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The Constrained-Routing and Spectrum Assignment Problem: Polyhedral Analysis and Algorithms

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Abstract

In this thesis[1], we study a variant of the Routing and Spectrum Assignment problem (RSA), namely the Constrained-Routing and Spectrum Assignment (C-RSA). The C-RSA problem is a key issue when dimensioning and managing a new generation of optical networks, called spectrally flexible optical networks. The C-RSA can be stated as follows. Given an undirected, loopless, and connected graph G, an optical spectrum $\mathbb{S}$ of available contiguous frequency slots, and a multiset of traffic demands K between pairs of origins and destinations, the C-RSA consists of assigning for each traffic demand $k \in K$ a path in G between its origin and destination, and an interval of contiguous frequency slots in $\mathbb{S}$ so that some technological constraints are satisfied, and some linear objective function is optimized. First, we propose an integer linear programming formulation for the C-RSA. We identify several families of valid inequalities for the associated polytope. Some of these inequalities are obtained by using the so-called conflict graphs. Moreover, we prove that these inequalities are facet-defining for the associated polytope under some necessary and sufficient conditions. In addition, we develop separation algorithms for these inequalities. Using these results, we devise a Branch-and-Cut (B&C) algorithm for the problem, and discuss experimental results using this algorithm. A second part of the thesis is devoted to an extended formulation for the C-RSA. A column generation algorithm is developed to solve its linear relaxation. We prove that the related pricing problem is equivalent to the so-called resource constrained shortest path problem, which is well known to be NP-hard. For this, we propose a pseudo-polynomial time algorithm using dynamic programming. Using this, we devise Branch-and-Price (B&P) and Branch-and-Cut-and-Price (B&C&P) algorithms to solve the problem. An extensive experimental study with comparisons between the different B&C, B&P, and B&C&P algorithms is also presented.
Next, we turn our attention to the Spectrum Assignment (SA) sub-problem. It has been shown to be equivalent to the problems of wavelength assignment, interval coloring, and dynamic storage allocation that are well known to be NP-hard. To the best of our knowledge, a polyhedral approach to the SA problem has not been considered before, even to its equivalent problems. For this, first, we propose an integer linear programming compact formulation and investigate the facial structure of the associated polytope. Moreover, we identify several classes of valid inequalities for the polytope and prove that these inequalities are facet-defining. We further discuss their separation problems. Based on these results, we devise a Branch-and-Cut (B&C) algorithm for the SA problem, along with some computational results are presented.

Keywords: optical network, network design, integer programming, polyhedron, facet, separation, branch-and-cut, branch-and-price, branch-and-cut-and-price, dynamic programming.

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Introduction

The global Internet Protocol (IP) traffic is expected to reach 396 exabytes per month by 2022, up from 194.4 Exabytes per month in 2020 [33]. Optical transport networks are then facing a serious challenge related to continuous growth in bandwidth capacity due to the growth of global communication services and networking: mobile internet network (e.g., 5th generation mobile network), cloud computing (e.g., data centers), Full High-definition (HD) interactive video (e.g., TV channel, social networks) [28], etc... as shown in Figure 5.1. To sustain the network operators face this trend of increase in bandwidth, a new generation of optical transport network architecture called Spectrally Flexible Optical Networks (SFONs) (called also FlexGrid Optical Networks) has been introduced as promising technology because of their flexibility, scalability, efficiency, reliability, and survivability [26][28] compared with the traditional FixedGrid Optical Wavelength Division Multiplexing (WDM)[137][138]. In SFONs the optical spectrum is divided into small spectral units, called frequency slots [148]. They have the same frequency of 12.5 GHz where WDM uses 50 GHz [163] as recommended by ITU-T [4]. This can be seen as an improvement in resource utilization.

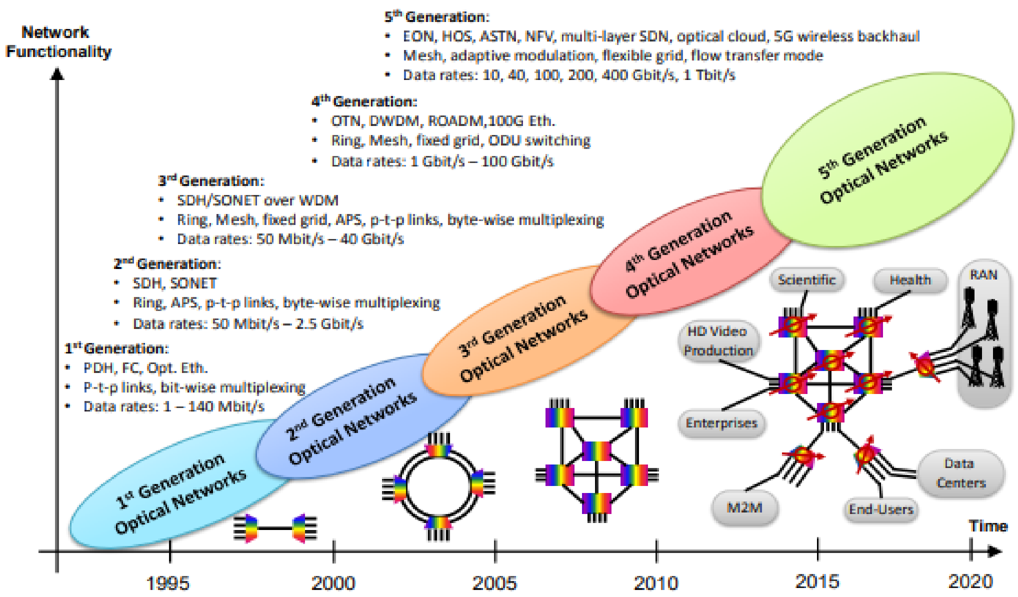


Figure 1: Historical Evolution of Optical Transport Networks [160].

The concept of slots was proposed initially by Masahiko Jinno et al. in 2008 [83], and later explored by the same authors in 2010 [174]. We refer the reader to [96] for more information about the architectures, technologies, and control of SFONs.
The Routing and Spectrum Assignment (RSA) problem plays a primary role when dimen-

sioning and designing of SFONs which is the main task for the development of this next generation of optical networks. It consists of assigning for each traffic demand, a physical optical path, and an interval of contiguous slots (called also channels) while optimizing some linear objective(s) and satisfying the following constraints [69]:

a) *spectrum contiguity*: an interval of contiguous slots should be allocated to each demand k with a width equal to the number of slots requested by demand k;

b) *spectrum continuity*: the interval of contiguous slots allocated to each traffic demand stills the same along the chosen path;

c) *non-overlapping spectrum*: the intervals of contiguous slots of demands whose paths are not edge-disjoints in the network cannot share any slot over the shared edges.

Numerous research studies have been conducted on the RSA problem since its first appearance. The RSA is known to be an NP-hard problem [155] [165], and more complex than the historical Routing and Wavelength Assignment (RWA) problem [74]. Various (mixed) integer linear programming (ILP) formulations and algorithms have been proposed to solve it. A detailed survey of spectrum management techniques for SFONs is presented in [165] where authors classified variants of the RSA problem into: offline RSA which has been initiated in [126], and online or dynamic RSA which has been initiated in [175] and recently developed in [117] and [189]. Numerous aspects are investigated in the tutorial [25]. This work focuses on the offline RSA problem. There exist two classes of ILP formulations used to solve the RSA problem, called edge-path and edge-node formulations. The ILP edge-path formulation is majorly used in the literature where variables are associated with all possible physical optical paths inducing an explosion of a number of variables and constraints which grow exponentially and in parallel with the growth of the instance size: number of demands, the total number of slots, and topology size: number of links and nodes [69]. To the best of our knowledge, we observe that several papers which use the edge-path formulation as an ILP formulation to solve the RSA problem, use a set of precomputed-paths without guaranty of optimality e.g. in [31], [126], [127], [128], [172], [192], and recently in [146]. On the other hand, column generation techniques have been used by Klinkowski et al. in [143], Jaumard et al. in [80], and recently by Enoch in [49] to solve the relaxation of the RSA taking into account all the possible paths for each traffic demand. To improve the LP bounds of the RSA relaxation, Klinkowsky et al. proposed in [130] a valid inequality based on clique inequality separable using a branch-and-bound algorithm. On the other hand, Klinkowski et al. in [131] propose a branch-and-cut-and-price method based on an edge-path formulation for the RSA problem. Recently, Fayez et al. [53], and Xuan et al. [179], they proposed a decomposition approach to solve the RSA separately (i.e., R+SA) based on a recursive algorithm and an ILP edge-path formulation.
To overcome the drawbacks of the edge-path formulation usage, a compact edge-node formulation has been introduced as an alternative for it. It holds a polynomial number of variables and constraints that grow only polynomially with the size of the instance. We found just a few works in the literature that use the edge-node formulation to solve the RSA problem e.g. [17], [172], [192]. Bertero et al. in [14] give a comparative study between several edge-node formulations and introduce new ILP formulations adapted from the existing ILP formulations in the literature.
On the other front, and due to the NP-hardness of the C-RSA problem, we found that several heuristics [45],[106],[148], and recently in [77], greedy algorithms [98], metaheuristics as tabu search in [65], simulated annealing in [131], genetic algorithms in [61], [73], [74], [44], ant colony algorithms in [87], and a hybrid meta-heuristic approach in [142], have been used to

solve large sized instances of the RSA problem. Furthermore, some researches start using some artificial intelligence algorithms [141], see for example [92] and [94], and some deep-learning algorithms [27], and also machine-learning algorithms in [147][191], and recently in [187] and [67] to get more perefermonce. Selvakumar et al. gives a survey in [152] in which they summarise the most contributions done for the RSA problem before 2019.
In this paper, we are interested in the resolution of a complex variant of the RSA problem, called the Constrained-Routing and Spectrum Assignment (C-RSA) problem. Here we suppose that the network should also satisfy the transmission-reach constraint for each traffic demand according to the actual service requirements. To the best of our knowledge a few related works on the RSA, to say the least, take into account this additional constraint s.t. the length of the chosen path for each traffic demand should not exceed a certain length (in kms). Recently, Hadhbi et al. in [69] and [70] introduced a novel tractable ILP based on the cut formulation for the C-RSA problem with a polynomial number of variables and an exponential number of constraints that are separable in polynomial time using network flow algorithms. Computational results show that their cut formulation solves larger instances compared with those of Velasco et al. in [172] and Cai et al. [17]. It has been used also as a basic formulation in the study of Colares et al. in [34], and also by Chouman et al. in [29] and [30] to show the impact of several objective functions on the optical network state. Note that Velasco et al. in [172], Cai et al. [17], and Bertero et al. in [14], did not take into account the transmission-reach constraint.
However, so far the exact algorithms proposed in the literature could not solve large-sized instances. We believe that a cutting-plane-based approach could be powerful for the problem. To the best of our knowledge, such an approach has not been yet considered except the works done by Bianchetti et al. in [15] for the RSA problem. For that, the main aim of this work is to investigate thoroughly the theoretical properties of the C-RSA problem. To this end, we aim to provide a deep polyhedral analysis of the C-RSA problem, and based on this, devise exacts algorithms based on branch-and-cut and branch-and-cut-and-price algorithms for solving the problem considering large-scale networks that are often used. Our contribution is then to introduce a new ILP formulation called *cut formulation* for the C-RSA problem which can be seen as an improved formulation for the one introduced by Hadhbi et al. in [69] and [70]. We investigate the facial structure of the associated polytope. We further identify several classes of valid inequalities to obtain tighter LP bounds. Some of these inequalities are obtained by using conflict graphs related to the problem: clique inequalities, odd-hole, and lifted odd-hole inequalities. We also use the Chvatal-Gomory procedure to generate larger classes of inequalities. We then devise their separation procedures and give sufficient conditions under which these inequalities are facet defining. On the other hand, we introduce extended ILP formulation based on path variables, called *path formulation*. It can be seen as a reformulation of the first *cut formulation*. This formulation has an exponential number of variables. A column generation algorithm is then used to solve its linear relaxation. We further adapt the valid inequalities proposed for the *cut formulation* to obtain also tighter bounds for the *path formulation*. Using the polyhedral results and the separation procedures, we develop a Branch-and-Cut (B&C) and Branch-and-Cut-and-Price (B&C&P) algorithms to solve the problem. Moreover, we boost its effectiveness through some enhancements to obtain tighter primal bounds based on a warm-start algorithm using some metaheuristics: simulated annealing, tabu search, and genetic algorithms useful to push a feasible integral solution (if possible) in the root of the B&C and B&C&P algorithms before the start of the resolution of C-RSA, and also a primal-heuristic based on a hybrid method between a greedy algorithm and a local search algorithm to construct a feasible integral solution from a given fractional solution in each node of the B&C and B&C&P trees. We provide at the end a

detailed comparative study between the B&C and B&C&P algorithms by using two types of instances: random and realistic ones. They are composed of two types of graphs: real graphs, and realistic ones from SND-LIB. The results show that the B&C&P algorithm is able to provide optimal solutions for several instances, which is not the case for the B&C algorithm within the CPU time limit (5 hours). Furthermore, we have studied the influence of the valid inequalities. The results show that some of them, in particular, clique and cover inequalities are efficient. However, some instances are still difficult to solve with both B&C and B&C&P algorithms.

Several concepts are exploited throughout this dissertation. We start this dissertation by presenting the basic notions of combinatorial optimization, complexity, graph theory, and further give some notations that are useful through this manuscript.

In Chapter 2, we present the C-RSA problem studied in this work. We then introduce an integer linear programming formulation namely cut formulation. We carry out an investigation of the related polytope defined by the convex hull of all its solutions. Moreover, we identify several classes of valid inequalities for the polytope and study their facial structure. Moreover, we introduce symmetry-breaking inequalities that are used to remove the equivalents sub-problems for the problem in question.

In Chapter 3, we propose a Branch-and-Cut algorithm for the cut formulation and describe the separation procedure of the valid inequalities introduced in the Chapter 2. A detailed comparative study is proposed at the end of this chapter by showing the impact of the additional valid inequalities using several mixed-integer linear program solvers.

On the other hand, in Chapter 4, we propose an extended ILP formulation based on path formulation for the C-RSA problem. We develop a column generation algorithm to solve its linear relaxation. Using this, we devise Branch-and-Price (B&P) and Branch-and-Cut-and-Price (B&C&P) algorithms to solve the problem, along with some computational results are presented. In the end of this chapter, we provide an extensive comparative analysis of performance between the B&C, B&P and B&C&P algorithms using two types of instances: random and realistic ones with $|K|$ up to 300 and $|\mathbb{S}|$ up to 320. They are composed of two types of graphs (topologies): real graphs and realistic ones from SND-LIB with $|V|$ up to 161 and $|E|$ up to $|166|$.

As a third part, in the chapter 5 we focus on the Spectrum Assignment (SA) sub-problem. It is well known to be NP-hard problem [13]. First, we propose an integer linear programming compact formulation. We investigate the facial structure of the associated polytope. Fuerthremore, we describe several valid inequalities, some of them come from those that are already proposed for the C-RSA. We also give sufficient conditions under which these inequalities are facet defining. Based on these results, we develop a B&C algorithm to solve the problem. On the other hand, we have noticed also that several symmetrical solutions may appear given that there exist several feasible equivalent solutions that have the same value, and they can be found by doing some permutations between the slots assigned to some demands while satisfying the SA constraints. For that, we derive some symmetry-breaking inequalities for the SA in order to well manage the equivalent sub-problems in the B&C tree. Moreover, we provide some lower bounds obtained by using some properties of the conflict graph. Based on all this, we present an extensive experimental study while showing the impact of the valid inequalities and symmetry-breaking inequalities on the effectiveness of the B&C algorithm.

Chapter 1

Preliminary Notions and State-of-the-Art

In this chapter, we present some preliminary notions related to combinatorial optimization, optimization algorithms and polyhedra approaches. We give also an overview for some exacts methods based on Branch-and-Cut and Branch-and-Cut-and-Price algorithms by explaining the principles of each method. We further give some definitions related to graph theory that are very useful throughout our study. We end this chapter by introducing some notions related to flexible optical networks to introduce the application case and express our motivations.

1.1 Combinatorial Optimization

Operational research is a discipline related to computer science and applied mathematics. In this dissertation, we are interested in one of its branches, called *combinatorial optimization.* The optimization problems related to combinatorial optimization can be formulated as follows. Let $E = \{e_1, ..., e_n\}$ be a finite set, namely basic set. We associated with each element e_i a weight $c(e_i) \in \mathbb{R}$ with $i \in \{1, ..., n\}$. Let F denote a family of subsets of E. The problem aims to identify one subset $\mathcal{F}$ from F with the smaller or larger weight given by the sum $\sum_{e_i \in \mathcal{F}} c(e_i)$. Such a problem is known under the name *combinatorial optimization problem* where the set F represents the set of all feasible solutions of the problem in question. In general, the set F is discrete or can be reduced to a discrete one, hence combinatorial word is referred. On the other hand, the term optimization means that we are looking for the identification of the best element $\mathcal{F}$ from the set of all feasible solutions F. In general, the set F contains an exponential number of feasible solutions. As result, it's known to be very hard to solve such combinatorial optimization problem by enumerating all its feasible solutions. To do so, various approaches have been developed and applied to solve combinatorial optimization problems. They are based on graph theory, linear and non-linear programming, integer programming, mixed integer programming, and polyhedral approach. These approaches have been shown to be very efficient from a complexity point of view. For this, we discuss in the next section some fundamental algorithmic and complexity theories that are related to combinatorial optimization.

1.2 Algorithmic and Complexity Theory

Several researchers in computer science and mathematics are interested in working on the classification of problems into easy or hard problems, and further on the algorithmic com-

plexity study whose objective is to find the most efficient algorithm among a set of proposed algorithms. This has been initiated by Cook [36], Edmonds [48] and Karp [123].
The theory of complexity [Garey and Johnson, 1979] [59] classifies problems into two essential classes: the P (polynomial time) class, and the NP (Non-deterministic polynomial time) class. In addition, the problems of the NP class are shared into two subclasses: the class of NP-Complete problems, and the class of NP-hard problems.
Before defining each class, we first give a general definition of a problem. In general, a *problem* is a question having parameters given in input such that an answer is needed for it, called *solution*. A problem is described by giving: a general description of all its parameters, and a listing of the properties that the solution must satisfy, known under the name *constraints*. An instance of a problem is obtained by specifying the value of each input parameter of the problem. For this, one can propose an algorithm to solve the problem in question. An algorithm for solving a given problem is a procedure that is decomposable into a sequence of finite operations. It allows giving a solution for each instance of the problem. In general, the complexity of an algorithm depends on the *size* of a problem that reflects the number of parameters needed to describe an instance. It can be shown polynomial if the maximum number of its operations necessary to solve an instance of size n is bounded by a polynomial function f in n (i.e., $f(n)$. This means that there exists a scalar c such that the number of its operations necessary is equal to $c.f(n)$. As a result, the notation big $\mathcal{O}$ is appeared to express the complexity of an algorithm.
There exists two type of problems in operation research: optimization problems and decision problems. In the context of optimization problems, we want to minimize (or maximize) a function while satisfying a set of constraints. On the other hand, in the context of decision problems, the solution is binary like yes / no or 0/1.
An easy problem that can be solved by a polynomial algorithm with respect to its size, is called a problem of class P. One can judge that a problem is part of NP class if we can verify in polynomial time that a solution of each instance of the problem is feasible. On the other hand, the NP-Complete class groups the decision problems for which there is no algorithm allowing their resolution in a polynomial time. According to Garey and Johnson [59], a Q problem is a NP-Complete problem if it belongs to the NP class, and there exists a P problem also belongs to the NP class such that can be reduced to the Q problem in polynomial time [59].
The Satisfiability Problem (SAT) is the first problem that has been demonstrated to be NP-Complete. This was proved in 1971 by Stephen Cook [36] [60].
The NP-hard problem class includes most of the decision problems and optimization problems. NP-hard problems are indeed difficult as $NP-$Complete problems. If a decision problem associated with a P optimization problem is NP-Complete then P is an NP-hard. [60]. Furthermore, note that every problem of class P is a problem of class NP ($P \subseteq NP$) as shown in Figure 1.2.
However, the reciprocal represents a well-known mathematical problem which is part of the 7 problems of the millennium prize. The question $P = NP$? is one of the most important questions that has not yet been solved. The answer to this question by ”yes” is to prove that all the problems of the NP class are in the P class. Cook has proved in [Cook, 1971] that all the problems of the NP class are reducible to the SAT problem, which means that if someone finds a polynomial algorithm for this problem, the question $P = NP$? is then solved ![60], i.e. we will be able to solve all NP-Complete problems in polynomial time.

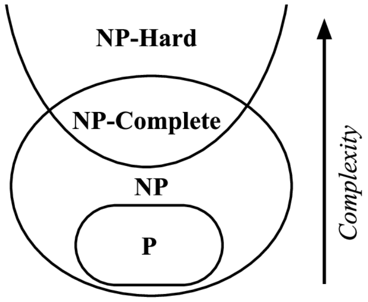


Figure 1.1: Relation between P, NP, NP-Complete and NP-hard problems [107].

1.3 Polyhedral Approach and Branch-and-Cut Algorithm

1.3.1 Elements of the Polyhedral Theory

In this section, we will introduce some definitions and properties of polyhedron theory. Schrijver in 1986 [150], Nemhauser and Wolsey in 1988 [109], Wolsey in 1998 [177] and Schrijver in 2003 [151] are the most useful references on polyhedron study [188].
Let x be a vector in $\mathbb{R}^n$, with n a positive integer. x is said a linear combination of vectors $x_1, x_2, .., x_k \in \mathbb{R}^n$ if there exist k scaler $\lambda_1, \lambda_2.., \lambda_k$ such that

$$x = \sum_{\imath=1}^{k} \lambda_i x_i \text{ with } \lambda_i \in \mathbb{R} \text{ for all } i \in \{1, ..., k\}.$$

Furthermore, if $\sum_{\imath=1}^{k} \lambda_i = 1$. Then x is said an affine combination of vectors $x_1, x_2, .., x_k$.
We say that x is convex combination of vectors $x_1, x_2, .., x_k \in \mathbb{R}^n$ if x is an affine combination of vectors $x_1, x_2, .., x_k \in \mathbb{R}^n$ and each scaler λ_i for $i \in \{1, ..., k\}$ is positive, i.e., $\lambda_i \in \mathbb{R}_+$ and $\sum_{\imath=1}^{k} \lambda_i = 1$. A set of vectors is said to be linearly independent with a vector x if x cannot be written as a linear combination of all the vectors in the set.
Given a set $S = \{x_1, ..., x_k\} \in \mathbb{R}^{n*m}$, the convex hull of a finite set of incidence vectors in S, denoted by $conv(\mathcal{S})$, is the set of all vectors that are a convex combination of vectors in S. We have

$$conv(\mathcal{S}) = \{x \in \mathbb{R}^n \text{ with } x = \textstyle\sum_{i=1}^{k} \lambda_i x_i, \forall \lambda_i \geq 0 \text{ and } \sum_i \lambda_i = 1\}.$$

This definition ensures that $S \subset conv(\mathcal{S})$.

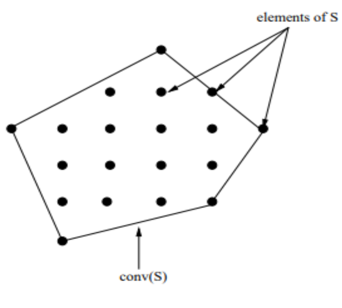


Figure 1.2: conv(S) vs S [11][42][168].

A polyhedron P is the set of solutions of a given problem described by a linear system $Ax \leq b$. It's denoted as $P = \{x \in \mathbb{R}^n | Ax \leq b\}$, where A is the matrix constraint characterized by m-row and n-columns, and $b \in \mathbb{R}^m$. Each point x of the polyhedron P represents a solution of P. Furthermore, If P is bounded so it defines a bounded polyhedron which is called polytope. The dimension of polyhedron P is one less than the maximum number of vectors of solution in P that are affinely independent. We distinguish the following cases:

a) If all the vectors solutions in P are independents then we call P full-dimensional polyhedron,

b) If there exists a submatrix $A^=$ of A of inequalities that are all verified with equality by all the solutions of P, and its associated equations system is of full rank, then $dim(P) = n - rank(A^=)$,

c) $dim(x) = 0$ for each $x \in P$,

d) $dim(\varnothing) = -1$.

An inequality $ax \leq \alpha$ is valid for the polyhedron P if and only if for every solution $\bar{x} \in P$, $a\bar{x} \leq \alpha$. It is said to be violated by a solution $\bar{x}$ if $a\bar{x} > \alpha$. The set $F \subset P$ is called face if there exists a valid inequality $ax \leq \alpha$ for the polyhedron P such that

$$F = \{x \in P, ax = \alpha\}.$$

We say that the valid inequality $ax \leq \alpha$ supports the face F if and only if $F \neq \varnothing$.
If $F \neq \varnothing$ and $F \neq P$, we call F a non trivial or proper face. If F is a proper face and its dimension is exactly one dimension smaller than P, i.e., $dim(F) = dim(P) - 1$, then F is called a facet of polyhedron P.
A facet F of polyhedron P is a non trivial face of the polyhedron P if there doesn't exist any proper face F' of P containing the face F. Otherwise, we say that its associated valid inequalities are redundants. To verify so, if P is full-dimensional polyhedron, then $ax \leq \alpha$ is a facet of polyhedron P if and only if F is a proper face and there exists a facet of P induced by $bx \leq \beta$ and a scalar$\rho \neq 0$ such that $F \subset \{x \in P | bx = \beta\}$ and $b = \rho a$. Otherwise, if P is not full dimensional polyhedron, then $ax \leq \alpha$ is a facet of polyhedron P if and only if F is a proper face and there exists a facet of P induced by $bx \leq \beta$, a scalar $\rho \neq 0$ and $\lambda \in \mathbb{R}^{n*rank(A^=)}$ such that $F \subset \{x \in P | bx = \beta\}$ and $b = \rho a + \lambda A^=$.
A solution $x \in P$ is an extreme point of P if x is a face of P of dimension 0. Furthermore, it cannot be written as a convex combination of other points in P. Figure 1.3 shows a geometric interpretation for the polyhedron P, valid inequality, face, facet and extreme point.
A vector solution x is called integer or integral if each of its components are integers.
The integral hull of the polyhedron P is the convex hull of integer vectors solution in P. For our case $conv(P \cap \{0, 1\})$ is an integral hull of polyhedron P, and it contains all the integer solutions for problem (P). Figure (1.4) gives a geometric interpretation for an integral hull $P_I = conv(P \cap \mathbb{Z}_+)$ of the polyhedron P.

1.3.2 Cutting Plane Method

Let P be a combinatorial optimization problem and $\mathcal{S}$ the set of its feasible solutions. The problem P can be written as $\min\{cx | x \in \mathcal{S}\}$, where c denotes the weight vector associated with the variables x of the problem in question. Consider the convex hull $conv(\mathcal{S})$ of the feasible solutions of P. The problem P is then equivalent to the linear program $\min\{cx | x \in conv(\mathcal{S})\}$. The polyhedral approach, introduced by Edmonds [48] consists in describing the polyhedron

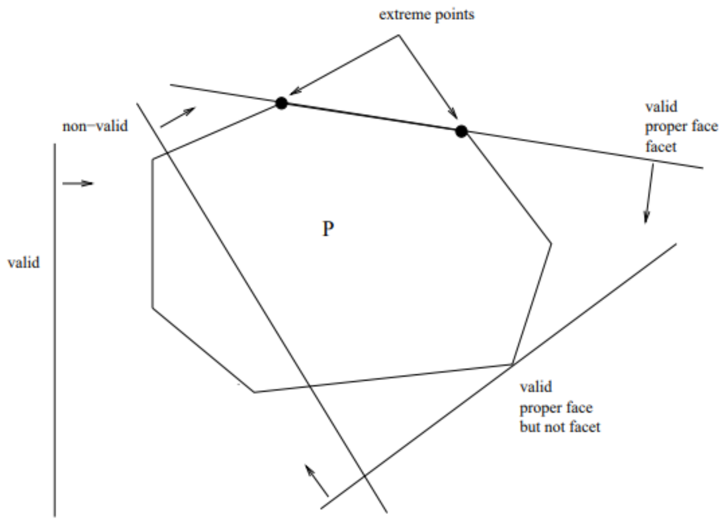


Figure 1.3: Geometric interpretation for the polyhedron P, valid inequality, face, facet and extreme point [168].

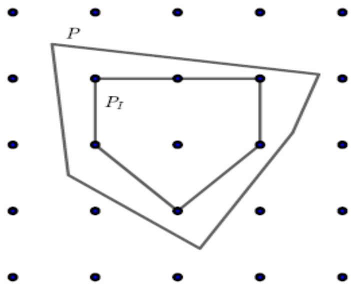


Figure 1.4: Geometric interpretation for an integral hull P_I of the polyhedron P [188].

$conv(\mathcal{S})$ by a set of linear inequalities that are facet-defining inequalities. This reduces the problem P to solving a linear program. As a result, one can solve the problem P using linear programming algorithms [40][122][124] that can be performed in polynomial time [122][124]. However, a complete description of the polyhedron may contain an exponential number of linear inequalities. The optimization problem on the polyhedron $conv(\mathcal{S})$ can therefore not be solved as a linear program having all its linear inequalities. However, one can reduce the number of these inequalities without guaranteeing a complete characterization of the polyhedron $conv(\mathcal{S})$. This may be sufficient to solve the problem using the so-called cutting-plane method. This method is based on the so-called separation problem defined as follows. Let C be a class of valid inequalities for the polyhedron $conv(\mathcal{S})$. The separation problem associated with C consists in deciding whether a given solution x satisfies all inequalities of C, and to find an inequality of C violated by x if not. To do so, Grötschel, Losvàsz, and Schrijver [66] have shown that a combinatorial optimization problem for C can be solved in polynomial time if and only if the separation problem associated with C can be solved in polynomial time. This allows solving a combinatorial optimization problem in polynomial time if we know how to solve in polynomial time the separation problem for a set of valid inequalities for the polyhedron $conv(\mathcal{S})$ using a cutting-plane method by solving a sequence of linear programs. For this, we start by solving a linear program containing a subset of constraints of $conv(\mathcal{S})$. Let us denote by x the optimal solution obtained. By applying the separation problems associated with the different classes of valid inequalities for the polyhedron $conv(\mathcal{S})$, we check if x satisfies all the constraints of $conv(\mathcal{S})$. If it is the case, then x is the solution to the problem. Otherwise, the constraints violated by x identified, and should be added to the linear program. We repeat this process until the optimal solution x^* belongs to the polyhedron $conv(\mathcal{S})$, i.e., x satisfies all the constraints of $conv(\mathcal{S})$.

1.3.3 Branch-and-Cut Algorithm

Note that a cutting-plane method alone may provide only an optimal solution for the linear relaxation of the problem in question. This solution may be not integer which means that it is not feasible for the original problem. In this case, we pass to the branching step which consists in dividing the problem into several Sub-problems that can be done by choosing a fractional variable x_i from the set of variables x, and considering several Sub-problems of the current problem by setting x_i to one of its allowed integer values (i.e., if x is binary, we create 2 Sub-problems respectively by setting x to 0 and 1 respectively). We then apply the method of cutting-plane for each of the Sub-problem. We continue this process until an optimal solution is obtained for the problem. This method is known under the name Branch-and-Cut by combining a branching method with a cutting plane method at each node of the tree.

1.4 Column Generation and Branch-and-Cut-and-Price Algorithms

On the other hand, there exists some mathematical formulations containning a huge number of variables that can be exponential in the worst case. They are known under the name "extended formulation". Their associated lineaire relaxation cannot be solved by a linear solver as simplex algorithm. To manage that, we use a column generation algorithm to solve its linear relaxation. To do so, we begin the algorithm with a restricted linear program of the formulation by considering a feasible subset of variables (columns). For that, we first generate a subset of variables inducing a feasible basis for the restricted linear program. This means that there exists at least one feasible solution for the restricted linear program. Based on this, we derive the so-called "Restricted Master Problem". At each iteration, the column generation algorithm checks if there exists a variable having a negative reduced cost using the solution of the dual problem, and adds it to the current restricted linear program. This procedure is based on solving the so-called "Pricing Problem". The pricing problem consists in identifying a new variable having a negative reduced cost using the optimal solution of the dual problem. We repeat this procedure in each iteration of our column generation until no new column is found. As a result, the final solution is optimal for the linear relaxation. Furthermore, if it is integral, then it is optimal for the C-RSA problem. Otherwise, we create two subproblems called children by branching on some fractional variables (variable branching rule) or on some constraints using the Ryan & Foster branching rule [145] (constraint branching rule). This is known under the name of Branch-and-Price by combining a branching method with a column generation algorithm at each node of the tree. One can strengthen such formulation by introducing several class of valid inequalities for the associated polyhedron. It is based on the so-called Branch-and-Cut-and-Price algorithm by combining a Branch-and-Price algorithm with a cutting-plane based algorithm adding several valid inequalities that are very useful to obtain tighter bounds at each node of the tree, and improve the effectivness of the algorithm.

1.5 Graph Theory

In this section, we introduce some elementary definitions in graph theory that are very useful throughout the proofs and algorithms description, especially in some separation problems and complexity studies. Therefore, Diestel in [43] and Golumbic in 2004 [63] are the most useful references on graph theory [188].

A graph is a pair $G = (V, E)$, where V is a finite set of nodes (called also vertex or point) linked by a set of edges (called also links) E which can be oriented or not oriented.

An edge (a, b) that connects the nodes a and b, it is said to be edge-incident with the node a and b. Two vertices a and b are adjacent if there is an edge connecting a and b, i.e., $\exists e \in E, e = (a, b)$. The number of edges incident with a node v is called the degree of v denoted by $\delta(v)$. The set of directional links going out from node v is denoted as $\delta^+(v)$, and the set of directional links coming into node v is denoted as $\delta^-(v)$.

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. G' is said to be induced by V if $E = E'$ and $V' \subset V$.

A path p in the graph $G = (V, E)$ from node a to node b, is a sequence of nodes that for each pair of successive nodes v_i v_{i+1}, there exist an edge e equals to $(v_i, v_{i+1}) \in E$. Finding a path from a source to a destination with a positive weight edge can be solved using well-known algorithms such as the Dijkstra algorithm and the Bellman-Ford algorithm.

A graph chain G is a sequence of nodes and edges in which each node is adjacent with the two nodes immediately preceding and following it.

A connected graph is connected if and only if there is a path from any point to any other point in the graph.

A loop is an edge that connects a node v to itself. Two distinct edges that have the same end nodes are parallel. A graph is simple if it has no loops or parallel edges, and particularly a loopless graph when there is no loop in the graph.

The adjacency matrix of an undirected graph $G = (V, E)$ is the matrix of size $|V| * |V|$, where the value in position (a, b) is the number of edges connecting a and b. So G is simple if and only if its adjacency matrix is a $(0, 1)-$matrix.

A vertex coloring of G is an assignment of colors to the vertices of G so that two adjacent vertices v and v' cannot get the same color. Same rule for edges, an edge coloring of G is an assignment of colors to the edges of G so that two adjacent edges e and e' cannot get the same color. We say that graph G is t-colorable if no more than t different colors assigned in G.

G' is called a weighted graph if each node in G' is associated with weight.

An interval t-coloring of a weighted graph $G' = (V, E, w)$ is a function $c : V- > \{1, 2, ..., t\}$ such that $c(v) + w(v) - 1 \leq t$. We assign an interval $[c(v), ..., c(v) + w(v) - 1]$ of consecutive integers satisfying $w(v)$ of each vertex v that the intervals of colors assigned to two adjacent vertices do not overlap. If interval t-coloring is feasible for a graph G' then G' is said to be interval t-colorable [155]. The interval chromatic number of G', denoted by χ is the least integer number t such that G' has a interval t-coloring [155].

Let us describe some graphs that may appear as sub-graph in different families of conflict graphs

a) Hole graph is called also chordless cycle is defined as graph cycle of number of links at least four in which two non-consecutive nodes are not linked.

b) Anti-hole graph is the graph complement of a hole graph.

c) Wheel graph of set of nodes $\{1, 2, ..., n\}$ is a graph that contains a hole of $\{1, 2, ..., n-1\}$ nodes, and for which every node in the hole $\{1, 2, ..., n-1\}$ is connected to one other node $\{n\}$ which represent the hub of the wheel.

d) Web graph $W(p, q)$ is obtained from a hole graph with p number of nodes by linking each node i of this hole with a node $j \in \{i + q, ..., i - q\}$.

e) Anti-Web graph is the graph complement of a web graph.

f) Anti-Web-Wheel graph of set of nodes $\{1, 2, ..., n\}$ is a graph that contains an anti-web of $\{1, 2, ..., n - 1\}$ nodes, and for which every node in the anti-web $\{1, 2, ..., n - 1\}$ is connected to one other node $\{n\}$ which represent the hub of the anti-web graph.

On the other hand, for any subset of nodes $X \subseteq V$ with $X \neq \emptyset$, let $\delta(X)$ denote the set of edges having one extremity in X and the other one in $\bar{X} = V \setminus X$ which is called a cut. When X is a singleton (i.e., $X = \{v\}$), we use $\delta(v)$ instead of $\delta(\{v\})$ to denote the set of edges incidents with a node $v \in V$. The cardinality of a set K is denoted by $|K|$.

1.6 Flexible Optical Networks

We introduce in this section some elementary notions related to the flexible optical network, and further give an overview of the related works.

1.6.1 Optical Networks

Optical networks are the heart of long-distance telecommunication networks [153]. A network can be defined as a graph $G = (V, E)$ which can be directed or undirected, where V is a finite set of nodes, and E is a finite set of edges that link a pair of nodes of V. Each node of V represents an entity that can be hardware or human. A set of demands is made between different entities in G to exchange information and data by provisioning hardware and software resources. In this dissertation, we focus on optical networks in which data passes through a fiber optic cable (which can be seen as an edge in the associated graph) which transmits over longer distances a signal between its two extremities in the form of light or photons.
To identify which network we are working on, we have looked to operator network hierarchy that comprises three different parts, each of them having a specific design depending on traffic requirements and dimensioning context. The access network is the first part of a telecommunication network that gives the end-customer access to the telecommunications service(s). On the other hand, the metropolitan network interconnects customers with some services in a geographic area [4][73]. However, the core network offers numerous services to the customer that are interconnected by the access network [4][73]. In our project, we focused on the core network given the costs dedicated for this part of the network and the combinatorial optimization problems issue from it that are very interesting from theoretical and application cases point of view.
An optical network is composed of several pieces of equipment to manage differents exchanged signals. To combine several input signals, multiplexers are placed in the network which are hardware components that combine several input signals into an optical fiber. At the end of the receiver, the multiplexers are called demultiplexers performing an inverse function of the multiplexers such that the combined signals are separated into a separate signal.
The optical signal quality can deteriorate when it exceeds the maximum transmission distance, namely transmission reach. To reinforce the passive optical signal on fiber without converting it to an electrical signal, Optical Amplifiers (OAs) are placed on fibers to do that. Erbium-Doped Fiber Amplifiers EDFA are the most important fiber amplifiers that are

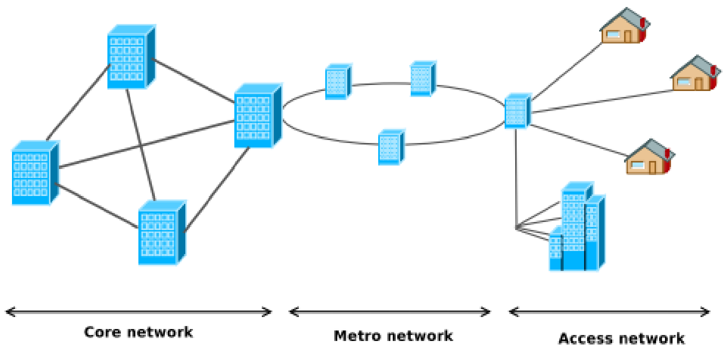


Figure 1.5: Schematic of Telecom Network [11].

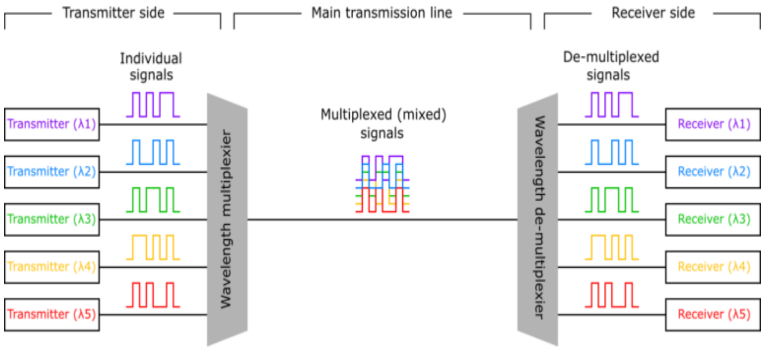


Figure 1.6: Wavelength Division Multiplexing [54].

used between long spans (the spaced line amplifiers between two consecutive amplifiers). An Amplifier produces the Spontaneous Emission Noise (ASE noise) to compensate the passive signal through a span. However, for each modulation format, we are limited by the maximum total cumulative spontaneous emission noise, i.e., it is necessary to regenerate the signal when we exceed a max number of amplifiers who are compensated the passive signal. The amplification site consists of an optical amplifier and section of Dispersion Compensating Fiber (DCF). As we said in the last paragraph, the amplification site compensates for the fiber absorption losses. Another component is placed on nodes to do better than this but they are more expensive called regenerators which can be represented as pair of transponders. A regenerator restores the optical signal quality. The signal regeneration is necessary to re-amplify, re-shape, and re-time (3R) the passive optical signal when the transmission reach of signals in an optical system is limited.
To manage the multiple signals passing through nodes in the network, an optical wwitch or add/drop multiplexers are essentials to add and drop individual optical signals without converting signals from optical to electrical in order to optimize the capacity and efficiency of optical networks. They are components of reconfigurable optical add/drop multiplexers (ROADM) which can take place on fibers and nodes.
In an optical network, we distinguish three layers, application layer, electronic switching, and multiplexing layer, and an optical layer. Application layer including all types of services, e.g., image, data, and videos... The electronic switching and multiplexing layer regroup data coming from the application layer and deliver it to its destination. This traffic will be aggregated into the optical layer, where they are carried by wavelengths [88]. In the optical

layer, the nodes and fibers are placed and the optical paths are established. As a result, a set of routers are placed and a virtual edge is established in the IP layer to link every two nodes for which there exists an optical path that connect them. Routers deliver the intended packets to their destination such that they have multiple input and output ports essential to perform the physical layer functionality, and the output port stores packets received from the switching fabric and transmit these packets on the outgoing link. They are equipped with a number of transmitters. Furthermore, interfaces are placed in the optical network to connect the routers to network nodes.

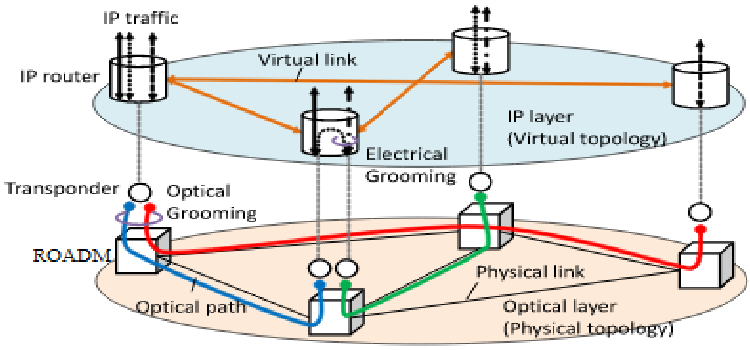


Figure 1.7: IP-Over-EON Network Architecture such that the virtual links in the IP layer correspond to light paths assigned in the optical layer by the RSA algorithm [167].

Note that an optical transport network is generally categorized in three modes, namely, opaque, transparent, and translucent which depend on the utilization of Optical-Electrical-Optical conversion [73]:

a) **Transparent Network:** in this case the signal keeps in the optical domain and at every node in the network we cannot regenerate the signal which means that the signal quality can degrade when it exceeds a certain distance (transmission-reach), and we fear can not find a modulation format compatible with the route of each traffic demand [73].

b) **Opaque Network:** in this mode of operation, we have in each node who has a degree plus than 2 (called an opaque node) an Optical-Electrical-Optical conversion of the signals such that each opaque node looks like the endpoint of each transmission signal where the signal terminated, regenerated and return from new to the next node over the route. So we need more transponders in this case that for each wavelength a couple of transponders (regenerator). This technique protects the signal against unexpected physical impairments [73]. However, it costs expensive due to the Capex and Opex costs dedicated for it.

c) **Translucent Network:** unlike opaque mode and transparent mode, the signal can be regenerated in the network before it exceeds transmission reach, so the optical signal keeps in the optical domain as far as possible before it exceeds transmission reach and needs to be regenerated. In context, we have two cases: nodes regenerators already exist or we have to place a regenerator in the network [73].

We focus in this work on a new variant of routing and resources allocation problems issue from the optical transport network design problems encountered when planning and dimensioning of a transparent optical transport network.

1.6.2 The Rise of Flexible Optical Networks

The two last decades of the new millennium saw a profound change in telecommunication networks with a continuous growth in demand. To face this trend of increase in bandwidth, network operators have had to make their network architectures and management evolve. To do so, two significant changes appeared recently in the optical network architecture. First the bandwidth-greedy *FixedGrid* architecture for Optical Wavelength Division Multiplexing (WDM) (called also wavelength routed network) [137] [138] based on fixed spectrum grid is being replaced by the *FlexGrid* architecture that is capable of supporting variable data rate (in Gb/s) through flexible spectrum. In these Spectrally Flexible Optical Networks (SFONs) (called also Elastic Optical Networks EONs) the optical spectrum is divided into slots having the same frequency of 12.5 GHz (where FixedGrid networks use 50 GHz, the width of a wavelength) as recommended by ITU-T [4]. See for example the figure 1.8 which shows that in the fixed-grid case we use 4 wavelengths of 50 GHz to serve 4 demandes of two of 10 Gb/s, one of 400 Gb/s, one of 1000 Gb/s. However, in the flex-grid we use just 9 slots of frequency 12.5 GHz to serve these demands.

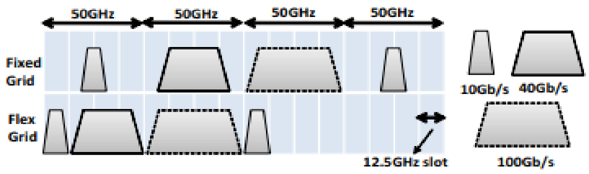


Figure 1.8: FixedGrid Vs FlexGrid [76].

The concept of slot was proposed initially by Masahiko Jinno et al. in 2008 [83], and lately explored by same authors in 2010 [174]. In SFONs any optical path can elastically span as many contiguous slots as needed. This technology provides a more efficient use of the spectral domain than the traditional Fixed Grid WDM. Secondly a new generation of transponders is becoming available namely, bandwidth-variable transponders (BV-Ts) and bandwidth variable wavelength cross-connects (BV-WXCs) [174]. They can manage data rates up to 400 Gb/s which cannot be accommodated by a 50 GHz wavelength, and restores the signal which is necessary to re-amplify, re-shape and re-time the passive optical signal (which is called (3R) signal regeneration rule) when the transmission-reach of signals is limited which represents the maximum length (in kms) for the routing of each traffic demand.

1.6.3 Flexible Optical Network Design Problems

The network operators have confronted several optimization problems in particular some variants of routing and resource allocation problems that appear when designing or planning optical networks. The historical Routing and Wavelength Assignment (RWA) problem is the key issue for routing and resource allocation problems to design a FixedGrid WDM networks. In this problem, we are given an optical network and a set of demands where each demand has an origin and destination. The task is to find a path for each demand and a wavelength such that a single 50 GHz wavelength is assigned to each demand. It was considered for the first time by Bal et al. in 1991, and extended by Chlamtac et al. in 1992 [24]. It is known to be a NP-hard problem [24] by showing the equivalence of the problem to the n-graph-coloring problem where the number of colors n corresponds to the number of wavelengths such that finding the minimal number of wavelengths to route all the traffic demands is equivalent to finding the chromatic number of conflict graph (where demands are represented by nodes such that two nodes are linked iff the final paths of the associated demands share an edge) when the paths are already established. It was considered also as a special case of the integer multicommodity flow (MCF) problem that some technologique specific constraints [16] are added and should be respected. Several mathematical models and algorithms have been proposed to solve the RWA problem. They are based on some ILP formulations as done in [16], [22], [38], [39], [78], [79], [85], [91], [90], [115], [158], decomposition-based methods [8], [158], [162], [183], and heuristics [9], [10], [58], [81], [99], [108], [157], [159], [161], [164],[184], [185].

In SFONs, RWA cannot handle the changes from wavelength to contiguous slots. As a result, the RWA has been replaced by the so-called Routing and Spectrum Assignment (RSA) problem. It can be stated as follows. Consider an optical network as an undirected, loopless, and connected graph $G = (V, E)$, which is specified by a set of nodes V, a multiset of links E, and a set of contiguous frequency slots $\{1, \ldots, \bar{s}\}$ with $\bar{s} \in \mathbb{Z}_+$. Each link $e = ij \in E$ is associated with a length $\ell_e \in \mathbb{R}_+$ (in kms), a cost $c_e \in \mathbb{R}_+$. Let K be a set of demands such that each demand $k \in K$ is specified by an origin node $o_k \in V$, a destination node $d_k \in V \setminus \{o_k\}$, and a slot-width $w_k \in \mathbb{Z}_+$. The RSA consists of determining for each $k \in K$, a (o_k,d_k)-path p_k (subset of edges) in G, and a subset of contiguous frequency slots $S_k \subset \{1, \ldots, \bar{s}\}$ (contiguity and continuity constraint) of width equal to w_k such that $S_k \cap S_{k'} = \emptyset$ for each pair of demands $k, k' \in K$ with $p_k \cap p_{k'} \neq \emptyset$ (non-overlapping constraint), while optimizing some linear objective function(s). The RSA problem is very harder compared with the RWA problem because of the continuity constraint that has not been taken into account when defining the RWA problem.

Today, SFONs use the Optical Orthogonal Frequency Division Multiplexing (O-OFDM) modulation technology which allocates optical spectrums with variable data rate (of the order of a few gigabits per second-Gb/s). In this context, a new modulation format constraint has been added to the routing and spectrum assignment Sub-problems. Hence, a new problem is appeared, called the Routing, Modulation and Spectrum Assignment (RMSA) problem.

There are 6 basic modulation formats, we mention x-Quadrature Amplitude Modulation x-QAM where x belongs to $\{8, 16, 32, 64\}$ [25]. This modulation format is used for the shorter distance lightpaths but with high transmission-reach and date rate. However, for longer distance lightpaths, we have Binary Phase-Shift Keying BPSK, Quadrature Phase-Shift Keying QPSK more robust modulation formats but less efficient compared to x-QAM modulation format [25]. Each one of them has a date rate (Gb/s), spectrum efficiency SE or number of bits per symbol measured in (b/s/Hz), capacity of one subcarrier or a signal speed for one frequency slot (in GHz) (in multiple of 12.5 GHz) and transmission-reach (kms). These modulation formats turn as a result a set of transponder configuration $\mathcal{F}$.

The Constrained-Routing and Spectrum Assignment problem is an hybridization between RSA and RMSA such that it looks like decomposition of RMSA into two Sub-problems Modulation Assignment Sub-problem (M) and then Routing and Spectrum Assignment problem (i.e., C-RSA=M+RSA), in which each traffic demand k is in format of data rate $b_k \in \mathbb{R}_+$ (in Gb/s) such that we suppose that the network operator has selected a multiset $\mathcal{F}_k$ of transponders configuration for each demand k such that each transponder configuration $f \in \mathcal{F}$ is characterized by a data-rate $r_f \in \mathbb{R}_+$ (in Gb/s), a number of slots $w_f \in \mathbb{N}_+$,

a transmission-reach $\bar{\ell}_f \in \mathbb{R}_+$ (in kms), a capex cost $cap_f \in \mathbb{R}_+$, and an opex cost $op_f \in \mathbb{R}_+$. Table below shows an example of a multiset of transponders configuration $\mathcal{F}$

The multiset $\mathcal{F}_k$ of selected transponder configurations for each traffic demand k should satisfy the data-rate constraint

$$\sum_{f \in F_k} r_f \geq b_k.$$

$\mathcal{F}$	data rate (Gb/s)	modulation format	spectrum-width (GHz)	number of slots	transmission-reach (kms)	capex cost	opex cost
1	100	DP-QPSK	37,5	3	3000	7	3
2	100	DP-QPSK	37,5	3	6000	17,5	6
3	200	DP-8QAM	62,5	5	1500	11	4
4	200	DP-8QAM	62,5	5	3000	27,5	9
5	200	DP-16QAM	37,5	3	1000	13	11
6	200	DP-16QAM	37,5	3	2000	32,5	15

Table 1.1: An example of a multiset of transponders configuration $\mathcal{F}$.

After this modulation assignment procedure, each demand k between origin node o_k to a destination node d_k is specified by a number of slots $w_k \in \mathbb{N}_+$, where $w_k = \sum_{f \in \mathcal{F}_k} w_f$, and a transmission-reach $\bar{\ell_k} \in \mathbb{R}_+$ (in kms), with $\bar{\ell_k} = \min_{f \in \mathcal{F}_k} \bar{l}_f$.
As a result of all this, we define a new variant of RSA and RMSA problems that we call Constrained-Routing and Spectrum Assignment such that respecting the transmission-reach constraint is added to the satisfaction of the three constraints of spectrum: contiguity, continuity and non-overlapping. In this dissertation, we are interested on the resolution of the C-RSA problem given that it satisfies all the real constraints required by a network operator compared with the existed variants like RSA and RMSA. There exist several use cases of this problem that are very meaningful for a network-operator today, we mention

a) Network planning and dimensioning Without or With survivability: as done in [3], [18], [21],[31], [61], [65], [80], [118],[127],[129] ,[130],[131],[132], [133],[134], [143], [156], [173], [192], and [180].

b) Regeneration placement problems: as done in [52], [64], [119], [89], and [88].

c) Dynamic networks: in this context, paths are already established and spectrums are already allocated for each traffic demand in K. The network operator has to satisfy new incoming demands one by one or all together which depends the network state, i.e., the availability of resources in the network. Several works related to the dynamic C-RSA problem are done in this context. We mention the works done by Castro et al. in [20], Hadi et al. in [71], Lohani et al. in [95], Wang et al. [176], Xu et al. in [178], Yin et al. [182].

d) Network restructuring: based on some fragmentation technics [2],[72],[154], [169], [181].

e) Network traffic prediction and security: face to uncertainty quantification of traffic [93], [100], [111].

f) Software-Defined Networking frameworks (SDN) to manage dynamic networks by integrating SFONs in SDN [135], [23], [160].

g) Fifth-generation (5G) optical transport networks: in the context of spectrum assignment management [152], [160].

h) Technological migration: for example, from FixedGrid optical network to Flexgrid optical. This can be seen as the issue of the day. Recent works have been done for this subject. We found the works done by Zhang et al. in [186].

This has caught our attention, and we then decided to focus on the Constrained-Routing and Spectrum Assignment problem.

Chapter 2

Cut Formulation and Polyhedra for the C-RSA Problem

2.1 The Constrained-Routing and Spectrum Assignment Problem

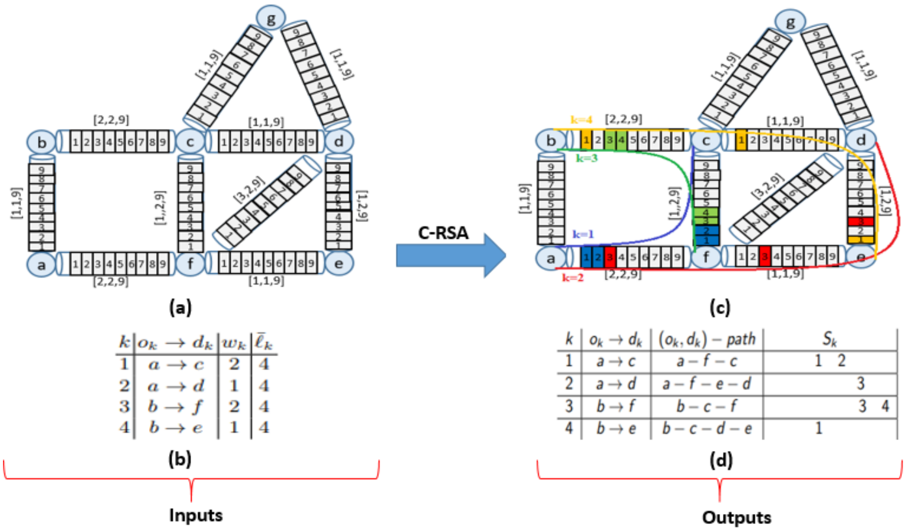
The Constrained-Routing and Spectrum Assignment Problem can be stated as follows. We consider a spectrally flexible optical networks as an undirected, loopless, and connected graph $G = (V, E)$, which is specified by a set of nodes V, and a multiset [1] E of links (optical-fibers). Each link $e = ij \in E$ is associated with a length $\ell_e \in \mathbb{R}_+$ (in kms), a cost $c_e \in \mathbb{R}_+$ s.t. each fiber-link $e \in E$ is divided into $\bar{s} \in \mathbb{N}_+$ slots. Let $\mathbb{S} = \{1, \dots, \bar{s}\}$ be an optical spectrum of available frequency slots with $\bar{s} \leq 320$ given that the maximum spectrum bandwidth of each fiber-link is 4000 GHz [82] (i.e., $320 = \frac{4000}{12.5}$), and K be a multiset [2] of demands s.t. each demand $k \in K$ is specified by an origin node $o_k \in V$, a destination node $d_k \in V \setminus \{o_k\}$, a slot-width $w_k \in \mathbb{Z}_+$, and a transmission-reach $\bar{\ell_k} \in \mathbb{R}_+$ (in kms). The C-RSA problem consists of determining for each demand $k \in K$, a (o_k,d_k)-path p_k in G s.t. $\sum_{e \in E(p_k)} l_e \leq \bar{l}_k$, where $E(p_k)$ denotes the set of edges belong the path p_k, and a subset of contiguous frequency slots $S_k \subset \mathbb{S}$ of width equal to w_k s.t. $S_k \cap S_{k'} = \emptyset$ for each pair of demands $k, k' \in K$ $(k \neq k')$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ so the total cost of the paths used for routing the demands (i.e., $\sum_{k \in K} \sum_{e \in E(p_k)} c_e$) is minimized.
Figure 2 shows the set of established paths and spectrums for the set of demands $\{k_1, k_2, k_3, k_4\}$ (Fig. 2(c) and Table 2(d)) of Table 2(b) in a graph G of 7 nodes and 10 edges (Fig. 2(a)) s.t. each edge e is characterized by a triplet $[l_e, c_e, \bar{s}]$, and optical spectrum $\mathbb{S} = \{1, 2, 3, ..., 8, 9\}$ with $\bar{s} = 9$.

2.2 Cut Formulation

Here we introduce our integer linear programming formulation based on cut formulation for the C-RSA problem which can be seen as a reformulation of the one introduced by Hadhbi et al. in [69]. For $k \in K$ and $e \in E$, let x^k_e be a variable which takes 1 if demand k goes through the edge e and 0 if not, and for $k \in K$ and $s \in \mathbb{S}$, let z^k_s be a variable which takes 1 if slot s is the last-slot allocated for the routing of demand k and 0 if not. The contiguous

[1] We take into account the presence of parallel fibers s.t. two edges e, e' which have the same extremities i and j are independents.

[2] We take into account that we can have several demands between the same origin-node and destination-node.



k	$o_k \to d_k$	w_k	$\bar{\ell}_k$
1	$a \to c$	2	4
2	$a \to d$	1	4
3	$b \to f$	2	4
4	$b \to e$	1	4

k	$o_k \to d_k$	(o_k, d_k) – path	S_k
1	$a \to c$	$a-f-c$	1 2
2	$a \to d$	$a-f-e-d$	3
3	$b \to f$	$b-c-f$	3 4
4	$b \to e$	$b-c-d-e$	1

Figure 2.1: Set of established paths and spectrums in graph G (Fig. 2(a)) for the set of demands $\{k_1, k_2, k_3, k_4\}$ defined in Table 2(b).

slots $s' \in \{s - w_k + 1, ..., s\}$ should be assigned to demand k whenever $z_s^k = 1$.
The C-RSA problem can be formulated as follows.

$$\min \sum_{k \in K} \sum_{e \in E} c_e x_e^k, \tag{2.1}$$

subject to

$$\sum_{e \in \delta(X)} x_e^k \geq 1, \forall k \in K, \forall X \subseteq V \text{ s.t. } |X \cap \{o_k, d_k\}| = 1, \tag{2.2}$$

$$\sum_{e \in E} l_e x_e^k \leq \bar{\ell}_k, \forall k \in K, \tag{2.3}$$

$$z_s^k = 0, \forall k \in K, \forall s \in \{1, ..., w_k - 1\}, \tag{2.4}$$

$$\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1, \forall k \in K, \tag{2.5}$$

$$x_e^k + x_e^{k'} + \sum_{s'=s}^{min(s+w_k-1, \bar{s})} z_{s'}^k + \sum_{s'=s}^{min(s+w_{k'}-1, \bar{s})} z_{s'}^{k'} \leq 3, \forall (e, k, k', s) \in Q, \tag{2.6}$$

$$0 \leq x_e^k \leq 1, \forall k \in K, \forall e \in E, \tag{2.7}$$

$$0 \leq z_s^k \leq 1, \forall k \in K, \forall s \in \mathbb{S}, \tag{2.8}$$

$$x_e^k \in \{0, 1\}, \forall k \in K, \forall e \in E, \tag{2.9}$$

$$z_s^k \in \{0, 1\}, \forall k \in K, \forall s \in \mathbb{S}. \tag{2.10}$$

where Q denotes the set of all the quadruples (e, k, k', s) for all $e \in E, k \in K, k' \in K \setminus \{k\}$, and $s \in \mathbb{S}$ with $(k, k') \notin K_c^e$.
Inequalities (2.2) ensure that there is an (o_k, d_k)-path between o_k and d_k for each demand k, and guarantee that all the demands should be routed. They are called cut inequalities. By optimizing the objective function (2.1), and given that the length of all edges are strictly

positives, this ensures that there is exactly one (o_k, d_k)-path between o_k and d_k which will be selected as optimal path for each demand k. We suppose that we have sufficient capacity in the network so that all the demands can be routed. This means that we have at least one feasible solution for the problem. Inequalities (2.3) express the length limit on the routing paths which is called "the transmission-reach constraint". Equations (2.4) express the fact that a demand k cannot use slot $s \leq w_k - 1$ as the last-slot. The slots $s \in \{1, ..., w_k - 1\}$ are called forbidden last-slots for demand k. Inequalities (2.5) should normally be an equation form ensuring that exactly one slot $s \in \{w_k, \ldots, \bar{s}\}$ must be assigned to demand k as last-slot . Here we relax this constraint. By a choice of the objective function, the equality is guaranteed at the optimum (e.g. $\min \sum_{k \in K} \sum_{s=w_k}^{\bar{s}} s.z_s^k$ or $\min \sum_{k \in K} \sum_{s=w_k}^{\bar{s}} s.w_k.z_s^k$). Inequalities (2.6) express the contiguity and non-overlapping constraints. Inequalities (2.7)-(2.8) are the trivial inequalities, and constraints (2.9)-(2.10) are the integrality constraints.
Note that the linear relaxation of the C-RSA can be solved in polynomial time given that inequalities (2.2) can be separated in polynomial time using network flows, see e.g. preflow algorithm of Goldberg and Tarjan introduced in [62] which can be run in $O(|V \setminus V_0^k|^3)$ time for each demand $k \in K$.

Proposition 2.2.1. *The formulation* (2.2)-(2.10) *is valid for the C-RSA problem.*

Proof. It is trivial given the definition of each constraint of the formulation (2.2)-(2.10) s.t. any feasible solution for this formulation is necessary a feasible solution for the C-RSA problem.

□

Proposition 2.2.2. *Every feasible solution of our cut formulation* (2.1)-(2.10) *is also feasible solution of multi-commodity flow problem.*

Proof. It is trivial given that any feasible solution of C-RSA problem ensures that there is a flow of w_k slots routed along a path p_k which links between the origin-node o_k and destination-node d_k for each demand $k \in K$ while satisfying the capacity of edges which equals to $\bar{s}$. □

Proposition 2.2.3. *Every feasible solution of multi-commodity flow problem is not necessary feasible for our cut formulation* (2.1)-(2.10)*.*

Proof. It is trivial given that the solution of the multi-commodity flow problem can easily violate the contiguity and continuity constraints of our C-RSA problem. This means that the w_k slots assigned to the demand k can be not contiguous in a feasible solution of multi-commodity flow problem, and also for example when the w_k slots can be not the same along the path p_k for the demand k. □

2.3 Associated Polytope

An instance of the C-RSA is defined by a triplet $(G, K, \mathbb{S})$. Let $P(G, K, \mathbb{S})$ be the polytope, convex hull of the solutions for the cut formulation (2.1)-(2.10). Throughout the proofs, we take into account that $x_e^k \leq 1$ for each demand $k \in K$ and edge $e \in E$, and $z_s^k \geq 0$ for each demand $k \in K$ and slot $s \in \mathbb{S}$. Note that a slot $s \in \mathbb{S}$ is assigned to a demand $k \in K$ iff $\sum_{s'=s}^{min(\bar{s}, s+w_k-1)} z_{s'}^k = 1$.
In this section, we discuss the facial structure of the polytope $P(G, K, \mathbb{S})$. First, we describe some structural properties. These will be used for determining the dimension of $P(G, K, \mathbb{S})$. For each demand k and each node v, one can compute a shortest path between each of the

pair of nodes (o_k, v), (v, d_k). If the lengths of the $(o_k, d_k)-$paths formed by the shortest paths (o_k, v) and (v, d_k) are both greater that $\bar{l}_k$ then node v cannot be in a path routing demand k, and we then say that v is a *forbidden node* for demand k due to the transmission-reach constraint. Let V_0^k denote the set of forbidden nodes for demand $k \in K$. Note that using Dijkstra's algorithm, one can identify in polynomial time the forbidden nodes V_0^k for each demand $k \in K$. On the other hand and regarding the edges, for each demand k and each edge $e = ij$, one can compute a shortest path between each of the pair of nodes (o_k, i), (j, d_k), (o_k, j) and (i, d_k). If the lengths of the $(o_k, d_k)-$paths formed by e together with the shortest (o_k, i) and (j, d_k) (resp. (o_k, j) and (i, d_k)) paths are both greater that $\bar{l}_k$ then edge ij cannot be in a path routing for demand k, and we then say that ij is a *forbidden edge* for demand k due to the transmission-reach constraint. Let E_t^k denote the set of forbidden edges due to the transmission-reach constraint for demand $k \in K$. Note that using Dijkstra's algorithm, one can identify in polynomial time the forbidden edges E_t^k for each demand $k \in K$. This allows us to create in polynomial time a proper topology G_k for each demand k by deleting the forbidden nodes V_0^k and forbidden edges E_t^k from the original graph G (i.e., $G_k = G(V \setminus V_0^k, E \setminus E_t^k)$). As a result, there may exist some forbidden-nodes due to the elementary-path constraint which means that all the $(o_k, d_k)-$paths passed through a node v are not elementary-paths. This can be done in polynomial time using Breadth First Search (BFS) algorithm of complexity $O(|E \setminus E_t^k| + |V \setminus V_0^k|)$ for each demand k. Note that we did not take into account this case in our study. Table 2.1 below shows the set of forbidden edges E_t^k and forbidden nodes V_0^k for each demand k in K already given in Fig. 2(b).

k	$o_k \to d_k$	w_k	$\bar{\ell}_k$	V_0^k	E_t^k
1	$a \to c$	2	4	$\{e, d, g\}$	$\{cg, dg, de, df, cd, ef\}$
2	$a \to d$	1,00	4	$\{g\}$	$\{cg, dg, df\}$
3	$b \to f$	2	4	$\{e, d, g\}$	$\{cg, dg, de, df, cd, ef\}$
4	$b \to e$	1,00	4	$\{g\}$	$\{cg, dg, df\}$

Table 2.1: Topology pre-processing for the set of demands K given in Fig. 2(b).

