will be referred to as *Restricted Master Problem* (RMP).

7.1.3.2 Pricing problems

Now, let us denote by (x^*, y^*, z^*) the solution given by the restricted master problem. We will denote by α, β, γ and δ the dual variables associated with inequalities (7.13)-(7.16) of the path formulation. These dual variables are such that $\alpha^k \in \mathbb{R}_+$ for each $k \in K$, $\beta^{ew} \in \mathbb{R}_-$ and $\gamma^{ew} \in \mathbb{R}_+$ for each $(e, w) \in A_1 \times W$, while $\delta^{aw} \in \mathbb{R}_-$ for each $a \in A_2$, $w \in W$. The reduced cost associated with each path variable $x^k(\pi)$, $k \in K$, $\pi \in \Pi_k$ is then denoted by rc^k, and given by the following expression

$$rc^k(\pi) = -(\alpha^k + \sum_{e \in A_1} \sum_{w \in W} a_k^{ew}(\pi)\beta^{ew}) \tag{7.23}$$

while the reduced cost related to each path variable $z^{ew}(p)$, where $(e, w) \in A_1 \times W$, $p \in P_{ew}$, is denoted by rc^{ew}, and is given by

$$rc^{ew}(p) = -(\gamma^{ew} + \sum_{a \in A_2} b_a^{ew}(p)\delta^{aw}) \tag{7.24}$$

Therefore, we define, for each commodity $k \in K$, the pricing problem, as looking for a path such that $rc^k = \min\{rc^k : \pi \in \Pi_k\}$ and $rc^k < 0$, or concluding that no such path exists. Observe that, for each $k \in K$, and for each path $\pi \in \Pi_k$, rc^k is composed by a fixed term, namely $-\alpha^k$ that depends only on k, and a second term, which is related to $(e, w) \in A_1 \times W$. Recall that a path in G_1 is supposed to be formed by a sequence of pairs $(e, w) \in A_1 \times W$, such that w is installed on e. Thus, one may consider every dual variable β^{ew} as a weight settled on the pair (e, w). In consequence, $\sum_{e \in A_1} \sum_{w \in W} \beta^{ew}$ might be viewed as the length of the path π. Since we are looking for a path in Π_k that minimizes the function rc^k, this problem can be seen as a *shortest path* problem in the graph G_1.

By the same way, we define the pricing problem related to subband path variables as follows. For each pair $(e, w) \in A_1 \times W$, we wish to identify a path such that $rc^{ew} = \min\{rc^{ew}(p) : p \in P_{ew}\}$ and $rc^{ew} < 0$, or concluding that no such path exists. Again, for each pair $(e, w) \in A_1 \times W$, and for each path $p \in P_{ew}$, rc^{ew} is composed by a fixed term $-\gamma^{ew}$, and a term depending on arcs of A_2. Dual variables δ may be viewed as weights impacted on arcs of A_1. Thus, the pricing problem in this case is equivalent to a shortest path problem in graph G_2.

Remark 7.1 *Both pricing problems for commodity and subband path variables can be solved in polynomial time.*

Indeed, since $\beta^{ew} < 0$ for all $(e, w) \in A_1 \times W$, and $\delta^{aw} < 0$, for all $a \in A_2$, the weights on pairs (e, w) and arcs a are non negative. Thus, both pricing problems can be solved efficiently by using Dijkstra's algorithm [41].

If the value of the shortest path in G_1 is such that $rc^k < 0$ for some $k \in K$, then, at least one commodity path variable should be added to the RMP. Similarly, if the shortest path in G_2 is such that $rc^{ew} < 0$ for some $(e, w) \in A_1 \times W$, then at least one subband path variable has to enter the current basis. If no path variable is identified by pricing problems ($rc^k > 0$, for all $k \in K$, and $rc^{ew} > 0$, for all $(e, w) \in A_1 \times W$), then the optimal solution of the current linear program is also optimal for the linear relaxation of path formulation.

Figure 7.2 shows an example of solution obtained by solving linear relaxation of path formulation. This instance includes a unique commodity going from v_1 to v_3. The path in G_1 associated with this commodity is given by $\{(e_1, w_2), (e_2, w_1)\}$. First section of this routing path, namely (e_1, w_2), is itself assigned the path $\{a_5, a_6, a_7\}$ in G_2. Now suppose that we are looking for new path variables to be added to the current linear programming formulation. Then, Figure 7.3 shows how dual variables may be distributed on both graphs G_1 and G_2 to solve the pricing problems.