Let $\delta_{G_k}(v)$ denote the set of edges incident with a node v for the demand k in G_k. Let $\delta^k(W)$ denote a cut for demand $k \in K$ in G_k s.t. $o_k \in W$ and $d_k \in V \setminus W$ where W is a subset of nodes in $V \setminus V_0^k$ of G_k. Let f be an edge in $\delta(W)$ s.t. all the edges $e \in \delta(W) \setminus \{f\}$ are forbidden for demand k. As a consequence, edge f is an *essential edge* for demand k. As the forbidden edges, the essential edges can be determined in polynomial time using network flows as follows.

a) we create a proper topology $G_k = G(V \setminus V_0^k, E \setminus E_t^k)$ for the demand k

b) we fix a weight equals to 1 for all the edges e in $E \setminus E_t^k$ for the demand k in G_k

c) we calculate $o_k - d_k$ Cut which separates o_k from d_k.

d) if $\delta_{G_k}(W) = \{e\}$ then the edge e is an essential edge for the demand k s.t. $o_k \in W$ and $d_k \in V \setminus W$. We increase the weight of the edge e by 1. Go to (3).

e) if $|\delta_{G_k}(W)| > 1$ then end of algorithm.

Let E_1^k denote the set of essential edges of demand k, and K_e denote a subset of demands in K s.t. edge e is an essential edge for each demand $k \in K_e$. Therefore,

$$x_e^k = 1, \text{ for all } k \in K \text{ and } e \in E_1^k. \tag{2.11}$$

In addition to the forbidden edges thus obtained due to the transmission-reach constraints, there may exist edges that may be forbidden because of lack of resources for demand k. This is the case when, for instance, the residual capacity of the edge in question does not allow a demand to use this edge for its routing, i.e., $w_k > \bar{s} - \sum_{k' \in K_e} w_{k'}$. Let E_c^k denote the set of forbidden edges for demand k, $k \in K$, due to the resource constraints. Note that the forbidden edges E_c^k and forbidden nodes v in V with $\delta(v) \subseteq E_t^k$, should also be deleted from the proper graph G_k of demand k, which means that G_k contains $|E \setminus |E_t^k|$ edges and $|V \setminus |\{v \in V, \delta(v) \subseteq E_t^k\}|$ nodes. Let $E_0^k = E_t^k \cup E_c^k$ denote the set of all forbidden edges for demand k that can be determined due to the transmission reach and resources constraints. Hence,

$$x_e^k = 0, \text{ for all } k \in K \text{ and } e \in E_0^k. \tag{2.12}$$

As a result of the pre-processing stage, some non-compatibility between demands may appear due to a lack of resources as follows.

Definition 2.3.1. *For an edge e, two demands k and k' with $e \notin E_0^k \cup E_1^k \cup E_0^{k'} \cup E_1^{k'}$, are said* non-compatible demands *because of lack of resources over the edge e iff the the residual capacity of the edge e does not allow to route the two demands k, k' together through e, i.e., $w_k + w_{k'} > \bar{s} - \sum_{k" \in K_e} w_{k"}$. Let K_c^e denote the set of pair of demands (k, k') in K that are non-compatibles for the edge e.*

2.3.1 Dimension

We first describe some properties that are useful to determine the dimension of $P(G, K, \mathbb{S})$.

Proposition 2.3.1. *The follows equation system* (2.13) *is of full rank*

$$\begin{cases} x_e^k = 0, \textit{ for all } k \in K \textit{ and } e \in E_0^k, \\ x_e^k = 1, \textit{ for all } k \in K \textit{ and } e \in E_1^k, \\ z_s^k = 0, \textit{ for all } k \in K \textit{ and } s \in \{1, ..., w_k - 1\}. \end{cases} \tag{2.13}$$

The rank of system (2.13) is given by

$$r = \sum_{k \in K} (|E_0^k| + |E_1^k| + (w_k - 1)).$$

Proof. Let Q denote a matrix associated with the system (2.13) which contains r lines linear independents. We distinguish 4 blocks of lines in Q as below

a) block Q^1 corresponds to the equations $x_e^k = 0$ for all $k \in K$ and all $e \in E_0^k$,

b) block Q^2 corresponds to the equations $x_e^k = 1$ for all $k \in K$ and all $e \in E_1^k$,

c) block Q^3 corresponds to the equations $z_s^k = 0$ for all $k \in K$ and all $s \in \{1, ..., w_k - 1\}$.

Note that the 3 blocks of the matrix Q are independents.

□

A solution of the C-RSA problem is given by two sets E_k and S_k for each demand $k \in K$ where E_k is a set of edges used for the routing of demand k which contains a path p_k satisfying the continuity of path p_k for the demand k (i.e., $E(p_k) \subseteq E_k$) s.t. $\sum_{e \in E_k} l_e \leq \bar{l}_k$ and $E_1^k \subseteq E_k$, and S_k is a set of slots which represent the set of last-slot of slot-disjoint (i.e., the intervals do not share some slots) selected for the demand k which forms a set of intervals of contiguous slots s.t. each interval contains w_k contiguous slots. To facilitate the understanding of proofs, we call by E_k a feasible path, and by S_k the last-slots assigned to the demand k.

Figure 2.2 shows the routing solutions for a demand k that are feasible for our problem throughout the proofs.

Below some genral hypotheses which will be used along the deffirents proofs for differents

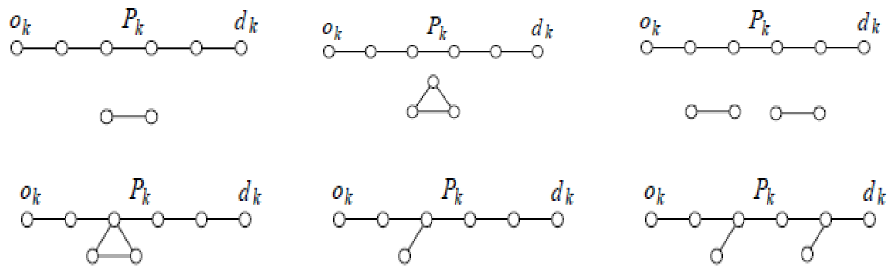


Figure 2.2: A set of edges E_k for a demand k containing an (o_k, d_k)-path P_k together with: isolated-edge, islated-cycle, two isolated-edges, linked-cycle, and linked edges.

propositions and theorems

a) We suppose that G_k is $o_k - d_k$ connected graph which ensure that there exists at least one feasible path which connects the origin node o_k with destination node d_k of demand k,

b) We suppose that $\bar{s}$ is sufficient to route all the demands which means that there is no demand rejected because a lack of resources on the links. This does not mean that we cannot have a forbidden edges because a lack of resources on the link i.e. there may exist some cases where $E_c^k \neq \emptyset$ for some demands k in K,

c) For each demand $k \in K$ and $e \in E \setminus (E_0^k \cup E_1^k)$, there exists at least a feasible route E_k between o_k and d_k s.t. $\sum_{e' \in E_k} l_{e'} + l_e \leq \bar{l}_k$, and for each $e' \in E_k$, the edges (e, e') are not non-compatible edges for the demand k.

Let $\mathcal{S}^i = (E^i, S^i)$ denote the set of edges and last-slots assigned to route the demands K in ith solution proposed for the C-RSA problem s.t. $E^i = (E_1^i, E_2^i, ..., E_{|K|-1}^i, E_{|K|}^i)$ and $S^i = (S_1^i, S_2^i, ..., S_{|K|-1}^i, S_{|K|}^i)$.

Proposition 2.3.2. *Consider an equation $\mu x + \sigma z = \lambda$ of $P(G, K, \mathbb{S})$. The C-RSA equation system* (2.13) *defines a minimal equation system for $P(G, K, \mathbb{S})$. As a consequence, we obtain that for each demand k*

a) $\sigma_s^k = 0$ for all slots $s \in \{w_k, ..., \bar{s}\}$,

b) $\mu_e^k = 0$ for all edges $e \in E \setminus (E_0^k \cup E_1^k)$,

and $\mu x + \sigma z = \lambda$ of $P(G, K, \mathbb{S})$ is a linear combination of equation system (2.13).

Proof. To prove that $\mu x + \sigma z$ is a linear combination of equations system (2.13), it is sufficient to prove that for each demand $k \in K$, there exists $\gamma_1^k \in \mathbb{R}^{|E_0^k|}, \gamma_2^k \in \mathbb{R}^{|E_1^k|}, \gamma_3^k \in \mathbb{R}^{w_k - 1}$ (given

that the matrix Q has 3 blocks) s.t. $(\mu, \sigma) = \gamma Q$.
Let $x^{\mathcal{S}}$ and $z^{\mathcal{S}}$ denote the incidence vector of a solution $\mathcal{S}$ of the C-RSA problem.
Let us show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$. Consider a demand k and a slot s in $\{w_k, ..., \bar{s}\}$. To do so, we consider a solution $\mathcal{S}^0 = (E^0, S^0)$ in which

a) a feasible path E_k^0 is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^0 is assigned to each demand $k \in K$ along each edge $e \in E_k^0$ with $|S_k^0| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^0$ and $s" \in S_{k'}^0$ with $E_k^0 \cap E_{k'}^0 \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^0} |\{s' \in S_k^0, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}^0$ with $E_k^0 \cap E_{k'}^0 \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s in the set of last-slots S_k^0 assigned to the demand k in the solution $\mathcal{S}^0$).

$\mathcal{S}^0$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}^0}, z^{\mathcal{S}^0})$ belongs to $P(G, K, \mathbb{S})$. Based on this, we derive a solution $\mathcal{S}^1 = (E^1, S^1)$ from the solution $\mathcal{S}^0$ by adding the slot s as last-slot to the demand k without modifying the paths assigned to the demands K in $\mathcal{S}^0$ (i.e., $E_k^1 = E_1^0$ for each $k \in K$), and the last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^0$ remain the same in the solution $\mathcal{S}^1$ i.e., $S_{k'}^0 = S_{k'}^1$ for each demand $k' \in K \setminus \{k\}$, and $S_k^1 = S_k^0 \cup \{s\}$ for the demand k. The solution $\mathcal{S}^1$ is feasible given that

a) a feasible path E_k^1 is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^1 is assigned to each demand $k \in K$ along each edge $e \in E_k^1$ with $|S_k^1| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^1$ and $s' \in S_{k'}^1$ with $E_k^1 \cap E_{k'}^1 \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^1} |\{s \in S_k^1, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^1}, z^{\mathcal{S}^1})$ belongs to $P(G, K, \mathbb{S})$. We then obtain that

$$\mu x^{\mathcal{S}^0} + \sigma z^{\mathcal{S}^0} = \mu x^{\mathcal{S}^1} + \sigma z^{\mathcal{S}^1} = \mu x^{\mathcal{S}^0} + \sigma z^{\mathcal{S}^0} + \sigma_s^k.$$

It follows that $\sigma_s^k = 0$ for demand k and a slot $s \in \{w_k, ..., \bar{s}\}$. The slot s is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k s.t. we find

$$\sigma_s^k = 0, \text{ for demand } k \text{ and all slots } s \in \{w_k, ..., \bar{s}\}$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\}$$

Consequently, we conclude that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\}$$

Next, we will show that $\mu_e^k = 0$ for all the demands $k \in K$ and all $e \in E \setminus (E_0^k \cup E_1^k)$. Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. For that, we consider a solution $\mathcal{S}'^0 = (E'^0, S'^0)$ in which

a) a feasible path E'^0_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S'^0_k is assigned to each demand $k \in K$ along each edge $e \in E'^0_k$ with $|S'^0_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S'^0_k$ and $s" \in S'^0_{k'}$ with $E'^0_k \cap E'^0_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E'^0_k} |\{s' \in S'^0_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s \in S'^0_k$ and $s' \in S'^0_{k'}$ with $(E'^0_k \cup \{e\}) \cap E'^0_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e in the set of edges E'^0_k selected to route the demand k in the solution $\mathcal{S}'^0$),

e) and the edge e is not non-compatible edge with the selected edges $e \in E'^0_k$ of demand k in the solution $\mathcal{S}'^0$, i.e., $\sum_{e' \in E'^0_k} l_{e'} + l_e \leq \bar{l}_k$. As a result, $E'^0_k \cup \{e\}$ is a feasible path for the demand k.

$\mathcal{S}'^0$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^0}, z^{\mathcal{S}'^0})$ belongs to $P(G, K, \mathbb{S})$. Based on this, we derive a solution $\mathcal{S}^2$ obtained from the solution $\mathcal{S}'^0$ by adding an unused edge $e \in E \setminus (E^k_0 \cup E^k_1)$ for the routing of demand k in K in the solution $\mathcal{S}'^0$ which means that $E^2_k = E'^0_k \cup \{e\}$, and removing slot s selected for the demand k in $\mathcal{S}'^0$ and replaced it by a new slot $s' \in \{w_k, ..., \mathbb{S}\}$ (i.e., $S^2_k = (S'^0_k \setminus \{s\}) \cup \{s'\}$ s.t. $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S'^0_{k'}$ with $E^2_k \cap E'^0_{k'} \neq \emptyset$. The last-slots and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^0$ remain the same in the solution $\mathcal{S}^2$, i.e., $S^2_{k'} = S'^0_{k'}$ and $E^2_{k'} = E'^0_{k'}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^2$ is clearly feasible given that

a) and a feasible path E^2_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^2_k is assigned to each demand $k \in K$ along each edge $e \in E^2_k$ with $|S^2_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S^2_k$ and $s' \in S^2_{k'}$ with $E^2_k \cap E^2_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E^2_k} |\{s \in S^2_k, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^2}, z^{\mathcal{S}^2})$ is belong to $P(G, K, \mathbb{S})$. It follows that

$$\mu x^{\mathcal{S}'^0} + \sigma z^{\mathcal{S}'^0} = \mu x^{\mathcal{S}^2} + \sigma z^{\mathcal{S}^2} = \mu x^{\mathcal{S}'^0} + \mu^k_e + \sigma z^{\mathcal{S}'^0} - \sigma^k_s + \sigma^k_{s'},$$

which implies that $\mu^k_e = 0$ for demand k and an edge e given that $\sigma^k_s = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$.
As e is chosen arbitrarily for the demand k with $e \notin E^k_0 \cup E^k_1$, we iterate the same procedure for all $e' \in E \setminus (E^k_0 \cup E^k_1 \cup \{e\})$. We conclude that for the demand k

$$\mu^k_e = 0, \text{ for all } e \in E \setminus (E^k_0 \cup E^k_1).$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e \in E \setminus (E^k_0 \cup E^k_1)$. We conclude at the end that

$$\mu^k_e = 0, \text{ for all } k \in K \text{ and all } e \in E \setminus (E^k_0 \cup E^k_1).$$

Therefore all the equations of the polytope $P(G,K,\mathbb{S})$ are given only in terms of the variables x_e^k with $e \in E_0^k \cup E_1^k$ and z_s^k with $s \in \{1,...,w_k\}$. Let $Q^k = \begin{pmatrix} Q_k^1 \\ Q_k^2 \\ Q_k^3 \end{pmatrix}$ be the submatrix of matrix Q associated to the equations (2.12) and (2.11) and involving variables x_e^k for all $e \in E_0^k \cup E_1^k$ and variables z_s^k with $s \in \{1,...,w_k\}$ for demand k. Note that a forbidden edge can never be an essential edge at the same time. Otherwise, the problem is infeasible. We want to show that $\mu^k = \gamma_1^k Q_k^1 + \gamma_2^k Q_k^2$ and $\sigma^k = \gamma_3^k Q_3^k$. For that, we first ensure that all the edges $e \in E_0^k$ for each demand k are independants s.t. for each demand $k \in K$ we have

$$\sum_{e \in E_0^k} \mu_e^k = \sum_{e \in E_0^k} \gamma_1^{k,e} \rightarrow \sum_{e \in E_0^k} (\mu_e^k - \gamma_1^{k,e}) = 0.$$

The only solution of this system is $\mu_e^k = \gamma_1^{k,e}$ for each $e \in E_0^k$ for the demand k. As k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$. We conclude that

$$\mu_e^k = \gamma_1^{k,e}, \text{ for all } k \in K \text{ and all } e \in E_0^k,$$

We re-do the same thing for the edges $e \in E_1^k$ for each demand k which are independants s.t. for each demand $k \in K$ we have

$$\sum_{e \in E_1^k} \mu_e^k = \sum_{e \in E_1^k} \gamma_2^{k,e} \rightarrow \sum_{e \in E_1^k} (\mu_e^k - \gamma_2^{k,e}) = 0$$

The only solution of this system is $\mu_e^k = \gamma_2^{k,e}$ for each $e \in E_1^k$ for the demand k. As k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$. We conclude that

$$\mu_e^k = \gamma_2^{k,e}, \text{ for all } k \in K \text{ and all } e \in E_1^k,$$

On the other hand, note that the slots $s \in \{1,...,w_k - 1\}$ for each demand k are independants s.t. for each demand $k \in K$, we have

$$\sum_{s=1}^{w_k-1} \sigma_s^k = \sum_{s=1}^{w_k-1} \gamma_3^{k,s} \rightarrow \sum_{s=1}^{w_k-1} (\sigma_s^k - \gamma_3^{k,s}) = 0$$

The only solution of this system is $\sigma_s^k = \gamma_3^{k,s}$ for each $s \in \{1,...,w_k - 1\}$ for the demand k. As k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$. We then get that

$$\sigma_s^k = \gamma_3^{k,s}, \text{ for all } k \in K \text{ and all } s \in \{1,...,w_k - 1\}. \tag{2.14}$$

We conclude at the end that for each $k \in K$ and $e \in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e}, \text{ if } e \in E_0^k \\ \gamma_2^{k,e}, \text{ if } e \in E_1^k \\ 0, otherwise \end{cases}$$

yielding

$$\mu^k = \gamma_1^k Q_k^1 + \gamma_2^k Q_k^2 \text{ for each } k \in K.$$

Moreover, for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\} \\ 0, otherwise \end{cases}$$

i.e., $\sigma^k = \gamma_3^k Q_k^3$.
As a result $(\mu, \sigma) = \gamma Q$ with $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ which ends the proof. □

Theorem 2.3.1. *The dimension of $P(G, K, \mathbb{S})$ is given by*

$$dim(P(G, K, \mathbb{S})) = |K| * (|E| + |\mathbb{S}|) - r.$$

Proof. Given the rank of the C-RSA equation system (2.13) and the proposition (2.3.2). □

2.3.2 Facial Investigation

In this section, we describe facets defining inequalities for the polytope $P(G, K, \mathbb{S})$ from the cut formulation (2.2)-(2.10), and the ones from the valid inequalities. First, we characterize when the basic inequalities (2.2)-(2.10) define facets.

Theorem 2.3.2. *Consider a demand $k \in K$, and an edge $e \in E \setminus (E_0^k, E_1^k)$. Then, the inequality $x_e^k \geq 0$ is facet defining for $P(G, K, \mathbb{S})$.*

Proof. Let's us denote F_e^k the face induced by the inequality $x_e^k \geq 0$, which is given by

$$F_e^k = \{(x, z) \in P(G, K, \mathbb{S}) : x_e^k = 0\}.$$

In order to prove that the inequality $x_e^k \geq 0$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that F_e^k is a proper face which means that it is not empty, and $F_e^k \neq P(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^3 = (E^3, S^3)$ as below

a) a feasible path E_k^3 is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^3 is assigned to each demand $k \in K$ along each edge $e' \in E_k^3$ with $|S_k^3| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^3$ and $s' \in S_{k'}^3$ with $E_k^3 \cap E_{k'}^3 \neq \emptyset$ (non-overlapping constraint),

d) and the edge e is not chosen to route the demand k in the solution $\mathcal{S}^3$, i.e., $e \notin E_k^3$.

Obviously, $\mathcal{S}^3$ is feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^3}, z^{\mathcal{S}^3})$ is belong to $P(G, K, \mathbb{S})$ and then to F_e^k given that it is composed by $x_e^k = 0$. As a result, F_e^k is not empty ($F_e^k \neq \emptyset$). Furthermore, given that $e \in E \setminus (E_0^k \cup E_1^k)$ for the demand k, this means that there exists at least one feasible path E_k for the demand k passed through the edge e which means that $F_e^k \neq P(G, K, \mathbb{S})$.
On the other hand, we know that all the solutions of F_e^k are in $P(G, K, \mathbb{S})$ which means that they verify the equations system (2.13) s.t. the new equations system (2.15) associated with F_e^k is written as below

$$\begin{cases} x_e^k = 0, \text{ s.t. } k \text{ and } e \text{ are chosen arbitrarily,} \\ x_e^k = 0, \text{ for all } k \in K \text{ and all } e \in E_0^k, \\ x_e^k = 1, \text{ for all } k \in K \text{ and all } e \in E_1^k, \\ z_s^k = 0, \text{ for all } k \in K \text{ and all } s \in \{1, ..., w_k - 1\}. \end{cases} \tag{2.15}$$

Given that the $e \in E \setminus (E_0^k \cup E_1^k)$, the system (2.15) shows that the equation $x_e^k = 0$ is not a result of equations of system (2.13) which means that the equation $x_e^k = 0$ is not redundant in the system (2.15), and hence the system is of full rank. As a result, the dimension of the face F_e^k is equal to

$$dim(F_e^k) = |K| * (|E| + |\mathbb{S}|) - rank(Q') = |K| * (|E| + |\mathbb{S}|) - (1 + r) = dim(P(G, K, \mathbb{S})) - 1,$$

where Q' is the matrix associated with the equation system (2.15). As a result, the face F_e^k is facet defining for $P(G, K, \mathbb{S})$. Furthermore, we strengthened the proof as follows using a technique called "proof by maximality". We denote the inequality $x_e^k \geq 0$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F_e^k \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and γ with $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (with $\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K} (w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. We will show that

a) $\mu_{e'}^k = 0$ for the demand k and all $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$,

b) and $\mu_{e'}^{k'} = 0$ for all demands $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E_0^{k'} \cup E_1^{k'})$,

c) and $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$.

First, let's show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$. Consider a demand k and a slot s in $\{w_k, ..., \bar{s}\}$. Based on this, we consider a solution $\mathcal{S}'^3 = (E'^3, S'^3)$ in which

a) a feasible path $E_k'^3$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S_k'^3$ is assigned to each demand $k \in K$ along each edge $e' \in E_k'^3$ with $|S_k'^3| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k'^3$ and $s" \in S_{k'}'^3$ with $E_k'^3 \cap E_{k'}'^3 \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E_k'^3} |\{s' \in S_k'^3, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}^3$ with $E_k'^3 \cap E_{k'}'^3 \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s in the set of last-slots $S_k'^3$ assigned to the demand k in the solution $\mathcal{S}'^3$),

e) and the edge e is not chosen to route the demand k in the solution $\mathcal{S}'^3$, i.e., $e \notin E_k'^3$.

$\mathcal{S}'^3$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^3}, z^{\mathcal{S}'^3})$ is belong to F and then to F_e^k given that it is also composed by $x_e^k = 0$. Based on this, we derive a solution $\mathcal{S}^4 = (E^4, S^4)$ from the solution $\mathcal{S}'^3$ by adding the slot s as last-slot to the demand k without modifying the paths assigned to the demands K in $\mathcal{S}'^3$ (i.e., $E_k^4 = E_1'^3$ for each $k \in K$), and the last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}'^3$ remain the same in the solution $\mathcal{S}^4$ i.e., $S_{k'}'^3 = S_{k'}^4$ for each demand $k' \in K \setminus \{k\}$, and $S_k^4 = S_k'^3 \cup \{s\}$ for the demand k. The solution $\mathcal{S}^4$ is feasible given that

a) a feasible path E_k^4 is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^4 is assigned to each demand $k \in K$ along each edge $e' \in E_k^4$ with $|S_k^4| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S^4_k$ and $s' \in S^4_{k'}$ with $E^4_k \cap E^4_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E^4_k} |\{s \in S^4_k, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint),

d) and the edge e is not chosen to route the demand k in the solution $\mathcal{S}^4$, i.e., $e \notin E^4_k$.

The corresponding incidence vector $(x^{\mathcal{S}^4}, z^{\mathcal{S}^4})$ is belong to F and then to F^k_e given that it is also composed by $x^k_e = 0$. We then obtain that

$$\mu x^{\mathcal{S}'^3} + \sigma z^{\mathcal{S}'^3} = \mu x^{\mathcal{S}^4} + \sigma z^{\mathcal{S}^4} = \mu x^{\mathcal{S}'^3} + \sigma z^{\mathcal{S}'^3} + \sigma^k_s.$$

It follows that $\sigma^k_s = 0$ for demand k and a slot $s \in \{w_k, ..., \bar{s}\}$.
The slot s is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k s.t. we find

$$\sigma^k_s = 0, \text{ for demand } k \text{ and all slots } s \in \{w_k, ..., \bar{s}\}$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma^{k'}_s = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\}$$

Consequently, we conclude that

$$\sigma^k_s = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\}$$

Next, we will show that $\mu^k_{e'} = 0$ for all the demands $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E^{k'}_0 \cup E^{k'}_1)$, and $\mu^k_{e'} = 0$ for the demand k and all $e' \in E \setminus (E^k_0 \cup E^k_1 \cup \{e\})$. Consider the demand $k \in K$ and an edge $e' \in E \setminus (E^k_0 \cup E^k_1 \cup \{e\})$ chosen arbitrarily. For that, we consider a solution $\mathcal{S}"^3 = (E"^3, S"^3)$ in which

a) a feasible path $E"^3_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^3_k$ is assigned to each demand $k \in K$ along each edge $e' \in E"^3_k$ with $|S"^3_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^3_k$ and $s" \in S"^3_{k'}$ with $E"^3_k \cap E"^3_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E"^3_k} |\{s' \in S"^3_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s \in S"^3_k$ and $s' \in S"^3_{k'}$ with $(E"^3_k \cup \{e'\}) \cap E"^3_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e' in the set of edges $E"^3_k$ selected to route the demand k in the solution $\mathcal{S}"^3$),

e) and the edge e is not chosen to route the demand k in the solution $\mathcal{S}"^3$, i.e., $e \notin E"^3_k$.

$\mathcal{S}"^3$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^3}, z^{\mathcal{S}"^3})$ is belong to F and then to F^k_e given that it is also composed by $x^k_e = 0$. Let $\mathcal{S}^5$ be a solution obtained from the solution $\mathcal{S}"^3$ by adding an unused edge $e' \in E \setminus (E^k_0 \cup E^k_1)$ for the routing of demand k in K in the solution $\mathcal{S}"^3$ which means that $E^5_k = E"^3_k \cup \{e'\}$, and removing slot s selected for the demand k in $\mathcal{S}"^3$ and replaced it by a new slot $s' \in \{w_k, ..., \mathbb{S}\}$ (i.e., $S^5_k = (S"^3_k \setminus \{s\}) \cup \{s'\}$ s.t. $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^3_{k'}$ with $E^5_k \cap E"^3_{k'} \neq \emptyset$). The last-slots and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}"^3$ remain the same in the solution $\mathcal{S}^5$, i.e., $S^5_{k'} = S"^3_{k'}$ and $E^5_{k'} = E"^3_{k'}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^5$ is clearly feasible given that

a) and a feasible path E^5_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^5_k is assigned to each demand $k \in K$ along each edge $e' \in E^5_k$ with $|S^5_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S^5_k$ and $s' \in S^5_{k'}$ with $E^5_k \cap E^5_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E^5_k} |\{s \in S^5_k, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint),

d) and the edge e is not chosen to route the demand k in the solution $\mathcal{S}^5$, i.e., $e \notin E^5_k$.

The corresponding incidence vector $(x^{\mathcal{S}^5}, z^{\mathcal{S}^5})$ is belong to F and then to F^k_e given that it is also composed by $x^k_e = 0$. It follows that

$$\mu x^{\mathcal{S}"^3} + \sigma z^{\mathcal{S}"^3} = \mu x^{\mathcal{S}^5} + \sigma z^{\mathcal{S}^5} = \mu x^{\mathcal{S}"^3} + \mu^k_{e'} + \sigma z^{\mathcal{S}"^3} - \sigma^k_s + \sigma^k_{s'}.$$

It follows that $\mu^k_{e'} = 0$ for demand k and an edge e' given that $\sigma^k_s = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$.
As e' is chosen arbitrarily for the demand k with $e' \notin E^k_0 \cup E^k_1 \cup \{e\}$, we iterate the same procedure for all $e' \in E \setminus (E^k_0 \cup E^k_1 \cup \{e\})$. We conclude that for the demand k

$$\mu^k_{e'} = 0, \text{ for all } e' \in E \setminus (E^k_0 \cup E^k_1 \cup \{e\}).$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E^k_0 \cup E^k_1)$. We conclude at the end that

$$\begin{aligned} \mu^{k'}_{e'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all } e' \in E \setminus (E^{k'}_0 \cup E^{k'}_1), \\ \mu^k_{e'} = 0, \text{ for all } e' \in E \setminus (E^k_0 \cup E^k_1 \cup \{e\}). \end{aligned}$$

On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu^{k'}_{e'} = \gamma^{k',e'}_1, \text{ for all } k' \in K \text{ and all } e' \in E^{k'}_0, \\ \mu^{k'}_{e'} = \gamma^{k',e'}_2, \text{ for all } k' \in K \text{ and all } e' \in E^{k'}_1, \\ \sigma^{k'}_{s'} = \gamma^{k',s'}_3, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k' \in K$ and $e' \in E$

$$\mu^{k'}_{e'} = \begin{cases} \gamma^{k',e'}_1, \text{ if } e' \in E^{k'}_0, \\ \gamma^{k',e'}_2, \text{ if } e' \in E^{k'}_1, \\ \rho, \text{ if } k' = k \text{ and } e' = e, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma^k_s = \begin{cases} \gamma^{k,s}_3, \text{if } s \in \{1, ..., w_k - 1\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \rho \alpha^k_e + \gamma Q$ which ends the proof. □

Theorem 2.3.3. *Consider a demand $k \in K$, and a slot $s \in \{w_k, .., \bar{s}\}$. Then, the inequality $z^k_s \geq 0$ is facet defining for $P(G, K, \mathbb{S})$.*

Proof. Let F_s^k denote the face induced by inequality $z_s^k \geq 0$, which is given by

$$F_s^k = \{(x,z) \in P(G,K,\mathbb{S}) : z_s^k = 0\}.$$

In order to prove that inequality $z_s^k \geq 0$ is facet defining for $P(G,K,\mathbb{S})$, we start checking that F_s^k is a proper face, and $F_s^k \neq P(G,K,\mathbb{S})$. We construct a solution $\mathcal{S}^6 = (E^6, S^6)$ as below

a) a feasible path E_k^6 is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^6 is assigned to each demand $k \in K$ along each edge $e' \in E_k^6$ with $|S_k^6| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^6$ and $s" \in S_{k'}^6$ with $E_k^6 \cap E_{k'}^6 \neq \emptyset$ (non-overlapping constraint),

d) and the slot s is not chosen to route the demand k in the solution $\mathcal{S}^6$, i.e., $s \notin S_k^6$.

Obviously, $\mathcal{S}^6$ is feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^6}, z^{\mathcal{S}^6})$ is belong to F and then to F_s^k given that it is composed by $z_s^k = 0$. As a result, F_s^k is not empty ($F_s^k \neq \emptyset$). Furthermore, given that $s \in \{w_k, ..., \bar{s}\}$ for the demand k, this means that there exists at least one feasible solution for the problem in which $s \in S_k$ for the demand k. As a result, $F_s^k \neq P(G,K,\mathbb{S})$.
On the other hand, we know that all the solutions of F_s^k are in $P(G,K,\mathbb{S})$ which means that they verify the equations system (2.13) s.t. the new equations system (2.20) associated with F_s^k is written as below

$$\begin{cases} z_s^k = 0, \text{ s.t. } k \text{ and } s \text{ are chosen arbitrarily}, \\ x_e^k = 0, \text{ for all } k \in K \text{ and all } e \in E_0^k, \\ x_e^k = 1, \text{ for all } k \in K \text{ and all } e \in E_1^k, \\ z_s^k = 0, \text{ for all } k \in K \text{ and all } s \in \{1, ..., w_k - 1\}. \end{cases} \tag{2.16}$$

The equation $z_s^k = 0$ is not result of equations of system (2.13) which means that the equation $z_s^k = 0$ is not redundant in the system (2.20). As a result, the system (2.20) is of full rank. As a result, the dimension of the face F_s^k is equal to

$$dim(F_s^k) = |K|*(|E|+|\mathbb{S}|) - rank(Q") = |K|*(|E|+|\mathbb{S}|) - (1+r) = dim(P(G,K,\mathbb{S})) - 1,$$

where $Q"$ denotes the matrix associated with the equation system (2.20). As a result, the face F_s^k is facet defining for $P(G,K,\mathbb{S})$. Furthermore, we strengthen the proof as follows. We denote the inequality $z_s^k \geq 0$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G,K,\mathbb{S})$. Suppose that $F_s^k \subset F = \{(x,z) \in P(G,K,\mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and γ with $\gamma = (\gamma_1, \gamma_2, ..., \gamma_4)$ ($\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma_{s'}^k = 0$ for demand k and all slots $s' \in \{w_k, ..., \bar{s}\} \setminus \{s\}$,

b) and $\sigma_{s'}^{k'} = 0$ for all demands $k' \in K \setminus \{k\}$ and all slots $s' \in \{w_{k'}, ..., \bar{s}\}$,

c) and $\mu_e^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$.

First, let's us show that $\mu_e^k = 0$ for all the demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$. Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. For that, we consider a solution $\mathcal{S}'^6 = (E'^6, S'^6)$ in which

a) a feasible path $E_k'^6$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S_k'^6$ is assigned to each demand $k \in K$ along each edge $e \in E_k'^6$ with $|S_k'^6| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k'^6$ and $s" \in S_{k'}'^6$ with $E_k'^6 \cap E_{k'}'^6 \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k'^6} |\{s' \in S_k'^6, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s' \in S_k'^6$ and $s" \in S_{k'}'^6$ with $(E_k'^6 \cup \{e\}) \cap E_{k'}'^6 \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e in the set of edges $E_k'^6$ selected to route the demand k in the solution $\mathcal{S}'^6$),

e) the edge e is not non-compatible edge with the selected edges $e \in E_k'^6$ of demand k in the solution $\mathcal{S}'^6$, i.e., $\sum_{e' \in E_k'^6} l_{e'} + l_e \leq \bar{l}_k$. As a result, $E_k'^6 \cup \{e\}$ is a feasible path for the demand k,

f) and the slot s is not chosen to route the demand k in the solution $\mathcal{S}"^6$, i.e., $s \notin S"^6_k$.

$\mathcal{S}'^6$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^6}, z^{\mathcal{S}'^6})$ is belong to F and then to F_s^k given that it is composed by $z_s^k = 0$. Based on this, we derive a solution $\mathcal{S}^7$ obtained from the solution $\mathcal{S}'^6$ by adding an unused edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in K in the solution $\mathcal{S}^6$ which means that $E_k^7 = E_k'^6 \cup \{e\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^6$ remain the same in the solution $\mathcal{S}^7$, i.e., $S_k^7 = S_k'^6$ for each $k \in K$, and $E_{k'}^7 = E_{k'}'^6$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^7$ is clearly feasible given that

a) and a feasible path E_k^7 is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^7 is assigned to each demand $k \in K$ along each edge $e \in E_k^7$ with $|S_k^7| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^7$ and $s" \in S_{k'}^7$ with $E_k^7 \cap E_{k'}^7 \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^7} |\{s' \in S_k^7, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and the slot s is not chosen to route the demand k in the solution $\mathcal{S}^7$, i.e., $s \notin S_k^7$.

The corresponding incidence vector $(x^{\mathcal{S}^7}, z^{\mathcal{S}^7})$ is belong to F and then to F_s^k given that it is composed by $z_s^k = 0$. It follows that

$$\mu x^{\mathcal{S}'^6} + \sigma z^{\mathcal{S}'^6} = \mu x^{\mathcal{S}^7} + \sigma z^{\mathcal{S}^7} = \mu x^{\mathcal{S}'^6} + \mu_e^k + \sigma z^{\mathcal{S}'^6}.$$

As a result, $\mu_e^k = 0$ for demand k and an edge e.
As e is chosen arbitrarily for the demand k with $e \notin E_0^k \cup E_1^k$, we iterate the same procedure for all $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$. We conclude that for the demand k

$$\mu_e^k = 0, \text{ for all } e \in E \setminus (E_0^k \cup E_1^k).$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e \in E \setminus (E_0^k \cup E_1^k)$. We conclude at the end that

$$\mu_e^k = 0, \text{ for all } k \in K \text{ and all } e \in E \setminus (E_0^k \cup E_1^k).$$

Next, we will show that, $\sigma_{s'}^{k'} = 0$ for all $k' \in K \setminus \{k\}$ and all $s' \in \{w_{k'}, ..., \bar{s}\}$, and $\sigma_{s'}^{k} = 0$ for all slots $s' \in \{w_k, ..., \bar{s}\} \setminus \{s\}$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\} \setminus \{s\}$. For that, we consider a solution $\mathcal{S}"^6 = (E"^6, S"^6)$ in which

a) a feasible path $E"^6_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^6_k$ is assigned to each demand $k \in K$ along each edge $e \in E"^6_k$ with $|S"^6_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^6_k$ and $s" \in S"^6_{k'}$ with $E"^6_k \cap E"^6_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E"^6_k} |\{s' \in S"^6_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^6_{k'}$ with $E"^6_k \cap E"^6_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^6_k$ assigned to the demand k in the solution $\mathcal{S}"^6$),

e) and the slot s is not chosen to route the demand k in the solution $\mathcal{S}"^6$, i.e., $s \notin S"^6_k$.

$\mathcal{S}"^6$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^6}, z^{\mathcal{S}"^6})$ is belong to F and then to F_s^k given that it is composed by $z_s^k = 0$. Based on this, we construct a solution $\mathcal{S}^8$ derived from the solution $\mathcal{S}"^6$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}"^6$ (i.e., $E^8_k = E"^6_k$ for each $k \in K \setminus \tilde{K}$, and $E^8_k \neq E"^6_k$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E^8_k is assigned to each demand $k \in \tilde{K}$ (routing constraint),

b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"^6_k$ and $s" \in S"^6_{k'}$ with $E^8_k \cap E"^6_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}, e \in E^8_k} |\{s' \in S"^6_k, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k \in K \setminus \tilde{K}, e \in E"^6_k} |\{s' \in S"^6_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"^6_{k"}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^6_k$ assigned to the demand k in the solution $\mathcal{S}"^6$),

d) and the slot s is not chosen to route the demand k in the solution $\mathcal{S}^8$, i.e., $s \notin S^8_k$.

The last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}"^6$ remain the same in $\mathcal{S}^8$, i.e., $S"^6_{k'} = S^8_{k'}$ for each demand $k' \in K \setminus \{k\}$, and $S^8_k = S"^6_k \cup \{s\}$ for the demand k. The solution $\mathcal{S}^8$ is clearly feasible given that

a) a feasible path E^8_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^8_k is assigned to each demand $k \in K$ along each edge $e \in E^8_k$ with $|S^8_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^8_k$ and $s" \in S^8_{k'}$ with $E^8_k \cap E^8_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E^8_k} |\{s' \in S^8_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and the slot s is not chosen to route the demand k in the solution $\mathcal{S}^8$, i.e., $s \notin S^8_k$.

The corresponding incidence vector $(x^{\mathcal{S}^8}, z^{\mathcal{S}^8})$ is belong to F and then to F^k_s given that it is composed by $z^k_s = 0$. We have so

$$\mu x^{\mathcal{S}"^6} + \sigma z^{\mathcal{S}"^6} = \mu x^{\mathcal{S}^8} + \sigma z^{\mathcal{S}^8} = \mu x^{\mathcal{S}"^6} + \sigma z^{\mathcal{S}"^6} + \sigma^k_{s'} - \sum_{\tilde{k} \in \tilde{K}} \sum_{e \in E"^6_k} \mu^{\tilde{k}}_e + \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E^8_k} \mu^{\tilde{k}}_{e'}.$$

It follows that $\sigma^k_{s'} = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\} \setminus \{s\}$ given that $\mu^k_e = 0$ for all the demands $k \in K$ and all edges $e \in E \setminus (E^k_0 \cup E^k_1)$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\} \setminus \{s\}$ of demand k s.t. we find

$$\sigma^k_{s'} = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \setminus \{s\}.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma^{k'}_{s'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s' \in \{w_{k'}, ..., \bar{s}\}$$

Consequently, we conclude that

$$\sigma^{k'}_{s'} = 0, \text{ for all } k' \in K \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } s \neq s' \text{ if } k = k'.$$

On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu^{k'}_{e'} &= \gamma^{k',e'}_1, \text{ for all } k' \in K \text{ and all } e' \in E^{k'}_0, \\ \mu^{k'}_{e'} &= \gamma^{k',e'}_2, \text{ for all } k' \in K \text{ and all } e' \in E^{k'}_1, \\ \sigma^{k'}_{s'} &= \gamma^{k',s'}_3, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k \in K$ and $e \in E$

$$\mu^k_e = \begin{cases} \gamma^{k,e}_1, \text{ if } e \in E^k_0, \\ \gamma^{k,e}_2, \text{ if } e' \in E^k_1, \\ 0, otherwise, \end{cases}$$

and for each $k' \in K$ and $s' \in \mathbb{S}$

$$\sigma^{k'}_{s'} = \begin{cases} \gamma^{k',s'}_3, \text{if } s' \in \{1, ..., w_{k'} - 1\}, \\ 0, \text{if } s' \in \{w_{k'}, ..., \bar{s}\} \text{ and } k' \neq k, \\ 0, \text{if } s' \in \{w_{k'}, ..., \bar{s}\} \setminus \{s\} \text{ and } k' = k, \\ \rho, \text{if } s' = s \text{ and } k' = k. \end{cases}$$

As a result $(\mu, \sigma) = \rho \beta^k_s + \gamma Q$ which ends our strengthening of the proof.

□

Definition 2.3.2. *For a demand k, two edges $e = ij \notin E_0^k \cap E_1^k, e' = lm \notin E_0^k \cap E_1^k$ are said* non-compatible edges *iff the lengths of (o_k, d_k)-paths formed by $e = ij$ and $e' = lm$ together are greater that $\bar{l}_k$.*

Note that we are able to determine the non-compatible edges for each demand k in polynomial time using shortest-path algorithms by verifying if the length of the following (o_k, d_k)-paths

a) (o_k, d_k)-path formed by e and e' together with the shortest (o_k, i), (j, l) and (m, d_k) paths,

b) (o_k, d_k)-path formed by e and e' together with the shortest (o_k, i), (j, m) and (l, d_k) paths,

c) (o_k, d_k)-path formed by e and e' together with the shortest (o_k, j), (i, l) and (m, d_k) paths,

d) (o_k, d_k)-path formed by e and e' together with the shortest (o_k, j), (i, m) and (l, d_k) paths,

e) (o_k, d_k)-path formed by e and e' together with the shortest (o_k, l), (m, i) and (j, d_k) paths,

f) (o_k, d_k)-path formed by e and e' together with the shortest (o_k, l), (m, j) and (i, d_k) paths,

g) (o_k, d_k)-path formed by e and e' together with the shortest (o_k, m), (l, i) and (j, d_k) paths,

h) (o_k, d_k)-path formed by e and e' together with the shortest (o_k, m), (l, j) and (i, d_k) paths,

are greater that $\bar{l}_k$.

Proposition 2.3.3. *Consider a demand $k \in K$. Let (e, e') be a pair of non-compatible edges for the demand k. Then, the inequality*

$$x_e^k + x_{e'}^k \leq 1, \tag{2.17}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It is trivial due to the transmission-reach constraint and given the definition of non-compatible edges for the demand k. □

Based on the definition of a non-compatible demands for an edge e, we introduce the following inequality.

Proposition 2.3.4. *Consider an edge $e \in E$. Let (k, k') be a pair of non-compatible demands for the edge e with $e \notin E_0^k \cup E_1^k \cup E_0^{k'} \cup E_1^{k'}$. Then, the inequality*

$$x_e^k + x_e^{k'} \leq 1, \tag{2.18}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of non-compatible demands for the edge e. □

Theorem 2.3.4. *Consider a demand $k \in K$, and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. Then, the inequality $x_e^k \leq 1$ is facet defining for $P(G, K, \mathbb{S})$ iff*

a) there does not exist a demand $k' \in K \setminus \{k\}$ s.t. the two demands k and k' are non-compatible demands for edge e,

b) there does not exist an edge $e' \in E \setminus (E_1^k \cup E_0^k \cup \{e\})$ s.t. the two edges e and e' are non-comptible edges for the demand k.

Proof. **Neccessity.**
For demand k and an edge $e \in E \setminus (E_0^k \cup E_1^k)$, if

a) there exists a demand $k' \in K \setminus \{k\}$ s.t. the two demands k and k' are non-compatible demands for edge e. Then, the inequality $x_e^k \leq 1$ is dominated by the inequality (2.18).

b) there exists an edge $e' \in E \setminus (E_1^k \cup E_0^k \cup \{e\})$ s.t. the two edges e and e' are non-comptible edges for the demand k. Then, the inequality $x_e^k \leq 1$ is dominated by the inequality (2.17).

As a result, the inequality $x_e^k \leq 1$ is not facet defining for $P(G, K, \mathbb{S})$.
Sufficiency.
Let $F_e'^k$ denote the face induced by inequality $x_e^k \leq 1$, which is given by

$$F_e'^k = \{(x, z) \in P(G, K, \mathbb{S}) : x_e^k = 1\}.$$

In order to prove that inequality $x_e^k \leq 1$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that $F_e'^k$ is a proper face, and $F_e'^k \neq P(G, K, \mathbb{S})$. We construct a solution $S^9 = (E^9, S^9)$ as below

a) a feasible path E_k^9 is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^9 is assigned to each demand $k \in K$ along each edge $e' \in E_k^9$ with $|S_k^9| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^9$ and $s' \in S_{k'}^9$ with $E_k^9 \cap E_{k'}^9 \neq \emptyset$ (non-overlapping constraint),

d) and the edge e is chosen to route the demand k in the solution $\mathcal{S}^9$, i.e., $e \in E_k^9$.

Obviously, $\mathcal{S}^9$ is feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^9}, z^{\mathcal{S}^9})$ is belong to $P(G, K, \mathbb{S})$ and then to $F_e'^k$ given that it is composed by $x_e^k = 1$. As a result, $F_e'^k$ is not empty ($F_e'^k \neq \emptyset$). Furthermore, given that $e \in E \setminus (E_0^k \cup E_1^k)$ for the demand k, this means that there exists at least one feasible path E_k for the demand k without passing through the edge e which means that $F_e'^k \neq P(G, K, \mathbb{S})$.
On the other hand, we know that all the solutions of $F_e'^k$ are in $P(G, K, \mathbb{S})$ which means that they verify the equations system (2.13) s.t. the new equations system (2.19) associated with $F_e'^k$ is written as below

$$\begin{cases} x_e^k = 1, \text{ s.t. } k \text{ and } e \text{ are chosen arbitrarily}, \\ x_e^k = 0, \text{ for all } k \in K \text{ and all } e \in E_0^k, \\ x_e^k = 1, \text{ for all } k \in K \text{ and all } e \in E_1^k, \\ z_s^k = 0, \text{ for all } k \in K \text{ and all } s \in \{1, ..., w_k - 1\}. \end{cases} \quad (2.19)$$

Given that the $e \in E \setminus (E_0^k \cup E_1^k)$, the system (2.19) shows that the equation $x_e^k = 1$ is not a result of equations of system (2.13) which means that the equation $x_e^k = 1$ is not redundant in the system (2.19). As a result, the system is of full rank. As a result, the dimension of the face $F_e'^k$ is equal to

$$dim(F_e'^k) = |K| * (|E| + |\mathbb{S}|) - rank(\tilde{Q}') = |K| * (|E| + |\mathbb{S}|) - (1 + r) = dim(P(G, K, \mathbb{S})) - 1,$$

where $\tilde{Q}'$ is the matrix associated with the equation system (2.19). As a result, the face $F_e'^k$ is facet defining for $P(G, K, \mathbb{S})$. Furthermore, we strengthened the proof as follows. We denote the inequality $x_e^k \leq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F_e'^k \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and γ with $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (with $\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K} (w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. We will show that

a) $\mu_{e'}^k = 0$ for the demand k and all $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$,

b) and $\mu_{e'}^{k'} = 0$ for all demands $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E_0^{k'} \cup E_1^{k'})$,

c) and $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$.

First, let's show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$. Consider a demand k and a slot s in $\{w_k, ..., \bar{s}\}$. To do so, we consider a solution $\mathcal{S}'^9 = (E'^9, S'^9)$ in which

a) a feasible path $E_k'^9$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S_k'^9$ is assigned to each demand $k \in K$ along each edge $e' \in E_k'^9$ with $|S_k'^9| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k'^9$ and $s" \in S_{k'}'^9$ with $E_k'^9 \cap E_{k'}'^9 \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E_k'^9} |\{s' \in S_k'^9, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}'^9$ with $E_k'^9 \cap E_{k'}'^9 \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s in the set of last-slots $S_k'^9$ assigned to the demand k in the solution $\mathcal{S}'^9$),

e) and the edge e is chosen to route the demand k in the solution $\mathcal{S}'^9$, i.e., $e \in E_k'^9$.

$\mathcal{S}'^9$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^9}, z^{\mathcal{S}'^9})$ is belong to $P(G, K, \mathbb{S})$. Based on this, we derive a solution $\mathcal{S}^{10} = (E^{10}, S^{10})$ from the solution $\mathcal{S}'^9$ by adding the slot s as last-slot to the demand k without modifying the paths assigned to the demands K in $\mathcal{S}'^9$ (i.e., $E_k^{10} = E_1'^9$ for each $k \in K$), and the last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}'^9$ remain the same in the solution $\mathcal{S}^{10}$ i.e., $S_{k'}'^9 = S_{k'}^{10}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{10} = S_k'^9 \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{10}$ is feasible given that

a) a feasible path E_k^{10} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{10} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{10}$ with $|S_k^{10}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{10}$ and $s' \in S_{k'}^{10}$ with $E_k^{10} \cap E_{k'}^{10} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E_k^{10}} |\{s \in S_k^{10}, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{10}}, z^{\mathcal{S}^{10}})$ is belong to F and then to $F_e'^k$ given that it is also composed by $x_e^k = 1$. We then obtain that

$$\mu x^{\mathcal{S}'^9} + \sigma z^{\mathcal{S}'^9} = \mu x^{\mathcal{S}^{10}} + \sigma z^{\mathcal{S}^{10}} = \mu x^{\mathcal{S}'^9} + \sigma z^{\mathcal{S}'^9} + \sigma_s^k.$$

It follows that $\sigma_s^k = 0$ for demand k and a slot $s \in \{w_k, ..., \bar{s}\}$.
The slot s is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k s.t. we find

$$\sigma_s^k = 0, \text{ for demand } k \text{ and all slots } s \in \{w_k, ..., \bar{s}\}$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\}$$

Consequently, we conclude that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\}$$

Next, we will show that $\mu_{e'}^k = 0$ for all the demands $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E_0^{k'} \cup E_1^{k'})$, and $\mu_{e'}^k = 0$ for the demand k and all $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$. Consider the demand $k \in K$ and an edge $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$ chosen arbitrarily. For that, we consider a solution $\mathcal{S}"^9 = (E"^9, S"^9)$ in which

a) a feasible path $E"^9_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^9_k$ is assigned to each demand $k \in K$ along each edge $e' \in E"^9_k$ with $|S"^9_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^9_k$ and $s" \in S"^9_{k'}$ with $E"^9_k \cap E"^9_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E"^9_k} |\{s' \in S"^9_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) the edge e' is not non-compatible edge with the selected edges $e" \in E"^9_k$ of demand k in the solution $\mathcal{S}"^9$, i.e., $\sum_{e" \in E"^9_k} l_{e"} + l_{e'} \leq \bar{l}_k$. As a result, $E"^9_k \cup \{e'\}$ is a feasible path for the demand k,

e) and the edge e is chosen to route the demand k in the solution $\mathcal{S}"^9$, i.e., $e \in E"^9_k$.

$\mathcal{S}"^9$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^9}, z^{\mathcal{S}"^9})$ is belong to F and then to $F_e'^k$ given that it is also composed by $x_e^k = 1$. Let $\mathcal{S}^{11}$ be a solution obtained from the solution $\mathcal{S}"^9$ by adding an unused edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in K in the solution $\mathcal{S}"^9$ which means that $E_k^{11} = E"^9_k \cup \{e\}$ s.t. $\{s - w_k + 1, ..., s\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K \setminus \{k\}$ and $s" \in S"^9_{k'}$ with $E_k^{11} \cap E"^9_{k'} \neq \emptyset$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}"^9$ remain the same in the solution $\mathcal{S}^{11}$, i.e., $S_k^{11} = S"^9_k$ for each $k \in K$, and $E_{k'}^{11} = E"^9_{k'}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{11}$ is clearly feasible given that

a) and a feasible path E_k^{11} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{11} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{11}$ with $|S_k^{11}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{11}$ and $s' \in S_{k'}^{11}$ with $E_k^{11} \cap E_{k'}^{11} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E_k^{11}} |\{s \in S_k^{11}, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{11}}, z^{\mathcal{S}^{11}})$ is belong to F and then to $F_e'^k$ given that it is also composed by $x_e^k = 1$. It follows that

$$\mu x^{\mathcal{S}"9} + \sigma z^{\mathcal{S}"9} = \mu x^{\mathcal{S}^{11}} + \sigma z^{\mathcal{S}^{11}} = \mu x^{\mathcal{S}"9} + \mu_{e'}^k + \sigma z^{\mathcal{S}"9}.$$

Hence, $\mu_{e'}^k = 0$ for demand k and an edge e'.
As e' is chosen arbitrarily for the demand k with $e' \notin E_0^k \cup E_1^k \cup \{e\}$, we iterate the same procedure for all $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$. We conclude that for the demand k

$$\mu_{e'}^k = 0, \text{ for all } e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\}).$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E_0^k \cup E_1^k)$. We conclude at the end that

$$\begin{aligned} \mu_{e'}^{k'} = 0, &\text{ for all } k' \in K \setminus \{k\} \text{ and all } e' \in E \setminus (E_0^{k'} \cup E_1^{k'}), \\ \mu_{e'}^k = 0, &\text{ for all } e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\}). \end{aligned}$$

On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k' \in K$ and $e' \in E$

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'}, \text{ if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'}, \text{ if } e' \in E_1^{k'}, \\ \rho, \text{ if } k' = k \text{ and } e' = e, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \rho \alpha_e^k + \gamma Q$ which ends the proof. □

Theorem 2.3.5. *Consider a demand $k \in K$, and a slot $s \in \{w_k, .., \bar{s}\}$. Then, the inequality $z_s^k \leq 1$ is facet defining for $P(G, K, \mathbb{S})$ if there does not exist a demand $k' \in K \setminus \{k\}$ with $E_1^k \cap E_1^{k'} \neq \emptyset$.*

Proof. **Neccessity.**
For a demand $k \in K$ and a slot $s \in \{w_k, .., \bar{s}\}$, if there exists a demand $k' \in K \setminus \{k\}$ with $E_1^k \cap E_1^{k'} \neq \emptyset$. Then, the inequality $z_s^k \leq 1$ is domined by the non-overlapping inequality (2.6) for each edge $e \in E_1^k \cap E_1^{k'}$. As a result, the inequality $z_s^k \leq 1$ is not facet defining for $P(G, K, \mathbb{S})$.

Sufficiency.
Let $F_s'^k$ denote the face induced by inequality $z_s^k \leq 1$, which is given by

$$F_s'^k = \{(x,z) \in P(G,K,\mathbb{S}) : z_s^k = 1\}.$$

In order to prove that inequality $z_s^k \leq 1$ is facet defining for $P(G,K,\mathbb{S})$, we start checking that $F_s'^k$ is a proper face, and $F_s'^k \neq P(G,K,\mathbb{S})$. We construct a solution $\mathcal{S}^{12} = (E^{12}, S^{12})$ as below

a) a feasible path E_k^{12} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{12} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{12}$ with $|S_k^{12}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{12}$ and $s" \in S_{k'}^{12}$ with $E_k^{12} \cap E_{k'}^{12} \neq \emptyset$ (non-overlapping constraint),

d) and the slot s is chosen to route the demand k in the solution $\mathcal{S}^{12}$, i.e., $s \notin S_k^{12}$.

Obviously, $\mathcal{S}^{12}$ is feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{12}}, z^{\mathcal{S}^{12}})$ is belong to F and then to $F_s'^k$ given that it is composed by $z_s^k = 1$. As a result, $F_s'^k$ is not empty ($F_s'^k \neq \emptyset$). Furthermore, given that $s \in \{w_k, ..., \bar{s}\}$ for the demand k, this means that there exists at least one feasible solution for the problem in which $s \notin S_k$ for the demand k. As a result, $F_s'^k \neq P(G,K,\mathbb{S})$.
On the other hand, we know that all the solutions of $F_s'^k$ are in $P(G,K,\mathbb{S})$ which means that they verify the equations system (2.13) s.t. the new equations system (2.20) associated with $F_s'^k$ is written as below

$$\begin{cases} z_s^k = 1, \text{ s.t. } k \text{ and } s \text{ are chosen arbitrarily}, \\ x_e^k = 0, \text{ for all } k \in K \text{ and all } e \in E_0^k, \\ x_e^k = 1, \text{ for all } k \in K \text{ and all } e \in E_1^k, \\ z_s^k = 0, \text{ for all } k \in K \text{ and all } s \in \{1, ..., w_k - 1\}. \end{cases} \tag{2.20}$$

The equation $z_s^k = 1$ is not result of equations of system (2.13) which means that the equation $z_s^k = 1$ is not redundant in the system (2.20). As a result, the system (2.20) is of full rank. As a result, the dimension of the face $F_s'^k$ is equal to

$$dim(F_s'^k) = |K| * (|E| + |\mathbb{S}|) - rank(\tilde{Q}) = |K| * (|E| + |\mathbb{S}|) - (1 + r) = dim(P(G,K,\mathbb{S})) - 1,$$

where $\tilde{Q}$ denotes the matrix associated with the equation system (2.20). As a result, the face $F_s'^k$ is facet defining for $P(G,K,\mathbb{S})$. Furthermore, we strengthen the proof as follows. We denote the inequality $z_s^k \leq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G,K,\mathbb{S})$. Suppose that $F_s'^k \subset F = \{(x,z) \in P(G,K,\mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and γ with $\gamma = (\gamma_1, \gamma_2, ..., \gamma_4)$ ($\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma_{s'}^k = 0$ for demand k and all slots $s' \in \{w_k, ..., \bar{s}\} \setminus \{s\}$,

b) and $\sigma_{s'}^{k'} = 0$ for all demands $k' \in K \setminus \{k\}$ and all slots $s' \in \{w_{k'}, ..., \bar{s}\}$,

c) and $\mu_e^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$.

First, let's us show that $\mu_e^k = 0$ for all the demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$. Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. For that, we consider a solution $\mathcal{S}'^{12} = (E'^{12}, S'^{12})$ in which

a) a feasible path $E_k'^{12}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S_k'^{12}$ is assigned to each demand $k \in K$ along each edge $e \in E_k'^{12}$ with $|S_k'^{12}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k'^{12}$ and $s" \in S_{k'}'^{12}$ with $E_k'^{12} \cap E_{k'}'^{12} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k'^{12}} |\{s' \in S_k'^{12}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s' \in S_k'^{12}$ and $s" \in S_{k'}'^{12}$ with $(E_k'^{12} \cup \{e\}) \cap E_{k'}'^{12} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e in the set of edges $E_k'^{12}$ selected to route the demand k in the solution $\mathcal{S}'^{12}$),

e) the edge e is not non-compatible edge with the selected edges $e \in E_k'^{12}$ of demand k in the solution $\mathcal{S}'^{12}$, i.e., $\sum_{e' \in E_k'^{12}} l_{e'} + l_e \leq \bar{l}_k$. As a result, $E_k'^{12} \cup \{e\}$ is a feasible path for the demand k,

f) and the slot s is chosen to route the demand k in the solution $\mathcal{S}"^{12}$, i.e., $s \notin S"^{12}_k$.

$\mathcal{S}'^{12}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{12}}, z^{\mathcal{S}'^{12}})$ is belong to F and then to $F_s'^k$ given that it is composed by $z_s^k = 1$. Based on this, we derive a solution $\mathcal{S}^{13}$ obtained from the solution $\mathcal{S}'^{12}$ by adding an unused edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in K in the solution $\mathcal{S}^{12}$ which means that $E_k^{13} = E_k'^{12} \cup \{e\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^{12}$ remain the same in the solution $\mathcal{S}^{13}$, i.e., $S_k^{13} = S_k'^{12}$ for each $k \in K$, and $E_{k'}^{13} = E_{k'}'^{12}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{13}$ is clearly feasible given that

a) and a feasible path E_k^{13} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{13} is assigned to each demand $k \in K$ along each edge $e \in E_k^{13}$ with $|S_k^{13}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{13}$ and $s" \in S_{k'}^{13}$ with $E_k^{13} \cap E_{k'}^{13} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{13}} |\{s' \in S_k^{13}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and the slot s is chosen to route the demand k in the solution $\mathcal{S}^{13}$, i.e., $s \notin S_k^{13}$.

The corresponding incidence vector $(x^{\mathcal{S}^{13}}, z^{\mathcal{S}^{13}})$ is belong to F and then to $F_s'^k$ given that it is composed by $z_s^k = 1$. It follows that

$$\mu x^{\mathcal{S}'^{12}} + \sigma z^{\mathcal{S}'^{12}} = \mu x^{\mathcal{S}^{13}} + \sigma z^{\mathcal{S}^{13}} = \mu x^{\mathcal{S}'^{12}} + \mu_e^k + \sigma z^{\mathcal{S}'^{12}}.$$

As a result, $\mu_e^k = 0$ for demand k and an edge e.
As e is chosen arbitrarily for the demand k with $e \notin E_0^k \cup E_1^k$, we iterate the same procedure for all $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$. We conclude that for the demand k

$$\mu_e^k = 0, \text{ for all } e \in E \setminus (E_0^k \cup E_1^k).$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e \in E \setminus (E_0^k \cup E_1^k)$. We conclude at the end that

$$\mu_e^k = 0, \text{ for all } k \in K \text{ and all } e \in E \setminus (E_0^k \cup E_1^k).$$

Next, we will show that, $\sigma_{s'}^{k'} = 0$ for all $k' \in K \setminus \{k\}$ and all $s' \in \{w_{k'}, ..., \bar{s}\}$, and $\sigma_{s'}^{k} = 0$ for all slots $s' \in \{w_k, ..., \bar{s}\} \setminus \{s\}$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\} \setminus \{s\}$. For that, we consider a solution $\mathcal{S}"^{12} = (E"^{12}, S"^{12})$ in which

a) a feasible path $E"^{12}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^{12}_k$ is assigned to each demand $k \in K$ along each edge $e \in E"^{12}_k$ with $|S"^{12}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{12}_k$ and $s" \in S"^{12}_{k'}$ with $E"^{12}_k \cap E"^{12}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E"^{12}_k} |\{s' \in S"^{12}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^{12}_{k'}$ with $E"^{12}_k \cap E"^{12}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{12}_k$ assigned to the demand k in the solution $\mathcal{S}"^{12}$),

e) and the slot s is chosen to route the demand k in the solution $\mathcal{S}"^{12}$, i.e., $s \notin S"^{12}_k$.

$\mathcal{S}"^{12}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{12}}, z^{\mathcal{S}"^{12}})$ is belong to F and then to $F_s'^k$ given that it is composed by $z_s^k = 1$. Based on this, we construct a solution $\mathcal{S}^{14}$ derived from the solution $\mathcal{S}"^{12}$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}"^{12}$ (i.e., $E_k^{14} = E"^{12}_k$ for each $k \in K \setminus \tilde{K}$, and $E_k^{14} \neq E"^{12}_k$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E_k^{14} is assigned to each demand $k \in \tilde{K}$ (routing constraint),

b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"^{12}_k$ and $s" \in S"^{12}_{k'}$ with $E_k^{14} \cap E"^{12}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}, e \in E_k^{14}} |\{s' \in S"^{12}_k, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k \in K \setminus \tilde{K}, e \in E"^{12}_k} |\{s' \in S"^{12}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"^{12}_{k"}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{12}_k$ assigned to the demand k in the solution $\mathcal{S}"^{12}$),

d) and the slot s is chosen to route the demand k in the solution $\mathcal{S}^{14}$, i.e., $s \notin S_k^{14}$.

The last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}"^{12}$ remain the same in $\mathcal{S}^{14}$, i.e., $S"^{12}_{k'} = S_{k'}^{14}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{14} = S"^{12}_k \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{14}$ is clearly feasible given that

a) a feasible path E_k^{14} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{14} is assigned to each demand $k \in K$ along each edge $e \in E_k^{14}$ with $|S_k^{14}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{14}$ and $s" \in S_{k'}^{14}$ with $E_k^{14} \cap E_{k'}^{14} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{14}} |\{s' \in S_k^{14}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and the slot s is chosen to route the demand k in the solution $\mathcal{S}^{14}$, i.e., $s \notin S_k^{14}$.

The corresponding incidence vector $(x^{\mathcal{S}^{14}}, z^{\mathcal{S}^{14}})$ is belong to F and then to $F_s'^k$ given that it is composed by $z_s^k = 1$. We have so

$$\mu x^{\mathcal{S}"^{12}} + \sigma z^{\mathcal{S}"^{12}} = \mu x^{\mathcal{S}^{14}} + \sigma z^{\mathcal{S}^{14}} = \mu x^{\mathcal{S}"^{12}} + \sigma z^{\mathcal{S}"^{12}} + \sigma_{s'}^k - \sum_{\tilde{k} \in \tilde{K}} \sum_{e \in E"_k^{12}} \mu_e^{\tilde{k}} + \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E_k^{14}} \mu_{e'}^{\tilde{k}}.$$

It follows that $\sigma_{s'}^k = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\} \setminus \{s\}$ given that $\mu_e^k = 0$ for all the demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\} \setminus \{s\}$ of demand k s.t. we find

$$\sigma_{s'}^k = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \setminus \{s\}.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma_{s'}^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s' \in \{w_{k'}, ..., \bar{s}\}$$

Consequently, we conclude that

$$\sigma_{s'}^{k'} = 0, \text{ for all } k' \in K \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } s \neq s' \text{ if } k = k'.$$

On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k \in K$ and $e \in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e}, \text{ if } e \in E_0^k, \\ \gamma_2^{k,e}, \text{ if } e' \in E_1^k, \\ 0, otherwise, \end{cases}$$

and for each $k' \in K$ and $s' \in \mathbb{S}$

$$\sigma_{s'}^{k'} = \begin{cases} \gamma_3^{k',s'}, \text{if } s' \in \{1, ..., w_{k'} - 1\}, \\ 0, \text{if } s' \in \{w_{k'}, ..., \bar{s}\} \text{ and } k' \neq k, \\ 0, \text{if } s' \in \{w_{k'}, ..., \bar{s}\} \setminus \{s\} \text{ and } k' = k, \\ \rho, \text{if } s' = s \text{ and } k' = k. \end{cases}$$

As a result $(\mu, \sigma) = \rho \beta_s^k + \gamma Q$ which ends our strengthening of the proof.

□

Theorem 2.3.6. *Consider a demand $k \in K$. Then, the inequality* (2.5), $\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1$, *is facet defining for $P(G, K, \mathbb{S})$.*

Proof. Let $F_{\mathbb{S}}^k$ denote the face induced by inequality $\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1$, which is given by

$$F_{\mathbb{S}}^k = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{s=w_k}^{\bar{s}} z_s^k = 1\}$$

In order to prove that inequality $\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that $F_{\mathbb{S}}^k$ is a proper face which means that it is not empty, and $F_{\mathbb{S}}^k \neq P(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{15} = (E^{15}, S^{15})$ as below

a) a feasible path E_k^{15} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{15} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{15}$ with $|S_k^{15}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{15}$ and $s" \in S_{k'}^{15}$ with $E_k^{15} \cap E_{k'}^{15} \neq \emptyset$ (non-overlapping constraint),

d) and one slot s from the set $\{w_k, ..., \bar{s}\}$ is chosen to route the demand k in the solution $\mathcal{S}^{15}$, i.e., $|S_k^{15}| = 1$.

Obviously, $\mathcal{S}^{15}$ is feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{15}}, z^{\mathcal{S}^{15}})$ is belong to F and then to $F_{\mathbb{S}}^k$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$. As a result, $F_{\mathbb{S}}^k$ is not empty ($F_{\mathbb{S}}^k \neq \emptyset$). Furthermore, given that $s \in \{w_k, ..., \bar{s}\}$ for the demand k, this means that there exists at least one feasible solution for the problem in which $|S_k| \geq 2$ for the demand k. As a result, $F_{\mathbb{S}}^k \neq P(G, K, \mathbb{S})$.
On the other hand, we know that all the solutions of $F_{\mathbb{S}}^k$ are in $P(G, K, \mathbb{S})$ which means that they verify the equations system (2.13) s.t. the following equations system (2.21) associated with $F_{\mathbb{S}}^k$ is written as below

$$\begin{cases} \sum_{s=w_k}^{\bar{s}} z_s^k = 1, \text{ s.t. } k \text{ is chosen arbitrarily}, \\ x_e^k = 0, \text{ for all } k \in K \text{ and all } e \in E_0^k, \\ x_e^k = 0, \text{ for all } k \in K \text{ and all } e \in E_c^k, \\ x_e^k = 1, \text{ for all } k \in K \text{ and all } e \in E_1^k, \\ z_s^k = 0, \text{ for all } k \in K \text{ and all } s \in \{1, ..., w_k - 1\}. \end{cases} \tag{2.21}$$

The system (2.21) shows that the equation $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$ is not result of equations of system (2.13) which means that the equation $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$ is not redundant in the system (2.21). As a result, the system (2.21) is in full rank which implies that the dimension of the face $F_{\mathbb{S}}^k$ is equal to

$$dim(F_{\mathbb{S}}^k) = |K| * (|E| + |\mathbb{S}|) - rank(M") = |K| * (|E| + |\mathbb{S}|) - (1 + r) = dim(P(G, K, \mathbb{S})) - 1,$$

where M" denotes the matrix associated with the equation system (2.21). As a result, the face $F^k_{\mathbb{S}}$ is facet defining for $P(G, K, \mathbb{S})$.

We strengthen the proof as follows. We denote the inequality $\sum_{s=w_k}^{\bar{s}} z^k_s \geq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that defines a facet F of $P(G, K, \mathbb{S})$. Suppose that $F^k_{\mathbb{S}} \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and γ with $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ ($\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E^k_0|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E^k_1|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma^{k'}_s = 0$ for all demands $k' \in K \setminus \{k\}$ and all slots $s \in \{w_k, ..., \bar{s}\}$,

b) and $\mu^k_e = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E^k_0 \cup E^k_1)$,

c) and all σ^k_s are equivalents for demand k and slots $s \in \{w_k, ..., \bar{s}\}$ for the demand k.

First, let's us show that $\mu^k_e = 0$ for all the demands $k \in K$ and all edges $e \in E \setminus (E^k_0 \cup E^k_1)$. Consider a demand $k \in K$ and an edge $e \in E \setminus (E^k_0 \cup E^k_1)$. For that, we consider a solution $\mathcal{S}'^{15} = (E'^{15}, S'^{15})$ in which

a) a feasible path E'^{15}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S'^{15}_k is assigned to each demand $k \in K$ along each edge $e \in E'^{15}_k$ with $|S'^{15}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S'^{15}_k$ and $s" \in S'^{15}_{k'}$ with $E'^{15}_k \cap E'^{15}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E'^{15}_k} |\{s' \in S'^{15}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) the edge e is not non-compatible edge with the selected edges $e \in E'^{15}_k$ of demand k in the solution $\mathcal{S}'^{15}$, i.e., $\sum_{e' \in E'^{15}_k} l_{e'} + l_e \leq \bar{l}_k$. As a result, $E'^{15}_k \cup \{e\}$ is a feasible path for the demand k,

e) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s \in S'^{15}_k$ and $s' \in S'^{15}_{k'}$ with $(E'^{15}_k \cup \{e\}) \cap E'^{15}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e in the set of edges E'^{15}_k selected to route the demand k in the solution $\mathcal{S}'^{15}$),

f) and one slot s from the set $\{w_k, ..., \bar{s}\}$ is chosen to route the demand k in the solution $\mathcal{S}'^{15}$, i.e., $|S'^{15}_k| = 1$.

$\mathcal{S}'^{15}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{15}}, z^{\mathcal{S}'^{15}})$ is belong to F and then to $F^k_{\mathbb{S}}$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z^k_s = 1$. Based on this, we derive a solution $\mathcal{S}^{16}$ obtained from the solution $\mathcal{S}'^{15}$ by adding an unused edge $e \in E \setminus (E^k_0 \cup E^k_1)$ for the routing of demand k in K in the solution $\mathcal{S}^{15}$ which means that $E^{16}_k = E'^{15}_k \cup \{e\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^{15}$ remain the same in the solution $\mathcal{S}^{16}$, i.e., $S^{16}_k = S'^{15}_k$ for each $k \in K$, and $E^{16}_{k'} = E'^{15}_{k'}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{16}$ is clearly feasible given that

a) and a feasible path E^{16}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{16}_k is assigned to each demand $k \in K$ along each edge $e \in E^{16}_k$ with $|S^{16}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s'-w_k+1,...,s'\}\cap\{s"-w_{k'}+1,...,s"\}=\emptyset$ for each $k,k'\in K$ and each $s'\in S^{16}_k$ and $s"\in S^{16}_{k'}$ with $E^{16}_k\cap E^{16}_{k'}\neq\emptyset$, i.e., for each edge $e\in E$ and each slot $s"\in\mathbb{S}$ we have $\sum_{k\in K,e\in E^{16}_k}|\{s'\in S^{16}_k, s"\in\{s'-w_k+1,...,s'\}|\leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{16}}, z^{\mathcal{S}^{16}})$ is belong to F and then to $F^k_{\mathbb{S}}$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z^k_s=1$. It follows that

$$\mu x^{\mathcal{S}'^{15}}+\sigma z^{\mathcal{S}'^{15}}=\mu x^{\mathcal{S}^{16}}+\sigma z^{\mathcal{S}^{16}}=\mu x^{\mathcal{S}'^{15}}+\mu^k_e+\sigma z^{\mathcal{S}'^{15}}.$$

As a result, $\mu^k_e=0$ for demand k and an edge e.
As e is chosen arbitrarily for the demand k with $e\notin E^k_0\cup E^k_1$, we iterate the same procedure for all $e'\in E\setminus(E^k_0\cup E^k_1\cup\{e\})$. We conclude that for the demand k

$$\mu^k_e=0,\ \text{for all } e\in E\setminus(E^k_0\cup E^k_1).$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k'\in K\setminus\{k\}$ and all $e\in E\setminus(E^k_0\cup E^k_1)$. We conclude at the end that

$$\mu^k_e=0,\ \text{for all } k\in K \text{ and all } e\in E\setminus(E^k_0\cup E^k_1).$$

Next, we will show that, $\sigma^{k'}_{s'}=0$ for all $k'\in K\setminus\{k\}$ and all $s'\in\{w_{k'},...,\bar{s}\}$. Consider the demand k' in $K\setminus\{k\}$ and a slot s' in $\{w_{k'},...,\bar{s}\}\setminus\{s\}$. For that, we consider a solution $\mathcal{S}"^{15}=(E"^{15},S"^{15})$ in which

a) a feasible path $E"^{15}_k$ is assigned to each demand $k\in K$ (routing constraint),

b) a set of last-slots $S"^{15}_k$ is assigned to each demand $k\in K$ along each edge $e\in E"^{15}_k$ with $|S"^{15}_k|\geq 1$ (contiguity and continuity constraints),

c) $\{s'-w_k+1,...,s'\}\cap\{s"-w_{k'}+1,...,s"\}=\emptyset$ for each $k,k'\in K$ and each $s'\in S"^{15}_k$ and $s"\in S"^{15}_{k'}$ with $E"^{15}_k\cap E"^{15}_{k'}\neq\emptyset$, i.e., for each edge $e\in E$ and each slot $s"\in\mathbb{S}$ we have $\sum_{k\in K,e\in E"^{15}_k}|\{s'\in S"^{15}_k, s"\in\{s'-w_k+1,...,s'\}|\leq 1$ (non-overlapping constraint),

d) and $\{s'-w_{k'}+1,...,s'\}\cap\{s"-w_k+1,...,s"\}=\emptyset$ for each $k\in K$ and $s"\in S"^{15}_k$ with $E"^{15}_k\cap E"^{15}_{k'}\neq\emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{15}_{k'}$ assigned to the demand k' in the solution $\mathcal{S}"^{15}$),

e) and $|S"^{15}_k|=1$ for the demand k.

$\mathcal{S}"^{15}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{15}}, z^{\mathcal{S}"^{15}})$ is belong to F and then to $F^k_{\mathbb{S}}$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z^k_s=1$. Based on this, we derive a solution $\mathcal{S}^{17}$ from the solution $\mathcal{S}"^{15}$ by adding the slot s' as last-slot to the demand k' with modifying the paths assigned to a subset of demands $\tilde{K}\subset K$ in $\mathcal{S}"^{15}$ (i.e., $E^{17}_k=E"^{15}_k$ for each $k\in K\setminus\tilde{K}$, and $E^{17}_k\neq E"^{15}_k$ for each $k\in\tilde{K}$) s.t.

a) a new feasible path E^{17}_k is assigned to each demand $k\in\tilde{K}$ (routing constraint),

b) and $\{s'-w_k+1,...,s'\}\cap\{s"-w_{k'}+1,...,s"\}=\emptyset$ for each $k\in\tilde{K}$ and $k'\in K\setminus\tilde{K}$ and each $s'\in S"^{15}_k$ and $s"\in S"^{15}_{k'}$ with $E^{17}_k\cap E"^{15}_{k'}\neq\emptyset$, i.e., for each edge $e\in E$ and each slot $s"\in\mathbb{S}$ we have $\sum_{k\in\tilde{K},e\in E^{17}_k}|\{s'\in S"^{15}_k, s"\in\{s'-w_k+1,...,s'\}|+\sum_{k\in K\setminus\tilde{K},e\in E"^{15}_k}|\{s'\in S"^{15}_k, s"\in\{s'-w_k+1,...,s'\}|\leq 1$ (non-overlapping constraint),

c) and $|S_k^{17}|$ for the demand k.

The last-slots assigned to the demands $K \setminus \{k'\}$ in $\mathcal{S}"^{15}$ remain the same in $\mathcal{S}^{17}$, i.e., $S"^{15}_k = S^{17}_k$ for each demand $k \in K \setminus \{k'\}$, and $S^{17}_{k'} = S"^{15}_{k'} \cup \{s'\}$ for the demand k'. The solution $\mathcal{S}^{17}$ is clearly feasible given that

a) a feasible path E_k^{17} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{17} is assigned to each demand $k \in K$ along each edge $e \in E_k^{17}$ with $|S_k^{17}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{17}$ and $s" \in S_{k'}^{17}$ with $E_k^{17} \cap E_{k'}^{17} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{17}} |\{s' \in S_k^{17}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $|S_k^{17}|$ for the demand k.

The corresponding incidence vector $(x^{\mathcal{S}^{17}}, z^{\mathcal{S}^{17}})$ is belong to F and then to $F_{\mathbb{S}}^k$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$. We have so

$$\mu x^{\mathcal{S}"^{15}} + \sigma z^{\mathcal{S}"^{15}} = \mu x^{\mathcal{S}^{17}} + \sigma z^{\mathcal{S}^{17}} = \mu x^{\mathcal{S}"^{15}} + \sigma z^{\mathcal{S}"^{15}} + \sigma_{s'}^{k'} - \sum_{\tilde{k} \in \tilde{K}} \sum_{e \in E"^{15}_k} \mu_e^{\tilde{k}} + \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E_k^{17}} \mu_{e'}^{\tilde{k}}.$$

It follows that $\sigma_{s'}^{k'} = 0$ for demand k' and a slot $s' \in \{w_{k'}, ..., \bar{s}\}$ given that $\mu_e^k = 0$ for all the demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$.

The slot s' is chosen arbitrarily for the demand k', we iterate the same procedure for all feasible slots in $\{w_{k'}, ..., \bar{s}\}$ of demand k' s.t. we find

$$\sigma_{s'}^{k'} = 0, \text{ for the demand } k' \text{ and all slots } s' \in \{w_{k'}, ..., \bar{s}\}.$$

Given that the demand k' is chosen arbitrarily. We iterate the same thing for all the demands $k"$ in $K \setminus \{k, k'\}$ such that

$$\sigma_s^{k"} = 0, \text{ for all } k" \in K \setminus \{k, k'\} \text{ and all slots } s \in \{w_{k"}, ..., \bar{s}\}.$$

Consequently, we conclude that

$$\sigma_{s'}^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s' \in \{w_{k'}, ..., \bar{s}\}.$$

Let's prove now that σ_s^k for demand k and slots s in $\{w_k, ..., \bar{s}\}$ are equivalent. Consider a slot $s' \in \{w_k, ..., \bar{s}\}$ s.t. $s' \notin S_k^{15}$. For that, we consider a solution $\tilde{\mathcal{S}}^{15} = (\tilde{E}^{15}, \tilde{S}^{15})$ in which

a) a feasible path $\tilde{E}_k^{15}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}_k^{15}$ is assigned to each demand $k \in K$ along each edge $e \in \tilde{E}_k^{15}$ with $|\tilde{S}_k^{15}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}_k^{15}$ and $s" \in \tilde{S}_{k'}^{15}$ with $\tilde{E}_k^{15} \cap \tilde{E}_{k'}^{15} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in \tilde{E}_k^{15}} |\{s' \in \tilde{S}_k^{15}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in \tilde{S}_{k'}^{15}$ with $\tilde{E}_k^{15} \cap \tilde{E}_{k'}^{15} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $\tilde{S}_k^{15}$ assigned to the demand k in the solution $\tilde{\mathcal{S}}^{15}$).,

e) and $|\tilde{S}_k^{15}| = 1$ for the demand k.

$\tilde{\mathcal{S}}^{15}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{15}}, z^{\tilde{\mathcal{S}}^{15}})$ is belong to F and then to $F_{\tilde{S}}^k$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$. Based on this, we construct a solution $\mathcal{S}^{18}$ derived from the solution $\tilde{\mathcal{S}}^{15}$ by adding the slot s' as last-slot to the demand k' in $\mathcal{S}_k^{18}$ and removing the last slot s assigned to k in $\tilde{S}_k^{15}$ (i.e., $S_k^{18} = (\tilde{S}_k^{15} \setminus \{s\}) \cup \{s'\}$ for the demand k) with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\tilde{\mathcal{S}}^{15}$ (i.e., $E_k^{18} = \tilde{E}_k^{15}$ for each $k \in K \setminus \tilde{K}$, and $E_k^{18} \neq \tilde{E}_k^{15}$ for each $k \in \tilde{K}$), and also the last-slots assigned to the demands $K \setminus \{k\}$ in $\tilde{\mathcal{S}}^{15}$ remain the same in $\mathcal{S}^{18}$. The solution $\mathcal{S}^{18}$ is clearly feasible given that

a) a feasible path E_k^{18} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{18} is assigned to each demand $k \in K$ along each edge $e \in E_k^{18}$ with $|S_k^{18}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{18}$ and $s" \in S_{k'}^{18}$ with $E_k^{18} \cap E_{k'}^{18} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{18}} |\{s' \in S_k^{18}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $|S_k^{18}| = 1$.

The corresponding incidence vector $(x^{\mathcal{S}^{18}}, z^{\mathcal{S}^{18}})$ is belong to F and then to $F_{\tilde{S}}^k$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{15}} + \sigma z^{\tilde{\mathcal{S}}^{15}} = \mu x^{\mathcal{S}^{18}} + \sigma z^{\mathcal{S}^{18}} = \mu x^{\tilde{\mathcal{S}}^{15}} + \sigma z^{\tilde{\mathcal{S}}^{15}} + \sigma_{s'}^k - \sigma_s^k + \sigma_{\tilde{s}}^k - \sum_{k \in \tilde{K}} \sum_{e \in \tilde{E}_k^{15}} \mu x^{\tilde{\mathcal{S}}^{15}} + \sum_{k \in \tilde{K}} \sum_{e \in E_k^{18}} \mu x^{\mathcal{S}^{18}}.$$

It follows that $\sigma_{s'}^k = \sigma_s^k$ for the demand k and a slots $s, s' \in \{w_k, ..., \bar{s}\}$ given that $\mu_e^k = 0$ for all $k \in K$ and all $e \in E \setminus (E_0^k \cup E_1^k)$.
The slot s is chosen arbitrarily for the demand k in $\{w_k, ..., \bar{s}\}$, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k s.t. we find

$$\sigma_{s'}^k = \sigma_s^k, \text{ for all slots } s, s' \in \{w_k, ..., \bar{s}\}.$$

Consequently, we obtain that $\sigma_s^k = \rho$ for demand k and slots s in $\{w_k, ..., \bar{s}\}$.
On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k' \in K$ and $e \in E$

$$\mu_e^{k'} = \begin{cases} \gamma_1^{k',e}, \text{ if } e \in E_0^{k'}, \\ \gamma_2^{k',e}, \text{ if } e \in E_1^{k'}, \\ \quad 0, otherwise, \end{cases}$$

and for each $k' \in K$ and $s \in \mathbb{S}$

$$\sigma_s^{k'} = \begin{cases} \gamma_3^{k',s}, \text{if } s \in \{1, ..., w_{k'} - 1\}, \\ \rho, \text{if } k' = k \text{ and } s \in \{w_{k'}, ..., \bar{s}\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{s=w_k}^{\bar{s}} \rho\beta_s^k + \gamma Q$ for the demand k which ends our strengthening of the proof. □

Theorem 2.3.7. *Consider a demand k and a subset of node $X \subset V$, with $|X \cap \{o_k, d_k\}| = 1$ and $\delta(X) \cap E_1^k = \emptyset$ s.t. $X \cap V_0^k = \emptyset$. Then, the inequality (2.2), $\sum_{e\in\delta(X)} x_e^k \geq 1$, is facet defining for $P(G, K, \mathbb{S})$.*

Proof. Let F_X^k denote the face induced by inequality $\sum_{e\in(\delta(X)\setminus E_0^k)} x_e^k \geq 1$, which is given by

$$F_X^k = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{e\in(\delta(X)\setminus E_0^k)} x_e^k = 1\}.$$

Let $X = \{o_k\}$. In order to prove that inequality $\sum_{e\in(\delta(X)\setminus E_0^k)} x_e^k \geq 1$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that F_X^k is a proper face which means that it is not empty, and $F_X^k \neq P(G, K, \mathbb{S})$.
We construct a solution $S^{19} = (E^{19}, S^{19})$ as below

a) a feasible path E_k^{19} is assigned to each demand $k \in K$ (routing constraint),
b) a set of last-slots S_k^{19} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{19}$ with $|S_k^{19}| \geq 1$ (contiguity and continuity constraints),
c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{19}$ and $s' \in S_{k'}^{19}$ with $E_k^{19} \cap E_{k'}^{19} \neq \emptyset$ (non-overlapping constraint),
d) and one edge e from $(\delta(X) \setminus E_0^k)$ is chosen to route the demand k in the solution $\mathcal{S}^{19}$, i.e., $|(\delta(X) \setminus E_0^k) \cap E_k^{19}| = 1$.

Obviously, $\mathcal{S}^{19}$ is feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{19}}, z^{\mathcal{S}^{19}})$ is belong to $P(G, K, \mathbb{S})$ and then to F_X^k given that it is composed by $\sum_{e\in(\delta(X)\setminus E_0^k)} x_e^k = 1$. As a result, F_X^k is not empty ($F_X^k \neq \emptyset$). Furthermore, given that $e \in E \setminus (E_0^k \cup E_1^k)$ for the demand k, this means that there exists at least one feasible path E_k for the demand k passed through the edge e which means that $F_X^k \neq P(G, K, \mathbb{S})$.
Let denote the inequality $\sum_{e\in(\delta(X)\setminus E_0^k)} x_e^k \geq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F_X^k \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and γ with $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (with $\gamma_1 \in \mathbb{R}^{\sum_{k\in K} |E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k\in K} |E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k\in K}(w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. We will show that

a) $\mu_{e'}^k = 0$ for the demand k and all $e' \in E \setminus (E_0^k \cup E_1^k \cup \delta(X))$,
b) and $\mu_{e'}^{k'} = 0$ for all demands $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E_0^{k'} \cup E_1^{k'})$,
c) and $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$,

d) and that μ_e^k are equivalent for all $e \in (\delta(X) \setminus E_0^k)$.

First, let's show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$. Consider a demand k and a slot s in $\{w_k, ..., \bar{s}\}$. For that, we consider a solution $\mathcal{S}'^{19} = (E'^{19}, S'^{19})$ in which

a) a feasible path $E_k'^{19}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S_k'^{19}$ is assigned to each demand $k \in K$ along each edge $e' \in E_k'^{19}$ with $|S_k'^{19}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k'^{19}$ and $s" \in S_{k'}'^{19}$ with $E_k'^{19} \cap E_{k'}'^{19} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E_k'^{19}} |\{s' \in S_k'^{19}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}'^{19}$ with $E_k'^{19} \cap E_{k'}'^{19} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s in the set of last-slots $S_k'^{19}$ assigned to the demand k in the solution $\mathcal{S}'^{19}$),

e) and one edge e from $(\delta(X) \setminus E_0^k)$ is chosen to route the demand k in the solution $\mathcal{S}'^{19}$, i.e., $|(\delta(X) \setminus E_0^k) \cap E_k'^{19}| = 1$.

$\mathcal{S}'^{19}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{19}}, z^{\mathcal{S}'^{19}})$ is belong to $P(G, K, \mathbb{S})$. Based on this, we derive a solution $\mathcal{S}^{20} = (E^{20}, S^{20})$ from the solution $\mathcal{S}'^{19}$ by adding the slot s as last-slot to the demand k without modifying the paths assigned to the demands K in $\mathcal{S}'^{19}$ (i.e., $E_k^{20} = E_1'^{19}$ for each $k \in K$), and the last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}'^{19}$ remain the same in the solution $\mathcal{S}^{20}$ i.e., $S_{k'}'^{19} = S_{k'}^{20}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{20} = S_k'^{19} \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{20}$ is feasible given that

a) a feasible path E_k^{20} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{20} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{20}$ with $|S_k^{20}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{20}$ and $s' \in S_{k'}^{20}$ with $E_k^{20} \cap E_{k'}^{20} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E_k^{20}} |\{s \in S_k^{20}, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{20}}, z^{\mathcal{S}^{20}})$ is belong to F and then to F_X^k given that it is also composed by $\sum_{e \in (\delta(X) \setminus E_0^k)} x_e^k = 1$. We then obtain that

$$\mu x^{\mathcal{S}'^{19}} + \sigma z^{\mathcal{S}'^{19}} = \mu x^{\mathcal{S}^{20}} + \sigma z^{\mathcal{S}^{20}} = \mu x^{\mathcal{S}'^{19}} + \sigma z^{\mathcal{S}'^{19}} + \sigma_s^k.$$

It follows that $\sigma_s^k = 0$ for demand k and a slot $s \in \{w_k, ..., \bar{s}\}$.
The slot s is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k s.t. we find

$$\sigma_s^k = 0, \text{ for demand } k \text{ and all slots } s \in \{w_k, ..., \bar{s}\}.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\}.$$

Consequently, we conclude that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\}.$$

Next, we will show that $\mu^k_{e'} = 0$ for all the demands $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E^{k'}_0 \cup E^{k'}_1)$, and $\mu^k_{e'} = 0$ for the demand k and all $e' \in E \setminus (E^k_0 \cup E^k_1 \cup \delta(X))$. Consider the demand $k \in K$ and an edge $e' \in E \setminus (E^k_0 \cup E^k_1 \cup \delta(X))$ chosen arbitrarily. For that, we consider a solution $\mathcal{S}"^{19} = (E"^{19}, S"^{19})$ in which

a) a feasible path $E"^{19}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^{19}_k$ is assigned to each demand $k \in K$ along each edge $e' \in E"^{19}_k$ with $|S"^{19}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{19}_k$ and $s" \in S"^{19}_{k'}$ with $E"^{19}_k \cap E"^{19}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E"^{19}_k} |\{s' \in S"^{19}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) the edge e' is not non-compatible edge with the selected edges $e" \in E"^{19}_k$ of demand k in the solution $\mathcal{S}"^{19}$, i.e., $\sum_{e" \in E"^{19}_k} l_{e"} + l_{e'} \leq \bar{l}_k$. As a result, $E"^{19}_k \cup \{e'\}$ is a feasible path for the demand k,

e) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s \in S"^{19}_k$ and $s' \in S"^{19}_{k'}$ with $(E"^{19}_k \cup \{e'\}) \cap E"^{19}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e' in the set of edges $E"^{19}_k$ selected to route the demand k in the solution $\mathcal{S}"^{19}$),

f) and one edge e from $(\delta(X) \setminus E^k_0)$ is chosen to route the demand k in the solution $\mathcal{S}"^{19}$, i.e., $|(\delta(X) \setminus E^k_0) \cap E"^{19}_k| = 1$.

$\mathcal{S}"^{19}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{19}}, z^{\mathcal{S}"^{19}})$ is belong to F and then to F^k_X given that it is also composed by $\sum_{e \in (\delta(X) \setminus E^k_0)} x^k_e = 1$. Let $\mathcal{S}^{21}$ be a solution obtained from the solution $\mathcal{S}"^{19}$ by adding an unused edge $e' \in E \setminus (E^k_0 \cup E^k_1)$ for the routing of demand k in K in the solution $\mathcal{S}"^{19}$ which means that $E^{21}_k = E"^{19}_k \cup \{e'\}$, and removing slot s selected for the demand k in $\mathcal{S}"^{19}$ and replaced it by a new slot $s' \in \{w_k, ..., \mathbb{S}\}$ (i.e., $S^{21}_k = (S"^{19}_k \setminus \{s\}) \cup \{s'\}$ s.t. $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^{19}_{k'}$ with $E^{21}_k \cap E"^{19}_{k'} \neq \emptyset$. The last-slots and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}"^{19}$ remain the same in the solution $\mathcal{S}^{21}$, i.e., $S^{21}_{k'} = S"^{19}_{k'}$ and $E^{21}_{k'} = E"^{19}_{k'}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{21}$ is clearly feasible given that

a) and a feasible path E^{21}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{21}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{21}_k$ with $|S^{21}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S^{21}_k$ and $s' \in S^{21}_{k'}$ with $E^{21}_k \cap E^{21}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E^{21}_k} |\{s \in S^{21}_k, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{21}}, z^{\mathcal{S}^{21}})$ is belong to F and then to F^k_X given that it is also composed by $\sum_{e \in (\delta(X) \setminus E^k_0)} x^k_e = 1$. It follows that

$$\mu x^{\mathcal{S}"^{19}} + \sigma z^{\mathcal{S}"^{19}} = \mu x^{\mathcal{S}^{21}} + \sigma z^{\mathcal{S}^{21}} = \mu x^{\mathcal{S}"^{19}} + \mu^k_{e'} + \sigma z^{\mathcal{S}"^{19}} - \sigma^k_s + \sigma^k_{s'}.$$

It follows that $\mu^k_{e'} = 0$ for demand k and an edge e' given that $\sigma^k_s = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$.

As e' is chosen arbitrarily for the demand k with $e' \notin E^k_0 \cup E^k_1 \cup \delta(X)$, we iterate the same procedure for all $e' \in E \setminus (E^k_0 \cup E^k_1 \cup \delta(X))$. We conclude that for the demand k

$$\mu^k_{e'} = 0, \text{ for all } e' \in E \setminus (E^k_0 \cup E^k_1 \cup \delta(X)).$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E_0^k \cup E_1^k)$. We conclude at the end that

$$\mu_{e'}^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all } e' \in E \setminus (E_0^{k'} \cup E_1^{k'}),$$
$$\mu_{e'}^{k} = 0, \text{ for all } e' \in E \setminus (E_0^{k} \cup E_1^{k} \cup \delta(X)).$$

Let's us prove that the μ_e^k for a demand k and edges $e \in (\delta(X) \setminus E_0^k)$ are equivalent. Consider an edge $e' \in (\delta(X) \setminus E_0^k)$ s.t. $e' \notin E_k^{19}$. For that, we consider a solution $\tilde{\mathcal{S}}^{19} = (\tilde{E}^{19}, \tilde{S}^{19})$ in which

a) a feasible path $\tilde{E}_k^{19}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}_k^{19}$ is assigned to each demand $k \in K$ along each edge $e \in \tilde{E}_k^{19}$ with $|\tilde{S}_k^{19}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}_k^{19}$ and $s" \in \tilde{S}_{k'}^{19}$ with $\tilde{E}_k^{19} \cap \tilde{E}_{k'}^{19} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in \tilde{E}_k^{19}} |\{s' \in \tilde{S}_k^{19}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and there is one edge e from $(\delta(X) \setminus E_0^k)$ selected for the routing of demand k in the solution $\tilde{\mathcal{S}}^{19}$, i.e., $|(\delta(X) \setminus E_0^k) \cap \tilde{E}_k^{19}| = 1$.

$\tilde{\mathcal{S}}^{19}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{19}}, z^{\tilde{\mathcal{S}}^{19}})$ is belong to F and then to F_X^k given that it is composed by $\sum_{e \in (\delta(X) \setminus E_0^k)} x_e^k = 1$. Based on this, we construct a solution $\mathcal{S}^{22}$ derived from the solution $\tilde{\mathcal{S}}^{19}$ by

a) modifying the path assigned to the demand k in $\tilde{\mathcal{S}}^{19}$ from $\tilde{E}_k^{19}$ to a path E_k^{22} passed through the edge e' with $|(\delta(X) \setminus E_0^k) \cap E_k^{22}| = 1$,

b) modifying the last-slots assigned to some demands $\tilde{K} \subset K$ from $\tilde{S}_{\tilde{k}}^{19}$ to $S_{\tilde{k}}^{22}$ for each $\tilde{k} \in \tilde{K}$ while satisfying non-overlapping constraint.

The paths assigned to the demands $K \setminus \{k\}$ in $\tilde{\mathcal{S}}^{19}$ remain the same in $\mathcal{S}^{22}$ (i.e., $E_{k"}^{22} = \tilde{E}_{k"}^{19}$ for each $k" \in K \setminus \{k\}$), and also without modifying the last-slots assigned to the demands $K \setminus \tilde{K}$ in $\tilde{\mathcal{S}}^{19}$, i.e., $\tilde{S}_k^{19} = S_k^{22}$ for each demand $k \in K \setminus \tilde{K}$. The solution $\mathcal{S}^{22}$ is clearly feasible given that

a) a feasible path E_k^{22} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{22} is assigned to each demand $k \in K$ along each edge $e \in E_k^{22}$ with $|S_k^{22}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{22}$ and $s" \in S_{k'}^{22}$ with $E_k^{22} \cap E_{k'}^{22} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{22}} |\{s' \in S_k^{22}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) $|(\delta(X) \setminus E_0^k) \cap E_k^{22}| = 1$.

The corresponding incidence vector $(x^{\mathcal{S}^{22}}, z^{\mathcal{S}^{22}})$ is belong to F and then to F_X^k given that it is composed by $\sum_{e \in (\delta(X) \setminus E_0^k)} x_e^k = 1$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{19}} + \sigma z^{\tilde{\mathcal{S}}^{19}} = \mu x^{\mathcal{S}^{22}} + \sigma z^{\mathcal{S}^{22}} = \mu x^{\tilde{\mathcal{S}}^{19}} + \sigma z^{\tilde{\mathcal{S}}^{19}} + \mu_{e'}^k - \mu_e^k + \sum_{\tilde{k} \in \tilde{K}} \sum_{s' \in S_{\tilde{k}}^{22}} \sigma_{s'}^{\tilde{k}} - \sum_{s \in \tilde{S}_{\tilde{k}}^{19}} \sigma_s^{\tilde{k}}$$
$$+ \sum_{e" \in E_k^{22} \setminus \{e'\}} \mu_{e"}^k - \sum_{e" \in \tilde{E}_k^{19} \setminus \{e\}} \mu_{e"}^k.$$

It follows that $\mu_{e'}^k = \mu_e^k$ for demand k and a edge $e' \in (\delta(X) \setminus E_0^k)$ given that $\mu_{e"}^k = 0$ for all $k \in K$ and all $e" \in E \setminus (E_0^k \cup E_1^k)$ with $e" \notin (\delta(X) \setminus E_0^k)$, and $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$.
Given that the pair of edges (e, e') are chosen arbitrary in $(\delta(X) \setminus E_0^k)$, we iterate the same procedure for all pairs $(e, e') \in (\delta(X) \setminus E_0^k)$ s.t. we find

$$\mu_e^k = \mu_{e'}^k, \text{for all pairs } e, e' \in (\delta(X) \setminus E_0^k).$$

Consequently, we obtain that $\mu_e^k = \rho$ for all $e \in (\delta(X) \setminus E_0^k)$.
On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k' \in K$ and $e \in E$

$$\mu_e^{k'} = \begin{cases} \gamma_1^{k',e}, \text{ if } e \in E_0^{k'}, \\ \gamma_2^{k',e}, \text{ if } e \in E_1^{k'}, \\ \rho, \text{ if } k = k' \text{ and } e \in (\delta(X) \setminus E_0^k), \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\}, \\ 0, otherwise. \end{cases}$$

We conclude that $(\mu, \sigma) = \rho \sum_{e \in (\delta(X) \setminus E_0^k)} \alpha_e^k + \gamma Q.$ □

Proposition 2.3.5. *Consider an edge $e \in E$, and an interval of contiguous slots $I = [s_i, s_j] \subset \mathbb{S}$. Let $k, k' \in K$ be pair of demands with $e \notin (E_0^k \cup E_0^{k'})$, $2w_k > |I|$, $2w_{k'} > |I|$, $w_{k'} + w_{k'} > |I|$, and k, k' are not non-compatible demands for the edge e. Then, the following inequality is valid for $P(G, K, \mathbb{S})$*

$$x_e^k + x_e^{k'} + \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s=s_i+w_{k'}-1}^{s_j} z_s^k \leq 3. \tag{2.22}$$

Proof. For each edge $e \in E$ and interval of contiguous slots $I \subseteq \mathbb{S}$, the inequality (2.22) ensures that if the two demands k, k' pass through edge e, they cannot share the interval $I = [s_i, s_j]$ over edge e. □

Theorem 2.3.8. *Consider an edge $e \in E$, and a slot $s \in \mathbb{S}$. Let k, k' be two demands in K with k, k' are not non-compatible demands for the edge e. Then, the inequality (2.6) is facet defining for $P(G, K, \mathbb{S})$ iff $K_e \setminus \{k, k'\} = \emptyset$, and there does not exist an interval of contiguous slots $I = [s_i, s_j]$ s.t.*

a) $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$,

b) *and* $|\{s_i + w_{k'} - 1, ..., s_j\}| \geq w_{k'}$, *c) and* $s \in \{s_i + \max(w_k, w_{k'}) - 1, ..., s_j - \max(w_k, w_{k'}) + 1\}$,

d) and $w_k + w_{k'} \geq |I| + 1$,

e) and $2w_k \geq |I| + 1$,

f) and $2w_{k'} \geq |I| + 1$.

Proof. Let $\tilde{K} = \{k, k'\}$.
Neccessity.
If $K_e \backslash \tilde{K} \neq \emptyset$, then the inequality (2.6) is dominated by the inequality (2.22) without changing its right hand side. Moreover, if there exists an interval of contiguous slots $I = [s_i, s_j]$ s.t.

a) $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ for each demand $k \in \tilde{K}$,

b) and $s \in \{s_i + \max\limits_{k' \in \tilde{K}} w_k - 1, ..., s_j - \max\limits_{k \in \tilde{K}} w_k + 1\}$,

c) and $w_k + w_{k'} \geq |I| + 1$ for each $k, k' \in \tilde{K}$,

d) and $2w_k \geq |I| + 1$ for each $k \in \tilde{K}$.

Then the inequality (2.6) is dominated by the inequality (2.22). Hence, the inequality (2.6) is not facet defining for $P(G, K, \mathbb{S})$.
Sufficiency.
Let $F^{e,s}_{\tilde{K}}$ denote the face induced by the inequality (2.6), which is given by

$$F^{e,s}_{\tilde{K}} = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{k \in \tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z^k_{s'} = |\tilde{K}| + 1\}.$$

In order to prove that inequality $\sum_{k \in \tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} \leq |\tilde{K}| + 1$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that $F^{e,s}_{\tilde{K}}$ is a proper face, and $F^{e,s}_{\tilde{K}} \neq P(G, K, \mathbb{S})$.
We construct a solution $\mathcal{S}^{31} = (E^{31}, S^{31})$ as below

a) a feasible path E^{31}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{31}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{31}_k$ with $|S^{31}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{31}_k$ and $s' \in S^{31}_{k'}$ with $E^{31}_k \cap E^{31}_{k'} \neq \emptyset$ (non-overlapping constraint),

d) and there is one demand k from the set of demands $\tilde{K}$ (i.e., $k \in \tilde{K}$ s.t. the demand k selects a slot s' as last-slot in the solution $\mathcal{S}^{31}$ with $s' \in \{s, ..., s + w_k - 1\}$, i.e., $s' \in S^{31}_k$ for a demand $k \in \tilde{K}$, and for each $s' \in S^{31}_{k'}$ for all $k' \in \tilde{K} \setminus \{k\}$ we have $s' \notin \{s, ..., s + w_{k'} - 1\}$,

e) and all the demands in $\tilde{K}$ pass through the edge e in the solution $\mathcal{S}^{31}$, i.e., $e \in E^{31}_k$ for each $k \in \tilde{K}$.

Obviously, $\mathcal{S}^{31}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{31}}, z^{\mathcal{S}^{31}})$ is belong to $P(G, K, \mathbb{S})$ and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k \in \tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. As a result, $F^{e,s}_{\tilde{K}}$ is not empty (i.e., $F^{e,s}_{\tilde{K}} \neq \emptyset$). Furthermore, given that $s \in \mathbb{S}$, this means that there exists at least one feasible slot assignment S_k for each demands k in $\tilde{K}$ with $S_k \cap \{s, ..., s + w_k - 1\} = \emptyset$. Hence, $F^{e,s}_{\tilde{K}} \neq P(G, K, \mathbb{S})$.

We denote the inequality $\sum_{k \in \tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} \leq |\tilde{K}| + 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F^{e,s}_{\tilde{K}} \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$

(s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k\in K}|E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k\in K}|E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k\in K}(w_k-1)}$) s.t. $(\mu,\sigma) = \rho(\alpha,\beta) + \gamma Q$, and that

a) $\sigma_{s'}^k = 0$ for all demands $k \in K$ and all slots $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s, ..., s+w_k-1\}$ if $k \in \tilde{K}$,

b) and $\sigma_{s'}^k$ are equivalents for all $k \in \tilde{K}$ and all $s' \in \{s, ..., s+w_k-1\}$,

c) and $\mu_{e'}^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e'$ if $k \in \tilde{K}$,

d) and all μ_e^k are equivalents for the set of demands in $\tilde{K}$,

e) and $\sigma_{s'}^k$ and μ_e^k are equivalents for all $k \in \tilde{K}$ and all $s' \in \{s, ..., s+w_k-1\}$.

We first show that $\mu_{e'}^k = 0$ for each edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$ with $e \neq e'$ if $k \in \tilde{K}$. Consider a demand $k \in K$ and an edge $e' \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e'$ if $k \in \tilde{K}$. For that, we consider a solution $\mathcal{S}'^{31} = (E'^{31}, S'^{31})$ in which

a) a feasible path $E_k'^{31}$ is assigned to each demand $k \in K$ (routing constraint),

b) and a set of last-slots $S_k'^{31}$ is assigned to each demand $k \in K$ along each edge $e' \in E_k'^{31}$ with $|S_k'^{31}| \geq 1$ (contiguity and continuity constraints),

c) and $\{s'-w_k+1, ..., s'\} \cap \{s"-w_{k'}+1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k'^{31}$ and $s" \in S_{k'}'^{31}$ with $E_k'^{31} \cap E_{k'}'^{31} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e'\in E_k'^{31}} |\{s' \in S_k'^{31}, s" \in \{s'-w_k+1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and there is one demand k from the set of demands $\tilde{K}$ (i.e., $k \in \tilde{K}$ s.t. the demand k selects a slot s' as last-slot in the solution $\mathcal{S}'^{31}$ with $s' \in \{s, ..., s+w_k-1\}$, i.e., $s' \in S_k'^{31}$ for a demand $k \in \tilde{K}$, and for each $s' \in S_{k'}'^{31}$ for all $k' \in \tilde{K} \setminus \{k\}$ we have $s' \notin \{s, ..., s+w_{k'}-1\}$,

e) and the edge e' is not non-compatible edge with the selected edges $e" \in E_k'^{31}$ of demand k in the solution $\mathcal{S}'^{31}$, i.e., $\sum_{e"\in E_k'^{31}} l_{e"} + l_{e'} \leq \bar{l}_k$. As a result, $E_k'^{31} \cup \{e'\}$ is a feasible path for the demand k,

f) and all the demands in $\tilde{K}$ pass through the edge e in the solution $\mathcal{S}'^{31}$, i.e., $e \in E_k'^{31}$ for each $k \in \tilde{K}$.

$\mathcal{S}'^{31}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{31}}, z^{\mathcal{S}'^{31}})$ is belong to F and then to $F_{\tilde{K}}^{e,s}$ given that it is composed by $\sum_{k\in\tilde{K}} x_{e'}^k + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k = 1$. Based on this, we derive a solution $\mathcal{S}^{32}$ obtained from the solution $\mathcal{S}'^{31}$ by adding an unused edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in K in the solution $\mathcal{S}^{31}$ which means that $E_k^{32} = E_k'^{31} \cup \{e'\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^{31}$ remain the same in the solution $\mathcal{S}^{32}$, i.e., $S_k^{32} = S_k'^{31}$ for each $k \in K$, and $E_{k'}^{32} = E_{k'}'^{31}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{32}$ is clearly feasible given that

a) and a feasible path E_k^{32} is assigned to each demand $k \in K$ (routing constraint),

b) and a set of last-slots S_k^{32} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{32}$ with $|S_k^{32}| \geq 1$ (contiguity and continuity constraints),

c) and $\{s'-w_k+1, ..., s'\} \cap \{s"-w_{k'}+1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{32}$ and $s" \in S_{k'}^{32}$ with $E_k^{32} \cap E_{k'}^{32} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e'\in E_k^{32}} |\{s' \in S_k^{32}, s" \in \{s'-w_k+1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{32}}, z^{\mathcal{S}^{32}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k\in\tilde{K}} x^k_{e'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. It follows that

$$\mu x^{\mathcal{S}'^{31}} + \sigma z^{\mathcal{S}'^{31}} = \mu x^{\mathcal{S}^{32}} + \sigma z^{\mathcal{S}^{32}} = \mu x^{\mathcal{S}'^{31}} + \mu^k_{e'} + \sigma z^{\mathcal{S}'^{31}}.$$

As a result, $\mu^k_{e'} = 0$ for demand k and an edge e'.
As e' is chosen arbitrarily for the demand k with $e \notin E^k_0 \cup E^k_1$ and $e \neq e'$ if $k \in \tilde{K}$, we iterate the same procedure for all $e" \in E \setminus (E^k_0 \cup E^k_1 \cup \{e'\})$ with $e \neq e"$ if $k \in \tilde{K}$. We conclude that for the demand k

$$\mu^k_{e'} = 0, \text{ for all } e' \in E \setminus (E^k_0 \cup E^k_1) \text{ with } e \neq e' \text{ if } k \in \tilde{K}.$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E^k_0 \cup E^k_1)$. We conclude at the end that

$$\mu^k_{e'} = 0, \text{ for all } k \in K \text{ and all } e' \in E \setminus (E^k_0 \cup E^k_1) \text{ with } e \neq e' \text{ if } k \in \tilde{K}.$$

Let's us show that $\sigma^k_{s'} = 0$ for all $k \in K$ and all $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s, ..., s + w_k - 1\}$ if $k \in \tilde{K}$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s, ..., s + w_k - 1\}$ if $k \in \tilde{K}$. For that, we consider a solution $\mathcal{S}"^{31} = (E"^{31}, S"^{31})$ in which

a) a feasible path $E"^{31}_k$ is assigned to each demand $k \in K$ (routing constraint),
b) a set of last-slots $S"^{31}_k$ is assigned to each demand $k \in K$ along each edge $e' \in E"^{31}_k$ with $|S"^{31}_k| \geq 1$ (contiguity and continuity constraints),
c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{31}_k$ and $s" \in S"^{31}_{k'}$ with $E"^{31}_k \cap E"^{31}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e'\in E"^{31}_k} |\{s' \in S"^{31}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),
d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^{31}_{k'}$ with $E"^{31}_k \cap E"^{31}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{31}_k$ assigned to the demand k in the solution $\mathcal{S}"^{31}$),
e) and there is one demand k from the set of demands $\tilde{K}$ (i.e., $k \in \tilde{K}$ s.t. the demand k selects a slot s' as last-slot in the solution $\mathcal{S}"^{31}$ with $s' \in \{s, ..., s + w_k - 1\}$, i.e., $s' \in S"^{31}_k$ for a demand $k \in \tilde{K}$, and for each $s' \in S"^{31}_{k'}$ for all $k' \in \tilde{K} \setminus \{k\}$ we have $s' \notin \{s, ..., s + w_{k'} - 1\}$,
f) and all the demands in $\tilde{K}$ pass through the edge e in the solution $\mathcal{S}"^{31}$, i.e., $e' \in E"^{31}_k$ for each $k \in \tilde{K}$.

$\mathcal{S}"^{31}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{31}}, z^{\mathcal{S}"^{31}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k\in\tilde{K}} x^k_{e'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. Based on this, we construct a solution $\mathcal{S}^{33}$ derived from the solution $\mathcal{S}"^{31}$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}"^{31}$ (i.e., $E^{33}_k = E"^{31}_k$ for each $k \in K \setminus \tilde{K}$, and $E^{33}_k \neq E"^{31}_k$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E^{33}_k is assigned to each demand $k \in \tilde{K}$ (routing constraint),
b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"^{31}_k$ and $s" \in S"^{31}_{k'}$ with $E^{33}_k \cap E"^{31}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in\tilde{K}, e'\in E^{33}_k} |\{s' \in S"^{31}_k, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k\in K\setminus\tilde{K}, e'\in E"^{31}_k} |\{s' \in S"^{31}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"^{31}_{k"}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{31}_k$ assigned to the demand k in the solution $\mathcal{S}"^{31}$).

The last-slots assigned to the demands $K \backslash \{k\}$ in $\mathcal{S}"^{31}$ remain the same in $\mathcal{S}^{33}$, i.e., $S"^{31}_{k'} = S^{33}_{k'}$ for each demand $k' \in K \setminus \{k\}$, and $S^{33}_k = S"^{31}_k \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{33}$ is clearly feasible given that

a) a feasible path E^{33}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{33}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{33}_k$ with $|S^{33}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{33}_k$ and $s" \in S^{33}_{k'}$ with $E^{33}_k \cap E^{33}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E^{33}_k} |\{s' \in S^{33}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{33}}, z^{\mathcal{S}^{33}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k \in \tilde{K}} x^k_{e'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. We have so

$$\mu x^{\mathcal{S}"^{31}} + \sigma z^{\mathcal{S}"^{31}} = \mu x^{\mathcal{S}^{33}} + \sigma z^{\mathcal{S}^{33}} = \mu x^{\mathcal{S}"^{31}} + \sigma z^{\mathcal{S}"^{31}} + \sigma^k_{s'} - \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E"^{31}_k} \mu^{\tilde{k}}_{e'} + \sum_{\tilde{k} \in \tilde{K}} \sum_{e" \in E^{33}_k} \mu^{\tilde{k}}_{e"}.$$

It follows that $\sigma^k_{s'} = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s, ..., s + w_k - 1\}$ if $k \in \tilde{K}$ given that $\mu^k_{e'} = 0$ for all the demand $k \in K$ and all edges $e' \in E \setminus (E^k_0 \cup E^k_1)$ with $e \neq e'$ if $k \in \tilde{K}$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $s' \notin \{s, ..., s + w_k - 1\}$ if $k \in \tilde{K}$ s.t. we find

$$\sigma^k_{s'} = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } s' \notin \{s, ..., s + w_k - 1\} \text{ if } k \in \tilde{K}.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma^{k'}_{s'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s' \in \{w_{k'}, ..., \bar{s}\} \text{ with } s' \notin \{s, ..., s + w_{k'} - 1\} \text{ if } k' \in \tilde{K}.$$

Consequently, we conclude that

$$\sigma^k_{s'} = 0, \text{ for all } k \in K \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } s' \notin \{s, ..., s + w_k - 1\} \text{ if } k \in \tilde{K}.$$

Let prove that $\sigma^k_{s'}$ for all $k \in \tilde{K}$ and all $s' \in \{s, ..., s + w_k - 1\}$ are equivalents. Consider a demand k' and a slot $s' \in \{s, ..., s + w_{k'} - 1\}$ with $k' \in \tilde{K}$. For that, we consider a solution $\tilde{\mathcal{S}}^{31} = (\tilde{E}^{31}, \tilde{S}^{31})$ in which

a) a feasible path $\tilde{E}^{31}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}^{31}_k$ is assigned to each demand $k \in K$ along each edge $e' \in \tilde{E}^{31}_k$ with $|\tilde{S}^{31}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}^{31}_k$ and $s" \in \tilde{S}^{31}_{k'}$ with $\tilde{E}^{31}_k \cap \tilde{E}^{31}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in \tilde{E}^{31}_k} |\{s' \in \tilde{S}^{31}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s \in S"^{31}_k$ with $\tilde{E}^{31}_k \cap \tilde{E}^{31}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{31}_{k'}$ assigned to the demand k' in the solution $\mathcal{S}"^{31}$),

e) and there is one demand k from the set of demands $\tilde{K}$ (i.e., $k \in \tilde{K}$ s.t. the demand k selects a slot s' as last-slot in the solution $\tilde{\mathcal{S}}^{31}$ with $s' \in \{s, ..., s+w_k-1\}$, i.e., $s' \in \tilde{S}^{31}_k$ for a demand $k \in \tilde{K}$, and for each $s' \in \tilde{S}^{31}_{k'}$ for all $k' \in \tilde{K} \setminus \{k\}$ we have $s' \notin \{s, ..., s+w_{k'}-1\}$,

f) and all the demands in $\tilde{K}$ pass through the edge e in the solution $\tilde{S}^{31}$, i.e., $e' \in \tilde{E}^{31}_k$ for each $k \in \tilde{K}$.

$\tilde{\mathcal{S}}^{31}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{31}}, z^{\tilde{\mathcal{S}}^{31}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k\in\tilde{K}} x^k_{e'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. Based on this, we construct a solution $\mathcal{S}^{34}$ derived from the solution $\tilde{\mathcal{S}}^{31}$ by adding the slot s' as last-slot to the demand k' with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\tilde{\mathcal{S}}^{31}$ (i.e., $E^{34}_k = \tilde{E}^{31}_k$ for each $k \in K \setminus \tilde{K}$, and $E^{34}_k \neq \tilde{E}^{31}_k$ for each $k \in \tilde{K}$), and also the last-slots assigned to the demands $K \setminus \{k, k'\}$ in $\tilde{\mathcal{S}}^{31}$ remain the same in $\mathcal{S}^{34}$, i.e., $\tilde{S}^{31}_{k''} = S^{34}_{k''}$ for each demand $k'' \in K \setminus \{k, k'\}$, and $S^{34}_{k'} = \tilde{S}^{31}_{k'} \cup \{s'\}$ for the demand k', and modifying the last-slots assigned to the demand k by adding a new last-slot $\tilde{s}$ and removing the last slot $s' \in \tilde{S}^{31}_k$ with $s' \in \{s_i + w_k + 1, ..., s_j\}$ and $\tilde{s} \notin \{s_i + w_k + 1, ..., s_j\}$ for the demand k with $k \in \tilde{K}$ s.t. $S^{34}_k = (\tilde{S}^{31}_k \setminus \{s\}) \cup \{\tilde{s}\}$ s.t. $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S^{34}_{k'}$ with $E^{34}_k \cap E^{34}_{k'} \neq \emptyset$. The solution $\mathcal{S}^{34}$ is clearly feasible given that

a) a feasible path E^{34}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{34}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{34}_k$ with $|S^{34}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s'' - w_{k'} + 1, ..., s''\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{34}_k$ and $s'' \in S^{34}_{k'}$ with $E^{34}_k \cap E^{34}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s'' \in \mathbb{S}$ we have $\sum_{k\in K, e' \in E^{34}_k} |\{s' \in S^{34}_k, s'' \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{34}}, z^{\mathcal{S}^{34}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k\in\tilde{K}} x^k_{e'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{31}} + \sigma z^{\tilde{\mathcal{S}}^{31}} = \mu x^{\mathcal{S}^{34}} + \sigma z^{\mathcal{S}^{34}} = \mu x^{\tilde{\mathcal{S}}^{31}} + \sigma z^{\tilde{\mathcal{S}}^{31}} + \sigma^{k'}_{s''} - \sigma^k_{s'} + \sigma^k_{\tilde{s}} \\ - \sum_{k\in\tilde{K}} \sum_{e'\in\tilde{E}^{31}_k} \mu^k_{e'} + \sum_{k\in\tilde{K}} \sum_{e'\in E^{34}_k} \mu^k_{e'}.$$

It follows that $\sigma^{k'}_{s''} = \sigma^k_{s'}$ for demand k' and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $k' \in \tilde{K}$ and $s' \in \{s, ..., s + w_{k'} - 1\}$ given that $\sigma^k_{\tilde{s}} = 0$ for $\tilde{s} \notin \{s, ..., s + w_k - 1\}$ with $k \in \tilde{K}$, and $\mu^k_{e'} = 0$ for all $k \in K$ and all $e' \in E \setminus (E^k_0 \cup E^k_1)$ with $e' \neq e$ if $k \in \tilde{K}$.
Given that the pair (k, k') are chosen arbitrary in the set of demands $\tilde{K}$, we iterate the same procedure for all pairs (k, k') s.t. we find

$$\sigma^k_{s'} = \sigma^{k'}_{s''}, \text{for all pairs } (k, k') \in \tilde{K}$$

with $s' \in \{s, ..., s + w_k - 1\}$ and $s' \in \{s, ..., s + w_{k'} - 1\}$. We re-do the same procedure for each two slots $s, s' \in \{s, ..., s + w_k - 1\}$ for each demand $k \in K$ with $k \in \tilde{K}$ s.t.

$$\sigma^k_{s'} = \sigma^k_{s''}, \text{for all } k \in \tilde{K} \text{ and } s, s' \in \{s, ..., s + w_k - 1\}.$$

Let us prove now that μ^k_e for all $k \in K$ with $k \in \tilde{K}$ are equivalents. For that, we consider a solution $\mathcal{S}^{35} = (E^{35}, S^{35})$ defined as below

a) a feasible path E^{35}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{35} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{35}$ with $|S_k^{35}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{35}$ and $s' \in S_{k'}^{35}$ with $E_k^{35} \cap E_{k'}^{35} \neq \emptyset$ (non-overlapping constraint),

d) and there is one demand k from the set of demands $\tilde{K}$ (i.e., $k \in \tilde{K}$ s.t. the demand k pass through the edge e in the solution $\mathcal{S}^{35}$, i.e., $e \in E_k^{35}$ for a demand $k \in \tilde{K}$, and $e \notin E_{k'}^{35}$ for all $k' \in \tilde{K} \setminus \{k\}$,

e) and all the demands in $\tilde{K}$ share the slot s over the edge e in the solution $\mathcal{S}^{35}$, i.e., $\{s_i + w_k + 1, ..., s_j\} \cap S_k^{35} \neq \emptyset$ for each $k \in \tilde{K}$.

Obviously, $\mathcal{S}^{35}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{35}}, z^{\mathcal{S}^{35}})$ is belong to $P(G, K, \mathbb{S})$ and then to $F_{\tilde{K}}^{e,s}$ given that it is composed by $\sum_{k \in \tilde{K}} x_e^k + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k = 1$.
Consider now a demand k' in $\tilde{K}$ s.t. $e \notin E_{k'}^{35}$. For that, we consider a solution $\tilde{\mathcal{S}}^{35} = (\tilde{E}^{35}, \tilde{S}^{35})$ in which

a) a feasible path $\tilde{E}_k^{35}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}_k^{35}$ is assigned to each demand $k \in K$ along each edge $e \in \tilde{E}_k^{35}$ with $|\tilde{S}_k^{35}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}_k^{35}$ and $s" \in \tilde{S}_{k'}^{35}$ with $\tilde{E}_k^{35} \cap \tilde{E}_{k'}^{35} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in \tilde{E}_k^{35}} |\{s' \in \tilde{S}_k^{35}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s' \in S_k^{35}$ with $\tilde{E}_k^{35} \cap \tilde{E}_{k'}^{35} \neq \emptyset$,

e) and there is one demand k from the set of demands $\tilde{K}$ (i.e., $k \in \tilde{K}$ s.t. the demand k pass through the edge e in the solution $\tilde{S}^{35}$, i.e., $e \in \tilde{E}_k^{35}$ for a demand $k \in \tilde{K}$, and $e \notin \tilde{E}_{k'}^{35}$ for all $k' \in \tilde{K} \setminus \{k\}$,

f) and all the demands in $\tilde{K}$ share the slot s over the edge e in the solution $\tilde{S}^{35}$, i.e., $\{s, ..., s + w_k - 1\} \cap \tilde{S}_k^{35} \neq \emptyset$ for each $k \in \tilde{K}$.

$\tilde{\mathcal{S}}^{35}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{35}}, z^{\tilde{\mathcal{S}}^{35}})$ is belong to F and then to $F_{\tilde{K}}^{e,s}$ given that it is composed by $\sum_{k \in \tilde{K}} x_e^k + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k = 1$. Based on this, we derive a solution $\mathcal{S}"^{36} = (E"^{36}, S"^{36})$ from the solution $\tilde{\mathcal{S}}^{35}$ by

a) the paths assigned to the demands $K \setminus \{k, k'\}$ in $\tilde{\mathcal{S}}^{35}$ remain the same in $\mathcal{S}"^{36}$ (i.e., $E"^{36}_{k"} = \tilde{E}_{k"}^{35}$ for each $k" \in K \setminus \{k, k'\}$),

b) without modifying the last-slots assigned to the demands K in $\tilde{\mathcal{S}}^{35}$, i.e., $\tilde{S}_k^{35} = S"^{36}_k$ for each demand $k \in K$,

c) modifying the path assigned to the demand k' in $\tilde{\mathcal{S}}^{35}$ from $\tilde{E}_{k'}^{35}$ to a path $E"^{36}_{k'}$ passed through the edge e (i.e., $e \in E"^{36}_{k'}$) with $k' \in \tilde{K}$ s.t. $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and each $s' \in \tilde{S}_{k'}^{35}$ and each $s' \in \tilde{S}_k^{35}$ with $\tilde{E}_k^{35} \cap E"^{36}_{k'} \neq \emptyset$,

d) modifying the path assigned to the demand k in $\tilde{\mathcal{S}}^{35}$ with $e \in \tilde{E}_k^{35}$ and $k \in \tilde{K}$ from $\tilde{E}_k^{35}$ to a path $E"^{36}_k$ without passing through the edge e (i.e., $e \notin E"^{36}_k$) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k"} + 1, ..., s'\} = \emptyset$ for each $k" \in K \setminus \{k, k'\}$ and each $s' \in \tilde{S}_k^{35}$ and each $s' \in \tilde{S}_{k"}^{35}$ with $\tilde{E}_{k"}^{35} \cap E"^{36}_k \neq \emptyset$, and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $s' \in \tilde{S}_k^{35}$ and each $s' \in \tilde{S}_{k'}^{35}$ with $E"^{36}_{k"} \cap E"^{36}_k \neq \emptyset$.

The solution $\mathcal{S}"^{36}$ is feasible given that

a) a feasible path $E"^{36}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^{36}_k$ is assigned to each demand $k \in K$ along each edge $e \in E"^{36}_k$ with $|S"^{36}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{36}_k$ and $s" \in S"^{36}_{k'}$ with $E"^{36}_k \cap E"^{36}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E"^{36}_k} |\{s' \in S"^{36}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}"^{36}}, z^{\mathcal{S}"^{36}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k \in \tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. We then obtain that

$$\mu x^{\tilde{\mathcal{S}}^{35}} + \sigma z^{\tilde{\mathcal{S}}^{35}} = \mu x^{\mathcal{S}^{36}} + \sigma z^{\mathcal{S}^{36}} = \mu x^{\tilde{\mathcal{S}}^{35}} + \sigma z^{\tilde{\mathcal{S}}^{35}} + \mu^{k'}_e - \mu^k_e$$
$$+ \sum_{e" \in E"^{36}_{k'} \setminus \{e\}} \mu^{k'}_{e"} - \sum_{e" \in \tilde{E}^{35}_{k'}} \mu^{k'}_{e"} + \sum_{e" \in E"^{36}_k} \mu^k_{e"} - \sum_{e" \in \tilde{E}^{35}_k \setminus \{e\}} \mu^k_{e"}.$$

It follows that $\mu^{k'}_e = \mu^k_e$ for demand k' and a edge $e' \in E \setminus (E^{k'}_0 \cup E^{k'}_1)$ with $v_{k',e'} \in \tilde{K}$ given that $\mu^k_{e"} = 0$ for all $k \in K$ and all $e" \in E \setminus (E^k_0 \cup E^k_1)$ with $k \in \tilde{K}$.
Given that the pair (k, k') are chosen arbitrary in the set of demands $\tilde{K}$, we iterate the same procedure for all pairs (k, k') s.t. we find

$$\mu^k_e = \mu^{k'}_e, \text{for all pairs } (k, k') \in \tilde{K}.$$

Furthermore, let prove that all $\sigma^k_{s'}$ and μ^k_e are equivalents for all $k \in \tilde{K}$ and $s' \in \{s, ..., s + w_k - 1\}$. For that, we consider for each demand k' with $k' \in \tilde{K}$, a solution $\mathcal{S}^{37} = (E^{37}, S^{37})$ derived from the solution $\tilde{\mathcal{S}}^{35}$ as below

a) the paths assigned to the demands $K \setminus \{k'\}$ in $\tilde{\mathcal{S}}^{35}$ remain the same in $\mathcal{S}^{37}$ (i.e., $E^{37}_{k"} = \tilde{E}^{35}_{k"}$ for each $k" \in K \setminus \{k'\}$),

b) without modifying the last-slots assigned to the demands $K \setminus \{k\}$ in $\tilde{\mathcal{S}}^{35}$, i.e., $\tilde{S}^{35}_{k"} = S^{37}_{k"}$ for each demand $k" \in K \setminus \{k\}$,

c) modifying the set of last-slots assigned to the demand k' in $\tilde{\mathcal{S}}^{35}$ from $\tilde{S}^{35}_{k'}$ to $S^{37}_{k'}$ s.t. $S^{37}_{k'} \cap \{s, ..., s + w_{k'} - 1\} = \emptyset$.

Hence, there are $|\tilde{K}| - 1$ demands from $\tilde{K}$ that share the slot s over the edge e (i.e., all the demands in $\tilde{K} \setminus \{k'\}$), and two demands $\{k, k'\}$ from $\tilde{K}$ that use the edge e in the solution $\mathcal{S}^{37}$. The solution $\mathcal{S}^{37}$ is then feasible given that

a) a feasible path E^{37}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{37}_k is assigned to each demand $k \in K$ along each edge $e \in E^{37}_k$ with $|S^{37}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{37}_k$ and $s" \in S^{37}_{k'}$ with $E^{37}_k \cap E^{37}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E^{37}_k} |\{s' \in S^{37}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\sum_{k \in \tilde{K}} |E^{37}_k \cap \{e\}| + |S^{37}_k \cap \{s, ..., s + w_k - 1\}| = |\tilde{K}| + 1$.