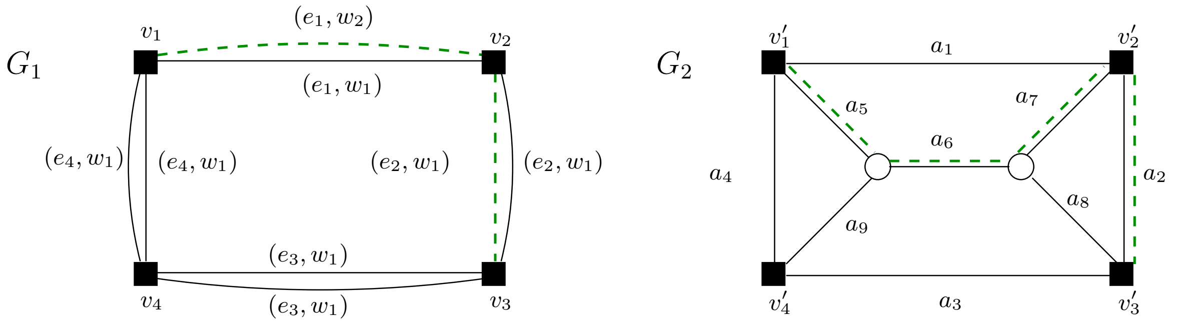


Figure 7.2: A solution of the path formulation

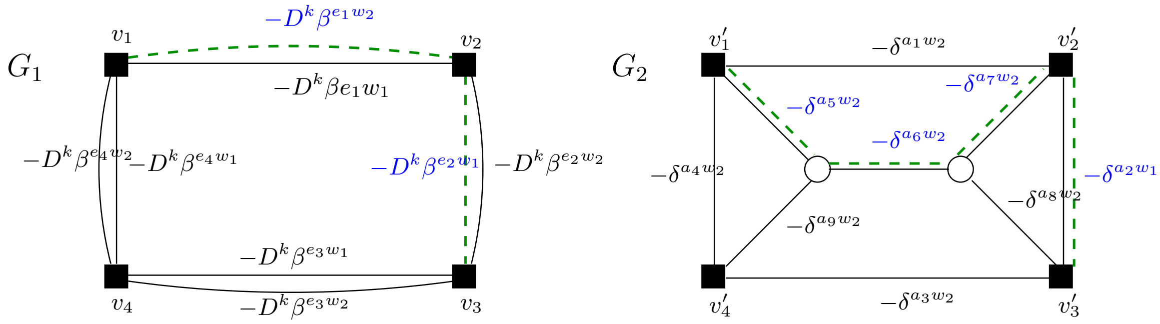


Figure 7.3: Graphs G_1 and G_2 with dual variables

Observe that, in G_1, the pairs (e_1, w_2), (e_2, w_1) that are involved in the routing of our commodity receive the weights $-D^k\beta^{e_1w_2}$ and $-D^k\beta^{e_2w_1}$. The path $\{(e_1, w_2), (e_2, w_1)\}$ then has a length given by $-D^k\beta^{e_1w_2} - D^k\beta^{e_2w_1}$. Note that only dual variables related to pairs $(e, w) \in A_1 \times W$ are distributed on G_1 since the fixed term $-\alpha^k$ can be considered after shortest path computation. Similarly, the section (e_1, w_1) for example is assigned a path in G_2 having weights $-\delta^{a_5w_2}$, $-\delta^{a_6w_2}$ and $-\delta_{a_7w_2}$. Again, the weights of arcs in G_2 are only given by dual variables related to arcs a. The fixed term $-\gamma^{ew}$ will also be added to the length of shortest path, after it is identified.

The solution provided by LP relaxation solved by column generation may not be integer. Therefore, it is not necessary a solution to OMBND problem. One has then to embed column generation procedure within a Branch-and-Bound algorithm in order to get an integer solution. This is known as a *Branch-and-Price* algorithm.

In section 7.3 we will describe a Branch-and-Price algorithm we have developed to solve OMBND problem. Before that, we present a new path formulation for the problem, which saves us the use of two independent pricing problems.

7.2 Aggregated path formulation

In this section describe a new approach to model OMBND problem using paths. This approach consists in first introducing an additional path formulation based on design variables together with commodity path variables. In addition, we use a new set of indicator coefficients that have a specific structure, so that they can express informations related to both graphs G_1 and G_2 simultaneously.

The objective here, is to attempts to overcome those two pricing problems that operate independently, and to get benefits from the relationship between G_1 and G_2 to embed a double information in a unique family of path variables. We introduce a two-stage procedure to price out those path variables, and present how the so-obtained column generation can be integrated within a Branch-and-Price framework (see section 7.3). Some experiments are conducted to show the performances of both Branch-and-Price algorithms, the numerical results are presented in section 7.4.

7.2.1 Path formulation

Consider the design variables y and commodity path variables x defined in the previous section. Recall that commodity path are computed in graph G_1. We will define a set of coefficients, denoted φ. Let k be a commodity of K and π a path of Π_k. For each pair $(e,w) \in A_1 \times W$ and each arc $a \in A_2$, $\varphi_a^{ew}(\pi)$ is such that

$$\varphi_a^{ew}(\pi) = \begin{cases} 1, & \text{if } \pi \text{ uses the pair } (e,w) \text{ in } G_1 \text{ and it is assigned a path in } G_2 \text{ using } a, \\ 0, & \text{otherwise.} \end{cases}$$

Figure 7.4 depicts a path in G_1 between nodes v_1 and v_4, that will be denoted π. This path is composed by pairs (e_1,w_2), (e_2,w_1) and (e_3,w_2). Each section of π is itself associated with a path in G_2. For example, (e_2,w_1) is assigned the path $\{a_2,a_3\}$. In this example, coefficients φ will take the following values: $\varphi_{a_1}^{e_1w_2}(\pi) = 1$, $\varphi_{a_2}^{e_2w_1}(\pi) = \varphi_{a_3}^{e_2w_1}(\pi) = 1$, $\varphi_{a_4}^{e_3w_2} = 1$, while $\varphi_a^{ew}(\pi) = 0$ for the remaining entries.

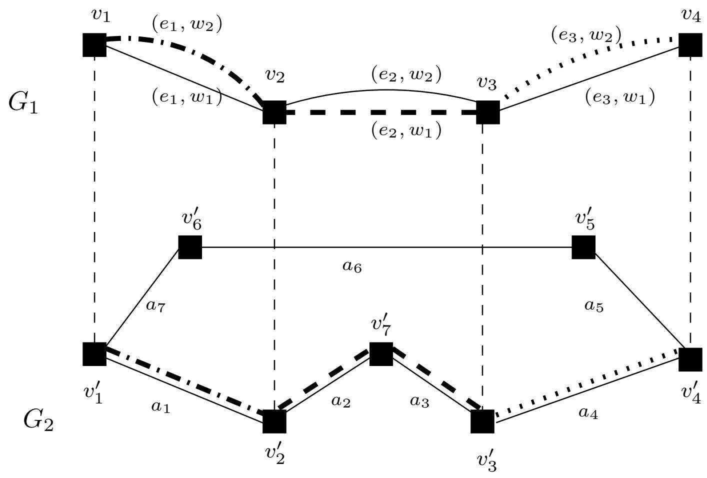


Figure 7.4: Two associated paths

Using this new coefficient, together with design and commodity path variables, we give the following integer linear programming formulation for OMBND problem:

$$\min \sum_{e \in A_1} \sum_{w \in W} c(w) y_{ew}$$

$$\sum_{\pi \in \Pi_k} x^k(\pi) \geq 1, \qquad \forall k \in K, \tag{7.25}$$

$$\sum_{k \in K} \sum_{\pi \in \Pi_k} \varphi_a^{ew}(\pi) D^k x^k(\pi) \leq C y_{ew}, \quad \forall e \in A_1, w \in W, a \in A_2, \tag{7.26}$$

$$\sum_{e \in A_1} \sum_{k \in K} \sum_{\pi \in \Pi_k} \varphi_a^{ew}(\pi) x^k(\pi) \leq 1, \qquad \forall a \in A_2, w \in W, \tag{7.27}$$

$$0 \leq x^k(\pi) \leq 1, x^k(\pi) \in \{0, 1\}, \qquad \forall k \in K, \pi \in \Pi_k, \tag{7.28}$$

$$0 \leq y_{ew} \leq 1, y_{ew} \in \{0, 1\}, \qquad \forall (e, w) \in A_1 \times W. \tag{7.29}$$

In this formulation there is a polynomial number of constraints and design variables, but a huge number of commodity path variables. Observe that all the constraints of the problem are expressed by formulation (7.25)-(7.29). Indeed, inequalities (7.25) are the commodity routing constraints. They ensure that a path in G_1 is associated with each commodity for its routing. Inequalities (7.26) are the capacity constraints for every pair (e, w) of $A_1 \times W$. Remark that they also appear for each $a \in A_2$, since a belong to the definition of coefficient φ. Inequalities (7.27) express indirectly the disjunction constraints for every arc $a \in A_2$ and every subband $w \in W$. In fact, each arc a used in a path associated with some section of π ($\pi \in \Pi_k$, for $k \in K$) is assigned at most once with subband w. This formulation will be referred to as *aggregated path formulation.*

Notice that, since we projected out subband path variables, the solution will be given by a set of subbands to install on G_1 as well as a set of paths for commodities routing. However, it is possible to reconstruct a complete description of the solution for OMBND problem, as coefficient φ will somehow bring out the path in G_2 associated with each pair $(e, w) \in A_1 \times W$ such that w is installed on e.

Similarly to formulation (7.13)-(7.19), the number of commodity path variables here may be exponential. Therefore, using column generation to solve the linear relaxation of (7.25)-(7.29) is required. In what follows, we describe the details of such procedure applied to aggregated path formulation.

7.2.2 Column generation

In this procedure, we solve the linear relaxation of (7.25)-(7.29) with an initial subset of paths (RMP). These path are computed in G_1 and generated using the procedure described in 7.1.3.1. Then we look for missing paths with negative reduced cost by solving a two-stage pricing problem. In such paths are identified, we add them to the RMP and repeat the process until no additional path may be generated.

Let us denote by α, β and γ the dual variables associated with the constraints (7.25)-(7.27), respectively. α is such that for each $k \in K$, $\alpha^k \in \mathbb{R}_-$, β is such that $\beta^{ewa} \in \mathbb{R}_+$, for each $e \in A_1$, $w \in W$ and $a \in A_2$. Finally, dual variables γ are such that $\gamma^{aw} \in \mathbb{R}_+$. Therefore, the reduced cost related to each commodity path variable $x^k(\pi)$, $k \in K$, $\pi \in \Pi_k$, is given by the following expression

$$rc^k(\pi) = -(\alpha^k + \sum_{e \in A_1} \sum_{w \in W} \sum_{a \in A_2} \varphi_a^{ew}(\pi)(D^k \beta^{ewa} + \gamma^{aw}))$$

Hence, we define for each commodity $k \in K$, the pricing problem, as trying to identify a path such that $rc^k = \min\{rc^k(\pi) : \pi \in \Pi_k\}$ and $rc^k < 0$. Note that here, this operation can be carried in two stages. First, dual variables γ are distributed on arcs of G_2, so that for each (e, w), every arc $a \in A_2$ receives $-\gamma^{aw}$. Then, for each (e, w), we compute the shortest path in G_2 using weights γ. Let us denote by p this path, and $l(e, w)$ its length. The second step consists in setting on each pair $(e, w) \in A_1 \times W$, a weight given by $-D^k \beta_a^{ew} + l(e, w)$, where $a \in p$. We then compute the shortest path in G_1 between nodes o_k and d_k. If the value of the shortest path in G_1 is such that $rc^k < 0$, then the corresponding commodity path variable should be added to the current linear program.

Note that, although the generated variable is related to a path in G_1, its reduced cost takes into account dual information impacted on both graphs G_1 and G_2.

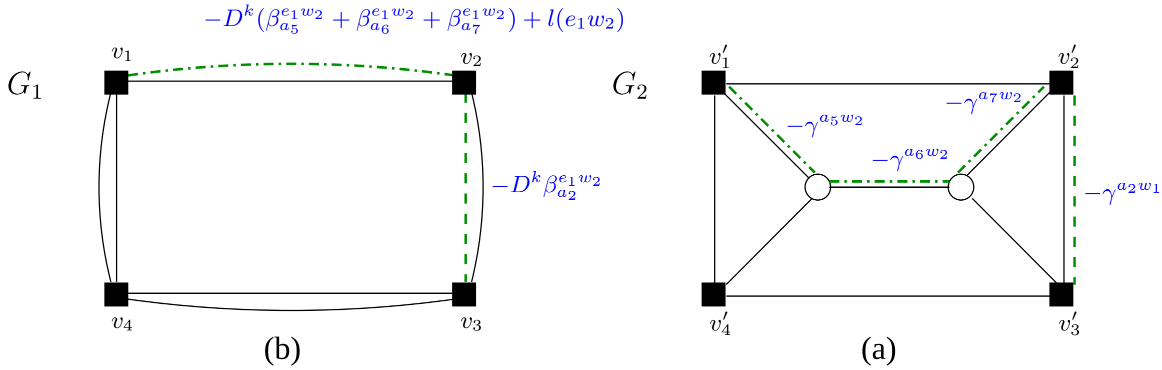


Figure 7.5: Graphs G_1 and G_2 with dual variables (from the aggregated path formulation)

Figure 7.5 shows an example of instance where each set of arcs carries its corresponding weight in terms of dual variables. In fact, we can see in Figure 7.5 (a) the first step of the pricing process, which consists in impacting weights based on γ dual variables on each arc of A_2. For example, the shortest path in G_2, corresponding to (e_1, w_2) is $\{a_5, a_6, a_7\}$. The length of this shortest path is a part of the weight assigned to pair (e_1, w_2), that receives $-D^k(\beta_{a_5}^{e_1w_2} + \beta_{a_6}^{e_1w_2} + \beta_{a_7}^{e_1w_2}) + l(e_1w_2)$, where $l(e_1w_2) = -(\gamma^{a_5w_2} + \gamma^{a_6w_2} + \gamma^{a_7w_2})$ (see Figure 7.5 (b)). It remains then to compute the shortest path in G_1, using weights based on the first step, together with dual variables β.

All the weights based on dual variables and impacted on arcs of G_1 and G_2 are positive, hence we can use Dijkstra's shortest path algorithm for both steps of the pricing procedure. Note that the column generation here does not allow to get a feasible solution for OMBND problem, since this solution might not be integer.

In what follows, we describe how both column generation procedures are embedded within Branch-and-Bound framework, to get the so-called Branch-and-Price algorithm, and to solve OMBND problem.

7.3 Branch-and-Price

We have developed two Branch-and-Price algorithms, based on path formulations proposed for OMBND problem. In next section, we will describe the framework of those algorithms.

7.3.1 Overview

Consider given two graphs G_1, G_2, a set of commodities K and a set of available subbands W. Also recall that a cost $c(w) > 0$ is associated with each subband of W. In both path formulations, we consider that this cost increases with the index of the subband. Typically, we let $c(w_1) \leq c(w_2) \leq c_{w_3} \leq \ldots \leq c(w_r)$, where $r = |W|$. This assumption comes from a practical requirement, that is subbands $i + 1$ should not be installed before subband i is installed. In some sense, this supposition is helpful for the model handling, since it also allows to break some symmetries on pairs (e, w).

To start the optimization, we set up both linear relaxations of (7.13)-(7.19) and (7.25)-(7.29), restricted to a subset of path variables. The initial subset of path variables is generated using the procedure described in section 7.1.3.1 for both formulations. Let us denote by $(\overline{x}, \overline{y}, \overline{z})$ the optimal solution of the restricted linear relaxation of path formulation (respectively aggregated path formulation). Then, we solve the two pricing problems (respectively the two stage pricing problem), and add the generated path variables to the current LP, if any.