The corresponding incidence vector $(x^{\mathcal{S}^{37}}, z^{\mathcal{S}^{37}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k\in\tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. We then obtain that

$$\mu x^{\tilde{\mathcal{S}}^{35}} + \sigma z^{\tilde{\mathcal{S}}^{35}} = \mu x^{\mathcal{S}^{37}} + \sigma z^{\mathcal{S}^{37}} = \mu x^{\tilde{\mathcal{S}}^{35}} + \sigma z^{\tilde{\mathcal{S}}^{35}} + \mu^{k'}_e - \sigma^{k'}_{s'} + \sum_{e"\in E^{37}_{k'}\setminus\{e\}} \mu^{k'}_{e"} - \sum_{e"\in \tilde{E}^{35}_{k'}} \mu^{k'}_{e"}.$$

It follows that $\mu^{k'}_e = \sigma^{k'}_{s'}$ for demand k' and slot $s' \in \{s, ..., s + w_{k'} - 1\}$ given that $\mu^k_{e"} = 0$ for all $k \in K$ and all $e" \in E \setminus (E^k_0 \cup E^k_1)$ with $e \neq e"$ if $k \in \tilde{K}$. Moreover, by doing the same thing over all slots $s' \in \{s, ..., s + w_{k'} - 1\}$, we found that

$$\mu^{k'}_e = \sigma^{k'}_{s'}, \text{for all } s' \in \{s, ..., s + w_{k'} - 1\}.$$

Given that k' is chosen arbitrarily in $\tilde{K}$, we iterate the same procedure for all $k \in \tilde{K}$ to show that

$$\mu^k_e = \sigma^k_{s'}, \text{for all } k \in \tilde{K} \text{ and all } s' \in \{s, ..., s + w_k - 1\}.$$

Based on this, and given that all μ^k_e are equivalents for all $k \in \tilde{K}$, and that $\sigma^k_{s'}$ are equivalents for all $k \in \tilde{K}$ and $s' \in \{s, ..., s + w_{k'} - 1\}$, we obtain that

$$\mu^k_e = \sigma^{k'}_{s'}, \text{ for all } k, k' \in \tilde{K} \text{ and all } s' \in \{s, ..., s + w_{k'} - 1\}.$$

Consequently, we conclude that

$$\mu^k_e = \sigma^{k'}_{s'} = \rho, \text{ for all } k, k' \in \tilde{K} \text{ and all } s' \in \{s, ..., s + w_{k'} - 1\}.$$

On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu^{k'}_{e'} &= \gamma^{k',e'}_1, \text{ for all } k' \in K \text{ and all } e' \in E^{k'}_0, \\ \mu^{k'}_{e'} &= \gamma^{k',e'}_2, \text{ for all } k' \in K \text{ and all } e' \in E^{k'}_1, \\ \sigma^{k'}_{s'} &= \gamma^{k',s'}_3, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k' \in K$ and $e' \in E$

$$\mu^{k'}_{e'} = \begin{cases} \gamma^{k',e'}_1, \text{ if } e' \in E^k_0, \\ \gamma^{k',e'}_2, \text{ if } e' \in E^k_1, \\ \rho, \text{ if } k' \in \tilde{K} \text{ and } e' = e, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s' \in \mathbb{S}$

$$\sigma^k_{s'} = \begin{cases} \gamma^{k,s'}_3, \text{if } s' \in \{1, ..., w_k - 1\} \\ \rho, \text{if } k \in \tilde{K} \text{ and } s' \in \{s, ..., s + w_k - 1\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{k\in\tilde{K}} \rho\alpha^k_e + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} \rho\beta^k_{s'} + \gamma Q.$ □

In what follows, we present several valid inequalities for $P(G, K, \mathbb{S})$.

2.4 Valid Inequalities and Facets

We start this section by introducing some classes of valid inequalities that can be defined using Chvatal-Gomory procedures.

2.4.1 Edge-Slot-Assignment Inequalities

Proposition 2.4.1. *Consider an edge $e \in E$ with $K_e \neq \emptyset$. Let s be a slot in $\mathbb{S}$. Then, the inequality*

$$\sum_{k" \in K_e} \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"} \leq 1, \tag{2.23}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. Inequality (2.23) ensures that the set of demands K_e cannot share the slot s over the edge e, which means that the slot s is assigned to at most one demand k from K_e over edge e. □

Inspiring from the inequality (2.23), we define the following inequality based on the non-overlapping inequality (2.6) and using the Chvatal-Gomory procedure.

Proposition 2.4.2. *Consider an edge $e \in E$. Let s be a slot in $\mathbb{S}$. Consider a triplet of demands $k, k', k" \in K$ with $e \notin E_0^k \cap E_0^{k'} \cap E_0^{k"}$, $(k, k') \notin K_c^e, (k, k") \notin K_c^e$, and $(k', k") \notin K_c^e$. Then, the inequality*

$$x_e^k + x_e^{k'} + x_e^{k"} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} + \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"} \leq 4, \tag{2.24}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. Consider an edge $e \in E$. Let s be a slot in $\mathbb{S}$. Inequality (2.24) ensures that if the three demands $k, k', k"$ pass through edge e, they cannot share the slot s.
Let's us show that the inequality (2.24) can be seen as Chvatal-Gomory cuts using Chvatal-Gomory procedure. We know from (2.26) that

$$x_e^k + x_e^{k'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} \leq 3,$$

$$x_e^k + x_e^{k"} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"} \leq 3,$$

$$x_e^{k'} + x_e^{k"} + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} + \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"} \leq 3.$$

By adding the three previous inequalities, we get the following inequality

$$2x_e^k + 2x_e^{k'} + 2x_e^{k"} + 2\sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + 2\sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} + 2\sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"} \leq 9.$$

By dividing the two sides of the previous inequality by 2, we obtain that

$$x_e^k + x_e^{k'} + x_e^{k"} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} + \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"} \leq \left\lfloor \frac{9}{2} \right\rfloor.$$

As a result,

$$x_e^k + x_e^{k'} + x_e^{k"} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} + \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"} \leq 4.$$

We conclude at the end that the inequality (2.24) is valid for $P(G, K, \mathbb{S})$. □

The inequality (2.24) can then be generalized for any subset of demand $\tilde{K} \subseteq K$ under certain conditions.

Proposition 2.4.3. *Consider an edge $e \in E$, and a slot s in $\mathbb{S}$. Let $\tilde{K}$ be a subset of demands of K with $e \notin E_0^k$ for each demand $k \in \tilde{K}$, $(k, k') \notin K_c^e$ for each pair of demands (k, k') in $\tilde{K}$, and $\sum_{k \in \tilde{K}} w_k \leq \bar{s} - \sum_{k" \in K_e \setminus \tilde{K}} w_{k"}$. Then, the inequality*

$$\sum_{k \in \tilde{K}} x_e^k + \sum_{k' \in \tilde{K}} \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} \leq |\tilde{K}| + 1, \tag{2.25}$$

is valid for $P(G, K, \mathbb{S})$.

Let $\binom{n}{k}$ denote the total number of possibilities to choose a k element in a set of n elements.

Proof. Inequality (2.25) ensures that if the demands $k \in \tilde{K}$ pass through edge e, they cannot share the slot s. For this, we use the Chvatal-Gomory and recurrence procedures to prove that (2.25) is valid for $P(G, K, \mathbb{S})$. For any subset of demands $\tilde{K} \subseteq K$ with $e \notin E_0^k$ for each demand $k \in \tilde{K}$, by recurrence procedures we get that for all demands $K' \subseteq \tilde{K}$ with $|K'| = |\tilde{K}| - 1$

$$\sum_{k \in K'} x_e^k + \sum_{k \in K'} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq |K'| + 1.$$

By adding the previous inequalities for all subset of demands $K' \subseteq \tilde{K}$ with $|K'| = |\tilde{K}| - 1$

$$\sum_{\substack{K' \subseteq \tilde{K} \\ |K'|=|\tilde{K}|-1}} \sum_{k \in K'} x_e^k + \sum_{\substack{K' \subseteq \tilde{K} \\ |K'|=|\tilde{K}|-1}} \sum_{k \in K'} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq \sum_{\substack{K' \subseteq \tilde{K} \\ |K'|=|\tilde{K}|-1}} (|K'| + 1).$$

Note that for each $k \in \tilde{K}$, the variable x_e^k and the sum $\sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k$ appear $(\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1)$ times in the previous sum. This implies that

$$\sum_{k \in \tilde{K}} (\binom{|\tilde{K}|}{|\tilde{K}| - 1} - 1) x_e^k + \sum_{k \in \tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} (\binom{|\tilde{K}|}{|\tilde{K}| - 1} - 1) z_{s'}^k \leq \binom{|\tilde{K}|}{|\tilde{K}| - 1} (|K'| + 1)$$

Given that $|K'| = |\tilde{K}| - 1$, this is equivalent to say that

$$\sum_{k\in\tilde{K}}(\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1)x_e^k + \sum_{k\in\tilde{K}}\sum_{s'=s}^{\min(s+w_k-1,\bar{s})}(\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1)z_{s'}^k \leq \binom{|\tilde{K}|}{|\tilde{K}|-1}|\tilde{K}|$$

Moreover, and taking into account that $(\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1) = |\tilde{K}| - 1$, we found that

$$\sum_{k\in\tilde{K}}(|\tilde{K}|-1)x_e^k + \sum_{k\in\tilde{K}}\sum_{s'=s}^{\min(s+w_k-1,\bar{s})}(|\tilde{K}|-1)z_{s'}^k \leq |\tilde{K}|^2$$

By dividing the two sides of the previous sum by $|\tilde{K}| - 1$, we have

$$\begin{aligned}\sum_{k\in\tilde{K}}x_e^k + \sum_{k\in\tilde{K}}\sum_{s'=s}^{\min(s+w_k-1,\bar{s})}z_{s'}^k \leq \left\lfloor\frac{|\tilde{K}|^2}{|\tilde{K}|-1}\right\rfloor &\Rightarrow \sum_{k\in\tilde{K}}x_e^k + \sum_{k\in\tilde{K}}\sum_{s'=s}^{\min(s+w_k-1,\bar{s})}z_{s'}^k \leq \left\lfloor|\tilde{K}|\frac{|\tilde{K}|}{|\tilde{K}|-1}\right\rfloor\\ &\Rightarrow \sum_{k\in\tilde{K}}x_e^k + \sum_{k\in\tilde{K}}\sum_{s'=s}^{\min(s+w_k-1,\bar{s})}z_{s'}^k \leq \left\lfloor|\tilde{K}|\frac{|\tilde{K}|-1+1}{|\tilde{K}|-1}\right\rfloor\\ &\Rightarrow \sum_{k\in\tilde{K}}x_e^k + \sum_{k\in\tilde{K}}\sum_{s'=s}^{\min(s+w_k-1,\bar{s})}z_{s'}^k \leq \left\lfloor|\tilde{K}|\frac{|\tilde{K}|-1}{|\tilde{K}|-1} + \frac{|\tilde{K}|}{|\tilde{K}|-1}\right\rfloor.\end{aligned}$$

After some simplifications, we found that

$$\sum_{k\in\tilde{K}}x_e^k+\sum_{k\in\tilde{K}}\sum_{s'=s}^{\min(s+w_k-1,\bar{s})}z_{s'}^k \leq |\tilde{K}|+\left\lfloor\frac{|\tilde{K}|}{|\tilde{K}|-1}\right\rfloor \Rightarrow \sum_{k\in\tilde{K}}x_e^k+\sum_{k\in\tilde{K}}\sum_{s'=s}^{\min(s+w_k-1,\bar{s})}z_{s'}^k \leq |\tilde{K}|+1,$$

given that $\left\lfloor\frac{|\tilde{K}|}{|\tilde{K}|-1}\right\rfloor = 1.$
We conclude at the end that the inequality (2.25) is valid for $P(G, K, \mathbb{S})$. □

The inequality (2.25) can be strengthened as follows. For that, and using the inequalities (2.23) and (2.6), we first show that the inequality (2.6) can be strengthened without modifying its right-hand side as follows.

Proposition 2.4.4. *Consider an edge $e \in E$. Let s be a slot in $\mathbb{S}$. Consider a pair of demands $k, k' \in K$ with $e \notin E_0^k \cap E_0^{k'}$ and $(k, k') \notin K_c^e$. Then, the inequality*

$$x_e^k + x_e^{k'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})}z_{s'}^k + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})}z_{s'}^{k'} + \sum_{k"\in K_e\setminus\{k,k'\}}\sum_{s'=s}^{\min(s+w_{k"}-1,\bar{s})}z_{s'}^{k"} \leq 3, \quad (2.26)$$

is valid for $P(G, K, \mathbb{S})$.

Proof. Consider an edge $e \in E$, and a pair of demands $k, k' \in K$. Let s be a slot in $\mathbb{S}$. Inequality (2.26) ensures that if the two demands k, k' pass through edge e, they cannot share the slot s with the set of demands in $K_e \setminus \{k, k'\}$. This can be seen as a partcular case for the inequality (2.23) induced by subset of demands $\tilde{K} = \{k, k'\} \cup K_e$.

We start the proof by assuming that the inequality (2.26) is not valid for $P(G, K, \mathbb{S})$. It follows that there exists a C-RSA solution S in which $s \notin S_{k"}$ for each demand $k" \in K_e \setminus \{k, k'\}$ s.t.

$$x_e^k(S) + x_e^{k'}(S) + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k(S) + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'}(S) + \sum_{k"\in K_e\setminus\{k,k'\}} \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"}(S) > 3.$$

Since $s \notin S_{k"}$ for each demand $k" \in K_e\setminus\{k, k'\}$ this means that $\sum_{k"\in K_e\setminus\{k,k'\}} \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"}(S) = 0$, and taking into account that $x_e^k(S) \leq 1$, $x_e^{k'}(S) \leq 1$, $\sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k(S) \leq 1$, and $\sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'}(S) \leq 1$, it follows that

$$x_e^k(S) + x_e^{k'}(S) + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k(S) + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'}(S) \leq 3,$$

which contradicts the inequality (2.26) for $\tilde{K} = \{k, k'\}$, and also what we supposed before, i.e., $x_e^k(S) + x_e^{k'}(S) + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k(S) + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'}(S) > 3$.

Hence $|E_k \cap \{e\}| + |E_{k'} \cap \{e\}| + |S_k \cap \{s\}| + |S_{k'} \cap \{s\}| + \sum_{k"\in K_e} |S_{k"} \cap \{s\}| \leq 3$.

□

Let's us generalize the inequality (2.26) for each edge e and all slot $s \in \mathbb{S}$ and any subset of demand $\tilde{K} \subseteq K$ under certain conditions.

Proposition 2.4.5. *Consider an edge $e \in E$, and a slot s in $\mathbb{S}$. Let $\tilde{K}$ be a subset of demands of K with $e \notin E_0^k$ for each demand $k \in \tilde{K}$, $(k, k') \notin K_c^e$ for each pair of demands (k, k') in $\tilde{K}$, and $\sum_{k\in\tilde{K}} w_k \leq \bar{s} - \sum_{k"\in K_e\setminus\tilde{K}} w_{k"}$. Then, the inequality*

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{k'\in K_e\setminus\tilde{K}} \sum_{s"=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s"}^{k'} \leq |\tilde{K}| + 1, \tag{2.27}$$

is valid for $P(G, K, \mathbb{S})$.

This can be seen as a strengthened version of the inequality (2.26).

Proof. Inequality (2.27) ensures that if the demands $k \in \tilde{K}$ pass through edge e, they cannot share the slot s with the set of demands in $K_e \setminus \tilde{K}$. This can be seen be a particular case the inequality (2.25) induced by $\tilde{K} \cup K_e$ for the slot s over the edge e.
We use the Chvatal-Gomory and recurrence procedures to prove that (2.27) is valid for $P(G, K, \mathbb{S})$. For any subset of demands $\tilde{K} \subseteq K$ with $e \notin E_0^k$ for each demand $k \in \tilde{K}$, by recurrence procedures we get that for all demands $K' \subseteq \tilde{K}$ with $|K'| = |\tilde{K}| - 1$

$$\sum_{k\in K'} x_e^k + \sum_{k\in K'} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{k"\in K_e\setminus K'} \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k'} \leq |K'| + 1.$$

By adding the previous inequalities for all $K' \subseteq \tilde{K}$ with $|K'| = |\tilde{K}| - 1$

$$\sum_{\substack{K'\subseteq\tilde{K}\\|K'|=|\tilde{K}|-1}} \sum_{k\in K'} x_e^k + \sum_{\substack{K'\subseteq\tilde{K}\\|K'|=|\tilde{K}|-1}} \sum_{k\in K'} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{\substack{K'\subseteq\tilde{K}\\|K'|=|\tilde{K}|-1}} \sum_{k"\in K_e\setminus\tilde{K}} \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"}$$
$$\leq \sum_{\substack{K'\subseteq\tilde{K}\\|K'|=|\tilde{K}|-1}} (|K'|+1).$$

Note that for each demand $k \in \tilde{K}$, the variable x_e^k and sum $\sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k$ appear $(\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1)$ times in the previous sum. It follows that

$$\sum_{k\in\tilde{K}} (\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1) x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} (\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1) z_{s'}^k$$
$$+ \sum_{k"\in K_e\setminus\tilde{K}} \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} \binom{|\tilde{K}|}{|\tilde{K}|-1} z_{s"}^{k"} \leq \binom{|\tilde{K}|}{|\tilde{K}|-1} (|K'|+1).$$

Given that $|K'| + 1 = |\tilde{K}|$ and $(\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1) = |\tilde{K}| - 1$, this means that

$$\sum_{k\in\tilde{K}} (|\tilde{K}|-1) x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} (|\tilde{K}|-1) z_{s'}^k + \sum_{k"\in K_e\setminus\tilde{K}} \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} |\tilde{K}| z_{s"}^{k"} \leq |\tilde{K}|^2.$$

By dividing the two sides of the previous sum by $|\tilde{K}| - 1$, we found that

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{k"\in K_e\setminus\tilde{K}} \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} \left\lfloor \frac{|\tilde{K}|}{|\tilde{K}|-1} \right\rfloor z_{s"}^{k"} \leq \left\lfloor \frac{|\tilde{K}|^2}{|\tilde{K}|-1} \right\rfloor.$$

After some simplifications, we found that

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}\cup(K_e\setminus\tilde{K})} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq |\tilde{K}| + \left\lfloor \frac{|\tilde{K}|}{|\tilde{K}|-1} \right\rfloor \Rightarrow \sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}\cup(K_e\setminus\tilde{K})} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq |\tilde{K}|+1,$$

given that $\left\lfloor \frac{|\tilde{K}|}{|\tilde{K}|-1} \right\rfloor = 1$. We conclude at the end that the inequality (2.27) is valid for $P(G, K, \mathbb{S})$. $\square$

Theorem 2.4.1. *Consider an edge $e \in E$, and a slot $s \in \mathbb{S}$. Let $\tilde{K}$ be a subset of demands in K with $|C| \geq 3$, and $\sum_{k\in\tilde{K}} w_k \leq \bar{s} - \sum_{k'\in K_e\setminus\tilde{K}} w_{k'}$. Then, the inequality (2.25) is facet defining for $P(G, K, \mathbb{S})$ iff $K_e \setminus \tilde{K} = \emptyset$, and there does not exist an interval of contiguous slots $I = [s_i, s_j]$ s.t.*

a) $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ *for each demand* $k \in \tilde{K}$, *b) and* $s \in \{s_i + \max\limits_{k' \in \tilde{K}} w_k - 1, ..., s_j - \max\limits_{k \in \tilde{K}} w_k + 1\}$,

c) and $w_k + w_{k'} \geq |I| + 1$ *for each* $k, k' \in \tilde{K}$,

d) and $2w_k \geq |I| + 1$ *for each* $k \in \tilde{K}$.

Proof. **Neccessity.**
If $K_e \setminus \tilde{K} \neq \emptyset$, then the inequality (2.25) is dominated by the inequality (2.27) without changing its right-hand side. Moreover, if there exists an interval of contiguous slots $I = [s_i, s_j]$ s.t.

a) $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ for each demand $k \in \tilde{K}$,

b) and $s \in \{s_i + \max\limits_{k' \in \tilde{K}} w_k - 1, ..., s_j - \max\limits_{k \in \tilde{K}} w_k + 1\}$,

c) and $w_k + w_{k'} \geq |I| + 1$ for each $k, k' \in \tilde{K}$,

d) and $2w_k \geq |I| + 1$ for each $k \in \tilde{K}$.

Then the inequality (2.25) is dominated by the inequality (2.32). Hence, the inequality (2.25) is not facet defining for $P(G, K, \mathbb{S})$.
Sufficiency.
Let $F^{e,s}_{\tilde{K}}$ denote the face induced by the inequality (2.25), which is given by

$$F^{e,s}_{\tilde{K}} = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{k \in \tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = |\tilde{K}| + 1\}.$$

In order to prove that inequality $\sum_{k \in \tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} \leq |\tilde{K}| + 1$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that $F^{e,s}_{\tilde{K}}$ is a proper face, and $F^{e,s}_{\tilde{K}} \neq P(G, K, \mathbb{S})$.
We construct a solution $\mathcal{S}^{38} = (E^{38}, S^{38})$ as below

a) a feasible path E^{38}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{38}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{38}_k$ with $|S^{38}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{38}_k$ and $s' \in S^{38}_{k'}$ with $E^{38}_k \cap E^{38}_{k'} \neq \emptyset$ (non-overlapping constraint),

d) and there is one demand k from the set of demands $\tilde{K}$ (i.e., $k \in \tilde{K}$ s.t. the demand k selects a slot s' as last-slot in the solution $\mathcal{S}^{38}$ with $s' \in \{s, ..., s + w_k - 1\}$, i.e., $s' \in S^{38}_k$ for a demand $k \in \tilde{K}$, and for each $s' \in S^{38}_{k'}$ for all $k' \in \tilde{K} \setminus \{k\}$ we have $s' \notin \{s, ..., s + w_{k'} - 1\}$,

e) and all the demands in $\tilde{K}$ pass through the edge e in the solution $\mathcal{S}^{38}$, i.e., $e \in E^{38}_k$ for each $k \in \tilde{K}$.

Obviously, $\mathcal{S}^{38}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{38}}, z^{\mathcal{S}^{38}})$ is belong to $P(G, K, \mathbb{S})$ and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k \in \tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. As a result, $F^{e,s}_{\tilde{K}}$ is not empty (i.e., $F^{e,s}_{\tilde{K}} \neq \emptyset$). Furthermore, given that $s \in \mathbb{S}$, this means that there exists at least one feasible slot assignment S_k for each demands k in $\tilde{K}$ with $S_k \cap \{s, ..., s + w_k - 1\} = \emptyset$. Hence, $F^{e,s}_{\tilde{K}} \neq P(G, K, \mathbb{S})$.

We denote the inequality $\sum_{k \in \tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} \leq |\tilde{K}| + 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F^{e,s}_{\tilde{K}} \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$

(s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k\in K}|E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k\in K}|E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k\in K}(w_k-1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma^k_{s'} = 0$ for all demands $k \in K$ and all slots $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s, ..., s + w_k - 1\}$ if $k \in \tilde{K}$,

b) and $\sigma^k_{s'}$ are equivalents for all $k \in \tilde{K}$ and all $s' \in \{s, ..., s + w_k - 1\}$,

c) and $\mu^k_{e'} = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E^k_0 \cup E^k_1)$ with $e \neq e'$ if $k \in \tilde{K}$,

d) and all μ^k_e are equivalents for the set of demands in $\tilde{K}$,

e) and $\sigma^k_{s'}$ and μ^k_e are equivalents for all $k \in \tilde{K}$ and all $s' \in \{s, ..., s + w_k - 1\}$.

We first show that $\mu^k_{e'} = 0$ for each edge $e' \in E \setminus (E^k_0 \cup E^k_1)$ for each demand $k \in K$ with $e \neq e'$ if $k \in \tilde{K}$. Consider a demand $k \in K$ and an edge $e' \in E \setminus (E^k_0 \cup E^k_1)$ with $e \neq e'$ if $k \in \tilde{K}$. For that, we consider a solution $\mathcal{S}'^{38} = (E'^{38}, S'^{38})$ in which

a) a feasible path E'^{38}_k is assigned to each demand $k \in K$ (routing constraint),

b) and a set of last-slots S'^{38}_k is assigned to each demand $k \in K$ along each edge $e' \in E'^{38}_k$ with $|S'^{38}_k| \geq 1$ (contiguity and continuity constraints),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S'^{38}_k$ and $s" \in S'^{38}_{k'}$ with $E'^{38}_k \cap E'^{38}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e'\in E'^{38}_k} |\{s' \in S'^{38}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and there is one demand k from the set of demands $\tilde{K}$ (i.e., $k \in \tilde{K}$ s.t. the demand k selects a slot s' as last-slot in the solution $\mathcal{S}'^{38}$ with $s' \in \{s, ..., s+w_k-1\}$, i.e., $s' \in S'^{38}_k$ for a demand $k \in \tilde{K}$, and for each $s' \in S'^{38}_{k'}$ for all $k' \in \tilde{K} \setminus \{k\}$ we have $s' \notin \{s, ..., s + w_{k'} - 1\}$,

e) and the edge e' is not non-compatible edge with the selected edges $e" \in E'^{38}_k$ of demand k in the solution $\mathcal{S}'^{38}$, i.e., $\sum_{e"\in E'^{38}_k} l_{e"} + l_{e'} \leq \bar{l}_k$. As a result, $E'^{38}_k \cup \{e'\}$ is a feasible path for the demand k,

f) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s \in S'^{38}_k$ and $s' \in S'^{38}_{k'}$ with $(E'^{38}_k \cup \{e'\}) \cap E'^{38}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e' in the set of edges E'^{38}_k selected to route the demand k in the solution $\mathcal{S}'^{38}$),

g) and all the demands in $\tilde{K}$ pass through the edge e in the solution $\mathcal{S}'^{38}$, i.e., $e \in E'^{38}_k$ for each $k \in \tilde{K}$.

$\mathcal{S}'^{38}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{38}}, z^{\mathcal{S}'^{38}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k\in\tilde{K}} x^k_{e'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. Based on this, we derive a solution $\mathcal{S}^{39}$ obtained from the solution $\mathcal{S}'^{38}$ by adding an unused edge $e' \in E \setminus (E^k_0 \cup E^k_1)$ for the routing of demand k in K in the solution $\mathcal{S}^{38}$ which means that $E^{39}_k = E'^{38}_k \cup \{e'\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^{38}$ remain the same in the solution $\mathcal{S}^{39}$, i.e., $S^{39}_k = S'^{38}_k$ for each $k \in K$, and $E^{39}_{k'} = E'^{38}_{k'}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{39}$ is clearly feasible given that

a) and a feasible path E^{39}_k is assigned to each demand $k \in K$ (routing constraint),

b) and a set of last-slots S^{39}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{39}_k$ with $|S^{39}_k| \geq 1$ (contiguity and continuity constraints),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{39}_k$ and $s" \in S^{39}_{k'}$ with $E^{39}_k \cap E^{39}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e'\in E^{39}_k} |\{s' \in S^{39}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{39}}, z^{\mathcal{S}^{39}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k\in\tilde{K}} x^k_{e'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. It follows that

$$\mu x^{\mathcal{S}'^{38}} + \sigma z^{\mathcal{S}'^{38}} = \mu x^{\mathcal{S}^{39}} + \sigma z^{\mathcal{S}^{39}} = \mu x^{\mathcal{S}'^{38}} + \mu^k_{e'} + \sigma z^{\mathcal{S}'^{38}}.$$

As a result, $\mu^k_{e'} = 0$ for demand k and an edge e'.
As e' is chosen arbitrarily for the demand k with $e \notin E^k_0 \cup E^k_1$ and $e \neq e'$ if $k \in \tilde{K}$, we iterate the same procedure for all $e" \in E \setminus (E^k_0 \cup E^k_1 \cup \{e'\})$ with $e \neq e"$ if $k \in \tilde{K}$. We conclude that for the demand k

$$\mu^k_{e'} = 0, \text{ for all } e' \in E \setminus (E^k_0 \cup E^k_1) \text{ with } e \neq e' \text{ if } k \in \tilde{K}.$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E^k_0 \cup E^k_1)$. We conclude at the end that

$$\mu^k_{e'} = 0, \text{ for all } k \in K \text{ and all } e' \in E \setminus (E^k_0 \cup E^k_1) \text{ with } e \neq e' \text{ if } k \in \tilde{K}.$$

Let's us show that $\sigma^k_{s'} = 0$ for all $k \in K$ and all $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s, ..., s + w_k - 1\}$ if $k \in \tilde{K}$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s, ..., s + w_k - 1\}$ if $k \in \tilde{K}$. For that, we consider a solution $\mathcal{S}"^{38} = (E"^{38}, S"^{38})$ in which

a) a feasible path $E"^{38}_k$ is assigned to each demand $k \in K$ (routing constraint),
b) a set of last-slots $S"^{38}_k$ is assigned to each demand $k \in K$ along each edge $e' \in E"^{38}_k$ with $|S"^{38}_k| \geq 1$ (contiguity and continuity constraints),
c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{38}_k$ and $s" \in S"^{38}_{k'}$ with $E"^{38}_k \cap E"^{38}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e'\in E"^{38}_k} |\{s' \in S"^{38}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),
d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^{38}_{k'}$ with $E"^{38}_k \cap E"^{38}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{38}_k$ assigned to the demand k in the solution $\mathcal{S}"^{38}$),
e) and there is one demand k from the set of demands $\tilde{K}$ (i.e., $k \in \tilde{K}$ s.t. the demand k selects a slot s' as last-slot in the solution $\mathcal{S}"^{38}$ with $s' \in \{s, ..., s + w_k - 1\}$, i.e., $s' \in S"^{38}_k$ for a demand $k \in \tilde{K}$, and for each $s' \in S"^{38}_{k'}$ for all $k' \in \tilde{K} \setminus \{k\}$ we have $s' \notin \{s, ..., s + w_{k'} - 1\}$,
f) and all the demands in $\tilde{K}$ pass through the edge e in the solution $\mathcal{S}"^{38}$, i.e., $e' \in E"^{38}_k$ for each $k \in \tilde{K}$.

$\mathcal{S}"^{38}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{38}}, z^{\mathcal{S}"^{38}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k\in\tilde{K}} x^k_{e'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. Based on this, we construct a solution $\mathcal{S}^{40}$ derived from the solution $\mathcal{S}"^{38}$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}"^{38}$ (i.e., $E^{40}_k = E"^{38}_k$ for each $k \in K \setminus \tilde{K}$, and $E^{40}_k \neq E"^{38}_k$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E^{40}_k is assigned to each demand $k \in \tilde{K}$ (routing constraint),
b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"^{38}_k$ and $s" \in S"^{38}_{k'}$ with $E^{40}_k \cap E"^{38}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in\tilde{K}, e'\in E^{40}_k} |\{s' \in S"^{38}_k, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k\in K\setminus\tilde{K}, e'\in E"^{38}_k} |\{s' \in S"^{38}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"^{38}_{k"}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{38}_k$ assigned to the demand k in the solution $\mathcal{S}"^{38}$).

The last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}"^{38}$ remain the same in $\mathcal{S}^{40}$, i.e., $S"^{38}_{k'} = S^{40}_{k'}$ for each demand $k' \in K \setminus \{k\}$, and $S^{40}_k = S"^{38}_k \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{40}$ is clearly feasible given that

a) a feasible path E^{40}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{40}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{40}_k$ with $|S^{40}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{40}_k$ and $s" \in S^{40}_{k'}$ with $E^{40}_k \cap E^{40}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E^{40}_k} |\{s' \in S^{40}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{40}}, z^{\mathcal{S}^{40}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k \in \tilde{K}} x^k_{e'} + \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z^k_{s'} = 1$. We have so

$$\mu x^{\mathcal{S}"^{38}} + \sigma z^{\mathcal{S}"^{38}} = \mu x^{\mathcal{S}^{40}} + \sigma z^{\mathcal{S}^{40}} = \mu x^{\mathcal{S}"^{38}} + \sigma z^{\mathcal{S}"^{38}} + \sigma^k_{s'} - \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E"^{38}_k} \mu^{\tilde{k}}_{e'} + \sum_{\tilde{k} \in \tilde{K}} \sum_{e" \in E^{40}_k} \mu^{\tilde{k}}_{e"}.$$

It follows that $\sigma^k_{s'} = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s, ..., s + w_k - 1\}$ if $k \in \tilde{K}$ given that $\mu^k_{e'} = 0$ for all the demands $k \in K$ and all edges $e' \in E \setminus (E^k_0 \cup E^k_1)$ with $e \neq e'$ if $k \in \tilde{K}$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $s' \notin \{s, ..., s + w_k - 1\}$ if $k \in \tilde{K}$ s.t. we find

$$\sigma^k_{s'} = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } s' \notin \{s, ..., s + w_k - 1\} \text{ if } k \in \tilde{K}.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma^{k'}_{s'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s' \in \{w_{k'}, ..., \bar{s}\} \text{ with } s' \notin \{s, ..., s + w_{k'} - 1\} \text{ if } k' \in \tilde{K}.$$

Consequently, we conclude that

$$\sigma^k_{s'} = 0, \text{ for all } k \in K \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } s' \notin \{s, ..., s + w_k - 1\} \text{ if } k \in \tilde{K}.$$

Let prove that $\sigma^k_{s'}$ for all $k \in \tilde{K}$ and all $s' \in \{s, ..., s + w_k - 1\}$ are equivalents. Consider a demand k' and a slot $s' \in \{s, ..., s + w_{k'} - 1\}$ with $k' \in \tilde{K}$. For that, we consider a solution $\tilde{\mathcal{S}}^{38} = (\tilde{E}^{38}, \tilde{S}^{38})$ in which

a) a feasible path $\tilde{E}^{38}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}^{38}_k$ is assigned to each demand $k \in K$ along each edge $e' \in \tilde{E}^{38}_k$ with $|\tilde{S}^{38}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}^{38}_k$ and $s" \in \tilde{S}^{38}_{k'}$ with $\tilde{E}^{38}_k \cap \tilde{E}^{38}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in \tilde{E}^{38}_k} |\{s' \in \tilde{S}^{38}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s \in S"^{38}_k$ with $\tilde{E}^{38}_k \cap \tilde{E}^{38}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{38}_{k'}$ assigned to the demand k' in the solution $\mathcal{S}"^{38}$),

e) and there is one demand k from the set of demands $\tilde{K}$ (i.e., $k \in \tilde{K}$ s.t. the demand k selects a slot s' as last-slot in the solution $\tilde{\mathcal{S}}^{38}$ with $s' \in \{s, ..., s+w_k-1\}$, i.e., $s' \in \tilde{S}^{38}_k$ for a demand $k \in \tilde{K}$, and for each $s' \in \tilde{S}^{38}_{k'}$ for all $k' \in \tilde{K} \setminus \{k\}$ we have $s' \notin \{s, ..., s+w_{k'}-1\}$),

f) and all the demands in $\tilde{K}$ pass through the edge e in the solution $\tilde{S}^{38}$, i.e., $e' \in \tilde{E}^{38}_k$ for each $k \in \tilde{K}$.

$\tilde{\mathcal{S}}^{38}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{38}}, z^{\tilde{\mathcal{S}}^{38}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k\in\tilde{K}} x^k_{e'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. Let $\mathcal{S}^{41}$ be a solution derived from the solution $\tilde{\mathcal{S}}^{38}$ by adding the slot s' as last-slot to the demand k' with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\tilde{\mathcal{S}}^{38}$ (i.e., $E^{41}_k = \tilde{E}^{38}_k$ for each $k \in K \setminus \tilde{K}$, and $E^{41}_k \neq \tilde{E}^{38}_k$ for each $k \in \tilde{K}$), and also the last-slots assigned to the demands $K \setminus \{k, k'\}$ in $\tilde{\mathcal{S}}^{38}$ remain the same in $\mathcal{S}^{41}$, i.e., $\tilde{S}^{38}_{k"} = S^{41}_{k"}$ for each demand $k" \in K \setminus \{k, k'\}$, and $S^{41}_{k'} = \tilde{S}^{38}_{k'} \cup \{s'\}$ for the demand k', and modifying the last-slots assigned to the demand k by adding a new last-slot $\tilde{s}$ and removing the last slot $s' \in \tilde{S}^{38}_k$ with $s' \in \{s_i+w_k+1, ..., s_j\}$ and $\tilde{s} \notin \{s_i+w_k+1, ..., s_j\}$ for the demand k with $k \in \tilde{K}$ s.t. $S^{41}_k = (\tilde{S}^{38}_k \setminus \{s\}) \cup \{\tilde{s}\}$ s.t. $\{\tilde{s}-w_k+1, ..., \tilde{s}\} \cap \{s'-w_{k'}+1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S^{41}_{k'}$ with $E^{41}_k \cap E^{41}_{k'} \neq \emptyset$. The solution $\mathcal{S}^{41}$ is clearly feasible given that

a) a feasible path E^{41}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{41}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{41}_k$ with $|S^{41}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s'-w_k+1, ..., s'\} \cap \{s"-w_{k'}+1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{41}_k$ and $s" \in S^{41}_{k'}$ with $E^{41}_k \cap E^{41}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e'\in E^{41}_k} |\{s' \in S^{41}_k, s" \in \{s'-w_k+1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{41}}, z^{\mathcal{S}^{41}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k\in\tilde{K}} x^k_{e'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{38}} + \sigma z^{\tilde{\mathcal{S}}^{38}} = \mu x^{\mathcal{S}^{41}} + \sigma z^{\mathcal{S}^{41}} = \mu x^{\tilde{\mathcal{S}}^{38}} + \sigma z^{\tilde{\mathcal{S}}^{38}} + \sigma^{k'}_{s"} - \sigma^k_{s'} + \sigma^k_{\tilde{s}} - \sum_{k\in\tilde{K}} \sum_{e'\in\tilde{E}^{38}_k} \mu^k_{e'} + \sum_{k\in\tilde{K}} \sum_{e'\in E^{41}_k} \mu^k_{e'}.$$

It follows that $\sigma^{k'}_{s"} = \sigma^k_{s'}$ for demand k' and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $k' \in \tilde{K}$ and $s' \in \{s, ..., s+w_{k'}-1\}$ given that $\sigma^k_{\tilde{s}} = 0$ for $\tilde{s} \notin \{s, ..., s+w_k-1\}$ with $k \in \tilde{K}$, and $\mu^k_{e'} = 0$ for all $k \in K$ and all $e' \in E \setminus (E^k_0 \cup E^k_1)$ with $e' \neq e$ if $k \in \tilde{K}$.
Given that the pair (k, k') are chosen arbitrary in the set of demands $\tilde{K}$, we iterate the same procedure for all pairs (k, k') s.t. we find

$$\sigma^k_{s'} = \sigma^{k'}_{s"}, \text{for all pairs } (k, k') \in \tilde{K}$$

with $s' \in \{s, ..., s+w_k-1\}$ and $s' \in \{s, ..., s+w_{k'}-1\}$. We re-do the same procedure for each two slots $s, s' \in \{s, ..., s+w_k-1\}$ for each demand $k \in K$ with $k \in \tilde{K}$ s.t.

$$\sigma^k_{s'} = \sigma^k_{s"}, \text{for all } k \in \tilde{K} \text{ and } s, s' \in \{s, ..., s+w_k-1\}.$$

Let us prove now that μ^k_e for all $k \in K$ with $k \in \tilde{K}$ are equivalents. For that, we consider a solution $\mathcal{S}^{42} = (E^{42}, S^{42})$ defined as below

a) a feasible path E^{42}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{42} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{42}$ with $|S_k^{42}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{42}$ and $s' \in S_{k'}^{42}$ with $E_k^{42} \cap E_{k'}^{42} \neq \emptyset$ (non-overlapping constraint),

d) and there is one demand k from the set of demands $\tilde{K}$ (i.e., $k \in \tilde{K}$ s.t. the demand k pass through the edge e in the solution $\mathcal{S}^{42}$, i.e., $e \in E_k^{42}$ for a demand $k \in \tilde{K}$, and $e \notin E_{k'}^{42}$ for all $k' \in \tilde{K} \setminus \{k\}$,

e) and all the demands in $\tilde{K}$ share the slot s over the edge e in the solution $\mathcal{S}^{42}$, i.e., $\{s_i + w_k + 1, ..., s_j\} \cap S_k^{42} \neq \emptyset$ for each $k \in \tilde{K}$.

Obviously, $\mathcal{S}^{42}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{42}}, z^{\mathcal{S}^{42}})$ is belong to $P(G, K, \mathbb{S})$ and then to $F_{\tilde{K}}^{e,s}$ given that it is composed by $\sum_{k \in \tilde{K}} x_e^k + \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z_{s'}^k = 1$.
Consider now a demand k' in $\tilde{K}$ s.t. $e \notin E_{k'}^{42}$. For that, we consider a solution $\tilde{\mathcal{S}}^{42} = (\tilde{E}^{42}, \tilde{S}^{42})$ in which

a) a feasible path $\tilde{E}_k^{42}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}_k^{42}$ is assigned to each demand $k \in K$ along each edge $e \in \tilde{E}_k^{42}$ with $|\tilde{S}_k^{42}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}_k^{42}$ and $s" \in \tilde{S}_{k'}^{42}$ with $\tilde{E}_k^{42} \cap \tilde{E}_{k'}^{42} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in \tilde{E}_k^{42}} |\{s' \in \tilde{S}_k^{42}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and there is one demand k from the set of demands $\tilde{K}$ (i.e., $k \in \tilde{K}$ s.t. the demand k pass through the edge e in the solution $\tilde{S}^{42}$, i.e., $e \in \tilde{E}_k^{42}$ for a demand $k \in \tilde{K}$, and $e \notin \tilde{E}_{k'}^{42}$ for all $k' \in \tilde{K} \setminus \{k\}$,

e) and all the demands in $\tilde{K}$ share the slot s over the edge e in the solution $\tilde{S}^{42}$, i.e., $\{s, ..., s + w_k - 1\} \cap \tilde{S}_k^{42} \neq \emptyset$ for each $k \in \tilde{K}$.

$\tilde{\mathcal{S}}^{42}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{42}}, z^{\tilde{\mathcal{S}}^{42}})$ is belong to F and then to $F_{\tilde{K}}^{e,s}$ given that it is composed by $\sum_{k \in \tilde{K}} x_e^k + \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z_{s'}^k = 1$. Based on this, we derive a solution $\mathcal{S}"^{43} = (E"^{43}, S"^{43})$ from the solution $\tilde{\mathcal{S}}^{42}$ by

a) the paths assigned to the demands $K \setminus \{k, k'\}$ in $\tilde{\mathcal{S}}^{42}$ remain the same in $\mathcal{S}"^{43}$ (i.e., $E"^{43}_{k"} = \tilde{E}_{k"}^{42}$ for each $k" \in K \setminus \{k, k'\}$),

b) without modifying the last-slots assigned to the demands K in $\tilde{\mathcal{S}}^{42}$, i.e., $\tilde{S}_k^{42} = S"^{43}_k$ for each demand $k \in K$,

c) modifying the path assigned to the demand k' in $\tilde{\mathcal{S}}^{42}$ from $\tilde{E}_{k'}^{42}$ to a path $E"^{43}_{k'}$ passed through the edge e (i.e., $e \in E"^{43}_{k'}$) with $k' \in \tilde{K}$ s.t. $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and each $s' \in \tilde{S}_{k'}^{42}$ and each $s' \in \tilde{S}_k^{42}$ with $\tilde{E}_k^{42} \cap E"^{43}_{k'} \neq \emptyset$,

d) modifying the path assigned to the demand k in $\tilde{\mathcal{S}}^{42}$ with $e \in \tilde{E}_k^{42}$ and $k \in \tilde{K}$ from $\tilde{E}_k^{42}$ to a path $E"^{43}_k$ without passing through the edge e (i.e., $e \notin E"^{43}_k$) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k"} + 1, ..., s'\} = \emptyset$ for each $k" \in K \setminus \{k, k'\}$ and each $s' \in \tilde{S}_k^{42}$ and each $s' \in \tilde{S}_{k"}^{42}$ with $\tilde{E}_{k"}^{42} \cap E"^{43}_k \neq \emptyset$, and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $s' \in \tilde{S}_k^{42}$ and each $s' \in \tilde{S}_{k'}^{42}$ with $E"^{43}_{k"} \cap E"^{43}_k \neq \emptyset$.

The solution $\mathcal{S}"^{43}$ is feasible given that

a) a feasible path $E"^{43}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^{43}_k$ is assigned to each demand $k \in K$ along each edge $e \in E"^{43}_k$ with $|S"^{43}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{43}_k$ and $s" \in S"^{43}_{k'}$ with $E"^{43}_k \cap E"^{43}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e\in E"^{43}_k} |\{s' \in S"^{43}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}"^{43}}, z^{\mathcal{S}"^{43}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k\in\tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. We then obtain that

$$\begin{aligned}\mu x^{\tilde{\mathcal{S}}^{42}} + \sigma z^{\tilde{\mathcal{S}}^{42}} &= \mu x^{\mathcal{S}^{43}} + \sigma z^{\mathcal{S}^{43}} = \mu x^{\tilde{\mathcal{S}}^{42}} + \sigma z^{\tilde{\mathcal{S}}^{42}} + \mu^{k'}_e - \mu^k_e \\ &+ \sum_{e"\in E"^{43}_{k'}\setminus\{e\}} \mu^{k'}_{e"} - \sum_{e"\in\tilde{E}^{42}_{k'}} \mu^{k'}_{e"} + \sum_{e"\in E"^{43}_k} \mu^k_{e"} - \sum_{e"\in\tilde{E}^{42}_k\setminus\{e\}} \mu^k_{e"}.\end{aligned}$$

It follows that $\mu^{k'}_e = \mu^k_e$ for demand k' and a edge $e' \in E \setminus (E^{k'}_0 \cup E^{k'}_1)$ with $v_{k',e'} \in \tilde{K}$ given that $\mu^k_{e"} = 0$ for all $k \in K$ and all $e" \in E \setminus (E^k_0 \cup E^k_1)$ with $k \in \tilde{K}$.
Given that the pair (k, k') are chosen arbitrary in the set of demands $\tilde{K}$, we iterate the same procedure for all pairs (k, k') s.t. we find

$$\mu^k_e = \mu^{k'}_e, \text{for all pairs } (k, k') \in \tilde{K}.$$

Furthermore, let prove that all $\sigma^k_{s'}$ and μ^k_e are equivalents for all $k \in \tilde{K}$ and $s' \in \{s, ..., s + w_k - 1\}$. For that, we consider for each demand k' with $k' \in \tilde{K}$, a solution $\mathcal{S}^{44} = (E^{44}, S^{44})$ derived from the solution $\tilde{\mathcal{S}}^{42}$ as below

a) the paths assigned to the demands $K \setminus \{k'\}$ in $\tilde{\mathcal{S}}^{42}$ remain the same in $\mathcal{S}^{44}$ (i.e., $E^{44}_{k"} = \tilde{E}^{42}_{k"}$ for each $k" \in K \setminus \{k'\}$),

b) without modifying the last-slots assigned to the demands $K \setminus \{k\}$ in $\tilde{\mathcal{S}}^{42}$, i.e., $\tilde{S}^{42}_{k"} = S^{44}_{k"}$ for each demand $k" \in K \setminus \{k\}$,

c) modifying the set of last-slots assigned to the demand k' in $\tilde{\mathcal{S}}^{42}$ from $\tilde{S}^{42}_{k'}$ to $S^{44}_{k'}$ s.t. $S^{44}_{k'} \cap \{s, ..., s + w_{k'} - 1\} = \emptyset$.

Hence, there are $|\tilde{K}| - 1$ demands from $\tilde{K}$ that share the slot s over the edge e (i.e., all the demands in $\tilde{K} \setminus \{k'\}$), and two demands $\{k, k'\}$ from $\tilde{K}$ that use the edge e in the solution $\mathcal{S}^{44}$. The solution $\mathcal{S}^{44}$ is then feasible given that

a) a feasible path E^{44}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{44}_k is assigned to each demand $k \in K$ along each edge $e \in E^{44}_k$ with $|S^{44}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{44}_k$ and $s" \in S^{44}_{k'}$ with $E^{44}_k \cap E^{44}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e\in E^{44}_k} |\{s' \in S^{44}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\sum_{k\in\tilde{K}} |E^{44}_k \cap \{e\}| + |S^{44}_k \cap \{s, ..., s + w_k - 1\}| = |\tilde{K}| + 1$.

The corresponding incidence vector $(x^{\mathcal{S}^{44}}, z^{\mathcal{S}^{44}})$ is belong to F and then to $F^{e,s}_{\tilde{K}}$ given that it is composed by $\sum_{k\in\tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z^k_{s'} = 1$. We then obtain that

$$\mu x^{\tilde{\mathcal{S}}^{42}} + \sigma z^{\tilde{\mathcal{S}}^{42}} = \mu x^{\mathcal{S}^{44}} + \sigma z^{\mathcal{S}^{44}} = \mu x^{\tilde{\mathcal{S}}^{42}} + \sigma z^{\tilde{\mathcal{S}}^{42}} + \mu^{k'}_e - \sigma^{k'}_{s'} + \sum_{e"\in E^{44}_{k'}\setminus\{e\}} \mu^{k'}_{e"} - \sum_{e"\in\tilde{E}^{42}_{k'}} \mu^{k'}_{e"}.$$

It follows that $\mu_e^{k'} = \sigma_{s'}^{k'}$ for demand k' and slot $s' \in \{s, ..., s + w_{k'} - 1\}$ given that $\mu_{e"}^k = 0$ for all $k \in K$ and all $e" \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e"$ if $k \in \tilde{K}$. Moreover, by doing the same thing over all slots $s' \in \{s, ..., s + w_{k'} - 1\}$, we found that

$$\mu_e^{k'} = \sigma_{s'}^{k'}, \text{for all } s' \in \{s, ..., s + w_{k'} - 1\}.$$

Given that k' is chosen arbitrarily in $\tilde{K}$, we iterate the same procedure for all $k \in \tilde{K}$ to show that

$$\mu_e^k = \sigma_{s'}^k, \text{for all } k \in \tilde{K} \text{ and all } s' \in \{s, ..., s + w_k - 1\}.$$

Based on this, and given that all μ_e^k are equivalents for all $k \in \tilde{K}$, and that $\sigma_{s'}^k$ are equivalents for all $k \in \tilde{K}$ and $s' \in \{s, ..., s + w_{k'} - 1\}$, we obtain that

$$\mu_e^k = \sigma_{s'}^{k'}, \text{ for all } k, k' \in \tilde{K} \text{ and all } s' \in \{s, ..., s + w_{k'} - 1\}.$$

Consequently, we conclude that

$$\mu_e^k = \sigma_{s'}^{k'} = \rho, \text{ for all } k, k' \in \tilde{K} \text{ and all } s' \in \{s, ..., s + w_{k'} - 1\}.$$

On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k' \in K$ and $e' \in E$

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'}, \text{ if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'}, \text{ if } e' \in E_1^{k'}, \\ \rho, \text{ if } k' \in \tilde{K} \text{ and } e' = e, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s' \in \mathbb{S}$

$$\sigma_{s'}^k = \begin{cases} \gamma_3^{k,s'}, \text{if } s' \in \{1, ..., w_k - 1\} \\ \rho, \text{if } k \in \tilde{K} \text{ and } s' \in \{s, ..., s + w_k - 1\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{k \in \tilde{K}} \rho \alpha_e^k + \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} \rho \beta_{s'}^k + \gamma Q.$ □

Theorem 2.4.2. *Consider an edge $e \in E$, and a slot $s \in \mathbb{S}$. Let $\tilde{K}$ be a subset of demands in K with $|\tilde{K}| \geq 3$, and $\sum_{k \in \tilde{K}} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus \tilde{K}} w_{k'}$. Then, the inequality (2.27) is facet defining for $P(G, K, \mathbb{S})$ iff there does not exist an interval of contiguous slots $I = [s_i, s_j]$ s.t.*

a) $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ for each demand $k \in \tilde{K}$,

b) and $s \in \{s_i + \max_{k' \in \tilde{K}} w_k - 1, ..., s_j - \max_{k \in \tilde{K}} w_k + 1\}$, *c) and* $w_k + w_{k'} \geq |I| + 1$ *for each* $k, k' \in \tilde{K}$*,*

d) and $w_k + w_{k'} \geq |I| + 1$ *for each* $k \in \tilde{K}$ *and each* $k' \in K_e \setminus \tilde{K}$*,*

e) and $2w_k \geq |I| + 1$ *for each* $k \in \tilde{K}$*,*

f) and $2w_{k'} \geq |I| + 1$ *for each* $k' \in K_e \setminus \tilde{K}$*.*

Proof. **Neccessity.**

If there exists an interval of contiguous slots $I = [s_i, s_j]$ s.t.

a) $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ for each demand $k \in \tilde{K}$,

b) and $s \in \{s_i + \max_{k' \in \tilde{K}} w_k - 1, ..., s_j - \max_{k \in \tilde{K}} w_k + 1\}$,

c) and $w_k + w_{k'} \geq |I| + 1$ for each $k, k' \in \tilde{K}$,

d) and $w_k + w_{k'} \geq |I| + 1$ for each $k \in \tilde{K}$ and each $k' \in K_e \setminus \tilde{K}$,

e) and $2w_k \geq |I| + 1$ for each $k \in \tilde{K}$,

f) and $2w_{k'} \geq |I| + 1$ for each $k' \in K_e \setminus \tilde{K}$.

Then the inequality (2.27) is dominated by the inequality (2.33) for for a clique $C = \tilde{K}$ and clique $C_e = K_e \setminus \tilde{K}$ in the conflict graph $\tilde{G}_I^e$. As result, the inequality (2.27) is not facet defining for $P(G, K, \mathbb{S})$.

Sufficiency.

Let's us denote $F'^{e,s}_{\tilde{K}}$ the face induced by the inequality (2.27), which is given by

$$F'^{e,s}_{\tilde{K}} = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{k \in \tilde{K}} x_e^k + \sum_{k \in \tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{K_e \setminus \tilde{K}} \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} = |\tilde{K}| + 1\}.$$

We denote the inequality $\sum_{k \in \tilde{K}} x_e^k + \sum_{k \in \tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{K_e \setminus \tilde{K}} \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} \leq |\tilde{K}| + 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F'^{e,s}_{\tilde{K}} \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K} (w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma_{s'}^k = 0$ for all demands $k \in K$ and all slots $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s, ..., s + w_k - 1\}$ if $k \in \tilde{K} \cup K_e$,

b) and $\sigma_{s'}^k$ are equivalents for all $k \in \tilde{K} \cup K_e$ and all $s' \in \{s, ..., s + w_k - 1\}$,

c) and $\mu_{e'}^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e'$ if $k \in \tilde{K}$,

d) and all μ_e^k are equivalents for the set of demands in $\tilde{K}$,

e) and $\sigma_{s'}^{k'}$ and μ_e^k are equivalents for all $k \in \tilde{K}$ and all $k' \in \tilde{K} \cup K_e$ and all $s' \in \{s, ..., s + w_{k'} - 1\}$.

We re-do the same technique of proof already detailed to prove that the inequality (2.25) is facet defining for $P(G, K, \mathbb{S})$ s.t. the solutions $\mathcal{S}^{38} - \mathcal{S}^{44}$ still feasible for $F'^{e,s}_{\tilde{K}}$ given that their incidence vector are composed by $\sum_{k \in \tilde{K}} x_e^k + \sum_{k \in \tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{K_e \setminus \tilde{K}} \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} \leq |\tilde{K}| + 1$. We conclude at the end that for each $k' \in K$ and $e' \in E$

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'}, \text{ if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'}, \text{ if } e' \in E_1^{k'}, \\ \rho, \text{ if } k' \in \tilde{K} \text{ and } e' = e, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s' \in \mathbb{S}$

$$\sigma^k_{s'} = \begin{cases} \gamma_3^{k,s'}, \text{if } s' \in \{1, ..., w_k - 1\} \\ \rho, \text{if } k \in \tilde{K} \cup K_e \text{ and } s' \in \{s, ..., s + w_k - 1\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{k \in \tilde{K}} \rho \alpha^k_e + \sum_{k \in \tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} \rho \beta^k_{s'} + \sum_{k \in K_e \setminus \tilde{K}} \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} \rho \beta^{k'}_{s'} + \gamma Q.$ □

2.4.2 Edge-Interval-Capacity-Cover Inequalities

Let's now introduce some valid inequalities which can be seen as cover inequalities using some notions of cover related to our problem.

Definition 2.4.1. *An interval $I = [s_i, s_j]$ represents a set of contiguous slots situated between the two slots s_i and s_j with $j \geq i + 1$ and $s_j \leq \bar{s}$.*

Definition 2.4.2. *For an interval of contiguous slots $I = [s_i, s_j]$, a subset of demands $K' \subseteq K$ is said a cover for the interval $I = [s_i, s_j]$ iff $\sum_{k \in \tilde{K}} w_k > |I|$ and $w_k < |I|$ for each $k \in \tilde{K}$.*

Definition 2.4.3. *For an interval of contiguous slots $I = [s_i, s_j]$, a cover $\tilde{K}$ is said a minimal cover if $\tilde{K} \setminus \{k\}$ is not a cover for interval $I = [s_i, s_j]$ for each demand $k \in \tilde{K}$, i.e., $\sum_{k' \in \tilde{K} \setminus \{k\}} w_{k'} \leq |I|$ for each demand $k \in \tilde{K}$.*

Based on these definitions, we introduce the following inequalities.

Proposition 2.4.6. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $j \geq i + 1$. Let $K' \subseteq K_e$ be a minimal cover for interval $I = [s_i, s_j]$ over edge e. Then, the inequality*

$$\sum_{k \in K'} \sum_{s=s_i+w_k-1}^{s_j} z^k_s \leq |K'| - 1, \tag{2.28}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. The interval $I = [s_i, s_j]$ can cover at most $|K'| - 1$ demands given that K' is a minimal cover for interval $I = [s_i, s_j]$ over edge e. We start the proof by assuming that the inequality (2.28) is not valid for $P(G, K, \mathbb{S})$. It follows that there exists a C-RSA solution S in which $\{s_i + w_k - 1, ..., s_j\} \cap S_k = \emptyset$ for a demand $k \in K'$ s.t.

$$\sum_{k \in K'} \sum_{s=s_i+w_k-1}^{s_j} z^k_s(S) > |K'| - 1.$$

Since $\{s_i + w_k - 1, ..., s_j\} \cap S_k = \emptyset$ for a demand $k \in K'$ this means that $\sum_{s=s_i+w_k-1}^{s_j} z^k_s(S) = 0$, and taking into account that K' is minimal cover for the interval $I = [s_i, s_j]$ over edge e, and $\sum_{s=s_i+w_k-1}^{s_j} z^k_s(S) \leq 1$ for each demand $k \in K'$, it follows that

$$\sum_{k' \in K' \setminus \{k\}} \sum_{s=s_i+w_{k'}-1}^{s_j} z^{k'}_s(S) \leq |K'| - 1,$$

which contradicts what we supposed before, i.e., $\sum_{k \in K'} \sum_{s=s_i+w_k-1}^{s_j} z^k_s(S) > |K'| - 1$.
Hence $\sum_{k \in K'} |S_k \cap \{s_i + w_k - 1, ..., s_j\}| \leq |K'| - 1$.
We conclude at the end that the inequality (2.28) is valid for $P(G, K, \mathbb{S})$. □

The inequality (2.28) can be strengthened using an extention of each minimal cover $K' \subset K_e$ for an interval I over edge e as follows.

Proposition 2.4.7. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$. Let $K' \subseteq K_e$ be a minimal cover for interval $I = [s_i, s_j]$ over edge e, and $\Xi(K')$ be a subset of demands in $K_e \setminus K'$ s.t. $\Xi(K') = \{k \in K_e \setminus K' \text{ s.t. } w_k \geq w_{k'} \quad \forall k' \in K'\}$. Then, the inequality*

$$\sum_{k \in K'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{k' \in \Xi(K')} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq |K'| - 1, \tag{2.29}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. The interval $I = [s_i, s_j]$ can cover at most $|K'| - 1$ demands from the demands in $K' \cup \Xi(K')$ given that K' is a minimal cover for interval $I = [s_i, s_j]$ over edge e and the definition of the set $\Xi(K')$ s.t. for each pair (k, k') with $k \in K'$ and $k' \in \Xi(K')$, the set $(K' \setminus \{k\}) \cup \{k'\}$ stills defining minimal cover for the interval I over the edge e. Furthermore, for each quadruplet $(k, k', \tilde{k}, \tilde{k}')$ with $k, k' \in K'$ and $\tilde{k}, \tilde{k}' \in \Xi(K')$, the set $(K' \setminus \{k, k'\}) \cup \{\tilde{k}, \tilde{k}'\}$ stills defining minimal cover for the interval I over the edge e given that $w_k + w_{k'} \leq w_{\tilde{k}} + w_{\tilde{k}'}$. We strengthen the proof as follows. Let's first suppose that the inequality (2.29) is not valid for $P(G, K, \mathbb{S})$. It follows that there exists a C-RSA solution S in which $\{s_i + w_{k'} - 1, ..., s_j\} \cap S_{k'} = \emptyset$ for each demand $k' \in \Xi(K')$ s.t.

$$\sum_{k \in K'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) > |K'| - 1.$$

Since $\{s_i + w_{k'} - 1, ..., s_j\} \cap S_{k'} = \emptyset$ for each demand $k' \in \Xi(K')$ this means that $\sum_{s=s_i+w_{k'}-1}^{s_j} z_s^{k'}(S) = 0$, and taking into account the inequality (2.28), and that K' is minimal cover for the interval $I = [s_i, s_j]$ over edge e, it follows that

$$\sum_{k \in K'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) \leq |K'| - 1,$$

which contradicts what we supposed before, i.e., $\sum_{k \in K'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) > |K'| - 1$. Hence $\sum_{k \in K'} |S_k \cap \{s_i + w_k - 1, ..., s_j\}| + \sum_{k' \in \Xi(K')} |S_{k'} \cap \{s_i + w_{k'} - 1, ..., s_j\}| \leq |K'| - 1$. We conclude at the end that the inequality (2.29) is valid for $P(G, K, \mathbb{S})$. $\square$

Moreover, the inequality (2.28) can be more strengthened using lifting procedures proposed by Nemhauser and Wolsey in [109] without modifying its right-hand side. Inspiring from the inequality (2.28), we define a new valid inequality as follows.

Proposition 2.4.8. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $j \geq i + 1$. Let $\tilde{K}$ be a minimal cover for the interval I s.t.*

a) $\sum_{k \in \tilde{K}} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus \tilde{K}} w_{k'}$,

b) $e \notin E_0^k$ *for each demand* $k \in \tilde{K}$,

c) $\tilde{K} \geq 3$,

d) $(k, k') \notin K_c^e$ *for each pair of demands* (k, k') *in* $\tilde{K}$.

Then, the inequality

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 2|\tilde{K}| - 1, \tag{2.30}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. The interval $I = [s_i, s_j]$ can cover at most $|\tilde{K}| - 1$ demands given that $\tilde{K}$ is a minimal cover for interval $I = [s_i, s_j]$ over edge. It follows that if the demands $\tilde{K}$ pass together through the edge e (i.e., $\sum_{k\in\tilde{K}} x_e^k = |\tilde{K}|$), there is at most $|\tilde{K}| - 1$ demands that can share the interval I over edge e.
We start the proof by assuming that the inequality (2.30) is not valid for $P(G, K, \mathbb{S})$. It follows that there exists a C-RSA solution S in which $\{s_i + w_k - 1, ..., s_j\} \cap S_k = \emptyset$ for a demand $k \in K'$ s.t.

$$\sum_{k\in K'} x_e^k(S) + \sum_{k'\in K'\setminus\{k\}} \sum_{s=s_i+w_{k'}-1}^{s_j} z_s^{k'}(S) \geq 2|K'|.$$

Since $\{s_i+w_k-1, ..., s_j\}\cap S_k = \emptyset$ for a demand $k \in K'$ this means that $\sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) = 0$, and taking into account that K' is minimal cover for the interval $I = [s_i, s_j]$ over edge e, it follows that

$$\sum_{k\in K'} x_e^k(S) + \sum_{k'\in K'\setminus\{k\}} \sum_{s=s_i+w_{k'}-1}^{s_j} z_s^{k'}(S) \leq 2|K'| - 1,$$

which contradicts what we supposed before, i.e., $\sum_{k\in K'} x_e^k(S)+\sum_{k'\in K'\setminus\{k\}} \sum_{s=s_i+w_{k'}-1}^{s_j} z_s^{k'}(S) \geq 2|K'|$.
One can imagine another case also when $K'\cap K_e = \emptyset$, it follows that there exists a C-RSA solution S' in which $E_k \cap \{e\} = \emptyset$ for each demand $k \in K'$, which means that $\sum_{k\in K'} x_e^k(S') = 0$ s.t.

$$\sum_{k\in K'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S') \geq 2|K'|.$$

Given that K' is a minimal cover for the interval I over edge I, it follows that

$$\sum_{k'\in K'\setminus\{k\}} \sum_{s=s_i+w_{k'}-1}^{s_j} z_s^{k'}(S') \leq 2|K'| - 1,$$

which contradicts our hypothesis, i.e., $\sum_{k\in K'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S') \geq 2|K'|$.
Hence $\sum_{k\in K'} |E_k \cap \{e\}| + \sum_{k\in K'} |S_k \cap \{s_i + w_k - 1, ..., s_j\}| \leq 2|K'| - 1$.
We conclude at the end that the inequality (2.30) is valid for $P(G, K, \mathbb{S})$. □

The inequality (2.30) can be strengthened by introducing its extended format of the minimal cover K' for the interval I over edge e as follows.

Proposition 2.4.9. *Consider an edge* $e \in E$. *Let* $I = [s_i, s_j]$ *be an interval of contiguous slots in* $[1, \bar{s}]$ *with* $j \geq i + 1$. *Let* $\tilde{K}$ *be a minimal cover for the interval* I, *and* $\tilde{K}_e$ *be a subset of demands in* $K_e \setminus \tilde{K}$ *s.t.*

a) $\sum_{k \in \tilde{K}} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus \tilde{K}} w_{k'}$,

b) $e \notin E_0^k$ *for each demand* $k \in \tilde{K}$,

c) $\tilde{K} \geq 3$,

d) $(k, k') \notin K_c^e$ *for each pair of demands* (k, k') *in* $\tilde{K}$,

e) $w_{k'} \geq w_k$ *for each* $k \in \tilde{K}$ *and each* $k' \in \tilde{K}_e$.

Then, the inequality

$$\sum_{k \in \tilde{K}} x_e^k + \sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{k' \in \tilde{K}_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq 2|\tilde{K}| - 1, \tag{2.31}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. The inequality (2.31) can be seen as a particular case for the inequality (2.30) induced by a set of demands $K' = \tilde{K} \cup \tilde{K}_e$ which stills defining a cover for the interval I over edge e. □

More general, a strengthened inequality based on the inequality (2.30) can be done using lifting procedures proposed by Nemhauser and Wolsey in [109] without modifying its right-hand side.

Remark 2.4.1. *Consider an edge* $e \in E$. *Let* $I = [s_i, s_j]$ *be an interval of contiguous slots with* $s_i + 1 \leq s_j$, s" *be a slot in* $\mathbb{S}$, *and* $\tilde{K}$ *be a subset of demands in* K *satisfying the conditions of the two inequalities* (2.27) *and* (2.30). *We ensure that the inequality* (2.27) *can never dominate the inequality* (2.30).

Let us denote by the symbol $a \preceq b$ iff b dominates a.