The main steps of Branch-and-Price algorithm for path formulation are summarized in Algorithm 7. Note that for the aggregated path formulation, steps 3 to 9 are replaced by solving the two stage pricing problem for all $k \in K$, and add the path minimizing $rc^k(\pi)$, $\pi \in \Pi_k$ and with $rc^k < 0$, if such path exists.

7.3.2 Branching

Let $(\mathcal{P})$ denote the linear program at a given node of the Branch-and-Price tree. Suppose that the optimal solution of linear relaxation of $(\mathcal{P})$ is fractional. Let $(\overline{x}, \overline{y}, \overline{z})$ be this fractional solution. The branching phase, consists in choosing a fractional variable say $\overline{x}^1$ among those in $(\overline{x}, \overline{y}, \overline{z})$, and create two sub-problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ by adding either constraint $\overline{x}^1 \leq \lfloor \overline{x}^1 \rfloor$ or $\overline{x}^1 \geq \lceil \overline{x}^1 \rceil$ to $(\mathcal{P})$. In our problem, it is to fix $\overline{x}^1$ either to 0 or 1.

Algorithm 7: Branch-and-Price algorithm for path formulation

Data : two graphs $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$, a set of commodities K, a set of available subbands W, and a cost vector $c \in I\!R^W$.
Output : optimal solution of OMBND problem, or best feasible upper bound.

1: LP $\leftarrow$ LP$_{initial}$;
2: solve the linear program LP;
   let $(\overline{x}, \overline{y}, \overline{z})$ be the optimal solution of LP;
3: Consider the dual variables and solve the two pricing problems;
4: **If** for all $(e, w) \in A_1 \times W$, $p \in P_{ew}$, $rc^{ew} > 0$ **then**
5: **If** for all $k \in K$, $\pi \in \Pi_k$, $rc^k > 0$ **then**
6: go to 10;
7: **else**
8: Add the variables induced by rc^{ew} and rc^{ew} with negative reduced cost;
9: go to 2
10: **If** $(\overline{x}, \overline{y}, \overline{z})$ is integer **then**
11: $(\overline{x}, \overline{y}, \overline{z})$ is optimal for OMBND. Stop;
12: **else**
13: Create two sub-problems by branching on design variables first;
14: **forall** open sub-problem **do**
15: go to 2;
16: **return** the best optimal solution for all sub-problems.

Several branching strategies have been developed to choose efficiently a fractional variables to branch on. In particular, most of the branching strategies proposed for path-based formulations are defined on original (arc flow) variables. In [16], Barnhart et al. propose a generalization of Ryan and Foster [97] branching rule for origin-destination integer multicommodity flow problems. This strategy consists in forbidding the use of some specific arcs in the considered paths. Such operation may be performed either by adding branching constraints that correspond to the forbidden arcs, or by removing those arcs from the graph when computing the shortest path (see [45] for a good tutorial on column generation and branch-and-price applied to vehicle routing problems). We refer the reader to [103, 105, 106] for more details on branching schemes in IP column generation.

In our case, we have observed that branching first on design variables was very strong, and only few path variables remain fractional after that, for both formulations. This can be explained by the close relationship between variables in both formulations.

Thus, we have used the following strategy. First we perform branching on fractional design variables y by choosing the variable with fraction close to 0.5 and high absolute objective function coefficient. Fixing design variables helps to get few remaining path variables that still fractional. If all the design variables are integer, then we perform branching on path variables by setting their value either to 0 or 1.

Based on these features, we devised two Branch-and-Price algorithms for OMBND problem by using the path and aggregated path formulations. We have tested our approaches on a set of random and realistic instances. The results are shown in the coming section.

7.4 Computational experiments

7.4.1 Implementation's feature

We have implemented the Branch-and-Price algorithms described in the previous section in C++ using ABACUS 3.2 [4] to handle the Branch-and-Price tree, and CPLEX 12.5 [2] as LP solver. Our approach was tested on a processor Intel Core i5-3210M CPU 2.50GHz $\times$ 4 with 3.7 Gb RAM, running under ubuntu 12.10 platform. We fixed the maximum CPU time to 3 hours.

Both algorithms were tested on random and realistic instances of network. The realistic instances are obtained from SNDlib data for instances *dfn_ bwin*, *dfn_ gwin*, *newyork* and *france*.

Note that we have performed the same data pre-processing as described in Chapter 4. The entries of the different tables presented in the sequel are the following:

V_2	:	number of nodes in G_2,
A_2	:	number of arcs,
K	:	number of commodities,
Gap	:	the relative error between the best upper bound (optimal solution if the problem has been solved to optimality) and the lower bound obtained provided by the compact formulation,
columns	:	number of generated path variables,
nodes	:	number of nodes in the Branch-and-Cut tree,
TT	:	total CPU time in h:m:s
TTpricing	:	CPU time spent in pricing out path variables (in %).

7.4.2 Managing infeasibility

Branching by setting variables to 0 or 1 may induce an infeasible linear program at a given level of the Branch-and-Price tree in ABACUS. Therefore, to avoid such situation, we have considered a set of "artificial" variables appearing in the critical constraints. We denote by τ and θ these variables and we let $\tau^k \in \mathbb{R}$, $0 \leq \tau^k \leq 1$, for each $k \in K$, and $\theta^{ew} \in \mathbb{R}$, $0 \leq \theta^{ew} \leq 1$, for each $(e, w) \in A_1 \times W$. Variables τ are involved in inequalities (7.13) (path formulation) and (7.25) (aggregated path formulation), while θ appears in inequality (7.15) in path formulation.

Notice that we do not use such variables in inequalities (7.14), (7.16), (7.26), and (7.27), since fixing variables to 0 does not affect feasibility of those constraints. We associate with artificial variables a large cost in the objective function, so that they penalizes its value if they are not equal to zero. However, these variables ensure that a feasible solution can always be identified, even if its cost is expensive.

7.4.3 Computational results

Our first series of experiments involve random instances, whose topologies as well as the commodities were randomly generated. We have considered graphs with 6 to 14 nodes, and at most 18 commodities per instance. Tables 7.1 and 7.2 report the results given by the column generation and the Branch-and-Price approaches on solving both path and aggregated path formulations, for random instances. The reported results concern 35 instances with a number of nodes in the physical layer (graph G_2) varying from 6 to 14 nodes, and a number of arcs varying from 16 to 40. We have considered up to 18 commodities for each kind of graph, and the number of available subbands is $|W| = 4$ except for the 14 nodes instances, where $|W| = 5$.

Table 7.1 shows in particular the results obtained by both column generation procedures for linear relaxation of formulations (7.13)-(7.19) and (7.25)-(7.29). The two last columns contain results provided by the compact formulation, namely the gap and CPU time computation. Note that the compact formulation is solved by Branch-and-Bound procedure. It appears from this table that gap provided by path formulation is equivalent to one of the compact formulation. Indeed, this shows empirically that both formulations have the same linear relaxations. We also remark that for most of the instances, the gap provided by path formulation is better than one of aggregated path formulation. In fact, except for instances with $|V_2| = 6$, $|K| = 8$, 10 and 11, and $|V_2| = 14$, $|K| = 8$, the gap value for path formulation is smaller than one of aggregated path formulation.