Proof. Assume that the inequality (2.27) dominates the inequality (2.30), this means that there exists a slot $s" \in \mathbb{S}$ s.t.

$$\sum_{k \in \tilde{K}} x_e^k + \sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \preceq \sum_{k \in \tilde{K}} x_e^k + \sum_{k \in \tilde{K}} \sum_{s'=s"}^{\min(s"+w_k-1, \bar{s})} z_{s'}^k \leq |\tilde{K}| + 1.$$

By removing the sum $\sum_{k \in \tilde{K}} x_e^k$ from the two sides of the previous comparison, we get

$$\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \preceq \sum_{k \in \tilde{K}} \sum_{s'=s"}^{\min(s"+w_k-1, \bar{s})} z_{s'}^k.$$

Given that the demands in $\tilde{K}$ are independants, we found that

$$\sum_{s=s_i+w_k-1}^{s_j} z_s^k \preceq \sum_{s'=s"}^{\min(s"+w_k-1, \bar{s})} z_{s'}^k \text{ for each } k \in \tilde{K}.$$

It follows that $\{s_i + w_k - 1, ..., s_j\} = [s_i + w_k - 1, s_j] \subseteq [s", min(s" + w_k - 1, \bar{s})]$ for each demand $k \in \tilde{K}$. Taking into account that $|\{s", ..., min(s" + w_k - 1, \bar{s})\}| \leq w_k$ for each $k \in \tilde{K}$, this means that

$$\begin{aligned} |\{s_i + w_k - 1, ..., s_j\}| = s_j - (s_i + w_k - 1) + 1 \leq w_k \implies s_j - s_i + 1 \leq 2 * w_k - 1 \text{ for each } k \in \tilde{K} \\ \implies |I| \leq 2 * w_k - 1 \text{ for each } k \in \tilde{K} \implies |I| \leq 2 * \min_{k \in \tilde{K}} w_k - 1 \end{aligned}$$

As a result, $w_k + w_{k'} \geq |I|$ for each pair of demand (k,k') in $\tilde{K}$ since that $w_k \geq \min\limits_{k"\in\tilde{K}} w_{k"}$ for each $k \in \tilde{K}$. This contradicts that the set of demand $\tilde{K}$ should satisfy that $\sum_{k\in\tilde{K}\setminus\{k'\}} w_k \leq |I|$ for each $k' \in \tilde{K}$. We conclude that the inequality (2.27) can never dominate the inequality (2.30) and satisfying the conditions of validity of the inequality (2.30) at the same time. □

Theorem 2.4.3. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $j \geq i+1$. Let $\tilde{K}$ be a subset of demands of K s.t.*

a) $\sum\limits_{k\in\tilde{K}} w_k \geq |I| + 1$,

b) $\sum\limits_{k\in\tilde{K}\setminus\{k'\}} w_k \leq |I|$ *for each* $k' \in \tilde{K}$,

c) $\sum\limits_{k\in\tilde{K}} w_k \leq \bar{s} - \sum\limits_{k'\in K_e\setminus\tilde{K}} w_{k'}$,

d) $e \notin E_0^k$ *for each demand* $k \in \tilde{K}$,

e) $\tilde{K} \geq 3$,

f) $(k,k') \notin K_c^e$ *for each pair of demands* (k,k') *in* $\tilde{K}$.

Then, the inequality (2.30) *is facet defining for the polytope $P(G,K,\mathbb{S},\tilde{K},I,e)$ iff there does not exist an interval of contiguous slots I' in $[1,\bar{s}]$ with $I \subset I'$ s.t. $\tilde{K}$ defines a minimal cover for the interval I', where*

$$P(G,K,\mathbb{S},\tilde{K},I,e) = \{(x,z) \in P(G,K,\mathbb{S}) : \sum_{k'\in K_e\setminus\tilde{K}} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = 0\}.$$

Proof. **Necessity**
If there exists an interval of contiguous slots I' in $[1,\bar{s}]$ with $I \subset I'$ s.t. $\tilde{K}$ defines a minimal cover for the interval I'. This means that $\{s_i + w_k - 1, ..., s_j\} \subset I'$. As a result, the inequality (2.30) induced by the minimal cover $\tilde{K}$ for the interval I, it is dominated by another inequality (2.30) induced by the same minimal cover $\tilde{K}$ for the interval I'. Hence, the inequality (2.30) cannot be facet defining for the polytope $P(G,K,\mathbb{S},\tilde{K},I,e)$.
Sufficiency.
Let $F_{\tilde{K}}^{\tilde{G}_I^e}$ denote the face induced by the inequality (2.30), which is given by

$$F_{\tilde{K}}^{\tilde{G}_I^e} = \{(x,z) \in P(G,K,\mathbb{S},\tilde{K},I,e) : \sum_{k\in\tilde{K}} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 2|\tilde{K}| - 1\}.$$

In order to prove that inequality $\sum_{k\in\tilde{K}} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 2|\tilde{K}| - 1$ is facet defining for $P(G,K,\mathbb{S},\tilde{K},I,e)$, we start checking that $F_{\tilde{K}}^{\tilde{G}_I^e}$ is a proper face, and $F_{\tilde{K}}^{\tilde{G}_I^e} \neq P(G,K,\mathbb{S},\tilde{K},I,e)$. We construct a solution $\mathcal{S}^{45} = (E^{45}, S^{45})$ as below

a) a feasible path E_k^{45} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{45} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{45}$ with $|S_k^{45}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{45}$ and $s' \in S_{k'}^{45}$ with $E_k^{45} \cap E_{k'}^{45} \neq \emptyset$ (non-overlapping constraint),

d) and there is $|\tilde{K}|-1$ demands from the minimal cover $\tilde{K}$ denoted by $\tilde{K}_{83}$ which are covered by the interval I (i.e., if $k \in \tilde{K}_{83}$, this means that the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{45}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{45}$ for each $k \in \tilde{K}_{83}$, and for each $s' \in S_{k'}^{45}$ for all $k' \in \tilde{K} \setminus \tilde{K}_{83}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$,

e) and all the demands in $\tilde{K}$ pass through the edge e in the solution $\mathcal{S}^{45}$, i.e., $e \in E_k^{45}$ for each $k \in \tilde{K}$.

Obviously, $\mathcal{S}^{45}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{45}}, z^{\mathcal{S}^{45}})$ is belong to $P(G, K, \mathbb{S}, \tilde{K}, I, e)$ and then to $F_{\tilde{K}}^{\tilde{G}_I^e}$ given that it is composed by $\sum_{k\in\tilde{K}} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 2|\tilde{K}|-1$. As a result, $F_{\tilde{K}}^{\tilde{G}_I^e}$ is not empty (i.e., $F_{\tilde{K}}^{\tilde{G}_I^e} \neq \emptyset$). Furthermore, given that $s \in \{s_i + w_k - 1, ..., s_j\}$ for each $k \in \tilde{K}$, this means that there exists at least one feasible slot assignment S_k for the demands k in $\tilde{K}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ for each $s \in S_k$ and each $k \in \tilde{K}$. This means that $F_{\tilde{K}}^{\tilde{G}_I^e} \neq P(G, K, \mathbb{S}, \tilde{K}, I, e)$.
We denote the inequality $\sum_{k\in\tilde{K}} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 2|\tilde{K}|-1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S}, \tilde{K}, I, e)$. Suppose that $F_{\tilde{K}}^{\tilde{G}_I^e} \subset F = \{(x, z) \in P(G, K, \mathbb{S}, \tilde{K}, I, e) : \mu x + \sigma z = \tau\}$. We show that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k\in K}|E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k\in K}|E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k\in K}(w_k-1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in \tilde{K}$,

b) and σ_s^k are equivalents for all $k \in \tilde{K}$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$,

c) and $\mu_{e'}^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e'$ if $k \in \tilde{K}$,

d) and all μ_e^k are equivalents for the set of demands in $\tilde{K}$,

e) and σ_s^k and μ_e^k are equivalents for all $k \in \tilde{K}$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$.

We first show that $\mu_{e'}^k = 0$ for each edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$ with $e \neq e'$ if $k \in \tilde{K}$. Consider a demand $k \in K$ and an edge $e' \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e'$ if $k \in \tilde{K}$. For that, we consider a solution $\mathcal{S}'^{45} = (E'^{45}, S'^{45})$ in which

a) a feasible path $E_k'^{45}$ is assigned to each demand $k \in K$ (routing constraint),

b) and a set of last-slots $S_k'^{45}$ is assigned to each demand $k \in K$ along each edge $e' \in E_k'^{45}$ with $|S_k'^{45}| \geq 1$ (contiguity and continuity constraints),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k'^{45}$ and $s" \in S_{k'}'^{45}$ with $E_k'^{45} \cap E_{k'}'^{45} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e'\in E_k'^{45}} |\{s' \in S_k'^{45}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) the edge e' is not non-compatible edge with the selected edges $e" \in E_k'^{45}$ of demand k in the solution $\mathcal{S}'^{45}$, i.e., $\sum_{e"\in E_k'^{45}} l_{e"} + l_{e'} \leq \bar{l}_k$. As a result, $E_k'^{45} \cup \{e'\}$ is a feasible path for the demand k,

e) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s \in S_k'^{45}$ and $s' \in S_{k'}'^{45}$ with $(E_k'^{45} \cup \{e'\}) \cap E_{k'}'^{45} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e' in the set of edges $E_k'^{45}$ selected to route the demand k in the solution $\mathcal{S}'^{45}$),

f) and there is $|\tilde{K}|-1$ demands from the minimal cover $\tilde{K}$ denoted by $\tilde{K}'_{83}$ which are covered by the interval I (i.e., if $k \in \tilde{K}'_{83}$, this means that the demand k selects a slot s as last-slot in the solution $\mathcal{S}'^{45}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S'^{45}_k$ for each $k \in \tilde{K}'_{83}$, and for each $s' \in S'^{45}_{k'}$ for all $k' \in \tilde{K} \setminus \tilde{K}'_{83}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$,

g) and all the demands in $\tilde{K}$ pass through the edge e in the solution $\mathcal{S}'^{45}$, i.e., $e \in E'^{45}_k$ for each $k \in \tilde{K}$.

$\mathcal{S}'^{45}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{45}}, z^{\mathcal{S}'^{45}})$ is belong to F and then to $F_{\tilde{K}}^{\tilde{G}_I^e}$ given that it is composed by $\sum_{k \in \tilde{K}} x^k_{e'} + \sum_{s=s_i+w_k-1}^{s_j} z^k_s = 2|\tilde{K}| - 1$. Based on this, we derive a solution $\mathcal{S}^{46}$ obtained from the solution $\mathcal{S}'^{45}$ by adding an unused edge $e' \in E \setminus (E^k_0 \cup E^k_1)$ for the routing of demand k in K in the solution $\mathcal{S}^{45}$ which means that $E^{46}_k = E'^{45}_k \cup \{e'\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^{45}$ remain the same in the solution $\mathcal{S}^{46}$, i.e., $S^{46}_k = S'^{45}_k$ for each $k \in K$, and $E^{46}_{k'} = E'^{45}_{k'}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{46}$ is clearly feasible given that

a) and a feasible path E^{46}_k is assigned to each demand $k \in K$ (routing constraint),

b) and a set of last-slots S^{46}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{46}_k$ with $|S^{46}_k| \geq 1$ (contiguity and continuity constraints),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{46}_k$ and $s" \in S^{46}_{k'}$ with $E^{46}_k \cap E^{46}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E^{46}_k} |\{s' \in S^{46}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{46}}, z^{\mathcal{S}^{46}})$ is belong to F and then to $F_{\tilde{K}}^{\tilde{G}_I^e}$ given that it is composed by $\sum_{k \in \tilde{K}} x^k_{e'} + \sum_{s=s_i+w_k-1}^{s_j} z^k_s = 2|\tilde{K}| - 1$. It follows that

$$\mu x^{\mathcal{S}'^{45}} + \sigma z^{\mathcal{S}'^{45}} = \mu x^{\mathcal{S}^{46}} + \sigma z^{\mathcal{S}^{46}} = \mu x^{\mathcal{S}'^{45}} + \mu^k_{e'} + \sigma z^{\mathcal{S}'^{45}}.$$

As a result, $\mu^k_{e'} = 0$ for demand k and an edge e'.
As e' is chosen arbitrarily for the demand k with $e \notin E^k_0 \cup E^k_1$ and $e \neq e'$ if $k \in \tilde{K}$, we iterate the same procedure for all $e" \in E \setminus (E^k_0 \cup E^k_1 \cup \{e'\})$ with $e \neq e"$ if $k \in \tilde{K}$. We conclude that for the demand k

$$\mu^k_{e'} = 0, \text{ for all } e' \in E \setminus (E^k_0 \cup E^k_1) \text{ with } e \neq e' \text{ if } k \in \tilde{K}.$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E^k_0 \cup E^k_1)$. We conclude at the end that

$$\mu^k_{e'} = 0, \text{ for all } k \in K \text{ and all } e' \in E \setminus (E^k_0 \cup E^k_1) \text{ with } e \neq e' \text{ if } k \in \tilde{K}.$$

Let's us show that $\sigma^k_s = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in \tilde{K}$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in \tilde{K}$. For that, we consider a solution $\mathcal{S}"^{45} = (E"^{45}, S"^{45})$ in which

a) a feasible path $E"^{45}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^{45}_k$ is assigned to each demand $k \in K$ along each edge $e' \in E"^{45}_k$ with $|S"^{45}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{45}_k$ and $s" \in S"^{45}_{k'}$ with $E"^{45}_k \cap E"^{45}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E"^{45}_k} |\{s' \in S"^{45}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^{45}_{k'}$ with $E"^{45}_k \cap E"^{45}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{45}_k$ assigned to the demand k in the solution $\mathcal{S}"^{45}$),

e) and there is $|\tilde{K}| - 1$ demands from the minimal cover $\tilde{K}$ denoted by $\tilde{K}"_{83}$ which are covered by the interval I (i.e., if $k \in \tilde{K}"_{83}$, this means that the demand k selects a slot s as last-slot in the solution $\mathcal{S}"^{45}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S"^{45}_k$ for each $k \in \tilde{K}"_{83}$, and for each $s' \in S"^{45}_{k'}$ for all $k' \in \tilde{K} \setminus \tilde{K}"_{83}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$,

f) and all the demands in $\tilde{K}$ pass through the edge e in the solution $\mathcal{S}"^{45}$, i.e., $e' \in E"^{45}_k$ for each $k \in \tilde{K}$.

$\mathcal{S}"^{45}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{45}}, z^{\mathcal{S}"^{45}})$ is belong to F and then to $F^{\tilde{G}^e_I}_{\tilde{K}}$ given that it is composed by $\sum_{k \in \tilde{K}} x^k_{e'} + \sum_{s=s_i+w_k-1}^{s_j} z^k_s = 2|\tilde{K}| - 1$. Based on this, we construct a solution $\mathcal{S}'^{47}$ derived from the solution $\mathcal{S}"^{45}$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}"^{45}$ (i.e., $E'^{47}_k = E"^{45}_k$ for each $k \in K \setminus \tilde{K}$, and $E'^{47}_k \neq E"^{45}_k$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E'^{47}_k is assigned to each demand $k \in \tilde{K}$ (routing constraint),

b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"^{45}_k$ and $s" \in S"^{45}_{k'}$ with $E'^{47}_k \cap E"^{45}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}, e' \in E'^{47}_k} |\{s' \in S"^{45}_k, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k \in K \setminus \tilde{K}, e' \in E"^{45}_k} |\{s' \in S"^{45}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"^{45}_{k"}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{45}_k$ assigned to the demand k in the solution $\mathcal{S}"^{45}$).

The last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}"^{45}$ remain the same in $\mathcal{S}'^{47}$, i.e., $S"^{45}_{k'} = S'^{47}_{k'}$ for each demand $k' \in K \setminus \{k\}$, and $S'^{47}_k = S"^{45}_k \cup \{s\}$ for the demand k. The solution $\mathcal{S}'^{47}$ is clearly feasible given that

a) a feasible path E'^{47}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S'^{47}_k is assigned to each demand $k \in K$ along each edge $e' \in E'^{47}_k$ with $|S'^{47}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S'^{47}_k$ and $s" \in S'^{47}_{k'}$ with $E'^{47}_k \cap E'^{47}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E'^{47}_k} |\{s' \in S'^{47}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}'^{47}}, z^{\mathcal{S}'^{47}})$ is belong to F and then to $F^{\tilde{G}^e_I}_{\tilde{K}}$ given that it is composed by $\sum_{k \in \tilde{K}} x^k_{e'} + \sum_{s=s_i+w_k-1}^{s_j} z^k_s = 2|\tilde{K}| - 1$. We have so

$$\mu x^{\mathcal{S}"^{45}} + \sigma z^{\mathcal{S}"^{45}} = \mu x^{\mathcal{S}'^{47}} + \sigma z^{\mathcal{S}'^{47}} = \mu x^{\mathcal{S}"^{45}} + \sigma z^{\mathcal{S}"^{45}} + \sigma^k_{s'} - \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E"^{45}_k} \mu^{\tilde{k}}_{e'} + \sum_{\tilde{k} \in \tilde{K}} \sum_{e" \in E'^{47}_k} \mu^{\tilde{k}}_{e"}.$$

It follows that $\sigma^k_{s'} = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in \tilde{K}$ given that $\mu^k_{e'} = 0$ for all the demands $k \in K$ and all edges $e' \in E \setminus (E^k_0 \cup E^k_1)$ with $e \neq e'$ if $k \in \tilde{K}$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in \tilde{K}$ s.t. we find

$$\sigma^k_{s'} = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } s' \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } k \notin \tilde{K}.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_{k'} - 1, ..., s_j\} \text{ if } k' \notin \tilde{K}.$$

Consequently, we conclude that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } k \notin \tilde{K}.$$

Let prove that σ_s^k for all $k \in \tilde{K}$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$ are equivalents. Consider a demand k' and a slot $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ with $k' \in \tilde{K}$. For that, we consider a solution $\tilde{\mathcal{S}}^{45} = (\tilde{E}^{45}, \tilde{S}^{45})$ in which

a) a feasible path $\tilde{E}_k^{45}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}_k^{45}$ is assigned to each demand $k \in K$ along each edge $e' \in \tilde{E}_k^{45}$ with $|\tilde{S}_k^{45}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}_k^{45}$ and $s" \in \tilde{S}_{k'}^{45}$ with $\tilde{E}_k^{45} \cap \tilde{E}_{k'}^{45} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in \tilde{E}_k^{45}} |\{s' \in \tilde{S}_k^{45}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s \in \tilde{S}_k^{45}$ with $\tilde{E}_k^{45} \cap \tilde{E}_{k'}^{45} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $\tilde{S}_{k'}^{45}$ assigned to the demand k' in the solution $\tilde{\mathcal{S}}^{45}$),

e) and there is $|\tilde{K}| - 1$ demands from the minimal cover $\tilde{K}$ denoted by $\tilde{K}_{86'}$ which are covered by the interval I (i.e., if $k \in \tilde{K}_{86'}$, this means that the demand k selects a slot s as last-slot in the solution $\tilde{\mathcal{S}}^{45}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in \tilde{S}_k^{45}$ for each $k \in \tilde{K}_{86'}$, and for each $s' \in \tilde{S}_{k'}^{45}$ for all $k' \in \tilde{K} \setminus \tilde{K}_{86'}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$,

f) and all the demands in $\tilde{K}$ pass through the edge e in the solution $\tilde{S}^{45}$, i.e., $e' \in \tilde{E}_k^{45}$ for each $k \in \tilde{K}$.

$\tilde{\mathcal{S}}^{45}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{45}}, z^{\tilde{\mathcal{S}}^{45}})$ is belong to F and then to $F_{\tilde{K}}^{\tilde{G}_I^e}$ given that it is composed by $\sum_{k \in \tilde{K}} x_{e'}^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 2|\tilde{K}| - 1$. Based on this, we construct a solution $\mathcal{S}'^{48}$ derived from the solution $\tilde{\mathcal{S}}^{45}$ by

a) with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\tilde{\mathcal{S}}^{45}$ (i.e., $E_k'^{48} = \tilde{E}_k^{45}$ for each $k \in K \setminus \tilde{K}$, and $E_k'^{48} \neq \tilde{E}_k^{45}$ for each $k \in \tilde{K}$),

b) and the last-slots assigned to the demands $K \setminus \{k, k'\}$ in $\tilde{\mathcal{S}}^{45}$ remain the same in $\mathcal{S}'^{48}$, i.e., $\tilde{S}_{k"}^{45} = S_{k"}'^{48}$ for each demand $k" \in K \setminus \{k, k'\}$,

c) and adding the slot s' as last-slot to the demand k', i.e., $S_{k'}'^{48} = \tilde{S}_{k'}^{45} \cup \{s'\}$ for the demand k',

d) and selecting a demand k from $\tilde{K}_{83}$ which allocates a last slot $s \in \tilde{S}_k^{45}$ with $s \in \{s_i + w_k + 1, ..., s_j\}$ in the solution $\tilde{\mathcal{S}}^{45}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $\tilde{S}_{k'}^{45}$ assigned to the demand k' in the solution $\tilde{\mathcal{S}}^{45}$),

e) and modifying the last-slots assigned to the demand k by adding a new last-slot $\tilde{s}$ and removing the last slot $s \in \tilde{S}_k^{45}$ with $s \in \{s_i + w_k + 1, ..., s_j\}$ and $\tilde{s} \notin \{s_i + w_k + 1, ..., s_j\}$ for the demand k with $k \in \tilde{K}$ s.t. $S_k'^{48} = (\tilde{S}_k^{45} \setminus \{s\}) \cup \{\tilde{s}\}$ s.t. $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}'^{48}$ with $E_k'^{48} \cap E_{k'}'^{48} \neq \emptyset$.

The solution $\mathcal{S}'^{48}$ is clearly feasible given that

a) a feasible path $E_k'^{48}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S'^{48}_k is assigned to each demand $k \in K$ along each edge $e' \in E'^{48}_k$ with $|S'^{48}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S'^{48}_k$ and $s" \in S'^{48}_{k'}$ with $E'^{48}_k \cap E'^{48}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E'^{48}_k} |\{s' \in S'^{48}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}'^{48}}, z^{\mathcal{S}'^{48}})$ is belong to F and then to $F^{\tilde{G}^e_I}_{\tilde{K}}$ given that it is composed by $\sum_{k \in \tilde{K}} x^k_{e'} + \sum_{s=s_i+w_k-1}^{s_j} z^k_s = 2|\tilde{K}| - 1$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{45}} + \sigma z^{\tilde{\mathcal{S}}^{45}} = \mu x^{\mathcal{S}'^{48}} + \sigma z^{\mathcal{S}'^{48}} = \mu x^{\tilde{\mathcal{S}}^{45}} + \sigma z^{\tilde{\mathcal{S}}^{45}} + \sigma^{k'}_{s'} - \sigma^k_s + \sigma^k_{\tilde{s}} - \sum_{k \in \tilde{K}} \sum_{e' \in \tilde{E}^{45}_k} \mu^k_{e'} + \sum_{k \in \tilde{K}} \sum_{e' \in E'^{48}_k} \mu^k_{e'}.$$

It follows that $\sigma^{k'}_{s'} = \sigma^k_s$ for demand k' and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $k' \in \tilde{K}$ and $s' \in \{s_i + w_{k'} + 1, ..., s_j\}$ given that $\sigma^k_{\tilde{s}} = 0$ for $\tilde{s} \notin \{s_i + w_k - 1, ..., s_j\}$ with $k \in \tilde{K}$, and $\mu^k_{e'} = 0$ for all $k \in K$ and all $e' \in E \setminus (E^k_0 \cup E^k_1)$ with $e' \neq e$ if $k \in \tilde{K}$.
Given that the pair (k, k') are chosen arbitrary in the minimal cover $\tilde{K}$, we iterate the same procedure for all pairs (k, k') s.t. we find

$$\sigma^k_s = \sigma^{k'}_{s'}, \text{for all pairs } (k, k') \in \tilde{K}$$

with $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. We re-do the same procedure for each two slots $s, s' \in \{s_i + w_k - 1, ..., s_j\}$ for each demand $k \in K$ with $k \in \tilde{K}$ s.t.

$$\sigma^k_s = \sigma^k_{s'}, \text{for all } k \in \tilde{K} \text{ and } s, s' \in \{s_i + w_k - 1, ..., s_j\}.$$

Let us prove now that μ^k_e for all $k \in K$ with $k \in \tilde{K}$ are equivalents. For that, we consider a solution $\mathcal{S}^{49} = (E^{49}, S^{49})$ defined as below

a) a feasible path E^{49}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{49}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{49}_k$ with $|S^{49}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S^{49}_k$ and $s' \in S^{49}_{k'}$ with $E^{49}_k \cap E^{49}_{k'} \neq \emptyset$ (non-overlapping constraint),

d) and there is one demand k from the minimal cover $\tilde{K}$ (i.e., $k \in \tilde{K}$ s.t. the demand k pass through the edge e in the solution $\mathcal{S}^{49}$, i.e., $e \in E^{49}_k$ for a node $k \in \tilde{K}$, and $e \notin E^{49}_{k'}$ for all $k' \in \tilde{K} \setminus \{k\}$,

e) and all the demands in $\tilde{K}$ are covered by the interval I in the solution $\mathcal{S}^{49}$, i.e., $\{s_i + w_k + 1, ..., s_j\} \cap S^{49}_k \neq \emptyset$ for each $k \in \tilde{K}$.

Obviously, $\mathcal{S}^{49}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{49}}, z^{\mathcal{S}^{49}})$ is belong to $P(G, K, \mathbb{S}, \tilde{K}, I, e)$ and then to $F^{\tilde{G}^e_I}_{\tilde{K}}$ given that it is composed by $\sum_{k \in \tilde{K}} x^k_e + \sum_{s=s_i+w_k-1}^{s_j} z^k_s = 2|\tilde{K}| - 1$.
Consider now a node k' in $\tilde{K}$ s.t. $e \notin E^{49}_{k'}$. For that, we consider a solution $\tilde{\mathcal{S}}^{49} = (\tilde{E}^{49}, \tilde{S}^{49})$ in which

a) a feasible path $\tilde{E}^{49}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}_k^{49}$ is assigned to each demand $k \in K$ along each edge $e \in \tilde{E}_k^{49}$ with $|\tilde{S}_k^{49}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}_k^{49}$ and $s" \in \tilde{S}_{k'}^{49}$ with $\tilde{E}_k^{49} \cap \tilde{E}_{k'}^{49} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in \tilde{E}_k^{49}} |\{s' \in \tilde{S}_k^{49}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and there is $|\tilde{K}| - 1$ demands from the minimal cover $\tilde{K}$ that use the edge e denoted by $\tilde{K}_{87}$ (i.e.,if $k \in \tilde{K}_{87}$, this means that the demand k pass through the edge e in the solution $\tilde{S}^{49}$, i.e., $e \in \tilde{E}_k^{49}$ for each $k \in \tilde{K}_{87}$, and $e \notin \tilde{E}_{k'}^{49}$ for all $k' \in \tilde{K}\tilde{K}_{87}$,

e) and all the demands in $\tilde{K}$ are covered by the interval I in the solution $\tilde{S}^{49}$, i.e., $\{s_i + w_k + 1, ..., s_j\} \cap \tilde{S}_k^{49} \neq \emptyset$ for each $k \in \tilde{K}$.

$\tilde{\mathcal{S}}^{49}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{S}^{49}}, z^{\tilde{S}^{49}})$ is belong to F and then to $F_{\tilde{K}}^{\tilde{G}_I^e}$ given that it is composed by $\sum_{k \in \tilde{K}} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 2|\tilde{K}| - 1$. Based on this, we derive a solution $\mathcal{S}"^{50} = (E"^{50}, S"^{50})$ from the solution $\tilde{\mathcal{S}}^{49}$ by

a) the paths assigned to the demands $K \setminus \{k, k'\}$ in $\tilde{\mathcal{S}}^{49}$ remain the same in $\mathcal{S}"^{50}$ (i.e., $E"^{50}_{k"} = \tilde{E}_{k"}^{49}$ for each $k" \in K \setminus \{k, k'\}$),

b) without modifying the last-slots assigned to the demands K in $\tilde{\mathcal{S}}^{49}$, i.e., $\tilde{S}_k^{49} = S"^{50}_k$ for each demand $k \in K$,

c) modifying the path assigned to the demand k' in $\tilde{\mathcal{S}}^{49}$ from $\tilde{E}_{k'}^{49}$ to a path $E"^{50}_{k'}$ passed through the edge e (i.e., $e \in E"^{50}_{k'}$) with $k' \in \tilde{K}$ s.t. $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and each $s' \in \tilde{S}_{k'}^{49}$ and each $s \in \tilde{S}_k^{49}$ with $\tilde{E}_k^{49} \cap E"^{50}_{k'} \neq \emptyset$,

d) selecting a demand k in $\tilde{K}_{87}$ which use the edge e in the solution $\mathcal{S}^{49}$,

e) modifying the path assigned to the selected demand k in $\tilde{\mathcal{S}}^{49}$ with $e \in \tilde{E}_k^{49}$ and $k \in \tilde{K}$ from $\tilde{E}_k^{49}$ to a path $E"^{50}_k$ without passing through the edge e (i.e., $e \notin E"^{50}_k$) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k"} + 1, ..., s'\} = \emptyset$ for each $k" \in K \setminus \{k, k'\}$ and each $s \in \tilde{S}_k^{49}$ and each $s' \in \tilde{S}_{k"}^{49}$ with $\tilde{E}_{k"}^{49} \cap E"^{50}_k \neq \emptyset$, and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $s \in \tilde{S}_k^{49}$ and each $s' \in \tilde{S}_{k'}^{49}$ with $E"^{50}_{k"} \cap E"^{50}_k \neq \emptyset$.

The solution $\mathcal{S}"^{50}$ is feasible given that

a) a feasible path $E"^{50}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^{50}_k$ is assigned to each demand $k \in K$ along each edge $e \in E"^{50}_k$ with $|S"^{50}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{50}_k$ and $s" \in S"^{50}_{k'}$ with $E"^{50}_k \cap E"^{50}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E"^{50}_k} |\{s' \in S"^{50}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}"^{50}}, z^{\mathcal{S}"^{50}})$ is belong to F and then to $F_{\tilde{K}}^{\tilde{G}_I^e}$ given that it is composed by $\sum_{k \in \tilde{K}} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 2|\tilde{K}| - 1$. We then obtain that

$$\mu x^{\tilde{S}^{49}} + \sigma z^{\tilde{S}^{49}} = \mu x^{\mathcal{S}^{50}} + \sigma z^{\mathcal{S}^{50}} = \mu x^{\tilde{S}^{49}} + \sigma z^{\tilde{S}^{49}} + \mu_e^{k'} - \mu_e^k$$
$$+ \sum_{e" \in E"^{50}_{k'} \setminus \{e\}} \mu_{e"}^{k'} - \sum_{e" \in \tilde{E}_{k'}^{49}} \mu_{e"}^{k'} + \sum_{e" \in E"^{50}_k} \mu_{e"}^k - \sum_{e" \in \tilde{E}_k^{49} \setminus \{e\}} \mu_{e"}^k.$$

It follows that $\mu_e^{k'} = \mu_e^k$ for demand k' and a edge $e' \in E \setminus (E_0^{k'} \cup E_1^{k'})$ with $v_{k',e'} \in \tilde{K}$ given that $\mu_{e"}^k = 0$ for all $k \in K$ and all $e" \in E \setminus (E_0^k \cup E_1^k)$ with $k \notin \tilde{K}$.

Given that the pair (k, k') are chosen arbitrary in the minimal cover $\tilde{K}$, we iterate the same procedure for all pairs (k, k') s.t. we find

$$\mu_e^k = \mu_e^{k'}, \text{for all pairs } (k, k') \in \tilde{K}.$$

Furthermore, let prove that all σ_s^k and μ_e^k are equivalents for all $k \in \tilde{K}$ and $s \in \{s_i + w_k - 1, ..., s_j\}$. For that, we consider for each demand k' with $k' \in \tilde{K}$, a solution $\mathcal{S}^{51} = (E^{51}, S^{51})$ derived from the solution $\tilde{\mathcal{S}}^{49}$ as below

a) the paths assigned to the demands $K \setminus \{k'\}$ in $\tilde{\mathcal{S}}^{49}$ remain the same in $\mathcal{S}^{51}$ (i.e., $E_{k''}^{51} = \tilde{E}_{k''}^{49}$ for each $k'' \in K \setminus \{k'\}$),

b) without modifying the last-slots assigned to the demands $K \setminus \{k\}$ in $\tilde{\mathcal{S}}^{49}$, i.e., $\tilde{S}_{k''}^{49} = S_{k''}^{51}$ for each demand $k'' \in K \setminus \{k\}$,

c) modifying the set of last-slots assigned to the demand k' in $\tilde{\mathcal{S}}^{49}$ from $\tilde{S}_{k'}^{49}$ to $S_{k'}^{51}$ s.t. $S_{k'}^{51} \cap \{s_i + w_{k'} - 1, ..., s_j\} = \emptyset$.

Hence, there are $|\tilde{K}| - 1$ demands from $\tilde{K}$ that are covered by the interval I (i.e., all the demands in $C \setminus \{k'\}$), and all the demands in $\tilde{K}$ use the edge e in the solution $\mathcal{S}^{51}$. The solution $\mathcal{S}^{51}$ is then feasible given that

a) a feasible path E_k^{51} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{51} is assigned to each demand $k \in K$ along each edge $e \in E_k^{51}$ with $|S_k^{51}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s'' - w_{k'} + 1, ..., s''\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{51}$ and $s'' \in S_{k'}^{51}$ with $E_k^{51} \cap E_{k'}^{51} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s'' \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{51}} |\{s' \in S_k^{51}, s'' \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\sum_{k \in \tilde{K}} |E_k^{51} \cap \{e\}| + |S_k^{51} \cap \{s_i + w_k - 1, ..., s_j\}| = 2|\tilde{K}| - 1$.

The corresponding incidence vector $(x^{\mathcal{S}^{51}}, z^{\mathcal{S}^{51}})$ is belong to F and then to $F_{\tilde{K}}^{\tilde{G}_I^e}$ given that it is composed by $\sum_{k \in \tilde{K}} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 2|\tilde{K}| - 1$. We then obtain that

$$\mu x^{\tilde{\mathcal{S}}^{49}} + \sigma z^{\tilde{\mathcal{S}}^{49}} = \mu x^{\mathcal{S}^{51}} + \sigma z^{\mathcal{S}^{51}} = \mu x^{\tilde{\mathcal{S}}^{49}} + \sigma z^{\tilde{\mathcal{S}}^{49}} + \mu_e^{k'} - \sigma_s^{k'} + \sum_{e'' \in E_{k'}^{51} \setminus \{e\}} \mu_{e''}^{k'} - \sum_{e'' \in \tilde{E}_{k'}^{49}} \mu_{e''}^{k'}.$$

It follows that $\mu_e^{k'} = \sigma_s^{k'}$ for demand k' and slot $s \in \{s_i + w_{k'} - 1, ..., s_j\}$ given that $\mu_{e''}^k = 0$ for all $k \in K$ and all $e'' \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e''$ if $k \in \tilde{K}$. Moreover, by doing the same thing over all slots $s \in \{s_i + w_{k'} - 1, ..., s_j\}$, we found that

$$\mu_e^{k'} = \sigma_s^{k'}, \text{for all } s \in \{s_i + w_{k'} - 1, ..., s_j\}.$$

Given that k' is chosen arbitrarily in $\tilde{K}$, we iterate the same procedure for all $k \in \tilde{K}$ to show that

$$\mu_e^k = \sigma_s^k, \text{for all } k \in \tilde{K} \text{ and all } s \in \{s_i + w_k - 1, ..., s_j\}.$$

Based on this, and given that all μ_e^k are equivalents for all $k \in \tilde{K}$, and that σ_s^k are equivalents for all $k \in \tilde{K}$ and $s \in \{s_i + w_{k'} - 1, ..., s_j\}$, we obtain that

$$\mu_e^k = \sigma_s^{k'}, \text{ for all } k, k' \in \tilde{K} \text{ and all } s \in \{s_i + w_{k'} - 1, ..., s_j\}.$$

Consequently, we conclude that

$$\mu_e^k = \sigma_s^{k'} = \rho, \text{ for all } k, k' \in \tilde{K} \text{ and all } s \in \{s_i + w_{k'} - 1, ..., s_j\}.$$

On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\mu_{e'}^{k'} = \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_0^{k'},$$
$$\mu_{e'}^{k'} = \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_1^{k'},$$
$$\sigma_{s'}^{k'} = \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}.$$

We conclude that for each $k' \in K$ and $e' \in E$

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'}, \text{ if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'}, \text{ if } e' \in E_1^{k'}, \\ \rho, \text{ if } k' \in \tilde{K} \text{ and } e' = e, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\} \\ \rho, \text{if } k \in \tilde{K} \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{k \in \tilde{K}} \rho \alpha_e^k + \sum_{s=s_i+w_k-1}^{s_j} \rho \beta_s^k + \gamma Q.$

□

Theorem 2.4.4. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $j \geq i + 1$. Let $\tilde{K}$ be a subset of demands of K, and $\tilde{K}_e$ be a subset of demands in $K_e \setminus \tilde{K}$ s.t.*

a) $\sum_{k \in \tilde{K}} w_k \geq |I| + 1,$

b) $\sum_{k \in \tilde{K} \setminus \{k'\}} w_k \leq |I|$ *for each* $k' \in \tilde{K}$,

c) $\sum_{k \in \tilde{K}} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus \tilde{K}} w_{k'},$

d) $e \notin E_0^k$ *for each demand* $k \in \tilde{K}$,

e) $\tilde{K} \geq 3$,

f) $(k, k') \notin K_c^e$ *for each pair of demands* (k, k') *in* $\tilde{K}$,

g) $w_{k'} \geq w_k$ *for each* $k \in \tilde{K}$ *and each* $k' \in \tilde{K}_e$.

Then, the inequality (2.31) is facet defining for the polytope $P(G, K, \mathbb{S}, \tilde{K}, \tilde{K}_e, I, e)$ iff there does not exist an interval of contiguous slots I' in $[1, \bar{s}]$ with $I \subset I'$ s.t. $\tilde{K}$ defines a minimal cover for the interval I', where

$$P(G, K, \mathbb{S}, \tilde{K}, \tilde{K}_e, I, e) = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{k' \in K_e \setminus (\tilde{K} \cup \tilde{K}_e)} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = 0\}.$$

Proof. **Necessity**
If there exists an interval of contiguous slots I' in $[1,\bar{s}]$ with $I \subset I'$ s.t. $\tilde{K}$ defines a minimal cover for the interval I'. This means that $\{s_i+w_k-1,...,s_j\} \subset I'$. As a result, the inequality (2.31) induced by the minimal cover $\tilde{K}$ for the interval I, it is dominated by another inequality (2.31) induced by the same minimal cover $\tilde{K}$ for the interval I'. Hence, the inequality (2.31) cannot be facet defining for the polytope $P(G,K,\mathbb{S},\tilde{K},\tilde{K}_e,I,e)$.
Sufficiency.
Let $F^{\tilde{G}_I^e}_{\tilde{K},\tilde{K}_e}$ denote the face induced by the inequality (2.30), which is given by

$$F^{\tilde{G}_I^e}_{\tilde{K},\tilde{K}_e} = \{(x,z) \in P(G,K,\mathbb{S},\tilde{K},\tilde{K}_e,I,e) : \sum_{k\in\tilde{K}} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{k'\in\tilde{K}_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = 2|\tilde{K}|-1\}.$$

In order to prove that inequality $\sum_{k\in\tilde{K}} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 2|\tilde{K}|-1$ is facet defining for $P(G,K,\mathbb{S},\tilde{K},\tilde{K}_e,I,e)$, we start checking that $F^{\tilde{G}_I^e}_{\tilde{K},\tilde{K}_e}$ is a proper face, and $F^{\tilde{G}_I^e}_{\tilde{K},\tilde{K}_e} \neq P(G,K,\mathbb{S},\tilde{K},\tilde{K}_e,I,e)$.
We construct a solution $\mathcal{S}^{52}=(E^{52},S^{52})$ as below

a) a feasible path E_k^{52} is assigned to each demand $k \in K$ (routing constraint),
b) a set of last-slots S_k^{52} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{52}$ with $|S_k^{52}| \geq 1$ (contiguity and continuity constraints),
c) $\{s-w_k+1,...,s\} \cap \{s'-w_{k'}+1,...,s'\} = \emptyset$ for each $k,k' \in K$ and each $s \in S_k^{52}$ and $s' \in S_{k'}^{52}$ with $E_k^{52} \cap E_{k'}^{52} \neq \emptyset$ (non-overlapping constraint),
d) and there is $|\tilde{K}|-1$ demands from the minimal cover $\tilde{K}$ denoted by $\tilde{K}_{90}$ which are covered by the interval I (i.e., if $k \in \tilde{K}_{90}$, this means that the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{52}$ with $s \in \{s_i+w_k-1,...,s_j\}$, i.e., $s \in S_k^{52}$ for each $k \in \tilde{K}_{90}$, and for each $s' \in S_{k'}^{52}$ for all $k' \in \tilde{K} \setminus \tilde{K}_{90}$ we have $s' \notin \{s_i+w_{k'}-1,...,s_j\}$,
e) and all the demands in $\tilde{K}$ pass through the edge e in the solution $\mathcal{S}^{52}$, i.e., $e \in E_k^{52}$ for each $k \in \tilde{K}$.

Obviously, $\mathcal{S}^{52}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{52}}, z^{\mathcal{S}^{52}})$ is belong to $P(G,K,\mathbb{S},\tilde{K},\tilde{K}_e,I,e)$ and then to $F^{\tilde{G}_I^e}_{\tilde{K},\tilde{K}_e}$ given that it is composed by $\sum_{k\in\tilde{K}} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 2|\tilde{K}|-1$. As a result, $F^{\tilde{G}_I^e}_{\tilde{K},\tilde{K}_e}$ is not empty (i.e., $F^{\tilde{G}_I^e}_{\tilde{K},\tilde{K}_e} \neq \emptyset$). Furthermore, given that $s \in \{s_i+w_k-1,...,s_j\}$ for each $k \in \tilde{K}$, this means that there exists at least one feasible slot assignment S_k for the demands k in $\tilde{K}$ with $s \notin \{s_i+w_k-1,...,s_j\}$ for each $s \in S_k$ and each $k \in \tilde{K}$. This means that $F^{\tilde{G}_I^e}_{\tilde{K},\tilde{K}_e} \neq P(G,K,\mathbb{S},\tilde{K},\tilde{K}_e,I,e)$.
We denote the inequality $\sum_{k\in\tilde{K}} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 2|\tilde{K}|-1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G,K,\mathbb{S},\tilde{K},\tilde{K}_e,I,e)$. Suppose that $F^{\tilde{G}_I^e}_{\tilde{K},\tilde{K}_e} \subset F = \{(x,z) \in P(G,K,\mathbb{S},\tilde{K},\tilde{K}_e,I,e) : \mu x + \sigma z = \tau\}$. We use the same proof of theorem 2.4.3 by showing that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1,\gamma_2,\gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k\in K}|E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k\in K}|E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k\in K}(w_k-1)}$) s.t. $(\mu,\sigma) = \rho(\alpha,\beta) + \gamma Q$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k,...,\bar{s}\}$ with $s \notin \{s_i+w_k-1,...,s_j\}$ if $k \in \tilde{K} \cup \tilde{K}_e$ as shown in the proof of theorem 2.4.3,
b) and σ_s^k are equivalents for all $k \in \tilde{K}$ and all $s \in \{s_i+w_k-1,...,s_j\}$ as shown in the proof of theorem 2.4.3,

c) and σ_s^k are equivalents for all $k \in \tilde{K}_e$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$ as shown in the proof of theorem 2.4.3,

d) and $\mu_{e'}^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e'$ if $k \in \tilde{K}$ as shown in the proof of theorem 2.4.3,

e) and all μ_e^k are equivalents for the set of demands in $\tilde{K}$ as shown in the proof of theorem 2.4.3,

f) and σ_s^k for all $k \in \tilde{K} \cup \tilde{K}_e$ $s \in \{s_i + w_k - 1, ..., s_j\}$ are equivalents with μ_e^k for all $k \in \tilde{K}$ as shown in the proof of theorem 2.4.3.

At the end, we concluded that for each $k' \in K$ and $e' \in E$

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'}, \text{ if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'}, \text{ if } e' \in E_1^{k'}, \\ \rho, \text{ if } k' \in \tilde{K} \text{ and } e' = e, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\} \\ \rho, \text{if } k \in \tilde{K} \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ \rho, \text{if } k \in \tilde{K}_e \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{k \in \tilde{K}} \rho \alpha_e^k + \sum_{s=s_i+w_k-1}^{s_j} \rho \beta_s^k + \sum_{k' \in \tilde{K}_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} \rho \beta_{s'}^{k'} + \gamma Q.$

□

2.4.3 Edge-Interval-Clique Inequalities

In what follows, we need to introduce some notions of graph theory related to conflict graphs to provide some valid inequalities for $P(G, K, \mathbb{S})$.

Definition 2.4.4. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$. Consider the conflict graph $\tilde{G}_I^e$ defined as follows. For each demand $k \in K$ with $w_k \leq |I|$ and $e \notin E_0^k$, consider a node v_k in $\tilde{G}_I^e$. Two nodes v_k and $v_{k'}$ are linked by an edge in $\tilde{G}_I^e$ if $w_k + w_{k'} > |I|$ and $(k, k') \notin K_c^e$. This is equivalent to say that two linked nodes v_k and $v_{k'}$ means that the two demands k, k' define a minimal cover for the interval I over edge e.*

For an edge $e \in E$, the conflict graph $\tilde{G}_I^e$ is a threshold graph with threshold value equals to $t = |I|$ s.t. for each node v_k with $e \notin E_0^k \cup E_1^k$, we associate a positive weight $\tilde{w}_{v_k} = w_k$ s.t. all two nodes v_k and $v_{k'}$ are linked by an edge if and only if $\tilde{w}_{v_k} + \tilde{w}_{v_{k'}} > t$ which is equivalent to the conflict graph $\tilde{G}_I^e$.

Proposition 2.4.10. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots. Let C be a clique in the conflict graph $\tilde{G}_I^e$ with $|C| \geq 3$, and $\sum_{v_k \in C} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus C} w_{k'}$. Then, the inequality*

$$\sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |C| + 1, \tag{2.32}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. For each edge $e \in E$ and interval of contiguous slots $I \subseteq \mathbb{S}$, the inequality (2.32) ensures that if the set of demands in the clique C pass through edge e, they cannot share the interval $I = [s_i, s_j]$ over edge e. This means that there is at most one demand from the demands in C that can be totally covered by the interval I over the edge e (i.e., all the slots assigned to the demand are in I). We start the proof by assuming that the inequality (2.32) is not valid for $P(G, K, \mathbb{S})$. It follows that there exists a C-RSA solution S in which $\{s_i + w_k - 1, ..., s_j\} \cap S_k = \emptyset$ for each demand $v_k \in C$ s.t.

$$\sum_{v_k \in C} x_e^k(S) + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) > |C| + 1.$$

Since $\{s_i+w_k-1, ..., s_j\} \cap S_k = \emptyset$ for each demand $v_k \in C$ this means that $\sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) = 0$, and taking into account that $x_e^k(S) \leq 1$ for each $v_k \in C$, it follows that

$$\sum_{v_k \in C} x_e^k(S) \leq |C| + 1,$$

which contradicts our hypothesis, i.e., $\sum_{v_k \in C} x_e^k(S) + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) > |C| + 1$.
On the other hand, one can imagine another case also when $\{k \in K \text{ s.t. } v_k \in C\} \cap K_e = \emptyset$, it follows that there exists a C-RSA solution S' in which $E_k \cap \{e\} = \emptyset$ for each demand $v_k \in C$, which means that $\sum_{v_k \in C} x_e^k(S') = 0$ s.t.

$$\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S') > |C| + 1.$$

Given that $2w_k > |I|$ for each $v_k \in C$. As a result, $\sum_{s=s_i+w_k-1}^{s_j} z_s^k(S') \leq 1$ for each demand $v_k \in C$. It follows that

$$\sum_{k' \in C \setminus \{k\}} \sum_{s=s_i+w_{k'}-1}^{s_j} z_s^{k'}(S') \leq |C| + 1,$$

which contradicts what we supposed before, i.e., $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S') > |C| + 1$.
Hence $\sum_{v_k \in C} |E_k \cap \{e\}| + \sum_{v_k \in C} |S_k \cap \{s_i + w_k - 1, ..., s_j\}| \leq |C| + 1$.
Furthermore, the inequality (2.32) can be shown as Chvatal-Gomory cuts using Chvatal-Gomory and recurrence procedures. For any subset of demands $C \subseteq K$ with $w_k + w_{k'} > |I|$ for each pair of demands $k, k' \in C$, and $e \notin E_0^k$, $w_k \leq |I|$ for each demand $v_k \in C$, and $\sum_{v_k \in C} w_k \leq \bar{s} - \sum_{v_{k'} \in K_e \setminus C} w_{k'}$, by recurrence procedure we get that for all $K' \subseteq C$ with $|K'| = |C| - 1$

$$\sum_{v_k \in C'} x_e^k + \sum_{v_k \in C'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |K'| + 1.$$

By adding the previous inequalities for all $K' \subseteq C$ with $|K'| = |C| - 1$, we get

$$\sum_{\substack{K' \subseteq C \\ |K'| = |C|-1}} \sum_{v_k \in C'} x_e^k + \sum_{\substack{K' \subseteq C \\ |K'| = |C|-1}} \sum_{v_k \in C'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \sum_{\substack{K' \subseteq C \\ |K'| = |C|-1}} (|K'| + 1).$$

Note that for each demand k with $v_k \in C$, the variable x_e^k and the sum $\sum_{s=s_i+w_k-1}^{s_j} z_s^k$ appear $(\binom{|C|}{|C|-1} - 1)$ times in the previous sum. It follows that

$$\sum_{v_k \in C} (\binom{|C|}{|C|-1} - 1) x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} .(\binom{|C|}{|C|-1} - 1) z_s^k \leq \binom{|C|}{|C|-1} |C|.$$

Given that $(\binom{|C|}{|C|-1} - 1) = |C| - 1$, we found that

$$\sum_{v_k \in C} (|C| - 1) x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} (|C| - 1) z_s^k \leq |C|^2.$$

By dividing the two sides of the previous sum by $|C| - 1$, we have

$$\sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \left\lfloor \frac{|C|^2}{|C|-1} \right\rfloor \Rightarrow \sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \left\lfloor |C| \frac{|C|}{|C|-1} \right\rfloor$$
$$\Rightarrow \sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \left\lfloor |C| \frac{|C|-1+1}{|C|-1} \right\rfloor.$$

By doing the following simplification

$$\sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \left\lfloor |C| \frac{|C|-1}{|C|-1} + \frac{|C|}{|C|-1} \right\rfloor \Rightarrow \sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \left\lfloor |C| + \frac{|C|}{|C|-1} \right\rfloor.$$

As a result,

$$\sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |C| + \left\lfloor \frac{|C|}{|C|-1} \right\rfloor \Rightarrow \sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |C| + 1$$
$$\text{given that } \left\lfloor \frac{|C|}{|C|-1} \right\rfloor = 1.$$

We conclude at the end that the inequality (2.32) is valid for $P(G, K, \mathbb{S})$. □

Remark 2.4.2. *Consider an edge e and an interval of contiguous slots $I = [s_i, s_j]$. Let $\tilde{K}$ be a subset of demands in K satisfying the conditions of validity of the inequalities (2.27) and (2.32). Then, the inequality (2.32) is dominated by the inequality (2.27) associated with slot $s" = s_i + \min_{k \in \tilde{K}} w_k + 1$ iff $|\{s_i + w_k, ., s_j\}| \leq w_k$ for each demand $k \in \tilde{K}$.*

Proof. We know from inequalities (2.27) and (2.32) that

$$\sum_{k \in \tilde{K}} x_e^k + \sum_{k \in \tilde{K}} \sum_{s'=s"}^{\min(s"+w_k-1, \bar{s})} z_{s'}^k \leq |\tilde{K}| + 1 \text{ and } \sum_{k \in \tilde{K}} x_e^k + \sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |\tilde{K}| + 1.$$

Sufficiency.
First, assume that the inequality (2.27) dominates the inequality (2.32). This means that there exists a slot $s" \in \mathbb{S}$ s.t.

$$\sum_{k \in \tilde{K}} x_e^k + \sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \preceq \sum_{k \in \tilde{K}} x_e^k + \sum_{k \in \tilde{K}} \sum_{s'=s"}^{\min(s"+w_k-1, \bar{s})} z_{s'}^k \leq |\tilde{K}| + 1.$$

By removing the sum $\sum_{k\in\tilde{K}} x_e^k$ from the two sides of the previous comparison

$$\sum_{k\in\tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \preceq \sum_{k\in\tilde{K}} \sum_{s'=s"}^{\min(s"+w_k-1,\bar{s})} z_{s'}^k.$$

Given that the demands $\tilde{K}$ are independants, we found that

$$\sum_{s=s_i+w_k-1}^{s_j} z_s^k \preceq \sum_{s'=s"}^{\min(s"+w_k-1,\bar{s})} z_{s'}^k \text{ for each } k\in\tilde{K}.$$

It follows that $\{s_i+w_k-1,...,s_j\} = [s_i+w_k-1, s_j] \subseteq [s", min(s"+w_k-1,\bar{s})]$ for each demand $k\in\tilde{K}$. Taking into account that $|\{s",...,min(s"+w_k-1,\bar{s})\}| \leq w_k$ for each $k\in\tilde{K}$, this means that

$$|\{s_i+w_k-1,...,s_j\}| = s_j-(s_i+w_k-1)+1 \leq w_k \text{ for each } k\in\tilde{K},$$

that which was to be demonstrated.
Neccessity.
Assume that $|\{s_i+w_k-1,...,s_j\}| \leq w_k$ for each demand $k\in\tilde{K}$. Given that $\{s_i+w_k-1,...,s_j\} = [s_i+w_k-1,s_j]$ and $s_i+w_k-1 \geq s_i+\min\limits_{k'\in\tilde{K}} w_{k'}-1$ for each demand $k\in\tilde{K}$, this means that $[s_i+w_k-1,s_j] \subseteq [s_i+\min\limits_{k'\in\tilde{K}} w_{k'}-1, s_j]$ for each demand $k\in\tilde{K}$.
Let $\tilde{k}$ be a demand in $\arg\min\{k\in\tilde{K}, w_k = \min\limits_{k'\in\tilde{K}} w_{k'}\}$. We know that $|I_{\tilde{k}}| \leq w_{\tilde{k}}$, i.e., $|\{s_i+\min\limits_{k'\in\tilde{K}} w_{k'}-1,s_j\}| = s_j-(s_i+\min\limits_{k'\in\tilde{K}} w_{k'}-1)+1 \leq w_k$ for each demand $k\in\tilde{K}$. This implies that $(s_i+\min\limits_{k'\in\tilde{K}} w_{k'}-1)+w_k-1 \geq s_j$ for each demand $k\in\tilde{K}$. It follows that $[s_i+\min\limits_{k'\in\tilde{K}} w_{k'}-1,s_j] \subseteq [s_i+\min\limits_{k'\in\tilde{K}} w_{k'}-1, s_i+\min\limits_{k'\in\tilde{K}} w_{k'}+w_k-2]$ for each demand $k\in\tilde{K}$. As a result, we obtain that for each demand $k\in\tilde{K}$

$$\begin{aligned} \{s_i+w_k-1,...,s_j\} = [s_i+w_k-1,s_j] &\subseteq [s_i+\min_{k'\in\tilde{K}} w_{k'}-1,s_j] \\ \text{and } [s_i+\min_{k'\in\tilde{K}} w_{k'}-1,s_j] &\subseteq [s_i+\min_{k'\in\tilde{K}} w_{k'}-1, s_i+\min_{k'\in\tilde{K}} w_{k'}+w_k-2] \\ \implies \{s_i+w_k-1,...,s_j\} = [s_i+w_k-1,s_j] &\subseteq [s_i+\min_{k'\in\tilde{K}} w_{k'}-1, s_i+\min_{k'\in\tilde{K}} w_{k'}+w_k-2]. \end{aligned}$$

By giving $s" = s_i+\min\limits_{k'\in\tilde{K}} w_{k'}-1$, it is equivalent to say that

$$\{s_i+w_k-1,...,s_j\} = [s_i+w_k-1,s_j] \subseteq [s", s"+w_k-1] \text{ for each } k\in\tilde{K}$$

We know from (2.27) that

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s"}^{\min(s"+w_k-1,\bar{s})} z_{s'}^k \leq |\tilde{K}|+1.$$

Taking into account that $[s",s"+w_k-1] = [s",s_i+w_k-2] \cup [s_i+w_k-1,s_j] \cup [s_j+1,s"+w_k-1]$

for each $k \in \tilde{K}$, it follows that

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s"}^{\min(s"+w_k-1,\bar{s})} z_{s'}^k = \sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} [\sum_{s'=s"}^{s_i+w_k-2} z_{s'}^k + \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + \sum_{s'=s_j+1}^{\min(s"+w_k-1,\bar{s})} z_{s'}^k] \leq |\tilde{K}|$$

$$\implies \sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s"}^{\min(s"+w_k-1,\bar{s})} z_{s'}^k = \sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s"}^{s_i+w_k-2} z_{s'}^k + \sum_{s'\in\{s_i+w_k-1,...,s_j\}} z_{s'}^k + \sum_{s'=s_j+1}^{\min(s"+w_k-1,\bar{s})} z_{s'}^k \leq |\tilde{K}|$$

$$\implies \sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s"}^{\min(s"+w_k-1,\bar{s})} z_{s'}^k = \sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'\in\{s_i+w_k-1,...,s_j\}} z_{s'}^k + \sum_{s'=s"}^{s_i+w_k-2} z_{s'}^k + \sum_{s'=s_j+1}^{\min(s"+w_k-1,\bar{s})} z_{s'}^k \leq |\tilde{K}| +$$

which shows that the inequality (2.27) dominates the inequality (2.32)

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s\in\{s_i+w_k-1,...,s_j\}} z_s^k \preceq \sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s"}^{\min(s"+w_k-1,\bar{s})} z_{s'}^k \leq |\tilde{K}| + 1.$$

□

Remark 2.4.3. *Consider an edge e and an interval of contiguous slots $I = [s_i, s_j]$. Let $\tilde{K}$ be a subset of demands in K satisfying the conditions of validity of the inequalities* (2.27) *and* (2.32). *Then, the inequality* (2.32) *dominates the inequality* (2.27) *associated with each slot $s" \in I$ iff $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ for each demand $k \in \tilde{K}$ and $s" \in \{s_i + \max_{k'\in\tilde{K}} w_k - 1, ..., s_j - \max_{k\in\tilde{K}} w_k + 1\}$.*

Proof. We know from inequalities (2.27) and (2.32) that

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq |\tilde{K}| + 1 \text{ and } \sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s\in\{s_i+w_k-1,...,s_j\}} z_s^k \leq |\tilde{K}| + 1.$$

Neccessity.
First, assume that $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ and $s" \in \{s_i + \max_{k'\in\tilde{K}} w_k - 1, ..., s_j - \max_{k\in\tilde{K}} w_k + 1\}$ for each demand $k \in \tilde{K}$, this means that

$$s" \geq s_i + w_k - 1 \text{ and } s" \leq s_j - w_k + 1 \text{ for each } k \in \tilde{K}$$
$$\implies s" \geq s_i + w_k - 1 \text{ and } s" + w_k - 1 \leq s_j \text{ for each } k \in \tilde{K}$$
$$\implies [s", s + w_k - 1] \subseteq [s_i + w_k - 1, s_j] \text{ for each } k \in \tilde{K}$$
$$\implies [s", s + w_k - 1] \subseteq \{s_i + w_k - 1, ..., s_j\} \text{ with } |\{s_i + w_k - 1, ..., s_j\}| \geq w_k \text{ for each } k \in \tilde{K}.$$

This means that $\{s_i + w_k - 1, ..., s_j\}$ can be written as unions of sub-intervals, i.e., $\{s_i + w_k - 1, ..., s_j\} = [s_i + w_k - 1, s" - 1] \cup [s", s" + w_k - 1] \cup [s" + w_k - 1, s_j]$. As a result,

$$\sum_{s\in\{s_i+w_k-1,...,s_j\}} z_s^k = \sum_{s=s_i+w_k-1}^{s"-1} z_{s'}^k + \sum_{s'=s"}^{s"+w_k-1} z_{s'}^k + \sum_{s'=s"+w_k}^{s_j} z_{s'}^k \text{ for each } k \in \tilde{K}.$$

By doing a sum over all the demands in $\tilde{K}$, it follows that

$$\sum_{k\in\tilde{K}} \sum_{s\in\{s_i+w_k-1,...,s_j\}} z_s^k = \sum_{k\in\tilde{K}} \sum_{s=s_i+w_k-1}^{s"-1} z_{s'}^k + \sum_{s'=s"}^{s"+w_k-1} z_{s'}^k + \sum_{s'=s"+w_k}^{s_j} z_{s'}^k.$$

As a result,

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s\in\{s_i+w_k-1,...,s_j\}} z_s^k = \sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s=s_i+w_k-1}^{s"-1} z_{s'}^k + \sum_{s'=s"}^{s"+w_k-1} z_{s'}^k + \sum_{s'=s"+w_k}^{s_j} z_{s'}^k \leq |\tilde{K}|+1$$
$$\implies \sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s"}^{s"+w_k-1} z_{s'}^k \leq \sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s\in\{s_i+w_k-1,...,s_j\}} z_s^k \leq |\tilde{K}|+1.$$

As a result, the inequality (2.32) dominates the inequality (2.27).
Sufficiency.
We assume that the inequality (2.32) dominates the inequality (2.27)

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s"}^{s"+w_k-1} z_{s'}^k \preceq \sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s\in\{s_i+w_k-1,...,s_j\}} z_s^k.$$

By removing the sum $\sum_{k\in\tilde{K}} x_e^k$ from two sides of the previous comparison, we found

$$\sum_{k\in\tilde{K}} \sum_{s'=s"}^{s"+w_k-1} z_{s'}^k \preceq \sum_{k\in\tilde{K}} \sum_{s\in\{s_i+w_k-1,...,s_j\}} z_s^k.$$

Taking into account that the demands in $\tilde{K}$ are indepedants, it follows that

$$\sum_{s'=s"}^{s"+w_k-1} z_{s'}^k \preceq \sum_{s\in\{s_i+w_k-1,...,s_j\}} z_s^k \text{ for each demand } k \in \tilde{K}.$$

Hence, $[s", s"+w_k-1] \subseteq [s_i+w_k-1, s_j]$ for each $k \in \tilde{K}$. This means that

$$|\{s_i+w_k-1,...,s_j\}| \geq w_k \text{ and } s" \geq s_i+w_k-1 \text{ and } s"+w_k-1 \leq s_j \text{ for each } k \in \tilde{K}$$
$$\implies s" \geq s_i + \max_{k\in\tilde{K}} w_k - 1 \text{ and } s" \leq s_j - \max_{k\in\tilde{K}} w_k + 1$$
$$\implies s" \in \{s_i + \max_{k\in\tilde{K}} w_k - 1, ..., s_j - \max_{k\in\tilde{K}} w_k + 1\}$$

As a result, $|\{s_i+w_k-1,...,s_j\}| \geq w_k$ for each demand $k \in \tilde{K}$, and $s" \in \{s_i + \max_{k\in\tilde{K}} w_k - 1, ..., s_j - \max_{k\in\tilde{K}} w_k + 1\}$ that which was to be demonstrated, and which ends the proof. □

Moreover, the inequality (2.32) can be strengthened as follows.

Proposition 2.4.11. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots. Let C be a clique in the conflict graph $\tilde{G}_I^e$ with $|C| \geq 3$, and $\sum_{v_k\in C} w_k \leq \bar{s} - \sum_{k'\in K_e\setminus C} w_{k'}$. Let $C_e \subseteq K_e \setminus C$ be a clique in the conflict graph $\tilde{G}_I^e$ s.t. $w_k + w_{k'} \geq |I|+1$ for each $v_k \in C$ and $v_{k'} \in C_e$. Then, the inequality*

$$\sum_{v_k\in C} x_e^k + \sum_{v_k\in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{v_{k'}\in C_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq |C|+1, \tag{2.33}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. For each edge $e \in E$ and interval of contiguous slots $I \subseteq \mathbb{S}$, the inequality (2.33) ensures that if the set of demands in the clique C pass through edge e, they cannot share the interval $I = [s_i, s_j]$ over edge e with a subset of demands in C_e. We first suppose that the inequality (2.33) is not valid for $P(G, K, \mathbb{S})$. It follows that there exists a C-RSA solution S in which $S_{k'} \cap \{s_i + w_{k'} - 1, ..., s_j\} = \emptyset$ for each demand $k' \in C_e$ s.t.

$$\sum_{v_k \in C} x_e^k(S) + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) > |C| + 1.$$

Since $S_{k'} \notin I$ for each demand $k' \in C_e$ this means that $\sum_{v_{k'} \in C_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'}(S) = 0$, and taking into account inequality (2.32) and that $x_e^k(S) \leq 1$ for each demand $v_k \in C$ and $\sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) \leq 1$ for each demand $v_k \in C$, it follows that

$$\sum_{v_k \in C} x_e^k(S) + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) \leq |C| + 1,$$

which contradicts what we supposed before.
On the other hand and when $C \cap K_e = \emptyset$, it follows that there exists a C-RSA solution S' in which $E_k \cap \{e\} = \emptyset$ and $S_{k'} \cap \{s_i + w_{k'} - 1, ..., s_j\} = \emptyset$ for each demand $k' \in C$ s.t.

$$\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S') > |C| + 1.$$

Given that $\sum_{s=s_i+w_k-1}^{s_j} z_s^k(S') \leq 1$ for each demand $k \in C$, it follows that

$$\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S') \leq |\tilde{K}| + 1,$$

which contradicts what we supposed before, i.e., $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S') > |C| + 1$.
As a result,

$$\sum_{v_k \in C} |E_k \cap \{e\}| + \sum_{v_k \in C} |S_k \cap \{s_i + w_k - 1, ..., s_j\}| + \sum_{k' \in C_e} |S_{k'} \cap \{s_i + w_{k'} - 1, ..., s_j\}| \leq |C| + 1.$$

□

Theorem 2.4.5. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots. Let C be a clique in the conflict graph $\tilde{G}_I^e$ with $|C| \geq 3$, and $\sum_{k \in C} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus C} w_{k'}$. Then, the inequality* (2.32) *is facet defining for $P(G, K, \mathbb{S})$ iff*

a) there does not exist a demand $k' \in K_e \setminus C$ with $w_k + w_{k'} > |I|$ and $w_{k'} \leq |I|$ and $2w_{k'} > |I|$,

b) and $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ for each demand k with $v_k \in C$,

c) and there does not exist an interval I' of contiguous slots with $I \subset I'$ s.t. C defines also a clique in the associated conflict graph $\tilde{G}_{I'}^e$.

Proof. **Neccessity.**
It is trivial given that

a) if

a) there does not exist a demand $k' \in K_e \setminus C$ with $w_k + w_{k'} > |I|$ and $w_{k'} \leq |I|$ and $2w_{k'} > |I|$,

b) and $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ for each demand k with $v_k \in C$.