Table 7.1: Comparing linear relaxations

				Path formulation		Aggregated path formulation		Compact formulation	
$\|V_2\|$	$\|A_2\|$	|W|	|K|	Gap (%)	Columns	Gap (%)	Columns	Gap (%)	TT
6	16	4	2	25.00	8	25.00	39	25.00	0:05:32
6	16	4	4	47.50	16	47.50	73	47.50	0:07:53
6	16	4	6	45.00	24	53.33	86	45.00	0:10:49
6	16	4	8	41.43	32	37.14	211	41.43	0:49:32
6	16	4	10	47.14	49	41.43	281	47.14	1:00:23
6	16	4	12	48.75	57	43.75	165	48.75	1:45:03
8	24	4	2	0.00	8	0.00	51	0.00	0:08:56
8	24	4	4	25.00	16	25.00	95	25.00	0:21:51
8	24	4	6	33.33	24	33.33	140	33.33	0:29:23
8	24	4	8	6.25	36	6.25	147	6.25	1:02:14
8	24	4	10	15.50	40	28.00	223	15.50	1:12:09
8	24	4	12	12.92	48	26.67	211	12.92	1:02:14
8	24	4	14	21.92	56	25.38	311	21.92	2:31:46
8	24	4	16	32.31	68	33.08	377	32.31	2:49:01
8	24	4	18	35.63	76	36.25	383	35.63	2:52:21
10	36	4	2	0.00	8	0.00	64	0.00	0:10:37
10	36	4	4	50.00	16	50.00	139	50.00	0:18:22
10	36	4	6	3.33	24	3.33	524	3.33	0:32:51
10	36	4	8	44.44	38	55.55	381	44.44	1:44:02
10	36	4	10	57.31	46	59.23	433	57.31	2:05:39
10	36	4	12	56.07	54	57.86	533	56.07	2:55:01
12	46	4	2	0.00	8	0.00	80	0.00	1:15:22
12	46	4	4	33.33	16	33.33	165	33.33	1:35:22
12	46	4	6	46.67	24	46.67	433	46.67	2:09:59
12	46	4	8	47.14	33	47.14	598	47.14	2:23:51
12	46	4	10	33.13	41	37.50	668	33.13	2:45:33
12	46	4	12	20.63	49	25.00	1047	20.63	3:00:00
14	40	5	2	0.00	11	25.00	218	0.00	1:49:32
14	40	5	4	0.00	21	12.50	768	0.00	2:33:01
14	40	5	6	14.29	31	14.29	799	14.29	3:00:00
14	40	5	8	44.40	46	41.11	693	44.40	3:00:00
14	40	5	10	37.51	50	39.23	1079	37.51	3:00:00
14	40	5	12	10.63	61	11.92	836	10.63	3:00:00
14	40	5	14	34.47	71	35.00	943	34.47	3:00:00
14	40	5	16	12.47	130	20.59	1103	12.47	3:00:00

We can see that column generation procedure do not perform the same way for both path formulations. Indeed, although the number of generated variables in the first procedure is not so important (less than 100 path variables, except for the last instance), it is significantly higher for the second procedure. This can be due to the fact that the aggregated approach might somehow induce a loss of information provided by the bi-layer structure of the problem, and the interaction between path variables in

				Path formulation				Aggregated path formulation			
$\|V_2\|$	$\|A_2\|$	|W|	|K|	columns	nodes	TT	TTpricing (%)	columns	nodes	TT	TTpricing (%)
6	16	4	2	8	3	0:00:01	0.00%	98	3	0:00:00	72.72%
6	16	4	4	24	107	0:00:02	17.16%	467	31	0:00:02	74.78%
6	16	4	6	36	219	0:00:23	19.91%	119	5	0:00:01	81.08%
6	16	4	8	39	403	0:00:04	18.13%	234	11	0:00:04	88.13%
6	16	4	10	399	3893	0:00:49	21.00%	457	11	0:00:03	93.85%
6	16	4	12	6249	24819	0:03:45	19.65%	357	11	0:00:02	92.16%
8	24	4	2	8	1	0:00:01	0.00%	51	1	0:00:01	54.54%
8	24	4	4	16	15	0:00:01	9.37%	97	15	0:00:01	63.63%
8	24	4	6	24	65	0:00:03	17.60%	143	65	0:00:03	87.83%
8	24	4	8	32	65	0:00:03	22.14%	415	3	0:00:02	92.64%
8	24	4	10	40	1189	0:00:26	18.40%	378	5	0:00:01	88.54 %
8	24	4	12	48	2585	0:00:59	19.84%	670	5	0:00:01	89.31 %
8	24	4	14	56	2048	0:08:31	18.87%	688	10	0:00:03	86.43 %
8	24	4	16	74	3280	0:16:44	18.52%	598	7	0:00:01	91.22 %
8	24	4	18	82	3580	0:17:00	19.42%	720	15	0:00:23	89.46 %
10	36	4	2	8	1	0:00:00	27.27%	64	1	0:00:00	82.92%
10	36	4	4	16	73	0:00:05	15.54%	150	9	0:00:04	88.54 %
10	36	4	6	62	127	0:00:06	18.94%	645	11	0:00:12	79.43 %
10	36	4	8	205	859	0:01:18	18.76%	436	17	0:00:20	82.09 %
10	36	4	10	481	3559	0:03:06	20.79%	543	23	0:00:57	86.55 %
10	36	4	12	1060	18527	0:28:46	19.70%	712	159	0:01:39	88.63 %
12	46	4	2	8	1	0:00:00	20.00%	80	1	0:00:01	87.80%
12	46	4	4	16	73	0:00:11	14.39%	165	1	0:00:01	86.43 %
12	46	4	6	77	127	0:00:12	17.53%	650	17	0:00:04	87.32 %
12	46	4	8	52	801	0:01:17	17.00%	670	15	0:00:03	88.29 %
12	46	4	10	40	1695	0:02:44	18.05%	769	7	0:00:07	91.43 %
12	46	4	12	260	509	0:01:30	24.08%	2610	117	0:00:02	92.43 %
14	40	5	2	11	1	0:00:00	17.85%	218	1	0:00:00	79.33 %
14	40	5	4	26	1	0:00:00	31.81%	932	179	0:00:58	85.34 %
14	40	5	6	36	17	0:00:08	13.28%	1079	237	0:01:01	92.12%
14	40	5	8	112	491	0:03:25	13.86%	1011	559	0:01:59	89.21 %
14	40	5	10	502	2771	0:18:49	15.82%	2392	3591	0:20:53	95.22 %
14	40	5	12	786	2771	0:19:12	18.06%	1221	2375	0:16:41	93.01 %
14	40	5	14	294	3479	0:20:45	17.90%	1079	3277	0:23:54	87.44 %
14	40	5	16	1722	2051	0:11:12	30.93%	2467	3559	0:28:37	88.65 %

Table 7.2: Branch-and-Price results for random instances

both graphs G_1 and G_2.