Then, the inequality (2.32) can never be dominated by another inequality without changing its right-hand side. Otherwise, if there exists a demand $k' \in K_e \setminus C$ with $w_k + w_{k'} > |I|$ and $w_{k'} \leq |I|$ and $2w_{k'} > |I|$, this implies that the inequality is dominated by (2.33). Moreover, if $|\{s_i + w_k - 1, ..., s_j\}| < w_k$ for each demand k with $v_k \in C$, then the inequality (2.32) is then dominated by the inequality (2.25) for a set of demands $\tilde{K} = \{k \in K \text{ s.t. } v_k \in C\}$ and slot $s = s_i + \min\limits_{k \in C} w_k + 1$ over edge e. Hence, the inequality (2.32) is not facet defining for $P(G, K, \mathbb{S})$.

b) if there exists an interval I' of contiguous slots with $I \subset I'$ s.t. C defines also a clique in the associated conflict graph $\tilde{G}^e_{I'}$. This implies that the inequality (2.32) induced by the clique C for the interval I is dominated by the inequality (2.32) induced by the same clique C for the interval I' given that $\{s_i + w_k - 1, ..., s_j\} \subset I'$ for each $k \in C$. As a result, the inequality (2.32) is not facet defining for $P(G, K, \mathbb{S})$.

Sufficiency.

Let $F_C^{\tilde{G}^e_I}$ denote the face induced by the inequality (2.32), which is given by

$$F_C^{\tilde{G}^e_I} = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = |C| + 1\}.$$

In order to prove that inequality $\sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |C| + 1$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that $F_C^{\tilde{G}^e_I}$ is a proper face, and $F_C^{\tilde{G}^e_I} \neq P(G, K, \mathbb{S})$.
We construct a solution $\mathcal{S}^{53} = (E^{53}, S^{53})$ as below

a) a feasible path E_k^{53} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{53} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{53}$ with $|S_k^{53}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{53}$ and $s' \in S_{k'}^{53}$ with $E_k^{53} \cap E_{k'}^{53} \neq \emptyset$ (non-overlapping constraint),

d) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{53}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{53}$ for a node $v_k \in C$, and for each $s' \in S_{k'}^{53}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$,

e) and all the demands in C pass through the edge e in the solution $\mathcal{S}^{53}$, i.e., $e \in E_k^{53}$ for each $k \in C$.

Obviously, $\mathcal{S}^{53}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{53}}, z^{\mathcal{S}^{53}})$ is belong to $P(G, K, \mathbb{S})$ and then to $F_C^{\tilde{G}^e_I}$ given that it is composed by $\sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. As a result, $F_C^{\tilde{G}^e_I}$ is not empty (i.e., $F_C^{\tilde{G}^e_I} \neq \emptyset$). Furthermore, given that $s \in \{s_i + w_k - 1, ..., s_j\}$ for each $v_k \in C$, this means that there exists at least one feasible slot assignment S_k for the demands k in C with $s \notin \{s_i + w_k - 1, ..., s_j\}$ for each $s \in S_k$ and each $v_k \in C$. This means that $F_C^{\tilde{G}^e_I} \neq P(G, K, \mathbb{S})$.
We denote the inequality $\sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |C| + 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F_C^{\tilde{G}^e_I} \subset F = \{(x, z) \in$

$P(G,K,\mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k\in K}|E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k\in K}|E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k\in K}(w_k-1)}$) s.t. $(\mu,\sigma) = \rho(\alpha,\beta) + \gamma Q$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in C$,
b) and σ_s^k are equivalents for all $v_k \in C$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$,
c) and $\mu_{e'}^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e'$ if $v_k \in C$,
d) and all μ_e^k are equivalents for the set of demands in C,
e) and σ_s^k and μ_e^k are equivalents for all $v_k \in C$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$.

We first show that $\mu_{e'}^k = 0$ for each edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$ with $e \neq e'$ if $k \in C$. Consider a demand $k \in K$ and an edge $e' \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e'$ if $k \in C$. For that, we consider a solution $\mathcal{S}'^{53} = (E'^{53}, S'^{53})$ in which

a) a feasible path $E_k'^{53}$ is assigned to each demand $k \in K$ (routing constraint),
b) and a set of last-slots $S_k'^{53}$ is assigned to each demand $k \in K$ along each edge $e' \in E_k'^{53}$ with $|S_k'^{53}| \geq 1$ (contiguity and continuity constraints),
c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k'^{53}$ and $s" \in S_{k'}'^{53}$ with $E_k'^{53} \cap E_{k'}'^{53} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e' \in E_k'^{53}} |\{s' \in S_k'^{53}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),
d) the edge e' is not non-compatible edge with the selected edges $e" \in E_k'^{53}$ of demand k in the solution $\mathcal{S}'^{53}$, i.e., $\sum_{e" \in E_k'^{53}} l_{e"} + l_{e'} \leq \bar{l}_k$. As a result, $E_k'^{53} \cup \{e'\}$ is a feasible path for the demand k,
e) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s \in S_k'^{53}$ and $s' \in S_{k'}'^{53}$ with $(E_k'^{53} \cup \{e'\}) \cap E_{k'}'^{53} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e' in the set of edges $E_k'^{53}$ selected to route the demand k in the solution $\mathcal{S}'^{53}$),
f) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}'^{53}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k'^{53}$ for a node $v_k \in C$, and for each $s' \in S_{k'}'^{53}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$,
g) and all the demands in C pass through the edge e in the solution $\mathcal{S}'^{53}$, i.e., $e \in E_k'^{53}$ for each $k \in C$.

$\mathcal{S}'^{53}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{53}}, z^{\mathcal{S}'^{53}})$ is belong to F and then to $F_C^{\tilde{G}_I^e}$ given that it is composed by $\sum_{v_k \in C} x_{e'}^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = |C| + 1$. Based on this, we derive a solution $\mathcal{S}^{54}$ obtained from the solution $\mathcal{S}'^{53}$ by adding an unused edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in K in the solution $\mathcal{S}^{53}$ which means that $E_k^{54} = E_k'^{53} \cup \{e'\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^{53}$ remain the same in the solution $\mathcal{S}^{54}$, i.e., $S_k^{54} = S_k'^{53}$ for each $k \in K$, and $E_{k'}^{54} = E_{k'}'^{53}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{54}$ is clearly feasible given that

a) and a feasible path E_k^{54} is assigned to each demand $k \in K$ (routing constraint),
b) and a set of last-slots S_k^{54} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{54}$ with $|S_k^{54}| \geq 1$ (contiguity and continuity constraints),
c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{54}$ and $s" \in S_{k'}^{54}$ with $E_k^{54} \cap E_{k'}^{54} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e' \in E_k^{54}} |\{s' \in S_k^{54}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{54}}, z^{\mathcal{S}^{54}})$ is belong to F and then to $F_C^{\tilde{G}_I^e}$ given that it is composed by $\sum_{v_k \in C} x_{e'}^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = |C| + 1$. It follows that

$$\mu x^{\mathcal{S}'^{53}} + \sigma z^{\mathcal{S}'^{53}} = \mu x^{\mathcal{S}^{54}} + \sigma z^{\mathcal{S}^{54}} = \mu x^{\mathcal{S}'^{53}} + \mu_{e'}^k + \sigma z^{\mathcal{S}'^{53}}.$$

As a result, $\mu_{e'}^k = 0$ for demand k and an edge e'.
As e' is chosen arbitrarily for the demand k with $e \notin E_0^k \cup E_1^k$ and $e \neq e'$ if $k \in C$, we iterate the same procedure for all $e" \in E \setminus (E_0^k \cup E_1^k \cup \{e'\})$ with $e \neq e"$ if $k \in C$. We conclude that for the demand k

$$\mu_{e'}^k = 0, \text{ for all } e' \in E \setminus (E_0^k \cup E_1^k) \text{ with } e \neq e' \text{ if } k \in C.$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E_0^k \cup E_1^k)$. We conclude at the end that

$$\mu_{e'}^k = 0, \text{ for all } k \in K \text{ and all } e' \in E \setminus (E_0^k \cup E_1^k) \text{ with } e \neq e' \text{ if } k \in C.$$

Let's us show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \notin C$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \notin C$. For that, we consider a solution $\mathcal{S}"^{53} = (E"^{53}, S"^{53})$ in which

a) a feasible path $E"^{53}_k$ is assigned to each demand $k \in K$ (routing constraint),
b) a set of last-slots $S"^{53}_k$ is assigned to each demand $k \in K$ along each edge $e' \in E"^{53}_k$ with $|S"^{53}_k| \geq 1$ (contiguity and continuity constraints),
c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{53}_k$ and $s" \in S"^{53}_{k'}$ with $E"^{53}_k \cap E"^{53}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E"^{53}_k} |\{s' \in S"^{53}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),
d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^{53}_{k'}$ with $E"^{53}_k \cap E"^{53}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{53}_k$ assigned to the demand k in the solution $\mathcal{S}"^{53}$),
e) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}"^{53}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S"^{53}_k$ for a node $v_k \in C$, and for each $s' \in S"^{53}_{k'}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$,
f) and all the demands in C pass through the edge e in the solution $\mathcal{S}"^{53}$, i.e., $e' \in E"^{53}_k$ for each $k \in C$.

$\mathcal{S}"^{53}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{53}}, z^{\mathcal{S}"^{53}})$ is belong to F and then to $F_C^{\tilde{G}_I^e}$ given that it is composed by $\sum_{v_k \in C} x_{e'}^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. Based on this, we construct a solution $\mathcal{S}^{55}$ derived from the solution $\mathcal{S}"^{53}$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}"^{53}$ (i.e., $E_k^{55} = E"^{53}_k$ for each $k \in K \setminus \tilde{K}$, and $E_k^{55} \neq E"^{53}_k$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E_k^{55} is assigned to each demand $k \in \tilde{K}$ (routing constraint),
b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"^{53}_k$ and $s" \in S"^{53}_{k'}$ with $E_k^{55} \cap E"^{53}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}, e' \in E_k^{55}} |\{s' \in S"^{53}_k, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k \in K \setminus \tilde{K}, e' \in E"^{53}_k} |\{s' \in S"^{53}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"^{53}_{k"}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{53}_k$ assigned to the demand k in the solution $\mathcal{S}"^{53}$).

The last-slots assigned to the demands $K\setminus\{k\}$ in $\mathcal{S}"^{53}$ remain the same in $\mathcal{S}^{55}$, i.e., $S"^{53}_{k'} = S^{55}_{k'}$ for each demand $k' \in K \setminus \{k\}$, and $S^{55}_k = S"^{53}_k \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{55}$ is clearly feasible given that

a) a feasible path E^{55}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{55}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{55}_k$ with $|S^{55}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{55}_k$ and $s" \in S^{55}_{k'}$ with $E^{55}_k \cap E^{55}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E^{55}_k} |\{s' \in S^{55}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{55}}, z^{\mathcal{S}^{55}})$ is belong to F and then to $F_C^{\tilde{G}^e_I}$ given that it is composed by $\sum_{v_k \in C} x^k_{e'} + \sum_{s=s_i+w_k-1}^{s_j} z^k_s = |C| + 1$. We have so

$$\mu x^{\mathcal{S}"^{53}} + \sigma z^{\mathcal{S}"^{53}} = \mu x^{\mathcal{S}^{55}} + \sigma z^{\mathcal{S}^{55}} = \mu x^{\mathcal{S}"^{53}} + \sigma z^{\mathcal{S}"^{53}} + \sigma^k_{s'} - \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E"^{53}_k} \mu^{\tilde{k}}_{e'} + \sum_{\tilde{k} \in \tilde{K}} \sum_{e" \in E^{55}_k} \mu^{\tilde{k}}_{e"}.$$

It follows that $\sigma^k_{s'} = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \notin C$ given that $\mu^k_{e'} = 0$ for all the demands $k \in K$ and all edges $e' \in E \setminus (E^k_0 \cup E^k_1)$ with $e \neq e'$ if $k \in C$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \notin C$ s.t. we find

$$\sigma^k_{s'} = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } s' \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } v_k \notin C.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma^{k'}_s = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_{k'} - 1, ..., s_j\} \text{ if } v_{k'} \notin C.$$

Consequently, we conclude that

$$\sigma^k_s = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } v_k \notin C.$$

Let prove that σ^k_s for all $v_k \in C$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$ are equivalents. Consider a demand k' and a slot $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ with $v_{k'} \in C$. For that, we consider a solution $\tilde{\mathcal{S}}^{53} = (\tilde{E}^{53}, \tilde{S}^{53})$ in which

a) a feasible path $\tilde{E}^{53}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}^{53}_k$ is assigned to each demand $k \in K$ along each edge $e' \in \tilde{E}^{53}_k$ with $|\tilde{S}^{53}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}^{53}_k$ and $s" \in \tilde{S}^{53}_{k'}$ with $\tilde{E}^{53}_k \cap \tilde{E}^{53}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in \tilde{E}^{53}_k} |\{s' \in \tilde{S}^{53}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s \in \tilde{S}^{53}_k$ with $\tilde{E}^{53}_k \cap \tilde{E}^{53}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $\tilde{S}^{53}_{k'}$ assigned to the demand k' in the solution $\tilde{\mathcal{S}}^{53}$),

e) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k selects a slot s as last-slot in the solution $\tilde{\mathcal{S}}^{53}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in \tilde{S}^{53}_k$ for a node $v_k \in C$, and for each $s' \in \tilde{S}^{53}_{k'}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$,

f) and all the demands in C pass through the edge e in the solution $\tilde{S}^{53}$, i.e., $e' \in \tilde{E}^{53}_k$ for each $k \in C$.

$\tilde{\mathcal{S}}^{53}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{53}}, z^{\tilde{\mathcal{S}}^{53}})$ is belong to F and then to $F_C^{\tilde{G}^e_I}$ given that it is composed by $\sum_{v_k \in C} x^k_{e'} + \sum_{s=s_i+w_k-1}^{s_j} z^k_s = |C| + 1$. Let $\mathcal{S}^{56}$ be a solution derived from the solution $\tilde{\mathcal{S}}^{53}$ by adding the slot s' as last-slot to the demand k' with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\tilde{\mathcal{S}}^{53}$ (i.e., $E^{56}_k = \tilde{E}^{53}_k$ for each $k \in K \setminus \tilde{K}$, and $E^{56}_k \neq \tilde{E}^{53}_k$ for each $k \in \tilde{K}$), and also the last-slots assigned to the demands $K \setminus \{k, k'\}$ in $\tilde{\mathcal{S}}^{53}$ remain the same in $\mathcal{S}^{56}$, i.e., $\tilde{S}^{53}_{k"} = S^{56}_{k"}$ for each demand $k" \in K \setminus \{k, k'\}$, and $S^{56}_{k'} = \tilde{S}^{53}_{k'} \cup \{s'\}$ for the demand k', and modifying the last-slots assigned to the demand k by adding a new last-slot $\tilde{s}$ and removing the last slot $s \in \tilde{S}^{53}_k$ with $s \in \{s_i + w_k + 1, ..., s_j\}$ and $\tilde{s} \notin \{s_i + w_k + 1, ..., s_j\}$ for the demand k with $v_k \in C$ s.t. $S^{56}_k = (\tilde{S}^{53}_k \setminus \{s\}) \cup \{\tilde{s}\}$ s.t. $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S^{56}_{k'}$ with $E^{56}_k \cap E^{56}_{k'} \neq \emptyset$. The solution $\mathcal{S}^{56}$ is clearly feasible given that

a) a feasible path E^{56}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{56}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{56}_k$ with $|S^{56}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{56}_k$ and $s" \in S^{56}_{k'}$ with $E^{56}_k \cap E^{56}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E^{56}_k} |\{s' \in S^{56}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{56}}, z^{\mathcal{S}^{56}})$ is belong to F and then to $F_C^{\tilde{G}^e_I}$ given that it is composed by $\sum_{v_k \in C} x^k_{e'} + \sum_{s=s_i+w_k-1}^{s_j} z^k_s = |C| + 1$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{53}} + \sigma z^{\tilde{\mathcal{S}}^{53}} = \mu x^{\mathcal{S}^{56}} + \sigma z^{\mathcal{S}^{56}} = \mu x^{\tilde{\mathcal{S}}^{53}} + \sigma z^{\tilde{\mathcal{S}}^{53}} + \sigma^{k'}_{s'} - \sigma^k_s + \sigma^k_{\tilde{s}}$$
$$- \sum_{k \in \tilde{K}} \sum_{e' \in \tilde{E}^{53}_k} \mu^k_{e'} + \sum_{k \in \tilde{K}} \sum_{e' \in E^{56}_k} \mu^k_{e'}.$$

It follows that $\sigma^{k'}_{s'} = \sigma^k_s$ for demand k' and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $v_{k'} \in C$ and $s' \in \{s_i + w_{k'} + 1, ..., s_j\}$ given that $\sigma^k_{\tilde{s}} = 0$ for $\tilde{s} \notin \{s_i + w_k - 1, ..., s_j\}$ with $v_k \in C$, and $\mu^k_{e'} = 0$ for all $k \in K$ and all $e' \in E \setminus (E^k_0 \cup E^k_1)$ with $e' \neq e$ if $k \in C$.
Given that the pair $(v_k, v_{k'})$ are chosen arbitrary in the clique C, we iterate the same procedure for all pairs $(v_k, v_{k'})$ s.t. we find

$$\sigma^k_s = \sigma^{k'}_{s'}, \text{for all pairs } (v_k, v_{k'}) \in C$$

with $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. We re-do the same procedure for each two slots $s, s' \in \{s_i + w_k - 1, ..., s_j\}$ for each demand $k \in K$ with $v_k \in C$ s.t.

$$\sigma^k_s = \sigma^k_{s'}, \text{for all } v_k \in C \text{ and } s, s' \in \{s_i + w_k - 1, ..., s_j\}.$$

Let us prove now that μ^k_e for all $k \in K$ with $v_k \in C$ are equivalents. For that, we consider a solution $\mathcal{S}^{57} = (E^{57}, S^{57})$ defined as below

a) a feasible path E^{57}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{57} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{57}$ with $|S_k^{57}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{57}$ and $s' \in S_{k'}^{57}$ with $E_k^{57} \cap E_{k'}^{57} \neq \emptyset$ (non-overlapping constraint),

d) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k pass through the edge e in the solution $\mathcal{S}^{57}$, i.e., $e \in E_k^{57}$ for a node $v_k \in C$, and $e \notin E_{k'}^{57}$ for all $v_{k'} \in C \setminus \{v_k\}$,

e) and all the demands in C are covered by the interval I in the solution $\mathcal{S}^{57}$, i.e., $\{s_i + w_k + 1, ..., s_j\} \cap S_k^{57} \neq \emptyset$ for each $k \in C$.

Obviously, $\mathcal{S}^{57}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{57}}, z^{\mathcal{S}^{57}})$ is belong to $P(G, K, \mathbb{S})$ and then to $F_C^{\tilde{G}_I^e}$ given that it is composed by $\sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = |C| + 1$.
Consider now a node $v_{k'}$ in C s.t. $e \notin E_{k'}^{57}$. For that, we consider a solution $\tilde{\mathcal{S}}^{57} = (\tilde{E}^{57}, \tilde{S}^{57})$ in which

a) a feasible path $\tilde{E}_k^{57}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}_k^{57}$ is assigned to each demand $k \in K$ along each edge $e \in \tilde{E}_k^{57}$ with $|\tilde{S}_k^{57}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}_k^{57}$ and $s" \in \tilde{S}_{k'}^{57}$ with $\tilde{E}_k^{57} \cap \tilde{E}_{k'}^{57} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in \tilde{E}_k^{57}} |\{s' \in \tilde{S}_k^{57}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k pass through the edge e in the solution $\tilde{S}^{57}$, i.e., $e \in \tilde{E}_k^{57}$ for a node $v_k \in C$, and $e \notin \tilde{E}_{k'}^{57}$ for all $v_{k'} \in C \setminus \{v_k\}$,

e) and all the demands in C are covered by the interval I in the solution $\tilde{S}^{57}$, i.e., $\{s_i + w_k + 1, ..., s_j\} \cap \tilde{S}_k^{57} \neq \emptyset$ for each $k \in C$.

$\tilde{\mathcal{S}}^{57}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{57}}, z^{\tilde{\mathcal{S}}^{57}})$ is belong to F and then to $F_C^{\tilde{G}_I^e}$ given that it is composed by $\sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = |C| + 1$. Based on this, we derive a solution $\mathcal{S}"^{58} = (E"^{58}, S"^{58})$ from the solution $\tilde{\mathcal{S}}^{57}$ by

a) the paths assigned to the demands $K \setminus \{k, k'\}$ in $\tilde{\mathcal{S}}^{57}$ remain the same in $\mathcal{S}"^{58}$ (i.e., $E"^{58}_{k"} = \tilde{E}_{k"}^{57}$ for each $k" \in K \setminus \{k, k'\}$),

b) without modifying the last-slots assigned to the demands K in $\tilde{\mathcal{S}}^{57}$, i.e., $\tilde{S}_k^{57} = S"^{58}_k$ for each demand $k \in K$,

c) modifying the path assigned to the demand k' in $\tilde{\mathcal{S}}^{57}$ from $\tilde{E}_{k'}^{57}$ to a path $E"^{58}_{k'}$ passed through the edge e (i.e., $e \in E"^{58}_{k'}$) with $v_{k'} \in C$ s.t. $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and each $s' \in \tilde{S}_{k'}^{57}$ and each $s \in \tilde{S}_k^{57}$ with $\tilde{E}_k^{57} \cap E"^{58}_{k'} \neq \emptyset$,

d) modifying the path assigned to the demand k in $\tilde{\mathcal{S}}^{57}$ with $e \in \tilde{E}_k^{57}$ and $v_k \in C$ from $\tilde{E}_k^{57}$ to a path $E"^{58}_k$ without passing through the edge e (i.e., $e \notin E"^{58}_k$) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k"} + 1, ..., s'\} = \emptyset$ for each $k" \in K \setminus \{k, k'\}$ and each $s \in \tilde{S}_k^{57}$ and each $s' \in \tilde{S}_{k"}^{57}$ with $\tilde{E}_{k"}^{57} \cap E"^{58}_k \neq \emptyset$, and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $s \in \tilde{S}_k^{57}$ and each $s' \in \tilde{S}_{k'}^{57}$ with $E"^{58}_{k"} \cap E"^{58}_k \neq \emptyset$.

The solution $\mathcal{S}"^{58}$ is feasible given that

a) a feasible path $E"^{58}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^{58}_k$ is assigned to each demand $k \in K$ along each edge $e \in E"^{58}_k$ with $|S"^{58}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{58}_k$ and $s" \in S"^{58}_{k'}$ with $E"^{58}_k \cap E"^{58}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E"^{58}_k} |\{s' \in S"^{58}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}"^{58}}, z^{\mathcal{S}"^{58}})$ is belong to F and then to $F_C^{\tilde{G}^e_I}$ given that it is composed by $\sum_{v_k \in C} x^k_e + \sum_{s=s_i+w_k-1}^{s_j} z^k_s = |C| + 1$. We then obtain that

$$\mu x^{\tilde{\mathcal{S}}^{57}} + \sigma z^{\tilde{\mathcal{S}}^{57}} = \mu x^{\mathcal{S}^{58}} + \sigma z^{\mathcal{S}^{58}} = \mu x^{\tilde{\mathcal{S}}^{57}} + \sigma z^{\tilde{\mathcal{S}}^{57}} + \mu^{k'}_e - \mu^k_e$$
$$+ \sum_{e" \in E"^{58}_{k'} \setminus \{e\}} \mu^{k'}_{e"} - \sum_{e" \in \tilde{E}^{57}_{k'}} \mu^{k'}_{e"} + \sum_{e" \in E"^{58}_k} \mu^k_{e"} - \sum_{e" \in \tilde{E}^{57}_k \setminus \{e\}} \mu^k_{e"}.$$

It follows that $\mu^{k'}_e = \mu^k_e$ for demand k' and a edge $e' \in E \setminus (E^{k'}_0 \cup E^{k'}_1)$ with $v_{k'} \in C$ given that $\mu^k_{e"} = 0$ for all $k \in K$ and all $e" \in E \setminus (E^k_0 \cup E^k_1)$ with $v_k \notin C$.
Given that the pair $(v_k, v_{k'})$ are chosen arbitrary in the clique C, we iterate the same procedure for all pairs $(v_k, v_{k'})$ s.t. we find

$$\mu^k_e = \mu^{k'}_e, \text{for all pairs } (v_k, v_{k'}) \in C.$$

Furthermore, let prove that all σ^k_s and μ^k_e are equivalents for all $k \in C$ and $s \in \{s_i + w_k - 1, ..., s_j\}$. For that, we consider for each demand k' with $v_{k'} \in C$, a solution $\mathcal{S}^{59} = (E^{59}, S^{59})$ derived from the solution $\tilde{\mathcal{S}}^{57}$ as below

a) the paths assigned to the demands $K \setminus \{k'\}$ in $\tilde{\mathcal{S}}^{57}$ remain the same in $\mathcal{S}^{59}$ (i.e., $E^{59}_{k"} = \tilde{E}^{57}_{k"}$ for each $k" \in K \setminus \{k'\}$),

b) without modifying the last-slots assigned to the demands $K \setminus \{k\}$ in $\tilde{\mathcal{S}}^{57}$, i.e., $\tilde{S}^{57}_{k"} = S^{59}_{k"}$ for each demand $k" \in K \setminus \{k\}$,

c) modifying the set of last-slots assigned to the demand k' in $\tilde{\mathcal{S}}^{57}$ from $\tilde{S}^{57}_{k'}$ to $S^{59}_{k'}$ s.t. $S^{59}_{k'} \cap \{s_i + w_{k'} - 1, ..., s_j\} = \emptyset$.

Hence, there are $|C| - 1$ demands from C that are covered by the interval I (i.e., all the demands in $C \setminus \{k'\}$), and two demands $\{k, k'\}$ from C that use the edge e in the solution $\mathcal{S}^{59}$. The solution $\mathcal{S}^{59}$ is then feasible given that

a) a feasible path E^{59}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{59}_k is assigned to each demand $k \in K$ along each edge $e \in E^{59}_k$ with $|S^{59}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{59}_k$ and $s" \in S^{59}_{k'}$ with $E^{59}_k \cap E^{59}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E^{59}_k} |\{s' \in S^{59}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\sum_{v_k \in C} |E^{59}_k \cap \{e\}| + |S^{59}_k \cap \{s_i + w_k - 1, ..., s_j\}| = |C| + 1$.

The corresponding incidence vector $(x^{\mathcal{S}^{59}}, z^{\mathcal{S}^{59}})$ is belong to F and then to $F_C^{\tilde{G}^e_I}$ given that it is composed by $\sum_{v_k \in C} x^k_e + \sum_{s=s_i+w_k-1}^{s_j} z^k_s = |C| + 1$. We then obtain that

$$\mu x^{\tilde{\mathcal{S}}^{57}} + \sigma z^{\tilde{\mathcal{S}}^{57}} = \mu x^{\mathcal{S}^{59}} + \sigma z^{\mathcal{S}^{59}} = \mu x^{\tilde{\mathcal{S}}^{57}} + \sigma z^{\tilde{\mathcal{S}}^{57}} + \mu^{k'}_e - \sigma^{k'}_s + \sum_{e" \in E^{59}_{k'} \setminus \{e\}} \mu^{k'}_{e"} - \sum_{e" \in \tilde{E}^{57}_{k'}} \mu^{k'}_{e"}.$$

It follows that $\mu_e^{k'} = \sigma_s^{k'}$ for demand k' and slot $s \in \{s_i + w_{k'} - 1, ..., s_j\}$ given that $\mu_{e"}^k = 0$ for all $k \in K$ and all $e" \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e"$ if $v_k \in C$. Moreover, by doing the same thing over all slots $s \in \{s_i + w_{k'} - 1, ..., s_j\}$, we found that

$$\mu_e^{k'} = \sigma_s^{k'}, \text{for all } s \in \{s_i + w_{k'} - 1, ..., s_j\}.$$

Given that k' is chosen arbitrarily in C, we iterate the same procedure for all $k \in C$ to show that

$$\mu_e^k = \sigma_s^k, \text{for all } v_k \in C \text{ and all } s \in \{s_i + w_k - 1, ..., s_j\}.$$

Based on this, and given that all μ_e^k are equivalents for all $v_k \in C$, and that σ_s^k are equivalents for all $v_k \in C$ and $s \in \{s_i + w_{k'} - 1, ..., s_j\}$, we obtain that

$$\mu_e^k = \sigma_s^{k'}, \text{ for all } k, k' \in C \text{ and all } s \in \{s_i + w_{k'} - 1, ..., s_j\}.$$

Consequently, we conclude that

$$\mu_e^k = \sigma_s^{k'} = \rho, \text{ for all } k, k' \in C \text{ and all } s \in \{s_i + w_{k'} - 1, ..., s_j\}.$$

On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k' \in K$ and $e' \in E$

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'}, \text{ if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'}, \text{ if } e' \in E_1^{k'}, \\ \rho, \text{ if } k' \in C \text{ and } e' = e, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\} \\ \rho, \text{if } v_k \in C \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{v_k \in C} \rho \alpha_e^k + \sum_{s=s_i+w_k-1}^{s_j} \rho \beta_s^k + \gamma Q.$

□

Theorem 2.4.6. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots. Let C be a clique in the conflict graph $\tilde{G}_I^e$ with $|C| \geq 3$, and $\sum_{v_k \in C} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus C} w_{k'}$. Let $C_e \subseteq K_e \setminus C$ be a clique in the conflict graph $\tilde{G}_I^e$ s.t. $w_k + w_{k'} \geq |I| + 1$ for each $v_k \in C$ and $v_{k'} \in C_e$. Then, the inequality (2.33) is facet defining for $P(G, K, \mathbb{S})$ if and only if*

a) there does not exist a demand $k" \in K_e \setminus C_e$ with $w_k + w_{k"} \geq |I| + 1$ for each $v_k \in C$, and $w_{k'} + w_{k"} \geq |I| + 1$ for each $v_{k'} \in C_e$.

b) and $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ for each demand k with $v_k \in C \cup C_e$,

c) and there does not exist an interval I' of contiguous slots with $I \subset I'$ s.t. $C \cup C_e$ defines also a clique in the associated conflict graph $\tilde{G}^e_{I'}$.

Proof. **Neccessity.**

a) If there exists a demand $k" \in K_e \setminus C_e$ with $w_k + w_{k"} \geq |I| + 1$ for each $v_k \in C$, and $w_{k'} + w_{k"} \geq |I| + 1$ for each $v_{k'} \in C_e$. Then, the inequality (2.33) is dominated by its lifted with $C'_e = C_e \cup \{k"\}$. Moreover, if $|\{s_i + w_k - 1, ..., s_j\}| < w_k$ for each demand k with $v_k \in C \cup C_e$, then the inequality (2.33) is then dominated by the inequality (2.27) for a set of demands $\tilde{K} = \{k \in K$ s.t. $v_k \in C\}$ and slot $s = s_i + \min\limits_{k \in C \cup C_e} w_k + 1$ over edge e. As a result, the inequality (2.33) is not facet defining for $P(G, K, \mathbb{S})$.

b) if there exists an interval I' of contiguous slots with $I \subset I'$ s.t. $C \cup C_e$ defines also a clique in the associated conflict graph $\tilde{G}^e_{I'}$. This implies that the inequality (2.33) induced by the clique $C \cup C_e$ for the interval I is dominated by the inequality (2.33) induced by the same clique $C \cup C_e$ for the interval I' given that $\{s_i + w_k - 1, ..., s_j\} \subset I'$ for each $k \in C \cup C_e$. As a result, the inequality (2.33) is not facet defining for $P(G, K, \mathbb{S})$.

Sufficiency.
Let $F'^{\tilde{G}^e_I}_C$ denote the face induced by the inequality (2.33), which is given by

$$F'^{\tilde{G}^e_I}_C = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{v_k \in C} x^k_e + \sum_{s = s_i + w_k - 1}^{s_j} z^k_s + \sum_{v_{k'} \in C_e} \sum_{s' = s_i + w_{k'} - 1}^{s_j} z^{k'}_{s'} = |C| + 1\}.$$

We denote the inequality $\sum_{v_k \in C} x^k_e + \sum_{s=s_i+w_k-1}^{s_j} z^k_s + \sum_{v_{k'} \in C_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} z^{k'}_{s'} \leq |C| + 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F'^{\tilde{G}^e_I}_C \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We use the same proof of the facial structure done for the inequality (2.32) in the proof of theorem 2.4.5 to prove that inequality $\sum_{v_k \in C} x^k_e + \sum_{s=s_i+w_k-1}^{s_j} z^k_s + \sum_{v_{k'} \in C_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} z^{k'}_{s'} \leq |C|+1$ is facet defining for $P(G, K, \mathbb{S})$. We first prove that $F'^{\tilde{G}^e_I}_C$ is a proper face based on the solution $\mathcal{S}^{53}$ defined in the proof of theorem 2.4.5 which stills feasible s.t. its corresponding incidence vector $(x^{\mathcal{S}^{53}}, z^{\mathcal{S}^{53}})$ is belong to F and then to $F'^{\tilde{G}^e_I}_C$ given that it is composed by $\sum_{v_k \in C} x^k_e + \sum_{s=s_i+w_k-1}^{s_j} z^k_s + \sum_{v_{k'} \in C_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} z^{k'}_{s'} = |C| + 1$. Furthermore, and based on the solutions $\mathcal{S}^{53}$ to $\mathcal{S}^{59}$ with corresponding incidence vector $(x^{\mathcal{S}^{53}}, z^{\mathcal{S}^{53}})$ to $(x^{\mathcal{S}^{59}}, z^{\mathcal{S}^{59}})$ are belong to F and then to $F'^{\tilde{G}^e_I}_C$ given that it is composed by $\sum_{v_k \in C} x^k_e + \sum_{s=s_i+w_k-1}^{s_j} z^k_s + \sum_{v_{k'} \in C_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} z^{k'}_{s'} = |C| + 1$, we showed that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E^k_0|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E^k_1|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma^k_s = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in C \cup C_e$,

b) and σ^k_s are equivalents for all $v_k \in C \cup C_e$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$,

c) and $\mu^k_{e'} = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E^k_0 \cup E^k_1)$ with $e \neq e'$ if $v_k \in C$,

d) and all μ^k_e are equivalents for the set of demands in C,

e) and $\sigma^{k'}_s$ and μ^k_e are equivalents for all $v_k \in C$ and all $v_{k'} \in C \cup C_e$ and all $s \in \{s_i + w_{k'} - 1, ..., s_j\}$.

At the end, we found that for each $k' \in K$ and $e' \in E$

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'}, \text{ if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'}, \text{ if } e' \in E_1^{k'}, \\ \rho, \text{ if } k' \in C \text{ and } e' = e, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\} \\ \rho, \text{if } v_k \in C \cup C_e \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{v_k \in C} \rho\alpha_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} \rho\beta_s^k + \sum_{v_{k'} \in C_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} \rho\beta_{s'}^{k'} + \gamma Q.$

□

2.4.4 Interval-Clique Inequalities

We have looked at the definition of the inequality (2.32), we detected that there may exist some cases that we can face which are not covered by the inequality (2.32). For this, we provide the following inequality and its generalization.

Proposition 2.4.12. *Consider an interval of contiguous slots $I = [s_i, s_j]$ in $\mathbb{S}$ with $s_i \leq s_j - 1$. Let k, k' be a pair of demands in K with $E_1^k \cap E_1^{k'} \neq \emptyset$, and $w_k \leq |I|$, and $w_{k'} \leq |I|$, and $w_k + w_{k'} > |I|$. Then, the inequality*

$$\sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq 1, \tag{2.34}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It is trivial given that the interval $I = [s_i, s_j]$ cannot cover the two demands k, k' shared an essential edge with total sum of number of slots exceeds $|I|$. Furthermore, the inequality (2.34) is a particular case of the inequality (2.32) for $\tilde{K} = \{k, k'\}$ over each edge $e \in E_1^k \cap E_1^{k'}$. However, it will be used for a generalized inequality using the following conflict graph . □

Definition 2.4.5. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$. Consider the conflict graph $\tilde{G}_I^E$ defined as follows. For each demand $k \in K$ with $w_k \leq |I|$, consider a node v_k in $\tilde{G}_I^E$. Two nodes v_k and $v_{k'}$ are linked by an edge in $\tilde{G}_I^E$ if $w_k + w_{k'} > |I|$ and $E_1^k \cap E_1^{k'} \neq \emptyset$.*

Proposition 2.4.13. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$, and C be a clique in the conflict graph $\tilde{G}_I^E$ with $|C| \geq 3$. Then, the inequality*

$$\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 1, \tag{2.35}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of clique set in the conflict graph $\tilde{G}_I^E$ s.t. for all two linked node v_k and $v_{k'}$ in $\tilde{G}_I^E$, we know from the inequality (2.34)

$$\sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq 1.$$

By adding the previous inequalities for all two linked node v_k and $v_{k'}$ in the clique set C, we get

$$\sum_{v_k}(|C|-1) \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |C|-1 \implies \sum_{v_k} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \frac{|C|-1}{|C|-1} \implies \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 1.$$

We conclude at the end that the inequality (2.35) is valid for $P(G, K, \mathbb{S})$. □

Theorem 2.4.7. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$, and C be a clique in the conflict graph $\tilde{G}_I^E$ with $|C| \geq 3$. Then, the inequality* (2.35) *is facet defining for $P(G, K, \mathbb{S})$ if and only if*

a) C is a maximal clique in the conflict graph $\tilde{G}_I^E$,

b) and there does not exist an interval of contiguous slots I' in $[1, \bar{s}]$ s.t. $I \subset I'$ with

  a) $w_k + w_{k'} \geq |I'|$ for each $k, k' \in C$,

  b) $w_k \leq |I'|$ and $2w_k \geq |I'| + 1$ for each $k \in C$.

Proof. **Neccessity.**
We distinguish two cases

a) if there exists a clique C' that contains all the demands $k \in C$. Then, the inequality (2.35) induced by the clique C is dominated by another inequality (2.35) induced by the clique C'. Hence, the inequality (2.35) cannot be facet defining for $P(G, K, \mathbb{S})$.

b) if there exists an interval of contiguous slots I' in $[1, \bar{s}]$ s.t. $I \subset I'$ with

  a) $w_k + w_{k'} \geq |I'|$ for each $k, k' \in C$,

  b) $w_k \leq |I'|$ and $2w_k \geq |I'| + 1$ for each $k \in C$.

This means that the inequality (2.35) induced by the clique C for the interval I is dominated by the inequality (2.35) induced by the clique C for the interval I'. Hence, the inequality (2.35) cannot be facet defining for $P(G, K, \mathbb{S})$.

Sufficiency.
Let $F_C^{\tilde{G}_I^E}$ denote the face induced by the inequality (2.35), which is given by

$$F_C^{\tilde{G}_I^E} = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1\}.$$

In order to prove that inequality $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 1$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that $F_C^{\tilde{G}_I^E}$ is a proper face, and $F_C^{\tilde{G}_I^E} \neq P(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{60} = (E^{60}, S^{60})$ as below

a) a feasible path E_k^{60} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{60}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{60}_k$ with $|S^{60}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S^{60}_k$ and $s' \in S^{60}_{k'}$ with $E^{60}_k \cap E^{60}_{k'} \neq \emptyset$ (non-overlapping constraint),

d) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{60}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S^{60}_k$ for a node $v_k \in C$, and for each $s' \in S^{60}_{k'}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

Obviously, $\mathcal{S}^{60}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{60}}, z^{\mathcal{S}^{60}})$ is belong to $P(G, K, \mathbb{S})$ and then to $F_C^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z^k_s = 1$. As a result, $F_C^{\tilde{G}_I^E}$ is not empty (i.e., $F_C^{\tilde{G}_I^E} \neq \emptyset$). Furthermore, given that $s \in \{s_i+w_k-1, ..., s_j\}$ for each $v_k \in C$, this means that there exists at least one feasible slot assignment S_k for the demands k in C with $s \notin \{s_i + w_k - 1, ..., s_j\}$ for each $s \in S_k$ and each $v_k \in C$. This means that $F_C^{\tilde{G}_I^E} \neq P(G, K, \mathbb{S})$.
We denote the inequality $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z^k_s \leq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F_C^{\tilde{G}_I^E} \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E^k_0|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E^k_1|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K} (w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma^k_s = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in C$,

b) and $\mu^k_e = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E^k_0 \cup E^k_1)$,

c) and σ^k_s are equivalents for all $v_k \in C$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$.

We first show that $\mu^k_e = 0$ for each edge $e \in E \setminus (E^k_0 \cup E^k_1)$ for each demand $k \in K$. Consider a demand $k \in K$ and an edge $e \in E \setminus (E^k_0 \cup E^k_1)$. For that, we consider a solution $\mathcal{S}'^{60} = (E'^{60}, S'^{60})$ in which

a) a feasible path E'^{60}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S'^{60}_k is assigned to each demand $k \in K$ along each edge $e \in E'^{60}_k$ with $|S'^{60}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S'^{60}_k$ and $s" \in S'^{60}_{k'}$ with $E'^{60}_k \cap E'^{60}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E'^{60}_k} |\{s' \in S'^{60}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S'13_{k'}$ with $E'13_k \cap E'13_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S'13_k$ assigned to the demand k in the solution $\mathcal{S}'^{60}$),

e) the edge e is not non-compatible edge with the selected edges $e \in E'^{60}_k$ of demand k in the solution $\mathcal{S}'^{60}$, i.e., $\sum_{e' \in E'^{60}_k} l_{e'} + l_e \leq \bar{l}_k$. As a result, $E'^{60}_k \cup \{e\}$ is a feasible path for the demand k,

f) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}'^{60}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S'^{60}_k$ for a node $v_k \in C$, and for each $s' \in S'^{60}_{k'}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}'^{60}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{60}}, z^{\mathcal{S}'^{60}})$ is belong to F and then to $F_C^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. Based on this, we derive a solution $\mathcal{S}^{61}$ obtained from the solution $\mathcal{S}'^{60}$ by adding an unused edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in K in the solution $\mathcal{S}^{60}$ which means that $E_k^{61} = E_k'^{60} \cup \{e\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^{60}$ remain the same in the solution $\mathcal{S}^{61}$, i.e., $S_k^{61} = S_k'^{60}$ for each $k \in K$, and $E_{k'}^{61} = E_{k'}'^{60}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{61}$ is clearly feasible given that

a) and a feasible path E_k^{61} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{61} is assigned to each demand $k \in K$ along each edge $e \in E_k^{61}$ with $|S_k^{61}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{61}$ and $s" \in S_{k'}^{61}$ with $E_k^{61} \cap E_{k'}^{61} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{61}} |\{s' \in S_k^{61}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{61}}, z^{\mathcal{S}^{61}})$ is belong to F and then to $F_C^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. It follows that

$$\mu x^{\mathcal{S}'^{60}} + \sigma z^{\mathcal{S}'^{60}} = \mu x^{\mathcal{S}^{61}} + \sigma z^{\mathcal{S}^{61}} = \mu x^{\mathcal{S}'^{60}} + \mu_e^k + \sigma z^{\mathcal{S}'^{60}}.$$

As a result, $\mu_e^k = 0$ for demand k and an edge e.
As e is chosen arbitrarily for the demand k with $e \notin E_0^k \cup E_1^k$, we iterate the same procedure for all $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$. We conclude that for the demand k

$$\mu_e^k = 0, \text{ for all } e \in E \setminus (E_0^k \cup E_1^k).$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e \in E \setminus (E_0^k \cup E_1^k)$. We conclude at the end that

$$\mu_e^k = 0, \text{ for all } k \in K \text{ and all } e \in E \setminus (E_0^k \cup E_1^k).$$

Let's us show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in C$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in C$. For that, we consider a solution $\mathcal{S}"^{60} = (E"^{60}, S"^{60})$ in which

a) a feasible path $E"^{60}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^{60}_k$ is assigned to each demand $k \in K$ along each edge $e \in E"^{60}_k$ with $|S"^{60}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{60}_k$ and $s" \in S"^{60}_{k'}$ with $E"^{60}_k \cap E"^{60}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E"^{60}_k} |\{s' \in S"^{60}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^{60}_{k'}$ with $E"^{60}_k \cap E"^{60}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{60}_k$ assigned to the demand k in the solution $\mathcal{S}"^{60}$),

e) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}"^{60}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S"^{60}_k$ for a node $v_k \in C$, and for each $s' \in S"^{60}_{k'}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}"^{60}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{60}}, z^{\mathcal{S}"^{60}})$ is belong to F and then to $F_C^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. Based on this, we construct a solution $\mathcal{S}^{62}$ derived from the solution $\mathcal{S}"^{60}$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}"^{60}$ (i.e., $E_k^{62} = E"_k^{60}$ for each $k \in K \setminus \tilde{K}$, and $E_k^{62} \neq E"_k^{60}$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E_k^{62} is assigned to each demand $k \in \tilde{K}$ (routing constraint),
b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"_k^{60}$ and $s" \in S"_{k'}^{60}$ with $E_k^{62} \cap E"_{k'}^{60} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}, e \in E_k^{62}} |\{s' \in S"_k^{60}, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k \in K \setminus \tilde{K}, e \in E"_k^{60}} |\{s' \in S"_k^{60}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),
c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"_{k"}^{60}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"_k^{60}$ assigned to the demand k in the solution $\mathcal{S}"^{60}$).

The last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}"^{60}$ remain the same in $\mathcal{S}^{62}$, i.e., $S"_{k'}^{60} = S_{k'}^{62}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{62} = S"_k^{60} \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{62}$ is clearly feasible given that

a) a feasible path E_k^{62} is assigned to each demand $k \in K$ (routing constraint),
b) a set of last-slots S_k^{62} is assigned to each demand $k \in K$ along each edge $e \in E_k^{62}$ with $|S_k^{62}| \geq 1$ (contiguity and continuity constraints),
c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{62}$ and $s" \in S_{k'}^{62}$ with $E_k^{62} \cap E_{k'}^{62} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{62}} |\{s' \in S_k^{62}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{62}}, z^{\mathcal{S}^{62}})$ is belong to F and then to $F_C^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. We have so

$$\mu x^{\mathcal{S}"^{60}} + \sigma z^{\mathcal{S}"^{60}} = \mu x^{\mathcal{S}^{62}} + \sigma z^{\mathcal{S}^{62}} = \mu x^{\mathcal{S}"^{60}} + \sigma z^{\mathcal{S}"^{60}} + \sigma_{s'}^k - \sum_{\tilde{k} \in \tilde{K}} \sum_{e \in E"_{\tilde{k}}^{60}} \mu_e^{\tilde{k}} + \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E_{\tilde{k}}^{62}} \mu_{e'}^{\tilde{k}}.$$

It follows that $\sigma_{s'}^k = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in C$ given that $\mu_e^k = 0$ for all the demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in C$ s.t. we find

$$\sigma_{s'}^k = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } s' \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } v_k \in C.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_{k'} - 1, ..., s_j\} \text{ if } v_{k'} \in C.$$

Consequently, we conclude that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } v_k \in C.$$

Let prove that σ_s^k for all $v_k \in C$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$ are equivalents. Consider a demand k' and a slot $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ with $v_{k'} \in C$, and a solution $\tilde{\mathcal{S}}^{60} = (\tilde{E}^{60}, \tilde{S}^{60})$ in which

a) a feasible path $\tilde{E}_k^{60}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}_k^{60}$ is assigned to each demand $k \in K$ along each edge $e \in \tilde{E}_k^{60}$ with $|\tilde{S}_k^{60}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}_k^{60}$ and $s" \in \tilde{S}_{k'}^{60}$ with $\tilde{E}_k^{60} \cap \tilde{E}_{k'}^{60} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in \tilde{E}_k^{60}} |\{s' \in \tilde{S}_k^{60}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s-w_k+1, ..., s\} \cap \{s'-w_{k'}+1, ..., s'\} = \emptyset$ for each $k \in K$ and $s \in \tilde{S}_k^{60}$ with $\tilde{E}_k^{60} \cap \tilde{E}_{k'}^{60} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $\tilde{S}_{k'}^{60}$ assigned to the demand k' in the solution $\tilde{\mathcal{S}}^{60}$),

e) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k selects a slot s as last-slot in the solution $\tilde{\mathcal{S}}^{60}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in \tilde{S}_k^{60}$ for a node $v_k \in C$, and for each $s' \in \tilde{S}_{k'}^{60}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\tilde{\mathcal{S}}^{60}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{60}}, z^{\tilde{\mathcal{S}}^{60}})$ is belong to F and then to $F_C^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. Based on this, we construct a solution $\mathcal{S}^{63}$ derived from the solution $\tilde{\mathcal{S}}^{60}$ by adding the slot s' as last-slot to the demand k' with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\tilde{\mathcal{S}}^{60}$ (i.e., $E_k^{63} = \tilde{E}_k^{60}$ for each $k \in K \setminus \tilde{K}$, and $E_k^{63} \neq \tilde{E}_k^{60}$ for each $k \in \tilde{K}$), and also the last-slots assigned to the demands $K \setminus \{k, k'\}$ in $\tilde{\mathcal{S}}^{60}$ remain the same in $\mathcal{S}^{63}$, i.e., $\tilde{S}_{k"}^{60} = S_{k"}^{63}$ for each demand $k" \in K \setminus \{k, k'\}$, and $S_{k'}^{63} = \tilde{S}_{k'}^{60} \cup \{s'\}$ for the demand k', and modifying the last-slots assigned to the demand k by adding a new last-slot $\tilde{s}$ and removing the last slot $s \in \tilde{S}_k^{60}$ with $s \in \{s_i + w_k + 1, ..., s_j\}$ and $\tilde{s} \notin \{s_i + w_k + 1, ..., s_j\}$ for the demand k with $v_k \in C$ s.t. $S_k^{63} = (\tilde{S}_k^{60} \setminus \{s\}) \cup \{\tilde{s}\}$ s.t. $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}^{63}$ with $E_k^{63} \cap E_{k'}^{63} \neq \emptyset$. The solution $\mathcal{S}^{63}$ is clearly feasible given that

a) a feasible path E_k^{63} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{63} is assigned to each demand $k \in K$ along each edge $e \in E_k^{63}$ with $|S_k^{63}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{63}$ and $s" \in S_{k'}^{63}$ with $E_k^{63} \cap E_{k'}^{63} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{63}} |\{s' \in S_k^{63}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{63}}, z^{\mathcal{S}^{63}})$ is belong to F and then to $F_C^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{60}} + \sigma z^{\tilde{\mathcal{S}}^{60}} = \mu x^{\mathcal{S}^{63}} + \sigma z^{\mathcal{S}^{63}} = \mu x^{\tilde{\mathcal{S}}^{60}} + \sigma z^{\tilde{\mathcal{S}}^{60}} + \sigma_{s'}^{k'} - \sigma_s^k + \sigma_{\tilde{s}}^k - \sum_{k \in \tilde{K}} \sum_{e \in \tilde{E}_k^{60}} \mu x^{\tilde{\mathcal{S}}^{60}} + \sum_{k \in \tilde{K}} \sum_{e \in E_k^{63}} \mu x^{\mathcal{S}^{63}}.$$

It follows that $\sigma_{s'}^{k'} = \sigma_s^k$ for demand k' and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $v_{k'} \in C$ and $s' \in \{s_i + w_{k'} + 1, ..., s_j\}$ given that $\sigma_{\tilde{s}}^k = 0$ for $\tilde{s} \notin \{s_i + w_k - 1, ..., s_j\}$ with $v_k \in C$, and $\mu_e^k = 0$ for all $k \in K$ and all $e \in E \setminus (E_0^k \cup E_1^k)$.
Given that the pair $(v_k, v_{k'})$ are chosen arbitrary in the clique C, we iterate the same procedure for all pairs $(v_k, v_{k'})$ s.t. we find

$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all pairs } (v_k, v_{k'}) \in C,$$

with $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. We re-do the same procedure for each two slots $s, s' \in \{s_i + w_k - 1, ..., s_j\}$ for each demand $k \in K$ with $v_k \in C$ s.t.

$$\sigma_s^k = \sigma_{s'}^k, \text{for all } v_k \in C \text{ and } s, s' \in \{s_i + w_k - 1, ..., s_j\}.$$

Consequently, we obtain that $\sigma_s^k = \rho$ for all $v_k \in C$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$.
On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k \in K$ and $e \in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e}, \text{ if } e \in E_0^k, \\ \gamma_2^{k,e}, \text{ if } e \in E_1^k, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\} \\ \rho, \text{if } v_k \in C \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} \rho \beta_s^k + \gamma Q.$

□

Let $N(v)$ denote the set of neighbors of node v in a given graph.

Theorem 2.4.8. *Consider an interval of contiguous slots $I = [s_i, s_j]$, and a pair of demands $k, k' \in K$ with $(v_k, v_{k'})$ in G_I^E. Then, the inequality (2.34) is facet defining for $P(G, K, \mathbb{S})$ iff*

a) $N(v_k) \cap N(v_{k'}) = \emptyset$ in the conflict graph $\tilde{G}_I^E$,

b) and there does not exist an interval of contiguous slots I' in $[1, \bar{s}]$ s.t. $I \subset I'$ with

a) $w_k + w_{k'} \geq |I'|$,right-hand

b) $w_k \leq |I'|$ and $2w_k \geq |I'| + 1$,

c) $w_{k'} \leq |I'|$ and $2w_{k'} \geq |I'| + 1$.

Proof. **Neccessity.**
We distinguish two cases:

a) if $N(v_k) \cap N(v_{k'}) \neq \emptyset$ in the conflict graph $\tilde{G}_I^E$, this means that there exists a clique C in the conflict graph $\tilde{G}_I^E$ of cardinality equals to $|C| \geq 3$ with $k, k' \in C$. As a result, the inequality (2.34) is dominated by the inequality (2.35) induced by the clique C. Hence, the inequality (2.34) is not facet defining for $P(G, K, \mathbb{S})$.

b) if there exists an interval of contiguous slots I' in $[1, \bar{s}]$ s.t. $I \subset I'$ with

a) $w_k + w_{k'} \geq |I'|$,

b) $w_k \leq |I'|$ and $2w_k \geq |I'| + 1$,

c) $w_{k'} \leq |I'|$ and $2w_{k'} \geq |I'| + 1$.

This means that the inequality (2.34) induced by the two demands k, k' for the interval I is dominated by the inequality (2.34) induced by the same demands for the interval I'.

Sufficiency.
We use the same proof of the theorem 2.4.7 for a clique $C = \{v_k, v_{k'}\}$ in the conflict graph $\tilde{G}_I^E$. □

2.4.5 Interval-Odd-Hole Inequalities

Proposition 2.4.14. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$, and H be an odd-hole H in the conflict graph $\tilde{G}_I^E$ with $|H| \geq 5$. Then, the inequality*

$$\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \frac{|H|-1}{2}, \tag{2.36}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of odd-hole set in the conflict graph $\tilde{G}_I^E$. We strengthen the proof as belows. For each pair of nodes $(v_k, v_{k'})$ linked in H by an edge, we know that $\sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq 1$. Given that H is an odd-hole which means that we have $|H| - 1$ pair of nodes $(v_k, v_{k'})$ linked in H, and by doing a sum for all pairs of nodes $(v_k, v_{k'})$ linked in H, it follows that

$$\sum_{(v_k, v_{k'}) \in E(H)} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq |H| - 1.$$

where $E(H)$ denotes the set of edges in the sub-graph of the conflict graph $\tilde{G}_I^E$ induced by H. Taking into account that each node v_k in H has two neighbors in H, this implies that $\sum_{s=s_i+w_k-1}^{s_j} z_s^k$ appears twice in the previous inequality. As a result,

$$\sum_{(v_k, v_{k'}) \in E(H)} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = \sum_{v_k \in H} 2 \sum_{s=s_i+w_k-1}^{s_j} z_s^k, \sum_{v_k \in H} 2 \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |H| - 1.$$

By dividing the two sides of the previous sum by 2, it follows that

$$\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \left\lfloor \frac{|H|-1}{2} \right\rfloor = \frac{|H|-1}{2} \text{ since } |H| \text{ is an odd number.}$$

We conclude at the end that the inequality (2.36) is valid for $P(G, K, \mathbb{S})$. □

The inequality (2.36) can be strengthened without modifying its right-hand side by combining the inequality (2.35) and (2.36) as follows.

Proposition 2.4.15. *Consider an interval of contiguous slots $I = [s_i, s_j] \subseteq \mathbb{S}$ with $s_i \leq s_j - 1$. Let H be an odd-hole H in the conflict graph $\tilde{G}_I^E$, and C be a clique in the conflict graph $\tilde{G}_I^E$ with* *a)* $|H| \geq 5$,

b) *and* $|C| \geq 3$,

c) *and* $H \cap C = \emptyset$,

d) *and the nodes* $(v_k, v_{k'})$ *are linked in* $\tilde{G}_I^E$ *for all* $v_k \in H$ *and* $v_{k'} \in C$.

Then, the inequality

$$\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq \frac{|H|-1}{2}, \tag{2.37}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of odd-hole set and clique set in the conflict graph $\tilde{G}_I^E$ s.t. if $\sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = 1$ for $v_{k'} \in C$, it forces the quantity $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k$ to be equal to 0. Otherwise, we know from the inequality (2.36) that the sum $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k$ is always smaller than $\frac{|H|-1}{2}$. We strengthen the proof by assuming that the inequality (2.37) is not valid for $P(G, K, \mathbb{S})$. It follows that there exists a C-RSA solution S in which each $s' \in \{s_i + w_{k'} - 1, ..., s_j\} \notin S_{k'}$ for each demand k' with node $v_{k'}$ in the clique C s.t.

$$\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'}(S) > \frac{|H|-1}{2}.$$

Since $\{s_i + w_{k'} - 1, ..., s_j\} \notin S_{k'}$ for each node $v_{k'}$ in the clique C, this means that $\sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'}(S) = 0$, and taking into account the inequality (2.36), and that $\sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) \leq 1$ for each $v_k \in H$ and $\sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'}(S) \leq 1$ for each $v_{k'} \in C$, it follows that $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) \leq \frac{|H|-1}{2}$, which contradicts that $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'}(S) > \frac{|H|-1}{2}$. Similar for a solution S' in which $s \in \{s_i + w_k - 1, ..., s_j\} \notin S'_k$ for each demand k with node v_k in the odd-hole H s.t.

$$\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S') + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'}(S') > \frac{|H|-1}{2}.$$

Since $\{s_i + w_k - 1, ..., s_j\} \notin S'_k$ for each node v_k in the odd-hole H, this means that $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S') = 0$, and taking into account the inequality (2.35), and that $\sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'}(S') \leq 1$ for each $v_{k'} \in C$. It follows that $\frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'}(S') \leq \frac{|H|-1}{2}$, which contradicts that $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S') + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'}(S') > \frac{|H|-1}{2}$. Hence $\sum_{v_k \in H} |S_k \cap \{s_i + w_k - 1, ..., s_j\}| + \sum_{v_{k'} \in C} |S_{k'} \cap \{s_i + w_{k'} - 1, ..., s_j\}| \leq \frac{|H|-1}{2}$. $\square$

Theorem 2.4.9. *Let* H *be an odd-hole in the conflict graph* $\tilde{G}_I^E$ *with* $|H| \geq 5$. *Then, the inequality* (2.36) *is facet defining for* $P(G, K, \mathbb{S})$ *if and only if*

a) *for each node* $v_{k'} \notin H$ *in* $\tilde{G}_I^E$, *there exists a node* $v_k \in H$ *s.t. the induced graph* $\tilde{G}_I^E((H \setminus \{v_k\}) \cup \{v_{k'}\})$ *does not contain an odd-hole* $H' = (H \setminus \{v_k\}) \cup \{v_{k'}\}$, *b) and there does not exist a node $v_{k'} \notin H$ in $\tilde{G}_I^E$ s.t. $v_{k'}$ is linked with all nodes $v_k \in H$,*

c) and there does not exist an interval I' of contiguous slots with $I \subset I'$ s.t. H defines also an odd-hole in the associated conflict graph $\tilde{G}_{I'}^E$.

Proof. **Neccessity.**

We distinguish the following cases:

a) if for a node $v_{k'} \notin H$ in $\tilde{G}_I^E$, there exists a node $v_k \in H$ s.t. the induced graph $\tilde{G}_I^E((H \setminus \{v_k\}) \cup \{v_{k'}\})$ contains an odd-hole $H' = (H \setminus \{v_k\}) \cup \{v_{k'}\}$. This implies that the inequality (2.36) can be dominated by doing some lifting procedures using the following valid inequalities

$$\sum_{v_k \in H} \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k \leq \frac{|H|-1}{2},$$
$$\sum_{v_{k'} \in H'} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq \frac{|H|-1}{2},$$

as follows

$$\sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} + 2 \sum_{v_{k"} \in H \setminus \{k,k'\}} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"} \leq |H| - 1.$$

By adding the sum $\sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'}$ to the previous inequality, we obtain

$$\sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + 2 \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} + 2 \sum_{v_{k"} \in H \setminus \{k,k'\}} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"} \leq |H| - 1 + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'}.$$

We know that $\sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq 1$, it follows that

$$\sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + 2 \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} + 2 \sum_{v_{k"} \in H \setminus \{k,k'\}} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"} \leq |H|.$$

By dividing the last inequality by 2, we obtain that

$$\sum_{s'=s_i+w_k-1}^{s_j} \frac{1}{2} z_{s'}^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} + \sum_{v_{k"} \in H \setminus \{k,k'\}} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"} \leq \left\lfloor \frac{|H|}{2} \right\rfloor.$$

Given that $H' = (H \setminus \{k\}) \cup \{k'\}$ s.t. $|H'| = |H|$, and $|H|$ is an odd number which implies that $\left\lfloor \frac{|H|}{2} \right\rfloor = \frac{|H|-1}{2}$. As a result

$$\sum_{s'=s_i+w_k-1}^{s_j} \frac{1}{2} z_{s'}^k + \sum_{v_{k'} \in H'} \sum_{s"=s_i+w_{k'}-1}^{s_j} z_{s"}^{k'} \leq \frac{|H'|-1}{2}.$$

That which was to be demonstrated.

b) if there exists a node $v_{k'} \in H$ in $\tilde{G}_I^E$ s.t. $v_{k'}$ is linked with all nodes $v_k \in H$. As a result, the inequality (2.36) is dominated by the following inequality

$$\sum_{v_k \in H} \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + \frac{|H|-1}{2} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq \frac{|H|-1}{2}.$$

c) if there exists an interval I' of contiguous slots with $I \subset I'$ s.t. H defines also an odd-hole in the associated conflict graph $\tilde{G}^E_{I'}$. This implies that the inequality (2.36) induced by the odd-hole H for the interval I is dominated by the inequality (2.36) induced by the same odd-hole H for the interval I' given that $\{s_i + w_k - 1, ..., s_j\} \subset I'$ for each $k \in H$. As a result, the inequality (2.36) is not facet defining for $P(G, K, \mathbb{S})$.

If no one of these two cases, the inequality (2.36) can never be dominated by another inequality without changing its right-hand side.

Sufficiency.

Let $F_H^{\tilde{G}^E_I}$ denote the face induced by the inequality (2.36), which is given by

$$F_H^{\tilde{G}^E_I} = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H| - 1}{2}\}.$$

In order to prove that inequality $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \frac{|H|-1}{2}$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that $F_H^{\tilde{G}^E_I}$ is a proper face, and $F_H^{\tilde{G}^E_I} \neq P(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{64} = (E^{64}, S^{64})$ as below

a) a feasible path E_k^{64} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{64} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{64}$ with $|S_k^{64}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{64}$ and $s' \in S_{k'}^{64}$ with $E_k^{64} \cap E_{k'}^{64} \neq \emptyset$ (non-overlapping constraint),

d) and there is $\frac{|H|-1}{2}$ demands $\tilde{H}$ from the odd-hole H (i.e., $v_k \in \tilde{H} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{64}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{64}$ for each node $v_k \in \tilde{H}$, and for each $s' \in S_{k'}^{64}$ for all $v_{k'} \in H \setminus \tilde{H}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

Obviously, $\mathcal{S}^{64}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{64}}, z^{\mathcal{S}^{64}})$ is belong to $P(G, K, \mathbb{S})$ and then to $F_H^{\tilde{G}^E_I}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}$. As a result, $F_H^{\tilde{G}^E_I}$ is not empty (i.e., $F_H^{\tilde{G}^E_I} \neq \emptyset$). Furthermore, given that $s \in \{s_i + w_k - 1, ..., s_j\}$ for each $v_k \in H$, this means that there exists at least one feasible slot assignment S_k for the demands k in H with $s \notin \{s_i + w_k - 1, ..., s_j\}$ for each $s \in S_k$ and each $v_k \in H$. This means that $F_H^{\tilde{G}^E_I} \neq P(G, K, \mathbb{S})$.

We denote the inequality $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \frac{|H|-1}{2}$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F_H^{\tilde{G}^E_I} \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K} (w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$,

b) and $\mu_e^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$,

c) and σ_s^k are equivalents for all $v_k \in H$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$.

We first show that $\mu_e^k = 0$ for each edge $e \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$. Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. For that, we consider a solution $\mathcal{S}'^{64} = (E'^{64}, S'^{64})$ in which

a) a feasible path $E_k'^{64}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S_k'^{64}$ is assigned to each demand $k \in K$ along each edge $e \in E_k'^{64}$ with $|S_k'^{64}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k'^{64}$ and $s" \in S_{k'}'^{64}$ with $E_k'^{64} \cap E_{k'}'^{64} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k'^{64}} |\{s' \in S_k'^{64}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) the edge e is not non-compatible edge with the selected edges $e' \in E_k'^{64}$ of demand k in the solution $\mathcal{S}'^{64}$, i.e., $\sum_{e' \in E_k'^{64}} l_{e'} + l_e \leq \bar{l}_k$. As a result, $E_k'^{64} \cup \{e\}$ is a feasible path for the demand k,

e) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s \in S_k'^{42}$ and $s' \in S_{k'}'^{42}$ with $(E_k'^{42} \cup \{e\}) \cap E_{k'}'^{42} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e in the set of edges $E_k'^{42}$ selected to route the demand k in the solution $\mathcal{S}'^{42}$),

f) and there is $\frac{|H|-1}{2}$ demands $\tilde{H}$ from the odd-hole H (i.e., $v_k \in \tilde{H} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}'^{64}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k'^{64}$ for each node $v_k \in \tilde{H}$, and for each $s' \in S_{k'}'^{64}$ for all $v_{k'} \in H \setminus \tilde{H}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

   $\mathcal{S}'^{64}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{64}}, z^{\mathcal{S}'^{64}})$ is belong to F and then to $F_H^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}$. Based on this, we derive a solution $\mathcal{S}^{65}$ obtained from the solution $\mathcal{S}'^{64}$ by adding an unused edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in K in the solution $\mathcal{S}^{64}$ which means that $E_k^{65} = E_k'^{64} \cup \{e\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^{64}$ remain the same in the solution $\mathcal{S}^{65}$, i.e., $S_k^{65} = S_k'^{64}$ for each $k \in K$, and $E_{k'}^{65} = E_{k'}'^{64}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{65}$ is clearly feasible given that

a) and a feasible path E_k^{65} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{65} is assigned to each demand $k \in K$ along each edge $e \in E_k^{65}$ with $|S_k^{65}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{65}$ and $s" \in S_{k'}^{65}$ with $E_k^{65} \cap E_{k'}^{65} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{65}} |\{s' \in S_k^{65}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{65}}, z^{\mathcal{S}^{65}})$ is belong to F and then to $F_H^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}$. It follows that

$$\mu x^{\mathcal{S}'^{64}} + \sigma z^{\mathcal{S}'^{64}} = \mu x^{\mathcal{S}^{65}} + \sigma z^{\mathcal{S}^{65}} = \mu x^{\mathcal{S}'^{64}} + \mu_e^k + \sigma z^{\mathcal{S}'^{64}}.$$

As a result, $\mu_e^k = 0$ for demand k and an edge e.
As e is chosen arbitrarily for the demand k with $e \notin E_0^k \cup E_1^k$, we iterate the same procedure for all $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$. We conclude that for the demand k

$$\mu_e^k = 0, \text{ for all } e \in E \setminus (E_0^k \cup E_1^k).$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e \in E \setminus (E_0^k \cup E_1^k)$. We conclude at the end that

$$\mu_e^k = 0, \text{ for all } k \in K \text{ and all } e \in E \setminus (E_0^k \cup E_1^k).$$

Let's us show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$. For that, we consider a solution $\mathcal{S}"^{64} = (E"^{64}, S"^{64})$ in which

a) a feasible path $E"^{64}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^{64}_k$ is assigned to each demand $k \in K$ along each edge $e \in E"^{64}_k$ with $|S"^{64}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{64}_k$ and $s" \in S"^{64}_{k'}$ with $E"^{64}_k \cap E"^{64}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E"^{64}_k} |\{s' \in S"^{64}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^{64}_{k'}$ with $E"^{64}_k \cap E"^{64}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{64}_k$ assigned to the demand k in the solution $\mathcal{S}"^{64}$),

e) and there is $\frac{|H|-1}{2}$ demands $\tilde{H}$ from the odd-hole H (i.e., $v_k \in \tilde{H} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}"^{64}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S"^{64}_k$ for each node $v_k \in \tilde{H}$, and for each $s' \in S"^{64}_{k'}$ for all $v_{k'} \in H \setminus \tilde{H}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}"^{64}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{64}}, z^{\mathcal{S}"^{64}})$ is belong to F and then to $F_H^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}$. Based on this, we construct a solution $\mathcal{S}^{66}$ derived from the solution $\mathcal{S}"^{64}$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}"^{64}$ (i.e., $E^{66}_k = E"^{64}_k$ for each $k \in K \setminus \tilde{K}$, and $E^{66}_k \neq E"^{64}_k$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E^{66}_k is assigned to each demand $k \in \tilde{K}$ (routing constraint),

b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"^{64}_k$ and $s" \in S"^{64}_{k'}$ with $E^{66}_k \cap E"^{64}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}, e \in E^{66}_k} |\{s' \in S"^{64}_k, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k \in K \setminus \tilde{K}, e \in E"^{64}_k} |\{s' \in S"^{64}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"^{64}_{k"}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{64}_k$ assigned to the demand k in the solution $\mathcal{S}"^{64}$).