Table 7.2 summarizes the results obtained by both Branch-and-Price algorithms for solving path and aggregated path formulations. We can see that all the instances presented in this table were solved to optimality by our Branch-and-Price algorithms within the time limit. In particular, note that the CPU time for both algorithms is smaller then one of the Branch-and-Bound algorithm (last column of Table 7.1). We can see for example that, even instances with $|V_2| = 14$ and $|K| = 6$ to 16, for which Branch-and-Bound algorithm could not prove the optimality of the identified solution within 3 hours, we could find the optimal solution in a few minutes. This clearly shows that a column generation based approach performs much more better than a classical Branch-and-Bound on the compact formulation. Note that, except for some instances, the number of variables generated within the second Branch-and-Price algorithm is still higher than one in the first Branch-and-Price. Also we can remark that most of the added variables are generated in the root node of the Branch-and-Price tree, for both algorithms. It should be pointed out that the number of nodes in the first Branch-and-Price tree is more important than in the second Branch-and-Price tree. In other words, we can observe that in the second algorithm, most of the columns are generated in the higher level nodes of the second tree, while only few columns are generated along a large-size tree for the first algorithm.

Our second series of experiments concern realistic instances based on data from SNDlib for networks *dfn_bwin*, *dfn_gwin*, *newyork* and *france*. Those instances have graphs with 10 to 25 nodes, while the number of commodities varies between 4 and 30 for dfn_gwin and newyork (we have considered up to 18 commodities for dfn_bwin and 16 commodities for france). The results of the Branch-and-Price algorithm based on the double column generation are summarized in Table 7.3. Table 7.4 shows the results provided by the Branch-and-Price algorithm using the two-stage column generation.

It appears from Table 7.3 that all the considered instances have been solved to optimality using the Branch-and-Price approach, within the fixed time limit. In fact, 30 instances have been solved to optimality in less than 10 minutes. Moreover, note that 11 among the 40 tested instances were solved to optimality at the root node. This can show that our data-preprocessing performs well on realistic instances. Due to the size and structure of some instances, we can observe that the CPU time spent by the algorithm in pricing operations increases compared to its average value for random instances (see Table 7.1). However, it seems that the number of generated columns in the whole tree is not so important regarding to the size of the instances. This is thank to our procedure to generate initial paths, that helps to identify a first set of interesting variables and thus to form a good initial basis. For the remaining instances, the number of generated path increases with the size of the instance. Yet our algorithm may perform some strange behaviour. Basically, more path variables are generated for instance newyork with $|K| = 25$, than for instance newyork with $|K| = 30$. We can explain such a result by the fact that the routing of some commodities may be challenged by the size (traffic amount) of other commodities. Indeed, the more commodities will be "conflictual" as they can not be packed together in the

Table 7.3: Branch-and-Price results for SNDlib-based instances - Path formulation

Instance	$\|V_2\|$	$\|A_2\|$	$\|W\|$	$\|K\|$	gap(%)	columns	nodes	TT	TTpricing (%)
dfn_bwin	10	90	4	2	25.00	8	3	0:00:00.28	28.5714%
dfn_bwin	10	90	4	4	12.50	16	3	0:00:00.28	28.5714%
dfn_bwin	10	90	4	6	8.33	41	3	0:00:00.37	45.9459%
dfn_bwin	10	90	4	8	43.75	166	397	0:00:26.60	31.8797%
dfn_bwin	10	90	4	10	40.00	247	859	0:00:44.09	32.4563%
dfn_bwin	10	90	4	12	29.17	49	381	0:00:19.66	29.5015%
dfn_bwin	10	90	4	14	27.59	81	2419	0:02:08.20	30.4992 %
dfn_bwin	10	90	4	16	27.27	510	4265	0:04:00.16	32.82 %
dfn_bwin	10	90	4	18	26.32	219	5913	0:05:42.10	31.48 %
dfn_gwin	11	94	4	2	0.00	10	1	0:00:00.44	36.36 %
dfn_gwin	11	94	4	4	0.00	20	1	0:00:00.44	34.0909%
dfn_gwin	11	94	4	6	0.00	36	1	0:00:00.40	52.5%
dfn_gwin	11	94	4	8	0.00	53	1	0:00:00.5	66.0714%
dfn_gwin	11	94	4	10	0.00	60	1	0:00:00.52	59.6154%
dfn_gwin	11	94	4	12	0.00	78	1	0:00:00.39	58.9744%
dfn_gwin	11	94	4	14	0.00	89	1	0:00:00.42	59.5238%
dfn_gwin	11	94	4	16	5.88	117	7	0:00:02.03	43.34 %
dfn_gwin	11	94	4	18	19.44	133	587	0:01:41.22	28.37 %
dfn_gwin	11	94	4	20	25.00	2499	2755	0:10:50.70	34.05 %
dfn_gwin	11	94	4	25	21.28	1620	2931	0:10:47.54	33.10 %
dfn_gwin	11	94	4	30	20.41	830	2931	0:10:32.60	31.08 %
newyork	16	92	5	2	0.00	10	1	0:00:00:10	28.5714%
newyork	16	92	5	4	0.00	20	1	0:00:00.72	33.33 %
newyork	16	92	5	6	0.00	30	1	0:00:00.77	36.36 %
newyork	16	92	5	8	37.50	567	807	0:08:09.32	26.46 %
newyork	16	92	5	10	40.00	172	2905	0:16:03.30	27.62 %
newyork	16	92	5	12	41.67	1358	6331	0:49:34.92	26.95 %
newyork	16	92	5	14	0.00	104	1	0:00:01.30	58.4615%
newyork	16	92	5	16	6.25	114	35	0:00:13.78	31.35 %
newyork	16	92	5	18	16.67	90	221	0:02:10.97	29.31 %
newyork	16	92	5	20	20.00	100	659	0:07:01.98	28.73 %
newyork	16	92	5	25	20.00	148	4165	0:29:09.84	31.84 %
newyork	16	92	5	30	20.00	100	659	0:06:44.59	29.06 %
france	25	90	5	2	50.00	10	23	0:00:35.99	11.86 %
france	25	90	5	4	37.50	20	91	0:02:25.19	13.75 %
france	25	90	5	6	41.67	30	147	0:06:33.75	15.57 %
france	25	90	5	8	37.50	40	511	0:25:54.95	17.28 %
france	25	90	5	10	40.00	50	2611	1:18:20.44	19.385%
france	25	90	5	12	33.33	60	1987	3:00:00	23.526%
france	25	90	5	14	21.43	70	2245	3:00:00	25.03 %
france	25	90	5	16	26.03	2639	16581	3:00:00	45.23 %

same subbands, the more instance will be difficult since many arcs might be saturated. Further path have then to be explored in order to identify relevant variables to introduce in the current linear program.

Table 7.4 shows the results of Branch-and-Price algorithm for the aggregated path formulation. We can see from this table that, this algorithm as well as the previous one allowed to solve to optimality all the tested instances within the CPU time limit. Observe that the gap values are quite similar to one in Table 7.3. Also remark that, similarly to column generation procedures, both Branch-and-Price algorithm do not work the same. In fact, the number of generated columns remains generally higher in the latter algorithm. However, it seems that from to a certain threshold of instance size and difficulty, the Branch-and-Price tree becomes slightly better manageable than in the former algorithm. Basically, instance dfn_gwin with $|K| = 20$ for example, where the number of nodes in the first Branch-and-Price tree is 2755, while it is 101 in the second Branch-and-Price tree. Also the two last rows given by instances france with $|K| = 14$ and 16, that are solved to optimality using the second approach, while the first algorithm could not complete the process within 3 hours. This can be explained by the fact that, in aggregated path formulation, a good trade-off between the number of generated columns and the size of the tree, can be achieved. Also, the branching scheme here induces some decisions that directly affect the size and the form of the tree. Indeed, the relationship between families of variables might make difficult to perform an efficient branching on the variables, and induce a large and unbalanced tree. In some sense, the aggregated formulation could help us to translate an explicit definition of path variables associated with both physical and virtual layer, to an embedded definition of variables. In other words, the aggregated path formulation performs better, since we handle one family of "double" path variables (defined in G_1 but implicitly related to a path in G_2), instead of two families, which is somehow easier.

Besides, these observations lead us to believe that branching on a subset of variables instead of fixing one variables per generated sub-problem may help considerably in enhancing the process. Also a primal heuristic should allow to prune much more efficiently the nodes of the tree whose exploration is not relevant.

7.5 Concluding remarks

In this chapter we have introduced a compact formulation for the OMBND problem. Based on this formulation, we have proposed two path-based formulations for the problem. The first path formulation considers an explicit decomposition approach, and

Table 7.4: Branch-and-Price results for SNDlib-based instances - Aggregated path formulation

Instance	$\lvert V_2 \rvert$	$\lvert A_2 \rvert$	$\lvert W \rvert$	$\lvert K \rvert$	gap(%)	columns	nodes	TT	TTpricing (%)
dfn_bwin	10	90	4	2	0.00	129	1	0:00:03.04	87.75%
dfn_bwin	10	90	4	4	12.50	340	3	0:00:04.55	92.70%
dfn_bwin	10	90	4	6	8.33	546	15	0:00:12.00	92.83%
dfn_bwin	10	90	4	8	43.75	588	23	0:00:57.00	89.17 %
dfn_bwin	10	90	4	10	40.00	724	23	0:01:33.00	92.52 %
dfn_bwin	10	90	4	12	29.17	873	35	0:01:44.00	93.46 %
dfn_bwin	10	90	4	14	27.59	1023	129	0:03:56.00	92.08 %
dfn_bwin	10	90	4	16	27.27	1165	253	0:16:32.00	90.38 %
dfn_bwin	10	90	4	18	26.32	876	311	0:20:31.00	94.17 %
dfn_gwin	11	94	4	2	0.00	241	1	0:00:05.47	94.14%
dfn_gwin	11	94	4	4	0.00	537	1	0:00:14.53	98.21%
dfn_gwin	11	94	4	6	0.00	448	1	0:00:09.38	96.80%
dfn_gwin	11	94	4	8	0.00	658	1	0:00:10.77	97.02%
dfn_gwin	11	94	4	10	10.23	785	3	0:00:19.07	95.96%
dfn_gwin	11	94	4	12	13.00	688	7	0:00:57.00	86.83%
dfn_gwin	11	94	4	14	8.73	843	7	0:01:39.00	87.92%
dfn_gwin	11	94	4	16	32.98	926	17	0:03:28.00	93.96%
dfn_gwin	11	94	4	18	5.88	1023	51	0:08:05.00	92.22%
dfn_gwin	11	94	4	20	19.44	876	101	0:10:55.00	88.9%
dfn_gwin	11	94	4	25	25.00	947	127	0:14:48.00	91.9%
dfn_gwin	11	94	4	30	21.28	1034	205	0:28:34.00	94.38%
newyork	16	92	5	2	20.41	526	3	0:00:37.80	95.74 %
newyork	16	92	5	4	12.50	830	7	0:00:50.26	95.80 %
newyork	16	92	5	6	33.2	2188	19	0:02:29.30	94.86 %
newyork	16	92	5	8	25.00	1634	239	0:03:28.00	88.9%
newyork	16	92	5	10	37.50	1435	431	0:07:31.00	89.32 %
newyork	16	92	5	12	40.00	1289	511	0:21:58.00	91.43 %
newyork	16	92	5	14	41.67	2076	873	0:00:12.00	88.74 %
newyork	16	92	5	16	12.50	2198	1021	0:16:53.00	91.28 %
newyork	16	92	5	18	6.25	4389	3287	0:20:42.00	90.33 %
newyork	16	92	5	20	16.67	3741	2719	0:26:18.00	91.28 %
newyork	16	92	5	25	20.00	3827	2501	1:08:37.00	92.33 %
newyork	16	92	5	30	20.00	4659	3283	1:40:53.00	88.84 %
france	25	90	5	2	20.00	51	1	0:00:03.00	90.33 %
france	25	90	5	4	50.00	88	3	0:01:48.00	94.37 %
france	25	90	5	6	37.50	103	7	0:01:44.00	95.12 %
france	25	90	5	8	41.67	114	7	0:00:53.00	94.22 %
france	25	90	5	10	37.50	2378	537	0:43:37.95	88.54 %
france	25	90	5	12	40.00	3439	721	1:40:20.44	89.17 %
france	25	90	5	14	33.33	4392	1077	2:37:48.76	90.27 %
france	25	90	5	16	21.43	5283	1259	2:10:30.09	95.39 %
france	25	90	5	18	27.44	6239	3423	3:00:00	88.28 %

induces a column generation procedure requiring two pricing sub-problems. The second model, namely aggregated path formulation, attempts to give an implicit decomposition of the problem, where the virtual layer includes informations of the physical layer, and this, using a family of variables having a specific structure. We have discussed the pricing problems for both path formulations, and we proved that they reduce to shortest path problem. We have devised a Branch-and-Price algorithm to solve each formulation and compared them, to show empirically that they are more efficient than a Branch-and-Bound for the compact formulation. Finally, we have given some numerical experiments to show the effectiveness of our approach and to compare both algorithms.