The last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}"^{64}$ remain the same in $\mathcal{S}^{66}$, i.e., $S"^{64}_{k'} = S^{66}_{k'}$ for each demand $k' \in K \setminus \{k\}$, and $S^{66}_k = S"^{64}_k \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{66}$ is clearly feasible given that

a) a feasible path E^{66}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{66}_k is assigned to each demand $k \in K$ along each edge $e \in E^{66}_k$ with $|S^{66}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{66}_k$ and $s" \in S^{66}_{k'}$ with $E^{66}_k \cap E^{66}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E^{66}_k} |\{s' \in S^{66}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{66}}, z^{\mathcal{S}^{66}})$ is belong to F and then to $F_H^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}$. We have so

$$\mu x^{\mathcal{S}"^{64}} + \sigma z^{\mathcal{S}"^{64}} = \mu x^{\mathcal{S}^{66}} + \sigma z^{\mathcal{S}^{66}} = \mu x^{\mathcal{S}"^{64}} + \sigma z^{\mathcal{S}"^{64}} + \sigma_{s'}^k - \sum_{\tilde{k} \in \tilde{K}} \sum_{e \in E"^{64}_k} \mu_e^{\tilde{k}} + \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E^{66}_k} \mu_{e'}^{\tilde{k}}.$$

It follows that $\sigma^k_{s'} = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$ given that $\mu^k_e = 0$ for all the demands $k \in K$ and all edges $e \in E \setminus (E^k_0 \cup E^k_1)$. The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$ s.t. we find

$$\sigma^k_{s'} = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } s' \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } v_k \in H.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma^{k'}_s = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_{k'} - 1, ..., s_j\} \text{ if } v_{k'} \in H.$$

Consequently, we conclude that

$$\sigma^k_s = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } v_k \in H.$$

Let prove that $\sigma^{k'}_{s'}$ for all $v_{k'} \in H$ and all $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ are equivalents. Consider a demand k' with $v_{k'} \in H$ and a slot $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. For that, we consider a solution $\mathcal{S}^{66} = (E^{66}, S^{66})$ in which

a) a feasible path E^{66}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{66}_k is assigned to each demand $k \in K$ along each edge $e \in E^{66}_k$ with $|S^{66}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{66}_k$ and $s" \in S^{66}_{k'}$ with $E^{66}_k \cap E^{66}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E^{66}_k} |\{s' \in S^{66}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_{k'} + 1, ..., s'\} \cap \{s" - w_k + 1, ..., s"\} = \emptyset$ for each $k \in K$ and $s" \in S^{66}_k$ with $E^{66}_k \cap E^{66}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S^{66}_{k'}$ assigned to the demand k' in the solution $\mathcal{S}^{66}$),

e) and there is $\frac{|H|-1}{2}$ demands $\tilde{H}$ from the odd-hole H (i.e., $v_k \in \tilde{H} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{66}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S^{66}_k$ for each node $v_k \in \tilde{H}$, and for each $s' \in S^{66}_{k'}$ for all $v_{k'} \in H \setminus \tilde{H}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

   $\mathcal{S}^{66}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}^{66}}, z^{\mathcal{S}^{66}})$ is belong to F and then to $F^{\tilde{G}^E_I}_H$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z^k_s = \frac{|H|-1}{2}$. Based on this, we construct a feasible solution $\mathcal{S}^{67}$ derived from the solution $\mathcal{S}^{66}$ as belows

a) without changing the established paths for the demands $K \setminus \tilde{K}$ in the solution $\mathcal{S}^{66}$, i.e., $E^{67}_k = E^{66}_k$ for each demand $k \in K \setminus \tilde{K}$,

b) and with changing the established paths for the demands $\tilde{K}$ in the solution $\mathcal{S}^{66}$ to a new paths E^{67}_k for each $k \in \tilde{K}$ s.t. $\{s" - w_{k"} - 1, ..., s"\} \cap \{s - w_k + 1, ..., s\} = \emptyset$ for each $k" \in K$ and $s" \in S^{66}_{k"}$ and $s \in S^{66}_k$ with $E^{67}_k \cap E^{67}_{k_i} \neq \emptyset$,

c) remove the last-slot $\tilde{s}$ totally covered by the interval I and which has been selected by a demand $k_i \in \{v_{k_1}, ..., v_{k_r}\}$ in the solution $\mathcal{S}^{66}$ (i.e., $\tilde{s} \in S^{66}_{k_i}$ and $\tilde{s}' \in \{s_i + w_{k_i} + 1, ..., s_j\}$) s.t. each pair of nodes $(v_{k'}, v_{k_j})$ are not linked in the odd-hole H with $j \neq i$,

d) and select a new last-slot $\tilde{s}' \notin \{s_i + w_{k_i} + 1, ..., s_j\}$ for the demand k_i i.e., $S^{67}_{k_i} = (S^{66}_{k_i} \setminus \{\tilde{s}\}) \cup \{\tilde{s}'\}$ s.t. $\{\tilde{s}' - w_{k_i} - 1, ..., \tilde{s}'\} \cap \{s - w_k + 1, ..., s\} = \emptyset$ for each $k \in K$ and $s \in S^{66}_k$ with $E^{67}_k \cap E^{67}_{k_i} \neq \emptyset$,

e) and add the slot s' to the set of last-slots $S^{66}_{k'}$ assigned to the demand k' in the solution $\mathcal{S}^{66}$, i.e., $S^{67}_{k'} = S^{66}_{k'} \cup \{s'\}$,

f) and without changing the set of last-slots assigned to the demands $K \setminus \{k', k_i\}$, i.e., $S_k^{67} = S_k^{66}$ for each demand $K \setminus \{k', k_i\}$.

The solution $\mathcal{S}^{67}$ is clearly feasible given that

a) a feasible path E_k^{67} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{67} is assigned to each demand $k \in K$ along each edge $e \in E_k^{67}$ with $|S_k^{67}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{67}$ and $s" \in S_{k'}^{67}$ with $E_k^{67} \cap E_{k'}^{67} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{67}} |\{s' \in S_k^{67}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{67}}, z^{\mathcal{S}^{67}})$ is belong to F and then to $F_H^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}$. It follows that

$$\mu x^{S^{66}} + \sigma z^{S^{66}} = \mu x^{S^{67}} + \sigma z^{S^{67}} = \mu x^{S^{66}} + \sigma z^{S^{66}} + \sigma_{s'}^{k'} + \sigma_{\tilde{s}'}^{k_i} - \sigma_{\tilde{s}}^{k_i} - \sum_{\tilde{k} \in \tilde{K}} \sum_{e \in E(p_{\tilde{k}})} \mu_e^{\tilde{k}} + \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E(p'_{\tilde{k}})} \mu_{e'}^{\tilde{k}}.$$

This implies that $\sigma_{\tilde{s}}^{k_i} = \sigma_{s'}^{k'}$ for $v_{k_i}, v_{k'} \in H$ given that $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k + 1, ..., s_j\}$ if $v_k \in H$, and $\mu_e^k = 0$ for all the demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$.
Given that the pair $(v_k, v_{k'})$ are chosen arbitrary in the odd-hole H, we iterate the same procedure for all pairs $(v_k, v_{k'})$ s.t. we find

$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all pairs } (v_k, v_{k'}) \in H.$$

Consequently, we obtain that $\sigma_s^k = \rho$ for all $v_k \in H$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$.
On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k \in K$ and $e \in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e}, \text{ if } e \in E_0^k, \\ \gamma_2^{k,e}, \text{ if } e \in E_1^k, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\}, \\ \rho, \text{if } v_k \in H \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} \rho \beta_s^k + \gamma Q.$ □

Theorem 2.4.10. *Let H be an odd-hole, and C be a clique in the conflict graph $\tilde{G}_I^E$ with* *a)* $|H| \geq 5$,

b) *and* $|C| \geq 3$,

c) *and* $H \cap C = \emptyset$,

d) *and the nodes* $(v_k, v_{k'})$ *are linked in* $\tilde{G}_I^E$ *for all* $v_k \in H$ *and* $v_{k'} \in C$.

Then, the inequality (2.37) *is facet defining for* $P(G, K, \mathbb{S})$ *if and only if*

a) for each node $v_{k"}$ *in* $\tilde{G}_I^E$ *with* $v_{k"} \notin H \cup C$ *and* $C \cup \{v_{k"}\}$ *is a clique in* $\tilde{G}_I^E$*, there exists a subset of nodes* $\tilde{H} \subseteq H$ *of size* $\frac{|H|-1}{2}$ *s.t.* $\tilde{H} \cup \{v_{k"}\}$ *is stable in* $\tilde{G}_I^E$*,*

b) and there does not exist an interval I' *of contiguous slots with* $I \subset I'$ *s.t.* H *and* C *define also an odd-hole and its connected clique in the associated conflict graph* $\tilde{G}_{I'}^E$*.*

Proof. **Neccessity.**

a) Note that if there exists a node $v_{k"} \notin H \cup C$ in $\tilde{G}_I^E$ s.t. $v_{k"}$ is linked with all nodes $v_k \in H$ and all nodes $v_{k'} \in C$. This implies that the inequality (2.37) is dominated by the following inequality

$$\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} + \frac{|H|-1}{2} \sum_{s'=s_i+w_{k"}-1}^{s_j} z_{s'}^{k"} \leq \frac{|H|-1}{2}.$$

b) if there exists an interval I' of contiguous slots with $I \subset I'$ s.t. H and C define also an odd-hole and its connected clique in the associated conflict graph $\tilde{G}_{I'}^E$. This implies that the inequality (2.37) induced by the odd-hole H and clique C for the interval I is dominated by the inequality (2.37) induced by the same odd-hole H and clique C for the interval I' given that $\{s_i + w_k - 1, ..., s_j\} \subset I'$ for each $k \in H$.

If these cases are not verified, we ensure that the inequality (2.37) can never be dominated by another inequality without modifying its right-hand side. Otherwise, the inequality (2.37) is not facet defining for $P(G, K, \mathbb{S})$.

Sufficiency.

Let $F_{H,C}^{\tilde{G}_I^E}$ denote the face induced by the inequality (2.37), which is given by

$$F_{H,C}^{\tilde{G}_I^E} = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = \frac{|H|-1}{2}\}.$$

In order to prove that inequality $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq \frac{|H|-1}{2}$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that $F_{H,C}^{\tilde{G}_I^E}$ is a proper face, and $F_{H,C}^{\tilde{G}_I^E} \neq P(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{68} = (E^{68}, S^{68})$ as below

a) a feasible path E_k^{68} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{68} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{68}$ with $|S_k^{68}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{68}$ and $s' \in S_{k'}^{68}$ with $E_k^{68} \cap E_{k'}^{68} \neq \emptyset$ (non-overlapping constraint),

d) and there is $\frac{|H|-1}{2}$ demands $\tilde{H}$ from the odd-hole H (i.e., $v_k \in \tilde{H} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{68}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{68}$ for each node $v_k \in \tilde{H}$, and for each $s' \in S_{k'}^{68}$ for all $v_{k'} \in H \setminus \tilde{H}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$,

e) and no demand from the clique C selects a last-slot s in the interval I in the solution $\mathcal{S}^{68}$, i.e., for each $k \in C$ and each $s \in S_k^{68}$ we have $s \notin \{s_i + w_k + 1, ..., s_j\}$.

Obviously, $\mathcal{S}^{68}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{68}}, z^{\mathcal{S}^{68}})$ is belong to $P(G, K, \mathbb{S})$ and then to $F_{H,C}^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = \frac{|H|-1}{2}$. As a result, $F_{H,C}^{\tilde{G}_I^E}$ is not empty (i.e., $F_{H,C}^{\tilde{G}_I^E} \neq \emptyset$). Furthermore, given that $s \in \{s_i + w_k - 1, ..., s_j\}$ for each $v_k \in H$, this means that there exists at least one feasible slot assignment S_k for the demands k in H with $s \notin \{s_i + w_k - 1, ..., s_j\}$ for each $s \in S_k$ and each $v_k \in H$. This means that $F_{H,C}^{\tilde{G}_I^E} \neq P(G, K, \mathbb{S})$.
We denote the inequality $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \frac{|H|-1}{2}$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F_{H,C}^{\tilde{G}_I^E} \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K} (w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H \cup C$ as we did in the proof of theorem 2.4.14,

b) and $\mu_e^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$ as we did in the proof of theorem 2.4.14,

c) and σ_s^k are equivalents for all $v_k \in H$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$ as we did in the proof of theorem 2.4.14,

s.t. the solutions $\mathcal{S}^{49}$–$\mathcal{S}^{69}$ still feasible for $F_{H,C}^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = \frac{|H|-1}{2}$. We should prove now that σ_s^k are equivalents for all $v_k \in C$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$. For that, we consider a node $v_k \in C$ and a slot $s \in \{s_i + w_k - 1, ..., s_j\}$. For that, we consider a solution $\mathcal{S}^{70} = (E^{70}, S^{70})$ in which

a) a feasible path E_k^{70} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{70} is assigned to each demand $k \in K$ along each edge $e \in E_k^{70}$ with $|S_k^{70}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{70}$ and $s" \in S_{k'}^{70}$ with $E_k^{70} \cap E_{k'}^{70} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{70}} |\{s' \in S_k^{70}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_{k'} + 1, ..., s'\} \cap \{s" - w_k + 1, ..., s"\} = \emptyset$ for each $k \in K$ and $s" \in S_k^{70}$ with $E_k^{70} \cap E_{k'}^{70} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s in the set of last-slots S_k^{70} assigned to the demand k in the solution $\mathcal{S}^{70}$),

e) and there is $\frac{|H|-1}{2}$ demands $\tilde{H}$ from the odd-hole H (i.e., $v_k \in \tilde{H} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{70}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{70}$ for each node $v_k \in \tilde{H}$, and for each $s' \in S_{k'}^{70}$ for all $v_{k'} \in H \setminus \tilde{H}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{70}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}^{70}}, z^{\mathcal{S}^{70}})$ is belong to F and then to $F_{H,C}^{\tilde{G}_I^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = \frac{|H|-1}{2}$. Based on this, we construct a solution $\mathcal{S}^{71}$ derived from the solution $\mathcal{S}^{70}$ as belows

a) without changing the established paths for the demands K in the solution $\mathcal{S}^{70}$, i.e., $E_k^{71} = E_k^{70}$ for each demand $k \in K$,

b) remove all the last-slots $\tilde{s}_i$ totally covered by the interval I and which has been selected by each demand $k_i \in \{v_{k_1}, ..., v_{k_r}\}$ in the solution $\mathcal{S}^{70}$ (i.e., $\tilde{s} \in S^{70}_{k_i}$ and $\tilde{s} \in \{s_i + w_{k_i} + 1, ..., s_j\}$) for each $k_i \in \{v_{k_1}, ..., v_{k_r}\}$,

c) and select a new last-slot $\tilde{s}'_i \notin \{s_i + w_{k_i} + 1, ..., s_j\}$ for each $k_i \in \{v_{k_1}, ..., v_{k_r}\}$ i.e., $S^{71}_{k_i} = (S^{70}_{k_i} \setminus \{\tilde{s}_i\}) \cup \{\tilde{s}'_i\}$ s.t. $\{\tilde{s}'_i - w_{k_i} - 1, ..., \tilde{s}'_i\} \cap \{s - w_k + 1, ..., s\} = \emptyset$ for each $k \in K$ and $s \in S^{70}_k$ with $E^{71}_k \cap E^{71}_{k_i} \neq \emptyset$ for each $k_i \in \{v_{k_1}, ..., v_{k_r}\}$,

d) and add the slot s' to the set of last-slots $S^{70}_{k'}$ assigned to the demand k' in the solution $\mathcal{S}^{70}$, i.e., $S^{71}_{k'} = S^{70}_{k'} \cup \{s'\}$,

e) without changing the set of last-slots assigned to the demands $K \setminus \{k', k_i\}$, i.e., $S^{71}_k = S^{70}_k$ for each demand $K \setminus \{k', k_i\}$.

The solution $\mathcal{S}^{71}$ is clearly feasible given that

a) a feasible path E^{71}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{71}_k is assigned to each demand $k \in K$ along each edge $e \in E^{71}_k$ with $|S^{71}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{71}_k$ and $s" \in S^{71}_{k'}$ with $E^{71}_k \cap E^{71}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E^{71}_k} |\{s' \in S^{71}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{71}}, z^{\mathcal{S}^{71}})$ is belong to F and then to $F^{\tilde{G}^E_I}_{H,C}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z^k_s + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z^{k'}_{s'} = \frac{|H|-1}{2}$. We have so

$$\mu x^{S^{70}} + \sigma z^{S^{70}} = \mu x^{S^{71}} + \sigma z^{S^{71}} = \mu x^{S^{70}} + \sigma z^{S^{70}} + \sigma^{k'}_{s'} + \sum_{i=1}^{r} \sigma^{k_i}_{\tilde{s}'_i} - \sum_{i=1}^{r} \sigma^{k_i}_{\tilde{s}_i}.$$

This implies that $\sum_{i=1}^r \sigma^{k_i}_{\tilde{s}_i} = \sigma^{k'}_{s'}$ for $v_{k'} \in H$ given that $\sigma^k_s = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k + 1, ..., s_j\}$ if $v_k \in H \cup C$.
Given that the $v_{k'}$ and $s' \in \{s_i + w_{k'} + 1, ..., s_j\}$ are chosen arbitrary in the clique C, we iterate the same procedure for all pairs $v_{k'} \in C$ and all $s' \in \{s_i + w_{k'} + 1, ..., s_j\}$ s.t. we find

$$\sigma^{k'}_{s'} = \rho \frac{|H|-1}{2}, \text{for all } v_{k'} \in C \text{ and } s' \in \{s_i + w_{k'} + 1, ..., s_j\}.$$

As a result,

$$\sigma^k_s = \sigma^{k'}_{s'}, \text{for all } (v_k, v_{k'}) \in C \text{ and } s \in \{s_i + w_k + 1, ..., s_j\} \text{ and } s' \in \{s_i + w_{k'} + 1, ..., s_j\}.$$

Consequently, we obtain that $\sigma^{k'}_{s'} = \rho \frac{|H|-1}{2}$ for all $v_{k'} \in C$ and all $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$.
On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\mu^{k'}_{e'} = \gamma^{k',e'}_1, \text{ for all } k' \in K \text{ and all } e' \in E^{k'}_0,$$
$$\mu^{k'}_{e'} = \gamma^{k',e'}_2, \text{ for all } k' \in K \text{ and all } e' \in E^{k'}_1,$$
$$\sigma^{k'}_{s'} = \gamma^{k',s'}_3, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}.$$

We conclude that for each $k \in K$ and $e \in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e}, \text{ if } e \in E_0^k, \\ \gamma_2^{k,e}, \text{ if } e \in E_1^k, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\}, \\ \rho, \text{if } v_k \in H \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ \rho \dfrac{|H| - 1}{2}, \text{if } v_k \in C \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} \rho\beta_s^k + \dfrac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} \rho\beta_{s'}^{k'} + \gamma Q.$ □

2.4.6 Slot-Assignment-Clique Inequalities

On the other hand, we detected that there may exist some cases that are not covered by the inequality (2.27) previously introduced. For this, we provide the following definition of a conflict graph and its associated inequality.

Definition 2.4.6. *Let $\tilde{G}_S^E$ be a conflict graph defined as follows. For all slot $s \in \{w_k, ..., \bar{s}\}$ and demand $k \in K$, consider a node $v_{k,s}$ in $\tilde{G}_S^E$. Two nodes $v_{k,s}$ and $v_{k',s'}$ are linked by an edge in $\tilde{G}_S^E$ iff $E_1^k \cap E_1^{k'} \neq \emptyset$ and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$.*

The conflict graph $\tilde{G}_S^E$ does not define an interval graph given that some nodes $v_{k,s}$ and $v_{k',s'}$ are not linked even if $\{s-w_k+1, ..., s\} \cap \{s'-w_{k'}+1, ..., s'\} \neq \emptyset$ (i.e., when $E_1^k \cap E_1^{k'} \neq \emptyset$).

Proposition 2.4.16. *Let C be a clique in the conflict graph $\tilde{G}_S^E$ with $|C| \geq 3$. Then, the inequality*

$$\sum_{v_{k,s} \in C} z_s^k \leq 1, \tag{2.38}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of a clique set in the conflict graph $\tilde{G}_S^E$ s.t. for each two linked nodes $v_{k,s}$ and $v_{k',s'}$ in $\tilde{G}_S^E$, we know that the inequality

$$z_s^k + z_{s'}^{k'} \leq 1,$$

is valid for $P(G, K, \mathbb{S})$. By adding the previous inequalities for all two linked nodes $v_{k,s}$ and $v_{k',s'}$ in $\tilde{G}_S^E$, we get

$$\sum_{v_{k,s}} (|C| - 1) z_s^k \leq |C| - 1 \implies \sum_{v_{k,s}} z_s^k \leq \frac{|C| - 1}{|C| - 1} \implies \sum_{v_{k,s}} z_s^k \leq 1,$$

which ends the proof. □

Remark 2.4.4. *The inequality* (2.38) *associated with a clique C, it is dominated by the inequality* (2.35) *associated with an interval $I = [s_i, s_j]$ and a subset of demands $\tilde{K}$ iff $[\min_{v_{k,s} \in C}(s - w_k + 1), \max_{v_{k,s} \in C} s] \subset I$ and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'}) \in C$, and $2w_k \geq |I| + 1$ and $w_k \leq |I|$ for each $v_k \in C$.*

Proof. Consider an interval of contiguous slots $I = [s_i, s_j] \subseteq [1, \bar{s}]$. Let C be a clique in the conflict graph $\tilde{G}_S^E$, and $\tilde{K} = \{k \in K \text{ s.t. } v_{k,s} \in C\}$ be a subset of demands in K with $\tilde{K}$ is a clique in the conflict graph $\tilde{G}_I^E$ for the interval $I = [s_i, s_j]$.
Neccessity.
First, assume that

a) $\tilde{s} \in \{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, s'\}$ for each pair of nodes $(v_{k,s}, v_{k',s'})$ in C,
b) and $[\min_{v_{k,s} \in C}(s - w_k + 1), \max_{v_{k,s} \in C} s] \subset I$.

Given that $s - w_k + 1 \geq \min_{v_{k',s'} \in C}(s' - w_{k'} + 1)$ and $s \leq \max_{v_{k',s'} \in C} s'$ for each $v_{k,s} \in C$, and that $|\{s - w_k + 1, ..., s\}| = w_k$ for each $v_{k,s} \in C$, it follows that $s \in \{s_i + w_k - 1, ..., s_j\} = [s_i + w_k - 1, s_j]$ for each $v_{k,s} \in C$ of demand $k \in \tilde{K}$. As a result, we get that

$$\sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k = \sum_{k \in \tilde{K}} z_s^k + \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\} \setminus \{s\}} z_{s'}^k. \tag{2.39}$$

Taking into account that $\tilde{K} = \{k \in K \text{ s.t. } v_{k,s} \in C\}$, this means that

$$\sum_{k \in \tilde{K}} z_s^k = \sum_{v_{k,s} \in C} z_s^k.$$

It follows that

$$\sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k = \sum_{v_{k,s} \in C} z_s^k + \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\} \setminus \{s\}} z_{s'}^k.$$

Given that all the variable z_s^k is positive for each $k \in K$ and $s \in \mathbb{S}$, this implies that

$$\sum_{v_{k,s} \in C} z_s^k \preceq \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k.$$

Hence, the inequality (2.38) is dominated by the inequality (2.35).
Sufficiency.
Assume that the inequality (2.38) is dominated by the inequality (2.35). It follows that

$$\sum_{v_{k,s} \in C} z_s^k \preceq \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k \implies \sum_{k \in \tilde{K}} z_s^k \preceq \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k$$

Given that the demands in $\tilde{K}$ are independants, this allows us to take that

$$z_s^k \preceq \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k \text{ for each } k \in \tilde{K}.$$

Given that the variable z_s^k is positive for each $k \in K$ and $s \in \mathbb{S}$, this means that

$$s \in \{s_i + w_k - 1, ..., s_j\} \text{ for each } k \in \tilde{K},$$

which is equivalent to say that

$$s \in \{s_i + w_k - 1, ..., s_j\} \text{ for each node } v_{k,s} \in C \implies s \in \{s_i + w_k - 1, ..., s_j\}.$$

It follows that

$$s - w_k + 1 \in I \text{ for each node } v_{k,s} \in C.$$

As a result,

$$\min_{v_{k,s} \in C}(s - w_k + 1) \in I \text{ and } \max_{v_{k,s} \in C} s \in I \text{ for each node } v_{k,s} \in C$$
$$\implies [\min_{v_{k,s} \in C}(s - w_k + 1), \max_{v_{k,s} \in C} s] \subseteq I.$$

Furthermore, and given that $w_k + w_{k'} > |I|$ for each pair of demands $k, k' \in \tilde{K}$, it follows that $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s\} \neq \emptyset$ for each $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ of each pair of demands $k, k' \in \tilde{K}$. Hence, $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s\} \neq \emptyset$ for each pair $(v_{k,s}, v_{k',s'}) \in C$ since $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. We conclude at the end that

a) $\tilde{s} \in \{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, s'\}$ for each pair of nodes $(v_{k,s}, v_{k',s'})$ in C,
b) and $[\min_{v_{k,s} \in C}(s - w_k + 1), \max_{v_{k,s} \in C} s] \subset I$,

which ends the proof. □

Theorem 2.4.11. *Consider a clique C in the conflict graph $\tilde{G}^E_S$. Then, the inequality (2.38) is facet defining for $P(G, K, \mathbb{S})$ iff C is a maximal clique in the conflict graph $\tilde{G}^E_S$, and there does not exist an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with*

a) $[\min_{v_{k,s} \in C}(s - w_k + 1), \max_{v_{k,s} \in C} s] \subset I$,
b) and $w_k + w_{k'} \geq |I| + 1$ *for each* $(v_k, v_{k'}) \in C$,
c) and $2w_k \geq |I| + 1$ *and* $w_k \leq |I|$ *for each* $v_k \in C$.

Proof. **Neccessity.**
If C is a not maximal clique in the conflict graph $\tilde{G}^E_S$, this means that the inequality (2.38) can be dominated by another inequality associated with a clique C' s.t. $C \subset C'$ without changing its right-hand side. Moreover, if there exists an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with

a) $[\min_{v_{k,s} \in C}(s - w_k + 1), \max_{v_{k,s} \in C} s] \subset I$,
b) and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'}) \in C$,
c) and $2w_k \geq |I| + 1$ and $w_k \leq |I|$ for each $v_k \in C$.

Then, the inequality (2.38) is dominated by the inequality (2.35). As a result, the inequality (2.38) cannot be facet defining for $P(G, K, \mathbb{S})$.
Sufficiency.
Let $F_C^{\tilde{G}^E_S}$ denote the face induced by the inequality (2.38), which is given by

$$F_C^{\tilde{G}^E_S} = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{v_{k,s} \in C} z^k_s = 1\}.$$

In order to prove that inequality $\sum_{v_{k,s}\in C} z_s^k \leq 1$ is facet defining for $P(G,K,\mathbb{S})$, we start checking that $F_C^{\tilde{G}_S^E}$ is a proper face, and $F_C^{\tilde{G}_S^E} \neq P(G,K,\mathbb{S})$. We construct a solution $\mathcal{S}^{72} = (E^{72}, S^{72})$ as below

a) a feasible path E_k^{72} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{72} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{72}$ with $|S_k^{72}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{72}$ and $s' \in S_{k'}^{72}$ with $E_k^{72} \cap E_{k'}^{72} \neq \emptyset$ (non-overlapping constraint),

d) and there is one pair of demand k and slot s from the clique C (i.e., $v_{k,s} \in C$ s.t. the demand k selects the slot s as last-slot in the solution $\mathcal{S}^{72}$, i.e., $s \in S_k^{72}$ for a node $v_{k,s} \in C$, and $s' \notin S_{k'}^{72}$ for all $v_{k',s'} \in C \setminus \{v_{k,s}\}$.

Obviously, $\mathcal{S}^{72}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{72}}, z^{\mathcal{S}^{72}})$ is belong to $P(G,K,\mathbb{S})$ and then to $F_C^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s}\in C} z_s^k = 1$. As a result, $F_C^{\tilde{G}_S^E}$ is not empty (i.e., $F_C^{\tilde{G}_S^E} \neq \emptyset$). Furthermore, given that $s \in \{w_k, ..., \bar{s}\}$ for each $v_{k,s} \in C$, this means that there exists at least one feasible slot assignment S_k for the demands k in C with $s \notin S_k$ for each $v_{k,s} \in C$. This means that $F_C^{\tilde{G}_S^E} \neq P(G,K,\mathbb{S})$.
Let denote the inequality $\sum_{v_{k,s}\in C} z_s^k \leq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G,K,\mathbb{S})$. Suppose that $F_C^{\tilde{G}_S^E} \subset F = \{(x,z) \in P(G,K,\mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k\in K}|E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k\in K}|E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k\in K}(w_k-1)}$) s.t. $(\mu,\sigma) = \rho(\alpha,\beta) + \gamma Q$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $v_{k,s} \notin C$,

b) and $\mu_e^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$,

c) and σ_s^k are equivalents for all $v_{k,s} \in C$.

We first show that $\mu_e^k = 0$ for each edge $e \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$. Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. For that, we consider a solution $\mathcal{S}'^{72} = (E'^{72}, S'^{72})$ in which

a) a feasible path $E_k'^{72}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S_k'^{72}$ is assigned to each demand $k \in K$ along each edge $e \in E_k'^{72}$ with $|S_k'^{72}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k'^{72}$ and $s" \in S_{k'}'^{72}$ with $E_k'^{72} \cap E_{k'}'^{72} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e\in E_k'^{72}} |\{s' \in S_k'^{72}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and each $s' \in S_k'^{72}$ and $s" \in S_{k'}'^{72}$ with $(E_k'^{72} \cup \{e\}) \cap E_{k'}'^{72} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e in the set of edges $E_k'^{72}$ selected to route the demand k in the solution $\mathcal{S}'^{72}$),

e) the edge e is not non-compatible edge with the selected edges $e \in E_k'^{72}$ of demand k in the solution $\mathcal{S}'^{72}$, i.e., $\sum_{e'\in E_k'^{72}} l_{e'} + l_e \leq \bar{l}_k$. As a result, $E_k'^{72} \cup \{e\}$ is a feasible path for the demand k,

f) and there is one pair of demand k and slot s from the clique C (i.e., $v_{k,s} \in C$ s.t. the demand k selects the slot s as last-slot in the solution $\mathcal{S}'^{72}$, i.e., $s \in S'^{72}_k$ for a node $v_{k,s} \in C$, and $s' \notin S'^{72}_{k'}$ for all $v_{k',s'} \in C \setminus \{v_{k,s}\}$.

$\mathcal{S}'^{72}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{72}}, z^{\mathcal{S}'^{72}})$ is belong to F and then to $F_C^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s} \in C} z_s^k = 1$. Based on this, we derive a solution $\mathcal{S}^{73}$ obtained from the solution $\mathcal{S}'^{72}$ by adding an unused edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in K in the solution $\mathcal{S}^{72}$ which means that $E_k^{73} = E'^{72}_k \cup \{e\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^{72}$ remain the same in the solution $\mathcal{S}^{73}$, i.e., $S_k^{73} = S'^{72}_k$ for each $k \in K$, and $E_{k'}^{73} = E'^{72}_{k'}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{73}$ is clearly feasible given that

a) and a feasible path E_k^{73} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{73} is assigned to each demand $k \in K$ along each edge $e \in E_k^{73}$ with $|S_k^{73}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{73}$ and $s" \in S_{k'}^{73}$ with $E_k^{73} \cap E_{k'}^{73} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{73}} |\{s' \in S_k^{73}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{73}}, z^{\mathcal{S}^{73}})$ is belong to F and then to $F_C^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s} \in C} z_s^k = 1$. It follows that

$$\mu x^{\mathcal{S}'^{72}} + \sigma z^{\mathcal{S}'^{72}} = \mu x^{\mathcal{S}^{73}} + \sigma z^{\mathcal{S}^{73}} = \mu x^{\mathcal{S}'^{72}} + \mu_e^k + \sigma z^{\mathcal{S}'^{72}}.$$

As a result, $\mu_e^k = 0$ for demand k and an edge e.
As e is chosen arbitrarily for the demand k with $e \notin E_0^k \cup E_1^k$, we iterate the same procedure for all $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$. We conclude that for the demand k

$$\mu_e^k = 0, \text{ for all } e \in E \setminus (E_0^k \cup E_1^k).$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e \in E \setminus (E_0^k \cup E_1^k)$. We conclude at the end that

$$\mu_e^k = 0, \text{ for all } k \in K \text{ and all } e \in E \setminus (E_0^k \cup E_1^k).$$

Let's us show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$ with $v_{k,s} \notin C$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$ with $v_{k,s'} \notin C$. For that, we consider a solution $\mathcal{S}"^{72} = (E"^{72}, S"^{72})$ in which

a) a feasible path $E"^{72}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^{72}_k$ is assigned to each demand $k \in K$ along each edge $e \in E"^{72}_k$ with $|S"^{72}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{72}_k$ and $s" \in S"^{72}_{k'}$ with $E"^{72}_k \cap E"^{72}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E"^{72}_k} |\{s' \in S"^{72}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S_{k'}^{72}$ with $E"^{72}_k \cap E"^{72}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{72}_k$ assigned to the demand k in the solution $\mathcal{S}"^{72}$),

e) and there is one pair of demand k and slot s from the clique C (i.e., $v_{k,s} \in C$ s.t. the demand k selects the slot s as last-slot in the solution $\mathcal{S}"^{72}$, i.e., $s \in S"^{72}_k$ for a node $v_{k,s} \in C$, and $s' \notin S"^{72}_{k'}$ for all $v_{k',s'} \in C \setminus \{v_{k,s}\}$.

$\mathcal{S}"^{72}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{72}}, z^{\mathcal{S}"^{72}})$ is belong to F and then to $F_C^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s}\in C} z_s^k = 1$. Based on this, we construct a solution $\mathcal{S}^{74}$ derived from the solution $\mathcal{S}"^{72}$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}"^{72}$ (i.e., $E_k^{74} = E"^{72}_k$ for each $k \in K \setminus \tilde{K}$, and $E_k^{74} \neq E"^{72}_k$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E_k^{74} is assigned to each demand $k \in \tilde{K}$ (routing constraint),

b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"^{72}_k$ and $s" \in S"^{72}_{k'}$ with $E_k^{74} \cap E"^{72}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in\tilde{K}, e\in E_k^{74}} |\{s' \in S"^{72}_k, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k\in K\setminus\tilde{K}, e\in E"^{72}_k} |\{s' \in S"^{72}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"^{72}_{k"}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{72}_k$ assigned to the demand k in the solution $\mathcal{S}"^{72}$).

The last-slots assigned to the demands $K\setminus\{k\}$ in $\mathcal{S}"^{72}$ remain the same in $\mathcal{S}^{74}$, i.e., $S"^{72}_{k'} = S_{k'}^{74}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{74} = S"^{72}_k \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{74}$ is clearly feasible given that

a) a feasible path E_k^{74} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{74} is assigned to each demand $k \in K$ along each edge $e \in E_k^{74}$ with $|S_k^{74}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{74}$ and $s" \in S_{k'}^{74}$ with $E_k^{74} \cap E_{k'}^{74} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e\in E_k^{74}} |\{s' \in S_k^{74}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{74}}, z^{\mathcal{S}^{74}})$ is belong to F and then to $F_C^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s}\in C} z_s^k = 1$. We have so

$$\mu x^{\mathcal{S}"^{72}} + \sigma z^{\mathcal{S}"^{72}} = \mu x^{\mathcal{S}^{74}} + \sigma z^{\mathcal{S}^{74}} = \mu x^{\mathcal{S}"^{72}} + \sigma z^{\mathcal{S}"^{72}} + \sigma_{s'}^k - \sum_{\tilde{k}\in\tilde{K}} \sum_{e\in E"^{72}_k} \mu_e^{\tilde{k}} + \sum_{\tilde{k}\in\tilde{K}} \sum_{e'\in E_k^{74}} \mu_{e'}^{\tilde{k}}.$$

It follows that $\sigma_{s'}^k = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $v_{k,s'} \notin C$ given that $\mu_e^k = 0$ for all the demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $v_{k,s'} \notin C$ s.t. we find

$$\sigma_{s'}^k = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s'} \notin C.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } v_{k',s} \notin C..$$

Consequently, we conclude that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s} \notin C.$$

Let's prove that σ_s^k for all $v_{k,s} \in C$ are equivalents. Consider a node $v_{k',s'}$ in C s.t. $s' \notin S_{k'}^{72}$. For that, we consider a solution $\tilde{\mathcal{S}}^{72} = (\tilde{E}^{72}, \tilde{S}^{72})$ in which

a) a feasible path $\tilde{E}_k^{72}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}_k^{72}$ is assigned to each demand $k \in K$ along each edge $e \in \tilde{E}_k^{72}$ with $|\tilde{S}_k^{72}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}_k^{72}$ and $s" \in \tilde{S}_{k'}^{72}$ with $\tilde{E}_k^{72} \cap \tilde{E}_{k'}^{72} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in \tilde{E}_k^{72}} |\{s' \in \tilde{S}_k^{72}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s \in \tilde{S}_k^{72}$ with $\tilde{E}_k^{72} \cap \tilde{E}_{k'}^{72} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $\tilde{S}_k^{72}$ assigned to the demand k' in the solution $\tilde{\mathcal{S}}^{72}$),

e) and there is one pair of demand k and slot s from the clique C (i.e., $v_{k,s} \in C$ s.t. the demand k selects the slot s as last-slot in the solution $\tilde{\mathcal{S}}^{72}$, i.e., $s \in S"_k^{72}$ for a node $v_{k,s} \in C$, and $s" \notin S"_{k'}^{72}$ for all $v_{k',s"} \in C \setminus \{v_{k,s}\}$.

$\tilde{\mathcal{S}}^{72}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{72}}, z^{\tilde{\mathcal{S}}^{72}})$ is belong to F and then to $F_C^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s} \in C} z_s^k = 1$. Based on this, we construct a solution $\mathcal{S}^{75}$ derived from the solution $\tilde{\mathcal{S}}^{72}$ by adding the slot s' as last-slot to the demand k' with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\tilde{\mathcal{S}}^{72}$ (i.e., $E_k^{75} = \tilde{E}_k^{72}$ for each $k \in K \setminus \tilde{K}$, and $E_k^{75} \neq \tilde{E}_k^{72}$ for each $k \in \tilde{K}$), and also the last-slots assigned to the demands $K \setminus \{k, k'\}$ in $\tilde{\mathcal{S}}^{72}$ remain the same in $\mathcal{S}^{75}$, i.e., $\tilde{S}_{k"}^{72} = S_{k"}^{75}$ for each demand $k" \in K \setminus \{k, k'\}$, and $S_{k'}^{75} = \tilde{S}_{k'}^{72} \cup \{s'\}$ for the demand k', and modifying the last-slots assigned to the demand k by adding a new last-slot $\tilde{s}$ and removing the last slot $s \in \tilde{S}_k^{72}$ with $v_{k,s} \in C$ and $v_{k,\tilde{s}} \notin C$ s.t. $S_k^{75} = (\tilde{S}_k^{72} \setminus \{s\}) \cup \{\tilde{s}\}$ for the demand k s.t. $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}^{75}$ with $E_k^{75} \cap E_{k'}^{75} \neq \emptyset$. The solution $\mathcal{S}^{75}$ is clearly feasible given that

a) a feasible path E_k^{75} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{75} is assigned to each demand $k \in K$ along each edge $e \in E_k^{75}$ with $|S_k^{75}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{75}$ and $s" \in S_{k'}^{75}$ with $E_k^{75} \cap E_{k'}^{75} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{75}} |\{s' \in S_k^{75}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{75}}, z^{\mathcal{S}^{75}})$ is belong to F and then to $F_C^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s} \in C} z_s^k = 1$. We have so

$$\begin{aligned} \mu x^{\tilde{\mathcal{S}}^{72}} + \sigma z^{\tilde{\mathcal{S}}^{72}} = \mu x^{\mathcal{S}^{75}} + \sigma z^{\mathcal{S}^{75}} &= \mu x^{\tilde{\mathcal{S}}^{72}} + \sigma z^{\tilde{\mathcal{S}}^{72}} + \sigma_{s'}^{k'} - \sigma_s^k + \sigma_{\tilde{s}}^k \\ &- \sum_{k \in \tilde{K}} \sum_{e \in \tilde{E}_k^{72}} \mu x^{\tilde{\mathcal{S}}^{72}} + \sum_{k \in \tilde{K}} \sum_{e \in E_k^{75}} \mu x^{\mathcal{S}^{75}}. \end{aligned}$$

It follows that $\sigma_{s'}^{k'} = \sigma_s^k$ for demand k' and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $v_{k',s'} \in C$ given that $\sigma_{\tilde{s}}^k = 0$ for $v_{k,\tilde{s}} \notin C$, and $\mu_e^k = 0$ for all $k \in K$ and all $e \in E \setminus (E_0^k \cup E_1^k)$.

Given that the pair $(v_{k,s}, v_{k',s'})$ are chosen arbitrary in the clique C, we iterate the same procedure for all pairs $(v_{k,\tilde{s}}, v_{k',s'})$ s.t. we find

$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all pairs } (v_{k,s}, v_{k',s'}) \in C.$$

Consequently, we obtain that $\sigma_s^k = \rho$ for all pairs $v_{k,s} \in C$.
On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned}
\mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_0^{k'}, \\
\mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_1^{k'}, \\
\sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}.
\end{aligned}$$

We conclude that for each $k \in K$ and $e \in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e}, \text{ if } e \in E_0^k \\ \gamma_2^{k,e}, \text{ if } e \in E_1^k \\ \quad 0, otherwise \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\} \\ \qquad \rho, \text{if } v_{k,s} \in C, \\ \qquad 0, \text{if } v_{k,s} \notin C. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{v_{k,s} \in C} \rho \beta_s^k + \gamma Q.$ □

2.4.7 Slot-Assignment-Odd-Hole Inequalities

We have observed that the conflict graph $\tilde{G}_S^E$ cannot define a interval graph graph given that it contains some nodes $v_{k,s}$ and $v_{k',s'}$ that are linked even if the $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$, i.e., when $k = k'$. As a result, one can strengthen the inequality (2.38) by introducing the following inequalities based on the so-called odd-hole inequalities.

Proposition 2.4.17. *Let H be an odd-hole in the conflict graph $\tilde{G}_S^E$ with $|H| \geq 5$. Then, the inequality*

$$\sum_{v_{k,s} \in H} z_s^k \leq \frac{|H| - 1}{2}, \tag{2.40}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of the odd-hole in the conflict graph $\tilde{G}_S^E$ s.t. for each pair of nodes $(v_{k,s}, v_{k',s'})$ linked in H by an edge, we know that $z_s^k + z_{s'}^{k'} \leq 1$. Given that H is an odd-hole which means that we have $|H| - 1$ pair of nodes $(v_{k,s}, v_{k',s'})$ linked in H, and by doing a sum over all pairs of nodes $(v_{k,s}, v_{k',s'})$ linked in H, it follows that

$$\sum_{(v_{k,s}, v_{k',s'}) \in E(H)} z_s^k + z_{s'}^{k'} \leq |H| - 1.$$

Taking into account that each node v_k in H has two neighbors in H, this implies that z_s^k appears twice in the previous inequality. As a result,

$$\sum_{(v_{k,s},v_{k',s'})\in E(H)} z_s^k + z_{s'}^{k'} = \sum_{v_{k,s}\in H} 2z_s^k \implies \sum_{v_{k,s}\in H} 2z_s^k \leq |H| - 1$$
$$\implies \sum_{v_{k,s}\in H} z_s^k \leq \left\lfloor \frac{|H|-1}{2} \right\rfloor = \frac{|H|-1}{2} \text{ since } |H| \text{ is an odd number.}$$

We conclude at the end that the inequality (2.40) is valid for $P(G, K, \mathbb{S})$. □

Remark 2.4.5. *The inequality* (2.40) *is dominated by the inequality* (2.36) *iff there exists an interval of contiguous slots* $I = [s_i, s_j] \subset [1, \bar{s}]$ *with*

a) $[\min_{v_{k,s}\in H}(s - w_k + 1), \max_{v_{k,s}\in H}] \subset I$*,*

b) and $w_k + w_{k'} \geq |I| + 1$ *for each* $(v_k, v_{k'})$ *linked in* H*,*

c) and $2w_k \geq |I| + 1$ *and* $w_k \leq |I|$ *for each* $v_k \in H$*.*

Proof. Consider an interval of contiguous slots $I = [s_i, s_j] \subseteq [1, \bar{s}]$. Let H be an odd-hole in the conflict graph $\tilde{G}_S^E$, and $\tilde{K} = \{k \in K \text{ s.t. } v_{k,s} \in H\}$ be a subset of demands in K with $\tilde{K}$ is an odd-hole in the conflict graph $\tilde{G}_I^E$ for the interval $I = [s_i, s_j]$.
Neccessity.
First, assume that

a) $\tilde{s} \in \{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, s'\}$ for each pair of nodes $(v_{k,s}, v_{k',s'})$ in H,

b) and $[\min_{v_{k,s}\in H}(s - w_k + 1), \max_{v_{k,s}\in H} s] \subset I$.

Given that $s - w_k + 1 \geq \min_{v_{k',s'}\in H}(s' - w_{k'} + 1)$ and $s \leq \max_{v_{k',s'}\in H} s'$ for each $v_{k,s} \in H$, and that $|\{s - w_k + 1, ..., s\}| = w_k$ for each $v_{k,s} \in H$, it follows that $s \in \{s_i + w_k - 1, ..., s_j\} = [s_i + w_k - 1, s_j]$ for each $v_{k,s} \in H$ of demand $k \in \tilde{K}$. As a result, we get that

$$\sum_{k\in\tilde{K}} \sum_{s'\in\{s_i+w_k-1,...,s_j\}} z_{s'}^k = \sum_{k\in\tilde{K}} z_s^k + \sum_{k\in\tilde{K}} \sum_{s'\in\{s_i+w_k-1,...,s_j\}\setminus\{s\}} z_{s'}^k. \tag{2.41}$$

Taking into account that $\tilde{K} = \{k \in K \text{ s.t. } v_{k,s} \in H\}$, this means that

$$\sum_{k\in\tilde{K}} z_s^k = \sum_{v_{k,s}\in H} z_s^k.$$

This implies that

$$\sum_{k\in\tilde{K}} \sum_{s'\in\{s_i+w_k-1,...,s_j\}} z_{s'}^k = \sum_{v_{k,s}\in H} z_s^k + \sum_{k\in\tilde{K}} \sum_{s'\in\{s_i+w_k-1,...,s_j\}\setminus\{s\}} z_{s'}^k$$
$$\implies \sum_{v_{k,s}\in H} z_s^k \preceq \sum_{k\in\tilde{K}} \sum_{s'\in\{s_i+w_k-1,...,s_j\}} z_{s'}^k \implies z_s^k \preceq \sum_{s'\in\{s_i+w_k-1,...,s_j\}} z_{s'}^k \text{ for each } v_{k,s} \in H.$$

Hence, the inequality (2.40) is dominated by the inequality (2.36).
Sufficiency.

Assume that the inequality (2.40) is dominated by the inequality (2.36) and given that $\tilde{K} = \{k \in K \text{ s.t. } v_{k,s} \in H\}$, this means that

$$\sum_{k \in \tilde{K}} z_s^k = \sum_{v_{k,s} \in H} z_s^k.$$

It follows that

$$\sum_{v_{k,s} \in H} z_s^k \preceq \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k \implies \sum_{k \in \tilde{K}} z_s^k \preceq \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k.$$

Given that the demands in $\tilde{K}$ are independants, this implies that

$$z_s^k \preceq \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k \text{ for each } k \in \tilde{K} \implies s \in \{s_i + w_k - 1, ..., s_j\} \text{ for each } k \in \tilde{K} \implies s \in \{s_i + w_k - 1, ...$$

As a result,

$$s - w_k + 1 \in I \text{ for each node } v_{k,s} \in H \implies \min_{v_{k,s} \in H} (s - w_k + 1) \in I$$

$$\text{and } \max_{v_{k,s} \in H} s \in I \text{ for each node } v_{k,s} \in H \implies [\min_{v_{k,s} \in H} (s - w_k + 1), \max_{v_{k,s} \in H} s] \subseteq I.$$

Furthermore, and given that $w_k + w_{k'} > |I|$ for each pair of demands $k, k' \in \tilde{K}$, it follows that $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s\} \neq \emptyset$ for each $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ of each pair of demands $k, k' \in \tilde{K}$. Hence, $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s\} \neq \emptyset$ for each pair $(v_{k,s}, v_{k',s'}) \in H$ since $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. We conclude at the end that

a) $\tilde{s} \in \{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, s'\}$ for each pair of nodes $(v_{k,s}, v_{k',s'})$ in H,
b) and $[\min_{v_{k,s} \in H} (s - w_k + 1), \max_{v_{k,s} \in H} s] \subset I$,

which ends the proof. □

Note that the inequality (2.40) can be strengthened without modifying its right-hand side by combining the inequality (2.40) and (2.38).

Proposition 2.4.18. *Let H be an odd-hole, and C be a clique in the conflict graph $\tilde{G}_S^E$ with*

a) $|H| \geq 5$,
b) and $|C| \geq 3$,
c) and $H \cap C = \emptyset$,
d) and the nodes $(v_{k,s}, v_{k',s'})$ are linked in $\tilde{G}_S^E$ for all $v_{k,s} \in H$ and $v_{k',s'} \in C$.

Then, the inequality

$$\sum_{v_{k,s} \in H} z_s^k + \frac{|H| - 1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'} \leq \frac{|H| - 1}{2}, \tag{2.42}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of the odd-hole and clique in $\tilde{G}_S^E$ s.t. if $\sum_{v_{k',s'} \in C} z_{s'}^{k'} = 1$ for a $v_{k',s'} \in C \in C$ which implies that the quantity $\sum_{v_{k,s} \in H} z_s^k$ is forced to be equal to 0. Otherwise, we know from the inequality (2.40) that the sum $\sum_{v_{k,s} \in H} z_s^k$ is always smaller than $\frac{|H|-1}{2}$. We strengthen the proof by assuming that the inequality (2.42) is not valid for $P(G, K, \mathbb{S})$. It follows that there exists a C-RSA solution S in which $s' \notin S_{k'}$ for each node $v_{k',s'}$ in the clique C s.t.

$$\sum_{v_{k,s} \in H} z_s^k(S) + \frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'}(S) > \frac{|H|-1}{2}.$$

Since $s' \notin S_{k'}$ for each node $v_{k',s'}$ in the clique C this means that $\sum_{v_{k',s'} \in C} z_{s'}^{k'}(S) = 0$, and taking into account the inequality (2.40), $z_s^k(S) \leq 1$ for each $v_{k,s} \in H$, and that $z_{s'}^{k'}(S) \leq 1$ for each $v_{k',s'} \in C$, it follows that

$$\sum_{v_{k,s} \in H} z_s^k(S) \leq \frac{|H|-1}{2},$$

which contradicts that $\sum_{v_{k,s} \in H} z_s^k(S) + \frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'}(S) > \frac{|H|-1}{2}$.
Similar for a solution S' in which $s \notin S'_k$ for each node $v_{k,s}$ in the odd-hole H s.t.

$$\sum_{v_{k,s} \in H} z_s^k(S') + \frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'}(S') > \frac{|H|-1}{2}.$$

Since $s \notin S'_k$ for each node $v_{k,s}$ in the odd-hole H this means that $\sum_{v_{k,s} \in H} z_s^k(S') = 0$, and taking into account the inequality (2.38), $z_{s'}^{k'}(S') \leq 1$ for each $v_{k',s'} \in C$, it follows that

$$\frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_s^k(S') \leq \frac{|H|-1}{2},$$

which contradicts that $\sum_{v_{k,s} \in H} z_s^k(S') + \frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'}(S') > \frac{|H|-1}{2}$.
Hence $\sum_{v_{k,s} \in H} |S_k \cap \{s\}| + \sum_{v_{k',s'} \in C} |S_{k'} \cap \{s'\}| \leq \frac{|H|-1}{2}$.

□

Remark 2.4.6. *The inequality* (2.42) *is dominated by the inequality* (2.37) *iff there exists an interval of contiguous slots* $I = [s_i, s_j] \subset [1, \bar{s}]$ *with*

a) $[\min_{v_{k,s} \in H \cup C} (s - w_k + 1), \max_{v_{k,s} \in H \cup C}] \subset I$,

b) and $w_k + w_{k'} \geq |I| + 1$ *for each* $(v_k, v_{k'})$ *linked in* H,

c) and $w_k + w_{k'} \geq |I| + 1$ *for each* $(v_k, v_{k'})$ *linked in* C,

d) and $w_k + w_{k'} \geq |I| + 1$ *for each* $v_k \in H$ *and* $v_{k'} \in C$,

e) and $2w_k \geq |I| + 1$ *and* $w_k \leq |I|$ *for each* $v_k \in H$,

f) and $2w_{k'} \geq |I| + 1$ *and* $w_{k'} \leq |I|$ *for each* $v_{k'} \in C$.

Proof. Similar to the proof of the remark 2.4.5. □

Theorem 2.4.12. *Let* H *be an odd-hole in the conflict graph* $\tilde{G}_S^E$ *with* $|H| \geq 5$. *Then, the inequality* (2.40) *is facet defining for* $P(G, K, \mathbb{S})$ *iff*

a) *for each node $v_{k',s'} \notin H$ in $\tilde{G}_S^E$, there exists a node $v_{k,s} \in H$ s.t. the induced graph $\tilde{G}_S^E((H \setminus \{v_{k,s}\}) \cup \{v_{k',s'}\})$ does not contain an odd-hole,*

b) *and there does not exist a node $v_{k',s'} \notin H$ in $\tilde{G}_S^E$ s.t. $v_{k',s'}$ is linked with all nodes $v_{k,s} \in H$,*

c) *and there does not exist an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with*

   a) *$[\min_{v_{k,s} \in H}(s - w_k + 1), \max_{v_{k,s} \in H}] \subset I$,*

   b) *and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'})$ linked in H,*

   c) *and $2w_k \geq |I| + 1$ and $w_k \leq |I|$ for each $v_k \in H$.*

Proof. **Neccessity.**
We distinguish the following cases:

a) if for a node $v_{k',s'} \notin H$ in $\tilde{G}_S^E$, there exists a node $v_{k,s} \in H$ s.t. the induced graph $\tilde{G}_S^E(H \setminus \{v_{k,s}\} \cup \{v_{k',s'}\})$ contains an odd-hole $H' = (H \setminus \{v_{k,s}\}) \cup \{v_{k',s'}\}$. This implies that the inequality (2.40) can be dominated using some technics of lifting based on the following two inequalities $\sum_{v_{k,s} \in H} z_s^k \leq \frac{|H|-1}{2}$, and $\sum_{v_{k',s'} \in H'} z_{s'}^{k'} \leq \frac{|H'|-1}{2}$.

b) if there exists a node $v_{k',s'} \notin H$ in $\tilde{G}_S^E$ s.t. $v_{k',s'}$ is linked with all nodes $v_{k,s} \in H$. This implies that the inequality (2.40) can be dominated by the following valid inequality

$$\sum_{v_{k,s} \in H} z_s^k + \frac{|H|-1}{2} z_{s'}^{k'} \leq \frac{|H|-1}{2}.$$

c) if there exists an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with

   a) $[\min_{v_{k,s} \in H}(s - w_k + 1), \max_{v_{k,s} \in H}] \subset I$,

   b) and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'})$ linked in H,

   c) and $2w_k \geq |I| + 1$ and $w_k \leq |I|$ for each $v_k \in H$.

This implies that the inequality (2.40) is dominated by the inequality (2.36).

If no one of these cases is verified, the inequality (2.40) can never be dominated by another inequality without changing its right-hand side. Otherwise, the inequality (2.40) cannot be facet defining for $P(G, K, \mathbb{S})$.
Sufficiency.
Let $F_H^{\tilde{G}_S^E}$ denote the face induced by the inequality (2.40), which is given by

$$F_H^{\tilde{G}_S^E} = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{v_{k,s} \in H} z_s^k = \frac{|H|-1}{2}\}.$$

In order to prove that inequality $\sum_{v_{k,s} \in H} z_s^k \leq \frac{|H|-1}{2}$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that $F_H^{\tilde{G}_S^E}$ is a proper face, and $F_H^{\tilde{G}_S^E} \neq P(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{76} = (E^{76}, S^{76})$ as below

a) a feasible path E_k^{76} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{76} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{76}$ with $|S_k^{76}| \geq 1$ (contiguity and continuity constraints),

c) $\{s-w_k+1,...,s\}\cap\{s'-w_{k'}+1,...,s'\}=\emptyset$ for each $k,k'\in K$ and each $s\in S_k^{76}$ and $s'\in S_{k'}^{76}$ with $E_k^{76}\cap E_{k'}^{76}\neq\emptyset$ (non-overlapping constraint),

d) and there is $\frac{|H|-1}{2}$ pairs of demand k and slot s from the odd-hole H (i.e., $v_{k,s}\in H$ s.t. the demand k selects the slot s as last-slot in the solution $\mathcal{S}^{76}$ denoted by $\tilde{H}_{76}$, i.e., $s\in S_k^{76}$ for each $v_{k,s}\in\tilde{H}_{76}$, and $s'\notin S_{k'}^{76}$ for all $v_{k',s'}\in H\setminus\tilde{H}_{76}$.

Obviously, $\mathcal{S}^{76}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{76}},z^{\mathcal{S}^{76}})$ is belong to $P(G,K,\mathbb{S})$ and then to $F_H^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s}\in H}z_s^k=\frac{|H|-1}{2}$. As a result, $F_H^{\tilde{G}_S^E}$ is not empty (i.e., $F_H^{\tilde{G}_S^E}\neq\emptyset$). Furthermore, given that $s\in\{w_k,...,\bar{s}\}$ for each $v_{k,s}\in H$, this means that there exists at least one feasible slot assignment S_k for the demands k in H with $s\notin S_k$ for each $v_{k,s}\in H$. This means that $F_H^{\tilde{G}_S^E}\neq P(G,K,\mathbb{S})$.
Let denote the inequality $\sum_{v_{k,s}\in H}z_s^k\leq\frac{|H|-1}{2}$ by $\alpha x+\beta z\leq\lambda$. Let $\mu x+\sigma z\leq\tau$ be a valid inequality that is facet defining F of $P(G,K,\mathbb{S})$. Suppose that $F_H^{\tilde{G}_S^E}\subset F=\{(x,z)\in P(G,K,\mathbb{S}):\mu x+\sigma z=\tau\}$. We show that there exist $\rho\in\mathbb{R}$ and $\gamma=(\gamma_1,\gamma_2,\gamma_3)$ (s.t. $\gamma_1\in\mathbb{R}^{\sum_{k\in K}|E_0^k|},\gamma_2\in\mathbb{R}^{\sum_{k\in K}|E_1^k|},\gamma_3\in\mathbb{R}^{\sum_{k\in K}(w_k-1)}$) s.t. $(\mu,\sigma)=\rho(\alpha,\beta)+\gamma Q$, and that

a) $\sigma_s^k=0$ for all demands $k\in K$ and all slots $s\in\{w_k,...,\bar{s}\}$ with $v_{k,s}\notin H$,

b) and $\mu_e^k=0$ for all demands $k\in K$ and all edges $e\in E\setminus(E_0^k\cup E_1^k)$,

c) and σ_s^k are equivalents for all $v_{k,s}\in H$.

We first show that $\mu_e^k=0$ for each edge $e\in E\setminus(E_0^k\cup E_1^k)$ for each demand $k\in K$. Consider a demand $k\in K$ and an edge $e\in E\setminus(E_0^k\cup E_1^k)$. For that, we consider a solution $\mathcal{S}'^{76}=(E'^{76},S'^{76})$ in which

a) a feasible path $E_k'^{76}$ is assigned to each demand $k\in K$ (routing constraint),

b) a set of last-slots $S_k'^{76}$ is assigned to each demand $k\in K$ along each edge $e\in E_k'^{76}$ with $|S_k'^{76}|\geq 1$ (contiguity and continuity constraints),

c) $\{s'-w_k+1,...,s'\}\cap\{s"-w_{k'}+1,...,s"\}=\emptyset$ for each $k,k'\in K$ and each $s'\in S_k'^{76}$ and $s"\in S_{k'}'^{76}$ with $E_k'^{76}\cap E_{k'}'^{76}\neq\emptyset$, i.e., for each edge $e\in E$ and each slot $s"\in\mathbb{S}$ we have $\sum_{k\in K,e\in E_k'^{76}}|\{s'\in S_k'^{76},s"\in\{s'-w_k+1,...,s'\}|\leq 1$ (non-overlapping constraint),

d) the edge e is not non-compatible edge with the selected edges $e"\in E_k'^{76}$ of demand k in the solution $\mathcal{S}'^{76}$, i.e., $\sum_{e"\in E_k'^{76}}l_{e"}+l_e\leq\bar{l}_k$. As a result, $E_k'^{76}\cup\{e'\}$ is a feasible path for the demand k,

e) $\{s-w_k+1,...,s\}\cap\{s'-w_{k'}+1,...,s'\}=\emptyset$ for each $k'\in K$ and $s\in S_k'^{76}$ and $s'\in S_{k'}'^{76}$ with $(E_k'^{76}\cup\{e'\})\cap E_{k'}'^{76}\neq\emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e' in the set of edges $E_k'^{76}$ selected to route the demand k in the solution $\mathcal{S}'^{76}$),

f) and there is $\frac{|H|-1}{2}$ pairs of demand k and slot s from the odd-hole H (i.e., $v_{k,s}\in H$ s.t. the demand k selects the slot s as last-slot in the solution $\mathcal{S}'^{76}$ denoted by $\tilde{H}'_{76}$, i.e., $s\in S_k'^{76}$ for each $v_{k,s}\in\tilde{H}'_{76}$, and $s'\notin S_{k'}'^{76}$ for all $v_{k',s'}\in H\setminus\tilde{H}'_{76}$.

$\mathcal{S}'^{76}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{76}},z^{\mathcal{S}'^{76}})$ is belong to F and then to $F_H^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s}\in H}z_s^k=\frac{|H|-1}{2}$. Based on this, we derive a solution $\mathcal{S}'^{77}$ obtained from the solution $\mathcal{S}'^{76}$ by adding an unused edge $e\in E\setminus(E_0^k\cup E_1^k)$

for the routing of demand k in K in the solution $\mathcal{S}^{76}$ which means that $E'^{77}_k = E'^{76}_k \cup \{e\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^{76}$ remain the same in the solution $\mathcal{S}'^{77}$, i.e., $S'^{77}_k = S'^{76}_k$ for each $k \in K$, and $E'^{77}_{k'} = E'^{76}_{k'}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}'^{77}$ is clearly feasible given that

a) and a feasible path E'^{77}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S'^{77}_k is assigned to each demand $k \in K$ along each edge $e \in E'^{77}_k$ with $|S'^{77}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S'^{77}_k$ and $s" \in S'^{77}_{k'}$ with $E'^{77}_k \cap E'^{77}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E'^{77}_k} |\{s' \in S'^{77}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}'^{77}}, z^{\mathcal{S}'^{77}})$ is belong to F and then to $F_H^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s} \in H} z^k_s = \frac{|H|-1}{2}$. It follows that

$$\mu x^{\mathcal{S}'^{76}} + \sigma z^{\mathcal{S}'^{76}} = \mu x^{\mathcal{S}'^{77}} + \sigma z^{\mathcal{S}'^{77}} = \mu x^{\mathcal{S}'^{76}} + \mu^k_e + \sigma z^{\mathcal{S}'^{76}}.$$

As a result, $\mu^k_e = 0$ for demand k and an edge e.