We could see that the Branch-and-Price algorithm brought out by the first path formulation performs generally better than one of the aggregated formulation. Indeed, although the latter explores less nodes in the Branch-and-Price tree, it spends a significant time in pricing out path variables, in particular at the root node. However, from a certain size of instance, both algorithms do not succeed to solve the problem to optimality. Several interesting perspectives can be considered to enhance their performances. In fact, we should turn ourselves to a more sophisticated branching strategy to handle the size of Branch-and-Price tree concerning the first path formulation. Besides, a deeper investigation of the pricing problem for the aggregated formulations should enable to better control the column generation procedure. Furthermore, take advantage of some of the valid inequalities introduced in Chapter 5, should be a promising prospect and yield an efficient Branch-and-Cut-and-Price algorithm.

Conclusion

In this dissertation, we have studied a capacitated network design problem, for single-layer and multilayer telecommunication networks, within a polyhedral context.

In the first part of the thesis, we considered the capacitated single-layer network design (CSLND) problem. We focused our attention on the arc-set polyhedron associated with this problem. We studied a set of functions that are, in fact, relaxations of the considered problem, when it is restricted to one arc. We investigated the basic properties of the polyhedra associated with these functions and derived new classes of valid inequalities. We then described necessary and sufficient conditions for theses inequalities to define facets. We presented an application of these results to the Bin-Packing function problem. The identified valid inequalities were thereafter used to devise a Branch-and-Cut algorithm for the CSLND problem. The later was implemented to solve instances from SNDlib with realistic and randomly generated traffic matrices. The experiments show in particular the efficiency of the valid inequalities and the separation procedure used in the Branch-and-Cut algorithm.

We studied afterwards a multilayer version of the problem that is OMBND, taking into account the relationship between both layers of the network. We introduced several integer linear programming formulations for the problem. In particular, we studied the polyhedron associated with the cut formulation, in an attempt to describe strong valid inequalities for the problem. We investigated the properties of this polyhedron as well as the facial structure of the basic inequalities. This led us to define several classes of valid inequalities, that are facets of polyhedron under certain necessary and sufficient conditions, that we described. These valid inequalities, as well as inequalities from CSLND polyhedral study, where incorporated within a Branch-and-Cut framework. The obtained algorithm allowed to solve the problem for realistic instances, and real instances provided by the french telecommunication operator Orange. We could measure the significant improvement enabled by the introduced inequalities on strengthening the linear relaxation of the problem.

Finally, in a last part of the dissertation, we discussed a compact formulation and two path decompositions for OMBND problem. In the former path formulation, we managed to consider explicitly both layers, by using two families of path variables (one for each layer). As the number of variables was exponential, we developed a column generation procedure using two pricing problems. In the later path formulation, the connection between the physical and virtual layers was addressed implicitly. In this case, we are dealing with a formulation having one family of exponential number path variables. A second procedure of column generation, with a two-stage pricing problem was proposed to tackle this formulation. Each of the two column generation procedures has been embedded within a Branch-and-Price framework. The experimental results show that both algorithms perform well, compared to the Branch-and-Bound approach.

There are many directions in which the research in this dissertation can be continued for both considered problems.

In the research of valid inequalities for CSLND problem, we considered an arc-set relaxation of the problem. Actually, a quite natural extension of this study is to consider the polyhedron associated with the cut-set relaxation of CSLND problem. In particular, it will be interesting to know how Min Set I and Min Set II inequalities can be generalized in the context of a cut-set polyhedron. We expect that their inclusion in a Branch-and-Cut framework will have a positive impact in enhancing the algorithm.

Concerning OMBND problem, most of the future work revolves around the algorithmic aspects. We need to develop more efficient separation heuristics for the Branch-and-Cut algorithm. It will be also interesting to focus on more sophisticated preprocessing methods in order facilitate the problem resolution. Besides, investigate the pricing problems associated with the proposed path formulations will help to improve the effectiveness of the Branch-and-Price approach. Implementing more elaborated branching strategies is also a possible direction for future study.

Furthermore, we need to develop stronger valid inequalities for the polyhedra of OMBND and CSLND problems. From a theoretical point of view, it would be interesting to address further relaxations of these problems and to characterize when the identified valid inequalities define facets of the underlying polyhedra.

After all, the complexity of optical networks and the relevance of current issues such as the energy-efficient networking, give several interesting extensions for both considered problems. Also it should be interesting to consider the robust version of the multilayer network design. Although, the single-layer network design under demand uncertainty recently started to be a field of interest for many researchers, as far as we know, there is no work treating the robust network design for two or more layers.

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abstract: A major challenge for nowadays telecommunication actors is to propose solutions to manage the traffic growth, and ensure a smart use of network resources. In this thesis, we address two dimensioning problems for both single-layer and multilayer telecommunication networks based on the multi-band OFDM technology, within a polyhedral framework. We give several integer linear programming formulations for the considered problems and investigate the properties of the associated polyhedra. We highlight the relationship between these problems and other well-know combinatorial optimization problems such as the Bin-Packing problem. In particular, this relationship is exploited to derive new classes of valid inequalities. We further carry on an investigation of the facial structure of these inequalities, and describe efficient algorithms for their separation. We then devise Branch-and-Cut and Branch-and- Price algorithms to solve both problems. Several series of experiments are conducted for random, realistic and real networks, of great interest for Orange Labs. The obtained results show empirically the efficiency of our approaches.

key words: optical multi-band networks, network design, polytope, facet, Branch-and-Cut algorithm, Branch-and-Price algorithm.

résumé: L'un des plus grands défis pour les acteurs de télécommunication actuels est de proposer des solutions afin de gérer la croissance du trafic, et d'assurer une utilisation intelligente des ressources existant dans un réseau. Dans cette thèse, nous étudions deux problèmes de dimensionnement de réseaux basés sur la technologie OFDM multi-bandes, dans un contexte polyédral. Nous proposons différents programmes linéaires en nombres entiers pour formuler les problèmes considérés et étudions les propriétés des polyèdres associés. Nous mettons en évidence la relation entre ces problèmes et d'autres problèmes classique d'optimisation combinatoire tel que le Bin Packing. En particulier, cette relation est exploitée afin de dériver de nouvelles classes d'inégalités valides. Nous menons alors une investigation sur la structure faciale des inégalités identifées et décrivons des algorithmes efficaces pour les problèmes de séparation associés. Nous concevons et développons des algorithmes de Coupes et Branchements et Génération de colonnes et Branchements pour résoudre les deux problèmes. Une phase d'expérimentation comprenant plusieurs séries de tests est ensuite conduite sur des instances aléatoires, réalistes et réelles, de grand intérêt pour Orange Labs. Les résultats de ces tests montrent de façon empirique l'efficacité de notre approche.

mots clés : réseaux optiques multi-bandes, conception de réseaux, polytope, facette, algorithme de coupes et branchements, algorithme de génération de colonnes et branchements.