As e is chosen arbitrarily for the demand k with $e \notin E^k_0 \cup E^k_1$, we iterate the same procedure for all $e' \in E \setminus (E^k_0 \cup E^k_1 \cup \{e\})$. We conclude that for the demand k

$$\mu^k_e = 0, \text{ for all } e \in E \setminus (E^k_0 \cup E^k_1).$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e \in E \setminus (E^k_0 \cup E^k_1)$. We conclude at the end that

$$\mu^k_e = 0, \text{ for all } k \in K \text{ and all } e \in E \setminus (E^k_0 \cup E^k_1).$$

Let's us show that $\sigma^k_s = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$ with $v_{k,s} \notin H$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$ with $v_{k,s'} \notin H$. For that, we consider a solution $\mathcal{S}"^{76} = (E"^{76}, S"^{76})$ in which

a) a feasible path $E"^{76}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^{76}_k$ is assigned to each demand $k \in K$ along each edge $e \in E"^{76}_k$ with $|S"^{76}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{76}_k$ and $s" \in S"^{76}_{k'}$ with $E"^{76}_k \cap E"^{76}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E"^{76}_k} |\{s' \in S"^{76}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^{76}_{k'}$ with $E"^{76}_k \cap E"^{76}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{76}_k$ assigned to the demand k in the solution $\mathcal{S}"^{76}$),

e) and there is one pair of demand k and slot s from the odd-hole H (i.e., $v_{k,s} \in H$ s.t. the demand k selects the slot s as last-slot in the solution $\mathcal{S}"^{76}$, i.e., $s \in S"^{76}_k$ for a node $v_{k,s} \in H$, and $s' \notin S"^{76}_{k'}$ for all $v_{k',s'} \in H \setminus \{v_{k,s}\}$.

$\mathcal{S}"^{76}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{76}}, z^{\mathcal{S}"^{76}})$ is belong to F and then to $F_H^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s} \in H} z^k_s = \frac{|H|-1}{2}$. Based on this, we construct a solution $\mathcal{S}^{78}$ derived from the solution $\mathcal{S}"^{76}$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}"^{76}$ (i.e., $E^{78}_k = E"^{76}_k$ for each $k \in K \setminus \tilde{K}$, and $E^{78}_k \neq E"^{76}_k$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E_k^{78} is assigned to each demand $k \in \tilde{K}$ (routing constraint),

b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"^{76}_k$ and $s" \in S"^{76}_{k'}$ with $E_k^{78} \cap E"^{76}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}, e \in E_k^{78}} |\{s' \in S"^{76}_k, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k \in K \setminus \tilde{K}, e \in E"^{76}_k} |\{s' \in S"^{76}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"^{76}_{k"}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{76}_k$ assigned to the demand k in the solution $\mathcal{S}"^{76}$).

The last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}"^{76}$ remain the same in $\mathcal{S}^{78}$, i.e., $S"^{76}_{k'} = S^{78}_{k'}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{78} = S"^{76}_k \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{78}$ is clearly feasible given that

a) a feasible path E_k^{78} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{78} is assigned to each demand $k \in K$ along each edge $e \in E_k^{78}$ with $|S_k^{78}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{78}$ and $s" \in S_{k'}^{78}$ with $E_k^{78} \cap E_{k'}^{78} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{78}} |\{s' \in S_k^{78}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{78}}, z^{\mathcal{S}^{78}})$ is belong to F and then to $F_H^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s} \in H} z_s^k = \frac{|H|-1}{2}$. We have so

$$\mu x^{\mathcal{S}"^{76}} + \sigma z^{\mathcal{S}"^{76}} = \mu x^{\mathcal{S}^{78}} + \sigma z^{\mathcal{S}^{78}} = \mu x^{\mathcal{S}"^{76}} + \sigma z^{\mathcal{S}"^{76}} + \sigma_{s'}^k - \sum_{\tilde{k} \in \tilde{K}} \sum_{e \in E"^{76}_k} \mu_e^{\tilde{k}} + \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E_k^{78}} \mu_{e'}^{\tilde{k}}.$$

It follows that $\sigma_{s'}^k = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $v_{k,s'} \notin H$ given that $\mu_e^k = 0$ for all the demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $v_{k,s'} \notin H$ s.t. we find

$$\sigma_{s'}^k = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s'} \notin H.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } v_{k',s} \notin H..$$

Consequently, we conclude that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s} \notin H.$$

Let's prove that σ_s^k for all $v_{k,s} \in H$ are equivalents. Consider a node $v_{k',s'}$ in H. For that, we consider a solution $\tilde{\mathcal{S}}^{76} = (\tilde{E}^{76}, \tilde{S}^{76})$ in which

a) a feasible path $\tilde{E}_k^{76}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}_k^{76}$ is assigned to each demand $k \in K$ along each edge $e \in \tilde{E}_k^{76}$ with $|\tilde{S}_k^{76}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}_k^{76}$ and $s" \in \tilde{S}_{k'}^{76}$ with $\tilde{E}_k^{76} \cap \tilde{E}_{k'}^{76} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in \tilde{E}_k^{76}} |\{s' \in \tilde{S}_k^{76}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s-w_k+1,...,s\}\cap\{s'-w_{k'}+1,...,s'\}=\emptyset$ for each $k\in K$ and $s\in\tilde{S}^{76}_k$ with $\tilde{E}^{76}_k\cap\tilde{E}^{76}_{k'}\neq\emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $\tilde{S}^{76}_{k'}$ assigned to the demand k' in the solution $\tilde{E}^{76}_k$),

e) and there is $\frac{|H|-1}{2}$ pairs of demand k and slot s from the odd-hole H (i.e., $v_{k,s}\in H$ s.t. the demand k selects the slot s as last-slot in the solution $\tilde{\mathcal{S}}^{76}$ denoted by $\tilde{H}'_{76}$, i.e., $s\in\mathcal{S}^{76}_k$ for each $v_{k,s}\in\tilde{H}'_{76}$, and $s'\notin\mathcal{S}^{76}_{k'}$ for all $v_{k',s'}\in H\setminus\tilde{H}'_{76}$.

$\tilde{\mathcal{S}}^{76}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{76}},z^{\tilde{\mathcal{S}}^{76}})$ is belong to F and then to $F_H^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s}\in H}z^k_s=\frac{|H|-1}{2}$. Based on this, we construct a solution $\mathcal{S}^{80}$ derived from the solution $\tilde{\mathcal{S}}^{76}$ by

a) modifying the paths assigned to a subset of demands $\tilde{K}\subset K$ in $\tilde{\mathcal{S}}^{76}$ (i.e., $E^{80}_k=\tilde{E}^{76}_k$ for each $k\in K\setminus\tilde{K}$, and $E^{80}_k\neq\tilde{E}^{76}_k$ for each $k\in\tilde{K}$),

b) and the last-slots assigned to the demands $K\setminus\{k,k'\}$ in $\tilde{\mathcal{S}}^{76}$ remain the same in $\mathcal{S}^{80}$, i.e., $\tilde{S}^{76}_{k"}=S'^{79}_{k"}$ for each demand $k"\in K\setminus\{k,k'\}$, where k is a demand with $v_{k,s}\in\tilde{H}_{76}$ and $s\in\tilde{S}^{76}_k$ s.t. $v_{k',s'}$ is not linked with any node $v_{k",s"}\in\tilde{H}_{76}\setminus\{v_{k,s}\}$,

c) and adding the slot s' as last-slot to the demand k', i.e., $S^{80}_{k'}=\tilde{S}^{76}_{k'}\cup\{s'\}$ for the demand k',

d) and modifying the last-slots assigned to the demand k by adding a new last-slot $\tilde{s}$ and removing the last slot $s\in\tilde{S}^{76}_k$ with $v_{k,s}\in H$ and $v_{k,\tilde{s}}\notin H$ s.t. $S^{80}_k=(\tilde{S}^{76}_k\setminus\{s\})\cup\{\tilde{s}\}$ for the demand k s.t. $\{\tilde{s}-w_k+1,...,\tilde{s}\}\cap\{s'-w_{k'}+1,...,s'\}=\emptyset$ for each $k'\in K$ and $s'\in S^{80}_{k'}$ with $E^{80}_k\cap E^{80}_{k'}\neq\emptyset$.

The solution $\mathcal{S}^{80}$ is clearly feasible given that

a) a feasible path E^{80}_k is assigned to each demand $k\in K$ (routing constraint),

b) a set of last-slots S^{80}_k is assigned to each demand $k\in K$ along each edge $e\in E^{80}_k$ with $|S^{80}_k|\geq 1$ (contiguity and continuity constraints),

c) $\{s'-w_k+1,...,s'\}\cap\{s"-w_{k'}+1,...,s"\}=\emptyset$ for each $k,k'\in K$ and each $s'\in S^{80}_k$ and $s"\in S^{80}_{k'}$ with $E^{80}_k\cap E^{80}_{k'}\neq\emptyset$, i.e., for each edge $e\in E$ and each slot $s"\in\mathbb{S}$ we have $\sum_{k\in K,e\in E^{80}_k}|\{s'\in S^{80}_k,s"\in\{s'-w_k+1,...,s'\}|\leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{80}},z^{\mathcal{S}^{80}})$ is belong to F and then to $F_H^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s}\in H}z^k_s=\frac{|H|-1}{2}$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{76}}+\sigma z^{\tilde{\mathcal{S}}^{76}}=\mu x^{\mathcal{S}^{80}}+\sigma z^{\mathcal{S}^{80}}=\mu x^{\tilde{\mathcal{S}}^{76}}+\sigma z^{\tilde{\mathcal{S}}^{76}}+\sigma^{k'}_{s'}-\sigma^k_s+\sigma^k_{\tilde{s}}$$
$$-\sum_{k\in\tilde{K}}\sum_{e\in\tilde{E}^{76}_k}\mu x^{\tilde{\mathcal{S}}^{76}}+\sum_{k\in\tilde{K}}\sum_{e\in E^{80}_k}\mu x^{\mathcal{S}^{80}}.$$

It follows that $\sigma^{k'}_{s'}=\sigma^k_s$ for demand k' and a slot $s'\in\{w_{k'},...,\bar{s}\}$ with $v_{k',s'}\in H$ given that $\sigma^k_{\tilde{s}}=0$ for $v_{k,\tilde{s}}\notin H$, and $\mu^k_e=0$ for all $k\in K$ and all $e\in E\setminus(E^k_0\cup E^k_1)$.
On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\mu^{k'}_{e'}=\gamma^{k',e'}_1,\ \text{for all } k'\in K \text{ and all } e'\in E^{k'}_0,$$
$$\mu^{k'}_{e'}=\gamma^{k',e'}_2,\ \text{for all } k'\in K \text{ and all } e'\in E^{k'}_1,$$
$$\sigma^{k'}_{s'}=\gamma^{k',s'}_3,\ \text{for all } k'\in K \text{ and all } s'\in\{1,...,w_{k'}-1\}.$$

We conclude that for each $k \in K$ and $e \in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e}, \text{ if } e \in E_0^k \\ \gamma_2^{k,e}, \text{ if } e \in E_1^k \\ 0, otherwise \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\} \\ \rho, \text{if } v_{k,s} \in H, \\ 0, \text{if } v_{k,s} \notin H. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{v_{k,s} \in H} \rho\beta_s^k + \gamma Q$.

□

Theorem 2.4.13. *Let H be an odd-hole, and C be a clique in the conflict graph $\tilde{G}_S^E$ with*

a) $|H| \geq 5$,

b) and $|C| \geq 3$,

c) and $H \cap C = \emptyset$,

d) and the nodes $(v_{k,s}, v_{k',s'})$ are linked in $\tilde{G}_S^E$ for all $v_{k,s} \in H$ and $v_{k',s'} \in C$.

Then, the inequality (2.42) is facet defining for $P(G, K, \mathbb{S})$ iff

a) for each node $v_{k",s"}$ in $\tilde{G}_S^E$ with $v_{k",s"} \notin H \cup C$ and $C \cup \{v_{k",s"}\}$ is a clique in $\tilde{G}_S^E$, there exists a subset of nodes $\tilde{H} \subseteq H$ of size $\frac{|H|-1}{2}$ s.t. $\tilde{H} \cup \{v_{k",s"}\}$ is stable in $\tilde{G}_S^E$,

b) and there does not exist an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with

a) $[\min_{v_{k,s} \in H \cup C}(s - w_k + 1), \max_{v_{k,s} \in H \cup C}] \subset I$,

b) and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'})$ linked in H,

c) and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'})$ linked in C,

d) and $w_k + w_{k'} \geq |I| + 1$ for each $v_k \in H$ and $v_{k'} \in C$,

e) and $2w_k \geq |I| + 1$ and $w_k \leq |I|$ for each $v_k \in H$,

f) and $2w_{k'} \geq |I| + 1$ and $w_{k'} \leq |I|$ for each $v_{k'} \in C$.

Proof. **Neccessity.**
We distinguish the following cases:

a) if there exists a node $v_{k",s"} \notin H \cup C$ in $\tilde{G}_S^E$ s.t. $v_{k",s"}$ is linked with all nodes $v_{k,s} \in H$ and also with all nodes $v_{k',s'} \in C$. This implies that the inequality (2.42) can be dominated by the following valid inequality

$$\sum_{v_{k,s} \in H} z_s^k + \frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'} + \frac{|H|-1}{2} z_{s"}^{k"} \leq \frac{|H|-1}{2}.$$

b) if there exists an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with

a) $[\min_{v_{k,s} \in H \cup C}(s - w_k + 1), \max_{v_{k,s} \in H \cup C}] \subset I$,

b) and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'})$ linked in H,

c) and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'})$ linked in C,

d) and $w_k + w_{k'} \geq |I| + 1$ for each $v_k \in H$ and $v_{k'} \in C$,

e) and $2w_k \geq |I| + 1$ and $w_k \leq |I|$ for each $v_k \in H$,

f) and $2w_{k'} \geq |I| + 1$ and $w_{k'} \leq |I|$ for each $v_{k'} \in C$.

This implies that the inequality (2.42) is dominated by the inequality (2.37).

If no one of these cases is verified, the inequality (2.37) can never be dominated by another inequality without changing its right-hand side. Otherwise, the inequality (2.42) cannot be facet defining for $P(G, K, \mathbb{S})$.

Sufficiency.

Let $F_{H,C}^{\tilde{G}_S^E}$ denote the face induced by the inequality (2.42), which is given by

$$F_{H,C}^{\tilde{G}_S^E} = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{v_{k,s} \in H} z_s^k + \frac{|H| - 1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'} = \frac{|H| - 1}{2}\}.$$

In order to prove that inequality $\sum_{v_{k,s} \in H} z_s^k + \frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'} \leq \frac{|H|-1}{2}$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that $F_{H,C}^{\tilde{G}_S^E}$ is a proper face, and $F_{H,C}^{\tilde{G}_S^E} \neq P(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{81} = (E^{81}, S^{81})$ as below

a) a feasible path E_k^{81} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{81} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{81}$ with $|S_k^{81}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{81}$ and $s' \in S_{k'}^{81}$ with $E_k^{81} \cap E_{k'}^{81} \neq \emptyset$ (non-overlapping constraint),

d) and there is $\frac{|H|-1}{2}$ pairs of demand k and slot s from the odd-hole H (i.e., $v_{k,s} \in H$ s.t. the demand k selects the slot s as last-slot in the solution $\mathcal{S}^{81}$ denoted by $\tilde{H}_{81}$, i.e., $s \in S_k^{81}$ for each $v_{k,s} \in \tilde{H}_{81}$, and $s' \notin S_{k'}^{81}$ for all $v_{k',s'} \in H \setminus \tilde{H}_{81}$.

Obviously, $\mathcal{S}^{81}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{81}}, z^{\mathcal{S}^{81}})$ is belong to $P(G, K, \mathbb{S})$ and then to $F_{H,C}^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s} \in H} z_s^k + \frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'} = \frac{|H|-1}{2}$. As a result, $F_{H,C}^{\tilde{G}_S^E}$ is not empty (i.e., $F_{H,C}^{\tilde{G}_S^E} \neq \emptyset$). Furthermore, given that $s \in \{w_k, ..., \bar{s}\}$ for each $v_{k,s} \in H$, this means that there exists at least one feasible slot assignment S_k for the demands k in H with $s \notin S_k$ for each $v_{k,s} \in H$. This means that $F_{H,C}^{\tilde{G}_S^E} \neq P(G, K, \mathbb{S})$.

Let denote the inequality $\sum_{v_{k,s} \in H} z_s^k + \frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'} \leq \frac{|H|-1}{2}$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F_{H,C}^{\tilde{G}_S^E} \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $v_{k,s} \notin H \cup C$ as done in the proof of theorem 2.4.12,

b) and $\mu_e^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$ as done in the proof of theorem 2.4.12,

c) and σ_s^k are equivalents for all $v_{k,s} \in H$ as done in the proof of theorem 2.4.12,

given that the solutions $\mathcal{S}^{65} - \mathcal{S}^{80}$ still feasible s.t. their corresponding incidence vectors are belong to $P(G, K, \mathbb{S})$, and then to $F_{H,C}^{\tilde{G}_{\mathbb{S}}^{E}}$ given that they are composed by $\sum_{v_{k,s} \in H} z_s^k + \frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'} = \frac{|H|-1}{2}$. In what follows, we prove that $\sigma_{s'}^{k'}$ are equivalents for all $v_{k',s'} \in C$. To do so, we consider a node $v_{s'}^{k'} \in C$, and a solution $\mathcal{S}^{82} = (E^{82}, S^{82})$ in which

a) a feasible path E_k^{82} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{82} is assigned to each demand $k \in K$ along each edge $e \in E_k^{82}$ with $|S_k^{82}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{82}$ and $s" \in S_{k'}^{82}$ with $E_k^{82} \cap E_{k'}^{82} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{82}} |\{s' \in S_k^{82}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s \in S_k^{82}$ with $E_k^{82} \cap E_{k'}^{82} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S_{k'}^{82}$ assigned to the demand k' in the solution $\mathcal{S}^{82}$),

e) and there is $\frac{|H|-1}{2}$ pairs of demand k and slot s from the odd-hole H (i.e., $v_{k,s} \in H$ s.t. the demand k selects the slot s as last-slot in the solution $\mathcal{S}^{82}$ denoted by H'_{71}, i.e., $s \in \mathcal{S}_k^{82}$ for each $v_{k,s} \in H'_{71}$, and $s' \notin \mathcal{S}_{k'}^{82}$ for all $v_{k',s'} \in H \setminus H'_{71}$.

$\mathcal{S}^{82}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}^{82}}, z^{\mathcal{S}^{82}})$ is belong to F and then to $F_{H,C}^{\tilde{G}_{\mathbb{S}}^{E}}$ given that it is composed by $\sum_{v_{k,s} \in H} z_s^k + \frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'} = \frac{|H|-1}{2}$. Based on this, we construct a solution $\mathcal{S}^{83}$ derived from the solution $\mathcal{S}^{82}$ by

a) with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}^{82}$ (i.e., $E_k^{83} = E_k^{82}$ for each $k \in K \setminus \tilde{K}$, and $E_k^{83} \neq E_k^{82}$ for each $k \in \tilde{K}$),

b) and the last-slots assigned to the demands $K \setminus (\{k \in K$ with $v_{k,s} \in \tilde{H}_{71}\} \cup \{k'\})$ in $\mathcal{S}^{82}$ remain the same in $\mathcal{S}^{83}$, i.e., $S_{k"}^{82} = S_{k"}^{83}$ for each demand $k" \in K \setminus (\{k \in K$ with $v_{k,s} \in \tilde{H}_{71}\} \cup \{k'\})$,

c) and adding the slot s' as last-slot to the demand k', i.e., $S_{k'}^{83} = S_{k'}^{82} \cup \{s'\}$ with $v_{k',s'} \in C$,

d) and modifying the last-slots assigned to each demand $k \in \{\tilde{k} \in K$ with $v_{\tilde{k},s} \in \tilde{H}_{71}\}$ by adding a new last-slot $\tilde{s}_k$ and removing the last slot $s_k \in S_k^{82}$ with $v_{k,s_k} \in H$ and $v_{k,\tilde{s}_k} \notin H \cup C$ s.t. $S_k^{83} = (S_k^{82} \setminus \{s_k\}) \cup \{\tilde{s}_k\}$ for each demand $k \in \{\tilde{k} \in K$ with $v_{\tilde{k},s} \in \tilde{H}_{71}\}$ s.t. $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}^{83}$ with $E_k^{83} \cap E_{k'}^{83} \neq \emptyset$.

The solution $\mathcal{S}^{83}$ is clearly feasible given that

a) a feasible path E_k^{83} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{83} is assigned to each demand $k \in K$ along each edge $e \in E_k^{83}$ with $|S_k^{83}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{83}$ and $s" \in S_{k'}^{83}$ with $E_k^{83} \cap E_{k'}^{83} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{83}} |\{s' \in S_k^{83}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{83}}, z^{\mathcal{S}^{83}})$ is belong to F and then to $F_{H,C}^{\tilde{G}_S^E}$ given that it is composed by $\sum_{v_{k,s}\in H} z_s^k + \frac{|H|-1}{2}\sum_{v_{k',s'}\in C} z_{s'}^{k'} = \frac{|H|-1}{2}$. We have so

$$\mu x^{\mathcal{S}^{82}} + \sigma z^{\mathcal{S}^{82}} = \mu x^{\mathcal{S}^{83}} + \sigma z^{\mathcal{S}^{83}} = \mu x^{\mathcal{S}^{82}} + \sigma z^{\mathcal{S}^{82}} + \sigma_{s'}^{k'} - \sum_{(k,s_k)\in\tilde{H}_{71}} \sigma_{s_k}^k$$
$$+ \sum_{k\in\{\tilde{k}\in K \text{ with } v_{\tilde{k},s}\in\tilde{H}_{71}\}} \sigma_{\tilde{s}_k}^k + \sum_{k\in\tilde{K}}\sum_{e\in E_k^{83}} \mu_e^k - \sum_{k\in\tilde{K}}\sum_{e\in E_k^{82}} \mu_e^k.$$

It follows that $\sigma_{s'}^{k'} = \sum_{(k,s_k)\in\tilde{H}_{71}} \sigma_{s_k}^k$ for demand k' and a slot $s'\in\{w_{k'},...,\bar{s}\}$ with $v_{k',s'}\in C$ given that $\sigma_{\tilde{s}_k}^k = 0$ for $v_{k,\tilde{s}_k}\notin H\cup C$, and $\mu_e^k = 0$ for all $k\in K$ and all $e\in E\setminus(E_0^k\cup E_1^k)$. As a result, $\sigma_{s'}^{k'} = \rho\frac{|H|-1}{2}$ given that σ_s^k are equivalent for all $v_{k,s}\in H$.
Given that the pair $v_{k',s'}$ is chosen arbitrary in the clique C, we iterate the same procedure for all $v_{k',s'}\in C$. Consequently, we obtain that $\sigma_{s'}^{k'} = \rho\frac{|H|-1}{2}$ for all $v_{k',s'}\in C$.
On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned}
\mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k'\in K \text{ and all } e'\in E_0^{k'},\\
\mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k'\in K \text{ and all } e'\in E_1^{k'},\\
\sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k'\in K \text{ and all } s'\in\{1,...,w_{k'}-1\}.
\end{aligned}$$

We conclude that for each $k\in K$ and $e\in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e}, \text{ if } e\in E_0^k,\\ \gamma_2^{k,e}, \text{ if } e\in E_1^k,\\ 0, otherwise,\end{cases}$$

and for each $k\in K$ and $s\in\mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s\in\{1,...,w_k-1\},\\ \rho, \text{if } v_{k,s}\in H,\\ \rho\dfrac{|H|-1}{2}, \text{if } v_{k,s}\in C,\\ 0, otherwise.\end{cases}$$

As a result $(\mu,\sigma) = \sum_{v_{k,s}\in H}\rho\beta_s^k + \dfrac{|H|-1}{2}\sum_{v_{k',s'}\in C}\rho\beta_{s'}^{k'} + \gamma Q.$

□

Let us now introduce some valid inequalities that are related to the routing Sub-problem due to the *transmission-reach* constraint.

2.4.8 Incompatibility-Clique Inequalities

Based on the inequalities (2.17) and (2.18), we introduce the following conflict graph.

Definition 2.4.7. *Let $\tilde{G}_E^K$ be a conflict graph defined as follows. For each demand k and edge $e\notin E_0^k\cup E_1^k$, consider a node v_e^k in $\tilde{G}_E^K$. Two nodes v_e^k and $v_{e'}^{k'}$ are linked by an edge in $\tilde{G}_E^K$*

a) if $k = k'$: e and e' are non compatible edges for demand k.

b) if $k \neq k'$: k and k' are non compatible demands for edge e.

Proposition 2.4.19. *Let C be a clique in $\tilde{G}^K_E$. Then, the inequality*

$$\sum_{v^k_e \in C} x^k_e \leq 1, \tag{2.43}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of a clique set in the conflict graph $\tilde{G}^K_E$ s.t. by adding the inequalities (2.17) or (2.18) for all pairs of nodes $(v^k_e, v^{k'}_{e'})$ in the clique C in $\tilde{G}^K_E$

$$\sum_{v^k_e \in C} (|C| - 1)x^k_e \leq (|C| - 1) \implies \sum_{v^k_e \in C} x^k_e \leq \frac{|C| - 1}{|C| - 1} \implies \sum_{v^k_e \in C} x^k_e \leq 1,$$

which ends the proof. □

Theorem 2.4.14. *Consider a clique C in the conflict graph $\tilde{G}^K_E$. Then, the inequality* (2.43) *is facet defining for $P(G, K, \mathbb{S})$ iff C is a maximal clique in the conflict graph $\tilde{G}^K_E$.*

Proof. It is trivial given that the inequality (2.43) can never be dominated by another inequality without changing its right-hand side.
Let $F_C^{\tilde{G}^K_E}$ denote the face induced by the inequality (2.43), which is given by

$$F_C^{\tilde{G}^K_E} = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{v_{k,e} \in C} x^k_e = 1\}.$$

In order to prove that inequality $\sum_{v_{k,e} \in C} x^k_e \leq 1$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that $F_C^{\tilde{G}^K_E}$ is a proper face, and $F_C^{\tilde{G}^K_E} \neq P(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{84} = (E^{84}, S^{84})$ as below

a) a feasible path E^{84}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{84}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{84}_k$ with $|S^{84}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S^{84}_k$ and $s' \in S^{84}_{k'}$ with $E^{84}_k \cap E^{84}_{k'} \neq \emptyset$ (non-overlapping constraint),

d) and there is one pair of demand k and edge e from the clique C (i.e., $v_{k,e} \in C$ s.t. the demand k selects the edge e for its routing in the solution $\mathcal{S}^{84}$, i.e., $e \in E^{84}_k$ for a node $v_{k,e} \in C$, and $e' \notin E^{84}_{k'}$ for all $v_{k',e'} \in C \setminus \{v_{k,e}\}$.

Obviously, $\mathcal{S}^{84}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{84}}, z^{\mathcal{S}^{84}})$ is belong to $P(G, K, \mathbb{S})$ and then to $F_C^{\tilde{G}^K_E}$ given that it is composed by $\sum_{v_{k,e} \in C} x^k_e = 1$. As a result, $F_C^{\tilde{G}^K_E}$ is not empty (i.e., $F_C^{\tilde{G}^K_E} \neq \emptyset$). Furthermore, given that $s \in \{w_k, ..., \bar{s}\}$ for each $v_{k,s} \in C$, this means that there exists at least one feasible slot assignment S_k for the demands k in C with $s \notin S_k$ for each $v_{k,s} \in C$. This means that $F_C^{\tilde{G}^K_E} \neq P(G, K, \mathbb{S})$.
Let denote the inequality $\sum_{v_{k,e} \in C} x^k_e \leq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F_C^{\tilde{G}^K_E} \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E^k_0|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E^k_1|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K} (w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$,

b) and $\mu_e^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$ with $v_{k,e} \notin C$,

c) and μ_e^k are equivalent for all $v_{k,e} \in C$.

We first show that $\mu_e^k = 0$ for each edge $e \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$ with $v_{k,e} \notin C$. Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. For that, we consider a solution $\mathcal{S}'^{84} = (E'^{84}, S'^{84})$ in which

a) a feasible path $E_k'^{84}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S_k'^{84}$ is assigned to each demand $k \in K$ along each edge $e \in E_k'^{84}$ with $|S_k'^{84}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k'^{84}$ and $s" \in S_{k'}'^{84}$ with $E_k'^{84} \cap E_{k'}'^{84} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k'^{84}} |\{s' \in S_k'^{84}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) the edge e is not non-compatible edge with the selected edges $e \in E_k'^{84}$ of demand k in the solution $\mathcal{S}'^{84}$, i.e., $\sum_{e' \in E_k'^{84}} l_{e'} + l_e \leq \bar{l}_k$. As a result, $E_k'^{84} \cup \{e\}$ is a feasible path for the demand k,

e) and there is one pair of demand k and edge e from the clique C (i.e., $v_{k,e} \in C$ s.t. the demand k selects the edge e for its routing in the solution $\mathcal{S}'^{84}$, i.e., $e \in E_k'^{9}$ for a node $v_{k,e} \in C$, and $e' \notin E_{k'}'^{9}$ for all $v_{k',e'} \in C \setminus \{v_{k,e}\}$.

$\mathcal{S}'^{84}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{84}}, z^{\mathcal{S}'^{84}})$ is belong to F and then to $F_C^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in C} x_e^k = 1$. Based on this, we derive a solution $\mathcal{S}^{85}$ obtained from the solution $\mathcal{S}'^{84}$ by adding an unused edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in K in the solution $\mathcal{S}^{84}$ which means that $E_k^{85} = E_k'^{84} \cup \{e\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^{84}$ remain the same in the solution $\mathcal{S}^{85}$, i.e., $S_k^{85} = S_k'^{84}$ for each $k \in K$, and $E_{k'}^{85} = E_{k'}'^{84}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{85}$ is clearly feasible given that

a) and a feasible path E_k^{85} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{85} is assigned to each demand $k \in K$ along each edge $e \in E_k^{85}$ with $|S_k^{85}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{85}$ and $s" \in S_{k'}^{85}$ with $E_k^{85} \cap E_{k'}^{85} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{85}} |\{s' \in S_k^{85}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{85}}, z^{\mathcal{S}^{85}})$ is belong to F and then to $F_C^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in C} x_e^k = 1$. It follows that

$$\mu x^{\mathcal{S}'^{84}} + \sigma z^{\mathcal{S}'^{84}} = \mu x^{\mathcal{S}^{85}} + \sigma z^{\mathcal{S}^{85}} = \mu x^{\mathcal{S}'^{84}} + \mu_e^k + \sigma z^{\mathcal{S}'^{84}}.$$

As a result, $\mu_e^k = 0$ for demand k and an edge e with $v_{k,e} \notin C$.
As e is chosen arbitrarily for the demand k with $e \notin E_0^k \cup E_1^k$ and $v_{k,e} \notin C$, we iterate the same procedure for all $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$ with $v_{k,e'} \notin C$. We conclude that for the demand k

$$\mu_e^k = 0, \text{ for all } e \in E \setminus (E_0^k \cup E_1^k) \text{ with } v_{k,e} \notin C.$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e \in E \setminus (E_0^k \cup E_1^k)$ with $v_{k',e} \notin C$. We conclude at the end that

$$\mu_e^k = 0, \text{ for all } k \in K \text{ and all } e \in E \setminus (E_0^k \cup E_1^k) \text{ with } v_{k,e} \notin C.$$

Let's us show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$, and a solution $\mathcal{S}"^{84} = (E"^{84}, S"^{84})$ in which

a) a feasible path $E"^{84}_k$ is assigned to each demand $k \in K$ (routing constraint),
b) a set of last-slots $S"^{84}_k$ is assigned to each demand $k \in K$ along each edge $e \in E"^{84}_k$ with $|S"^{84}_k| \geq 1$ (contiguity and continuity constraints),
c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{84}_k$ and $s" \in S"^{84}_{k'}$ with $E"^{84}_k \cap E"^{84}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E"^{84}_k} |\{s' \in S"^{84}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),
d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^{84}_{k'}$ with $E"^{84}_k \cap E"^{84}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{84}_k$ assigned to the demand k in the solution $\mathcal{S}"^{84}$),
e) and there is one pair of demand k and edge e from the clique C (i.e., $v_{k,e} \in C$ s.t. the demand k selects the edge e for its routing in the solution $\mathcal{S}"^{84}$, i.e., $e \in E"^{84}_k$ for a node $v_{k,e} \in C$, and $e' \notin E"^{84}_{k'}$ for all $v_{k',e'} \in C \setminus \{v_{k,e}\}$.

$\mathcal{S}"^{84}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{84}}, z^{\mathcal{S}"^{84}})$ is belong to F and then to $F_C^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in C} x_e^k = 1$. Based on this, we construct a solution $\mathcal{S}^{86}$ derived from the solution $\mathcal{S}"^{84}$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}"^{84}$ (i.e., $E_k^{86} = E"^{84}_k$ for each $k \in K \setminus \tilde{K}$, and $E_k^{86} \neq E"^{84}_k$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E_k^{86} is assigned to each demand $k \in \tilde{K}$ (routing constraint),
b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"^{84}_k$ and $s" \in S"^{84}_{k'}$ with $E_k^{86} \cap E"^{84}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}, e \in E_k^{86}} |\{s' \in S"^{84}_k, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k \in K \setminus \tilde{K}, e \in E"^{84}_k} |\{s' \in S"^{84}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),
c) and there is one pair of demand k and edge e from the clique C (i.e., $v_{k,e} \in C$ s.t. the demand k selects the edge e for its routing in the solution $\mathcal{S}^{86}$, i.e., $e \in E_k^{86}$ for a node $v_{k,e} \in C$, and $e' \notin E_{k'}^{86}$ for all $v_{k',e'} \in C \setminus \{v_{k,e}\}$,
d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"^{84}_{k"}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{84}_k$ assigned to the demand k in the solution $\mathcal{S}"^{84}$).

The last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}"^{84}$ remain the same in $\mathcal{S}^{86}$, i.e., $S"^{84}_{k'} = S_{k'}^{86}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{86} = S"^{84}_k \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{86}$ is clearly feasible given that

a) a feasible path E_k^{86} is assigned to each demand $k \in K$ (routing constraint),
b) a set of last-slots S_k^{86} is assigned to each demand $k \in K$ along each edge $e \in E_k^{86}$ with $|S_k^{86}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{86}$ and $s" \in S_{k'}^{86}$ with $E_k^{86} \cap E_{k'}^{86} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{86}} |\{s' \in S_k^{86}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{86}}, z^{\mathcal{S}^{86}})$ is belong to F and then to $F_C^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in C} x_e^k = 1$. We have so

$$\mu x^{\mathcal{S}"^{84}} + \sigma z^{\mathcal{S}"^{84}} = \mu x^{\mathcal{S}^{86}} + \sigma z^{\mathcal{S}^{86}} = \mu x^{\mathcal{S}"^{84}} + \sigma z^{\mathcal{S}"^{84}} + \sigma_{s'}^k - \sum_{\tilde{k} \in \tilde{K}} \sum_{e \in E"_k^{84}} \mu_e^{\tilde{k}} + \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E_k^{86}} \mu_{e'}^{\tilde{k}}.$$

It follows that $\sigma_{s'}^k = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ given that $\mu_e^k = 0$ for all the demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$ with $v_{k,e} \notin C$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $v_{k,s'} \notin C$ s.t. we find

$$\sigma_{s'}^k = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s'} \notin C.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } v_{k',s} \notin C..$$

Consequently, we conclude that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s} \notin C.$$

Let's prove that μ_e^k for all $v_{k,e}$ are equivalents. Consider a node $v_{k',e'}$ in C s.t. $e' \notin E_{k'}^{84}$. For that, we consider a solution $\tilde{\mathcal{S}}^{84} = (\tilde{E}^{84}, \tilde{S}^{84})$ in which

a) a feasible path $\tilde{E}_k^{84}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}_k^{84}$ is assigned to each demand $k \in K$ along each edge $e \in \tilde{E}_k^{84}$ with $|\tilde{S}_k^{84}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}_k^{84}$ and $s" \in \tilde{S}_{k'}^{84}$ with $\tilde{E}_k^{84} \cap \tilde{E}_{k'}^{84} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in \tilde{E}_k^{84}} |\{s' \in \tilde{S}_k^{84}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and there is one pair of demand k and edge e from the clique C (i.e., $v_{k,e} \in C$ s.t. the demand k selects the edge e for its routing in the solution $\tilde{\mathcal{S}}^{84}$, i.e., $e \in \tilde{E}_k^{84}$ for a node $v_{k,e} \in C$, and $e" \notin \tilde{E}_{k'}^{84}$ for all $v_{k',e"} \in C \setminus \{v_{k,e}\}$.

$\tilde{\mathcal{S}}^{84}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{84}}, z^{\tilde{\mathcal{S}}^{84}})$ is belong to F and then to $F_C^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in C} x_e^k = 1$. Based on this, we construct a solution $\mathcal{S}^{87}$ derived from the solution $\tilde{\mathcal{S}}^{84}$ by

a) modifying the path assigned to the demand k' in $\tilde{\mathcal{S}}^{84}$ from $\tilde{E}_{k'}^{84}$ to a path $E_{k'}^{87}$ passed through the edge e' with $v_{k',e'} \in C$,

b) modifying the path assigned to the demand k in $\tilde{\mathcal{S}}^{84}$ with $e \in \tilde{E}_k^{84}$ and $v_{k,e} \in C$ from $\tilde{E}_k^{84}$ to a path E_k^{87} without passing through any edge $e" \in E \setminus (E_0^k \cup E_1^k)$ s.t. $v_{k',e'}$ and $v_{k,e"}$ linked in C,

c) modifying the last-slots assigned to some demands $\tilde{K} \subset K$ from $\tilde{S}_{\tilde{k}}^{84}$ to $S_{\tilde{k}}^{87}$ for each $\tilde{k} \in \tilde{K}$ while satisfying non-overlapping constraint.

The paths assigned to the demands $K \setminus \{k, k'\}$ in $\tilde{\mathcal{S}}^{84}$ remain the same in $\mathcal{S}^{87}$ (i.e., $E^{87}_{k"} = \tilde{E}^{84}_{k"}$ for each $k" \in K \setminus \{k, k'\}$), and also without modifying the last-slots assigned to the demands $K \setminus \tilde{K}$ in $\tilde{\mathcal{S}}^{84}$, i.e., $\tilde{S}^{84}_k = S^{87}_k$ for each demand $k \in K \setminus \tilde{K}$. The solution $\mathcal{S}^{87}$ is clearly feasible given that

a) a feasible path E^{87}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{87}_k is assigned to each demand $k \in K$ along each edge $e \in E^{87}_k$ with $|S^{87}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{87}_k$ and $s" \in S^{87}_{k'}$ with $E^{87}_k \cap E^{87}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E^{87}_k} |\{s' \in S^{87}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{87}}, z^{\mathcal{S}^{87}})$ is belong to F and then to $F_C^{\tilde{G}^K_E}$ given that it is composed by $\sum_{v_{k,e} \in C} x^k_e = 1$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{84}} + \sigma z^{\tilde{\mathcal{S}}^{84}} = \mu x^{\mathcal{S}^{87}} + \sigma z^{\mathcal{S}^{87}} = \mu x^{\tilde{\mathcal{S}}^{84}} + \sigma z^{\tilde{\mathcal{S}}^{84}} + \mu^{k'}_{e'} - \mu^k_e + \sum_{\tilde{k} \in \tilde{K}} \sum_{s' \in S^{87}_{\tilde{k}}} \sigma^{\tilde{k}}_{s'} - \sum_{s \in \tilde{S}^{84}_{\tilde{k}}} \sigma^{\tilde{k}}_s$$
$$+ \sum_{e" \in E^{87}_{k'} \setminus \{e'\}} \mu^{k'}_{e"} - \sum_{e" \in \tilde{E}^{84}_{k'}} \mu^{k'}_{e"} + \sum_{e" \in E^{87}_k} \mu^k_{e"} - \sum_{e" \in \tilde{E}^{84}_k \setminus \{e\}} \mu^k_{e"}.$$

It follows that $\mu^{k'}_{e'} = \mu^k_e$ for demand k' and a edge $e' \in E \setminus (E^{k'}_0 \cup E^{k'}_1)$ with $v_{k',e'} \in C$ given that $\mu^k_{e"} = 0$ for all $k \in K$ and all $e" \in E \setminus (E^k_0 \cup E^k_1)$ with $v_{k,e"} \notin C$, and $\sigma^k_s = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$.
Given that the pair $(v_{k,e}, v_{k',e'})$ are chosen arbitrary in the clique C, we iterate the same procedure for all pairs $(v_{k,e}, v_{k',e'})$ s.t. we find

$$\mu^k_e = \mu^{k'}_{e'}, \text{for all pairs } (v_{k,e}, v_{k',e'}) \in C.$$

Consequently, we obtain that $\mu^k_e = \rho$ for all $v_{k,e} \in C$.
On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\mu^{k'}_{e'} = \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E^{k'}_0,$$
$$\mu^{k'}_{e'} = \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E^{k'}_1,$$
$$\sigma^{k'}_{s'} = \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}.$$

We conclude that for each $k \in K$ and $e \in E$

$$\mu^k_e = \begin{cases} \gamma_1^{k,e}, \text{ if } e \in E^k_0, \\ \gamma_2^{k,e}, \text{ if } e \in E^k_1, \\ \rho, \text{ if } v_{k,e} \in C, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma^k_s = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{v_{k,e} \in C} \rho \alpha^k_e + \gamma Q$.

□

2.4.9 Incompatibility-Odd-Hole Inequalities

Proposition 2.4.20. *Let H be an odd-hole in the conflict graph $\tilde{G}_E^K$ with $|H| \geq 5$. Then, the inequality*

$$\sum_{v_e^k \in H} x_e^k \leq \frac{|H|-1}{2}, \tag{2.44}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of the odd-hole in the conflict graph $\tilde{G}_E^K$. We strengthen the proof as belows. For each pair of nodes $(v_e^k, v_{e'}^{k'})$ linked in H by an edge, we know that $x_e^k + x_{e'}^{k'} \leq 1$. Given that H is an odd-hole which means that we have $|H| - 1$ pair of nodes $(v_e^k, v_{e'}^{k'})$ linked in H, and by doing a sum for all pairs of nodes $(v_e^k, v_{e'}^{k'})$ linked in H, it follows that

$$\sum_{(v_e^k, v_{e'}^{k'}) \in E(H)} x_e^k + x_{e'}^{k'} \leq |H| - 1.$$

Taking into account that each node v_e^k in H has two neighbors in H, this implies that x_e^k appears twice in the previous inequality. As a result,

$$\sum_{(v_e^k, v_{e'}^{k'}) \in E(H)} x_e^k + x_{e'}^{k'} = \sum_{v_e^k \in H} 2x_e^k \implies \sum_{v_e^k \in H} 2x_e^k \leq |H| - 1$$

$$\implies \sum_{v_e^k \in H} x_e^k \leq \left\lfloor \frac{|H|-1}{2} \right\rfloor = \frac{|H|-1}{2} \text{ since } |H| \text{ is an odd number.}$$

We conclude at the end that the inequality (2.44) is valid for $P(G, K, \mathbb{S})$. □

The inequality (2.44) can be strengthened without modifying its right-hand side by combining the inequality (2.44) and (2.43) as follows.

Proposition 2.4.21. *Let H be an odd-hole in the conflict graph $\tilde{G}_E^K$, and C be a clique in the conflict graph $\tilde{G}_E^K$ with*

a) $|H| \geq 5$,

b) and $|C| \geq 3$,

c) and $H \cap C = \emptyset$,

d) and the nodes $(v_e^k, v_{e'}^{k'})$ are linked in $\tilde{G}_E^K$ for all $v_e^k \in H$ and $v_{e'}^{k'} \in C$.

Then, the inequality

$$\sum_{v_e^k \in H} x_e^k + \frac{|H|-1}{2} \sum_{v_{e'}^{k'} \in C} x_{e'}^{k'} \leq \frac{|H|-1}{2}, \tag{2.45}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of the odd-hole and clique in $\tilde{G}_E^K$ s.t. if $\sum_{v_{e'}^{k'} \in C} x_{e'}^{k'} = 1$ for a $v_{e'}^{k'} \in C$, which implies that the quantity $\sum_{v_e^k \in H} x_e^k$ is forced to be equal to 0. Otherwise, we know from the inequality (2.44) that the sum $\sum_{v_e^k \in H} x_e^k$ should be smaller than $\frac{|H|-1}{2}$. We strengthen the proof by assuming first that the inequality (2.45) is not valid for $P(G, K, \mathbb{S})$. It follows that there exists a C-RSA solution S in which $e' \notin E_{k'}$ for each node $v_{e'}^{k'}$ in the clique C s.t.

$$\sum_{v_e^k \in H} x_e^k(S) + \frac{|H|-1}{2} \sum_{v_{e'}^{k'} \in C} x_{e'}^{k'}(S) > \frac{|H|-1}{2}.$$

Since $e' \notin E_{k'}$ for each node $v_{e'}^{k'}$ in the clique C this means that $\sum_{v_{e'}^{k'} \in C} x_{e'}^{k'}(S) = 0$, and taking into account the inequality (2.44), and that $x_e^k(S) \leq 1$ for each $v_e^k \in H$ and $x_{e'}^{k'}(S) \leq 1$ for each $v_{e'}^{k'} \in C$, it follows that

$$\sum_{v_e^k \in H} x_e^k(S) \leq \frac{|H|-1}{2},$$

which contradicts what we supposed before, i.e., $\sum_{v_e^k \in H} x_e^k(S) + \frac{|H|-1}{2} \sum_{v_{e'}^{k'} \in C} x_{e'}^{k'}(S) > \frac{|H|-1}{2}$. Similar for a solution S' in which $e \notin E_k'$ for each node v_e^k in the odd-hole H s.t.

$$\sum_{v_e^k \in H} x_e^k(S') + \frac{|H|-1}{2} \sum_{v_{e'}^{k'} \in C} x_{e'}^{k'}(S') > \frac{|H|-1}{2}.$$

Since $e \notin E_k'$ for each node v_e^k in the odd-hole H this means that $\sum_{v_e^k \in H} x_e^k(S') = 0$, and taking into account the inequality (2.43), and that $x_{e'}^{k'}(S') \leq 1$ for each $v_{e'}^{k'} \in C$, it follows that

$$\frac{|H|-1}{2} \sum_{v_{e'}^{k'} \in C} x_{e'}^{k'}(S') \leq \frac{|H|-1}{2},$$

which contradicts what we supposed before, i.e., $\sum_{v_e^k \in H} x_e^k(S') + \frac{|H|-1}{2} \sum_{v_{e'}^{k'} \in C} x_{e'}^{k'}(S') > \frac{|H|-1}{2}$.

Hence $\displaystyle\sum_{v_e^k \in H} |E_k \cap \{e\}| + \sum_{v_{e'}^{k'} \in C} |E_{k'} \cap \{e'\}| \leq \frac{|H|-1}{2}$.

We conclude at the end that the inequality (2.45) is valid for $P(G, K, \mathbb{S})$. $\square$

Theorem 2.4.15. *Let H be an odd-hole in the conflict graph $\tilde{G}_E^K$ with $|H| \geq 5$. Then, the inequality (2.44) is facet defining for $P(G, K, \mathbb{S})$ if and only if*

a) for each $v_{k',e'} \notin H$, there exists a node $v_{k,e} \in H$ s.t. the induced graph $\tilde{G}_E^K(H \setminus \{v_{k,e}\} \cup \{v_{k',e'}\})$ does not contain an odd-hole $H' = H \setminus \{v_{k,e}\} \cup \{v_{k',e'}\}$,

b) and there does not exist a node $v_{k',e'} \notin H$ in $\tilde{G}_E^K$ s.t. $v_{k',e'}$ is linked with all nodes $v_{k,e} \in H$.

Proof. **Neccessity.**
We distinguish the following cases:

a) if for a node $v_{k',e'} \notin H$ in $\tilde{G}_E^K$, there exists a node $v_{k,e} \in H$ s.t. the induced graph $\tilde{G}_E^K(H \setminus \{v_{k,e}\} \cup \{v_{k',e'}\})$ contains an odd-hole $H' = (H \setminus \{v_{k,e}\}) \cup \{v_{k',e'}\}$. This implies that the inequality (2.44) can be dominated using some technics of lifting based on the following two inequalities $\sum_{v_{k,e} \in H} x_e^k \leq \frac{|H|-1}{2}$, and $\sum_{v_{k',e'} \in H'} x_{e'}^{k'} \leq \frac{|H'|-1}{2}$.

b) if there exists a node $v_{k',e'} \notin H$ in $\tilde{G}_S^E$ s.t. $v_{k',e'}$ is linked with all nodes $v_{k,e} \in H$. This implies that the inequality (2.44) can be dominated by the following valid inequality

$$\sum_{v_{k,e} \in H} x_e^k + \frac{|H|-1}{2} x_{e'}^{k'} \leq \frac{|H|-1}{2}.$$

If no one of these cases is verified, the inequality (2.44) can never be dominated by another inequality without changing its right-hand side. Otherwise, the inequality (2.44) is not facet defining for $P(G, K, \mathbb{S})$.

Sufficiency.

Let $F_H^{\tilde{G}_E^K}$ denote the face induced by the inequality (2.44), which is given by

$$F_H^{\tilde{G}_E^K} = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{v_{k,e} \in H} x_e^k = \frac{|H|-1}{2}\}.$$

In order to prove that inequality $\sum_{v_{k,e} \in H} x_e^k = \frac{|H|-1}{2}$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that $F_H^{\tilde{G}_E^K}$ is a proper face, and $F_H^{\tilde{G}_E^K} \neq P(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{88} = (E^{88}, S^{88})$ as below

a) a feasible path E_k^{88} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{88} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{88}$ with $|S_k^{88}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{88}$ and $s' \in S_{k'}^{88}$ with $E_k^{88} \cap E_{k'}^{88} \neq \emptyset$ (non-overlapping constraint),

d) and there is $\frac{|H|-1}{2}$ pairs of demands edges (k, e) from the odd-hole H denoted by H_{88} (i.e., $v_{k,e} \in H_{88}$ s.t. the demand k selects the edge e for its routing in the solution $\mathcal{S}^{88}$, i.e., $e \in E_k^{88}$ for each node $v_{k,e} \in H_{88}$, and $e' \notin E_{k'}^{88}$ for all $v_{k',e'} \in H \setminus H_{88}$.

Obviously, $\mathcal{S}^{88}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{88}}, z^{\mathcal{S}^{88}})$ is belong to $P(G, K, \mathbb{S})$ and then to $F_H^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in H} x_e^k = \frac{|H|-1}{2}$. As a result, $F_H^{\tilde{G}_E^K}$ is not empty (i.e., $F_H^{\tilde{G}_E^K} \neq \emptyset$). Furthermore, given that $s \in \{w_k, ..., \bar{s}\}$ for each $v_{k,s} \in H$, this means that there exists at least one feasible slot assignment S_k for the demands k in H with $s \notin S_k$ for each $v_{k,s} \in H$. This means that $F_H^{\tilde{G}_E^K} \neq P(G, K, \mathbb{S})$.

Let denote the inequality $\sum_{v_{k,e} \in H} x_e^k \leq \frac{|H|-1}{2}$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F_H^{\tilde{G}_E^K} \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K} (w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$,

b) and $\mu_e^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$ with $v_{k,e} \notin H$,

c) and μ_e^k are equivalent for all $v_{k,e} \in H$.

We first show that $\mu_e^k = 0$ for each edge $e \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$ with $v_{k,e} \notin H$. Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. For that, we consider a solution $\mathcal{S}'^{88} = (E'^{88}, S'^{88})$ in which

a) a feasible path $E_k'^{88}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S_k'^{88}$ is assigned to each demand $k \in K$ along each edge $e \in E_k'^{88}$ with $|S_k'^{88}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k'^{88}$ and $s" \in S_{k'}'^{88}$ with $E_k'^{88} \cap E_{k'}'^{88} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k'^{88}} |\{s' \in S_k'^{88}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) the edge e is not non-compatible edge with the selected edges $e' \in E_k'^{88}$ of demand k in the solution $\mathcal{S}'^{88}$, i.e., $\sum_{e' \in E_k'^{88}} l_{e'} + l_e \leq \bar{l}_k$. As a result, $E_k'^{88} \cup \{e\}$ is a feasible path for the demand k,

e) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s \in S_k'^{88}$ and $s' \in S_{k'}'^{88}$ with $(E_k'^{88} \cup \{e'\}) \cap E_{k'}'^{88} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e in the set of edges $E_k'^{88}$ selected to route the demand k in the solution $\mathcal{S}'^{88}$),

f) and there is $\frac{|H|-1}{2}$ pairs of demand-edge (k, e) from the odd-hole H denoted by H'_{88} (i.e., $v_{k,e} \in H'_{88}$ s.t. the demand k selects the edge e for its routing in the solution $\mathcal{S}'^{88}$, i.e., $e \in E_k'^{88}$ for each node $v_{k,e} \in H'_{88}$, and $e' \notin E_{k'}'^{88}$ for all $v_{k',e'} \in H \setminus H'_{88}$.

$\mathcal{S}'^{88}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{88}}, z^{\mathcal{S}'^{88}})$ is belong to F and then to $F_H^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in H} x_e^k = \frac{|H|-1}{2}$. Based on this, we derive a solution $\mathcal{S}^{89}$ obtained from the solution $\mathcal{S}'^{88}$ by adding an unused edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in K in the solution $\mathcal{S}^{88}$ which means that $E_k^{89} = E_k'^{88} \cup \{e\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^{88}$ remain the same in the solution $\mathcal{S}^{89}$, i.e., $S_k^{89} = S_k'^{88}$ for each $k \in K$, and $E_{k'}^{89} = E_{k'}'^{88}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{89}$ is clearly feasible given that

a) and a feasible path E_k^{89} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{89} is assigned to each demand $k \in K$ along each edge $e \in E_k^{89}$ with $|S_k^{89}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{89}$ and $s" \in S_{k'}^{89}$ with $E_k^{89} \cap E_{k'}^{89} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{89}} |\{s' \in S_k^{89}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{89}}, z^{\mathcal{S}^{89}})$ is belong to F and then to $F_H^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in H} x_e^k = \frac{|H|-1}{2}$. It follows that

$$\mu x^{\mathcal{S}'^{88}} + \sigma z^{\mathcal{S}'^{88}} = \mu x^{\mathcal{S}^{89}} + \sigma z^{\mathcal{S}^{89}} = \mu x^{\mathcal{S}'^{88}} + \mu_e^k + \sigma z^{\mathcal{S}'^{88}}.$$

As a result, $\mu_e^k = 0$ for demand k and an edge e with $v_{k,e} \notin H$.
As e is chosen arbitrarily for the demand k with $e \notin E_0^k \cup E_1^k$ and $v_{k,e} \notin H$, we iterate the same procedure for all $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$ with $v_{k,e'} \notin H$. We conclude that for the demand k

$$\mu_e^k = 0, \text{ for all } e \in E \setminus (E_0^k \cup E_1^k) \text{ with } v_{k,e} \notin H.$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e \in E \setminus (E_0^k \cup E_1^k)$ with $v_{k',e} \notin H$. We conclude at the end that

$$\mu_e^k = 0, \text{ for all } k \in K \text{ and all } e \in E \setminus (E_0^k \cup E_1^k) \text{ with } v_{k,e} \notin H.$$

Let's us show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$. For that, we consider a solution $\mathcal{S}"^{88} = (E"^{88}, S"^{88})$ in which

a) a feasible path $E"^{88}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"^{88}_k$ is assigned to each demand $k \in K$ along each edge $e \in E"^{88}_k$ with $|S"^{88}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{88}_k$ and $s" \in S"^{88}_{k'}$ with $E"^{88}_k \cap E"^{88}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E"^{88}_k} |\{s' \in S"^{88}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^{88}_{k'}$ with $E"^{88}_k \cap E"^{88}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{88}_k$ assigned to the demand k in the solution $\mathcal{S}"^{88}$),

e) and there is $\frac{|H|-1}{2}$ pairs of demand-edge (k, e) from the odd-hole H denoted by $H"_{88}$ (i.e., $v_{k,e} \in H"_{88}$ s.t. the demand k selects the edge e for its routing in the solution $\mathcal{S}"^{88}$, i.e., $e \in E"^{88}_k$ for each node $v_{k,e} \in H"_{88}$, and $e' \notin E"^{88}_{k'}$ for all $v_{k',e'} \in H \setminus H"_{88}$.

   $\mathcal{S}"^{88}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{88}}, z^{\mathcal{S}"^{88}})$ is belong to F and then to $F_H^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in H} x^k_e = \frac{|H|-1}{2}$. Based on this, we construct a solution $\mathcal{S}^{90}$ derived from the solution $\mathcal{S}"^{88}$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}"^{88}$ (i.e., $E^{90}_k = E"^{88}_k$ for each $k \in K \setminus \tilde{K}$, and $E^{90}_k \neq E"^{88}_k$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E^{90}_k is assigned to each demand $k \in \tilde{K}$ (routing constraint),

b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"^{88}_k$ and $s" \in S"^{88}_{k'}$ with $E^{90}_k \cap E"^{88}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}, e \in E^{90}_k} |\{s' \in S"^{88}_k, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k \in K \setminus \tilde{K}, e \in E"^{88}_k} |\{s' \in S"^{88}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"^{88}_{k"}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{88}_k$ assigned to the demand k in the solution $\mathcal{S}"^{88}$).

   The last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}"^{88}$ remain the same in $\mathcal{S}^{90}$, i.e., $S"^{88}_{k'} = S^{90}_{k'}$ for each demand $k' \in K \setminus \{k\}$, and $S^{90}_k = S"^{88}_k \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{90}$ is clearly feasible given that

a) a feasible path E^{90}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{90}_k is assigned to each demand $k \in K$ along each edge $e \in E^{90}_k$ with $|S^{90}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{90}_k$ and $s" \in S^{90}_{k'}$ with $E^{90}_k \cap E^{90}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E^{90}_k} |\{s' \in S^{90}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

   The corresponding incidence vector $(x^{\mathcal{S}^{90}}, z^{\mathcal{S}^{90}})$ is belong to F and then to $F_H^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in H} x^k_e = \frac{|H|-1}{2}$. We have so

$$\mu x^{\mathcal{S}"^{88}} + \sigma z^{\mathcal{S}"^{88}} = \mu x^{\mathcal{S}^{90}} + \sigma z^{\mathcal{S}^{90}} = \mu x^{\mathcal{S}"^{88}} + \sigma z^{\mathcal{S}"^{88}} + \sigma^k_{s'} - \sum_{\tilde{k} \in \tilde{K}} \sum_{e \in E"^{88}_k} \mu^{\tilde{k}}_e + \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E^{90}_k} \mu^{\tilde{k}}_{e'}.$$

   It follows that $\sigma^k_{s'} = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ given that $\mu^k_e = 0$ for all the demands $k \in K$ and all edges $e \in E \setminus (E^k_0 \cup E^k_1)$ with $v_{k,e} \notin H$.

The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $v_{k,s'} \notin H$ s.t. we find

$$\sigma^k_{s'} = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s'} \notin H.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma^{k'}_s = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } v_{k',s} \notin H..$$

Consequently, we conclude that

$$\sigma^k_s = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s} \notin H.$$

Let's prove that μ^k_e for all $v_{k,e}$ are equivalents. Consider a node $v_{k',e'}$ in H s.t. $e' \notin E^{88}_{k'}$. For that, we consider a solution $\tilde{\mathcal{S}}^{88} = (\tilde{E}^{88}, \tilde{S}^{88})$ in which

a) a feasible path $\tilde{E}^{88}_k$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}^{88}_k$ is assigned to each demand $k \in K$ along each edge $e \in \tilde{E}^{88}_k$ with $|\tilde{S}^{88}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}^{88}_k$ and $s" \in \tilde{S}^{88}_{k'}$ with $\tilde{E}^{88}_k \cap \tilde{E}^{88}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in \tilde{E}^{88}_k} |\{s' \in \tilde{S}^{88}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and there is $\frac{|H|-1}{2}$ pairs of demand-edge (k, e) from the odd-hole H denoted by $\tilde{H}_{88}$ (i.e., $v_{k,e} \in \tilde{H}_{88}$ s.t. the demand k selects the edge e for its routing in the solution $\tilde{\mathcal{S}}^{88}$, i.e., $e \in \tilde{E}^{88}_k$ for each node $v_{k,e} \in \tilde{H}_{88}$, and $e" \notin \tilde{E}^{88}_{k'}$ for all $v_{k',e"} \in H \setminus \tilde{H}_{88}$.

$\tilde{\mathcal{S}}^{88}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{88}}, z^{\tilde{\mathcal{S}}^{88}})$ is belong to F and then to $F_H^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in H} x^k_e = \frac{|H|-1}{2}$. Based on this, we construct a solution $\mathcal{S}^{91}$ derived from the solution $\tilde{\mathcal{S}}^{88}$ by

a) modifying the path assigned to the demand k' in $\tilde{\mathcal{S}}^{88}$ from $\tilde{E}^{88}_{k'}$ to a path $E^{91}_{k'}$ passed through the edge e' with $v_{k',e'} \in H$,

b) and selecting a pair of demand-edge (k, e) from the set of pairs of demand-edge in H_{88} s.t. $v_{k',e'}$ is not linked with any node $v_{k",e"}$ in $H_{88} \setminus \{v_{k,e}\}$,

c) modifying the path assigned to the demand k in $\tilde{\mathcal{S}}^{88}$ with $e \in \tilde{E}^{88}_k$ and $v_{k,e} \in H$ from $\tilde{E}^{88}_k$ to a path E^{91}_k without passing through any edge $e" \in E \setminus (E^k_0 \cup E^k_1)$ s.t. $v_{k',e'}$ and $v_{k,e"}$ linked in H,

d) modifying the last-slots assigned to some demands $\tilde{K} \subset K$ from $\tilde{S}^{88}_{\tilde{k}}$ to $S^{91}_{\tilde{k}}$ for each $\tilde{k} \in \tilde{K}$ while satisfying non-overlapping constraint.

The paths assigned to the demands $K \setminus \{k, k'\}$ in $\tilde{\mathcal{S}}^{88}$ remain the same in $\mathcal{S}^{91}$ (i.e., $E^{91}_{k"} = \tilde{E}^{88}_{k"}$ for each $k" \in K \setminus \{k, k'\}$), and also without modifying the last-slots assigned to the demands $K \setminus \tilde{K}$ in $\tilde{\mathcal{S}}^{88}$, i.e., $\tilde{S}^{88}_k = S^{91}_k$ for each demand $k \in K \setminus \tilde{K}$. The solution $\mathcal{S}^{91}$ is clearly feasible given that

a) a feasible path E^{91}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{91}_k is assigned to each demand $k \in K$ along each edge $e \in E^{91}_k$ with $|S^{91}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{91}$ and $s" \in S_{k'}^{91}$ with $E_k^{91} \cap E_{k'}^{91} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{91}} |\{s' \in S_k^{91}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{91}}, z^{\mathcal{S}^{91}})$ is belong to F and then to $F_H^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in H} x_e^k = \frac{|H|-1}{2}$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{88}} + \sigma z^{\tilde{\mathcal{S}}^{88}} = \mu x^{\mathcal{S}^{91}} + \sigma z^{\mathcal{S}^{91}} = \mu x^{\tilde{\mathcal{S}}^{88}} + \sigma z^{\tilde{\mathcal{S}}^{88}} + \mu_{e'}^{k'} - \mu_e^k + \sum_{\tilde{k} \in \tilde{K}} \sum_{s' \in S_{\tilde{k}}^{91}} \sigma_{s'}^{\tilde{k}} - \sum_{s \in \tilde{S}_{\tilde{k}}^{88}} \sigma_s^{\tilde{k}}$$
$$+ \sum_{e" \in E_{k'}^{91} \setminus \{e'\}} \mu_{e"}^{k'} - \sum_{e" \in \tilde{E}_{k'}^{88}} \mu_{e"}^{k'} + \sum_{e" \in E_k^{91}} \mu_{e"}^{k} - \sum_{e" \in \tilde{E}_k^{88} \setminus \{e\}} \mu_{e"}^{k}.$$

It follows that $\mu_{e'}^{k'} = \mu_e^k$ for demand k' and a edge $e' \in E \setminus (E_0^{k'} \cup E_1^{k'})$ with $v_{k',e'} \in H$ given that $\mu_{e"}^k = 0$ for all $k \in K$ and all $e" \in E \setminus (E_0^k \cup E_1^k)$ with $v_{k,e"} \notin H$, and $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$.
Given that the pair $(v_{k,e}, v_{k',e'})$ are chosen arbitrary in the odd-hole H, we iterate the same procedure for all pairs $(v_{k,e}, v_{k',e'})$ s.t. we find

$$\mu_e^k = \mu_{e'}^{k'}, \text{for all pairs } (v_{k,e}, v_{k',e'}) \in H.$$

Consequently, we obtain that $\mu_e^k = \rho$ for all $v_{k,e} \in H$.
On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k \in K$ and $e \in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e}, \text{ if } e \in E_0^k, \\ \gamma_2^{k,e}, \text{ if } e \in E_1^k, \\ \rho, \text{ if } v_{k,e} \in H, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\}, \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{v_{k,e} \in H} \rho \alpha_e^k + \gamma Q.$

□

Theorem 2.4.16. *Let H be an odd-hole, and C be a clique in the conflict graph $\tilde{G}_E^K$ with*

a) $|H| \geq 5$,

b) and $|C| \geq 3$,

c) and $H \cap C = \emptyset$,

d) and the nodes $(v_{k,e}, v_{k',e'})$ are linked in $\tilde{G}_E^K$ for all $v_{k,e} \in H$ and $v_{k',e'} \in C$.

Then, the inequality (2.45) is facet defining for $P(G, K, \mathbb{S})$ iff for each node $v_{k",e"}$ in $\tilde{G}_E^K$ with $v_{k",e"} \notin H \cup C$ and $C \cup \{v_{k",e"}\}$ is a clique in $\tilde{G}_E^K$, there exists a subset of nodes $\tilde{H} \subseteq H$ of size $\frac{|H|-1}{2}$ s.t. $\tilde{H} \cup \{v_{k",e"}\}$ is stable in $\tilde{G}_E^K$.

Proof. **Neccessity.**
If there exists a node $v_{k",e"} \notin H \cup C$ in $\tilde{G}_E^K$ s.t. $v_{k",e"}$ is linked with all nodes $v_{k,e} \in H$ and also with all nodes $v_{k',e'} \in C$. This implies that the inequality (2.45) can be dominated by the following valid inequality

$$\sum_{v_{k,e} \in H} x_e^k + \frac{|H|-1}{2} \sum_{v_{k',e'} \in C} x_{e'}^{k'} + \frac{|H|-1}{2} x_{e"}^{k"} \leq \frac{|H|-1}{2}.$$

As a result, the inequality (2.45) is not facet defining for $P(G, K, \mathbb{S})$.
Sufficiency.
Let $F_{H,C}^{\tilde{G}_E^K}$ denote the face induced by the inequality (2.45), which is given by

$$F_{H,C}^{\tilde{G}_E^K} = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{v_{k,e} \in H} x_e^k + \frac{|H|-1}{2} \sum_{v_{k',e'} \in C} x_{e'}^{k'} = \frac{|H|-1}{2}\}.$$

In order to prove that inequality $\sum_{v_{k,e} \in H} x_e^k + \frac{|H|-1}{2} \sum_{v_{k',e'} \in C} x_{e'}^{k'} = \frac{|H|-1}{2}$ is facet defining for $P(G, K, \mathbb{S})$, we start checking that $F_{H,C}^{\tilde{G}_E^K}$ is a proper face, and $F_{H,C}^{\tilde{G}_E^K} \neq P(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{92} = (E^{92}, S^{92})$ as below

a) a feasible path E_k^{92} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{92} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{92}$ with $|S_k^{92}| \geq 1$ (contiguity and continuity constraints),

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{92}$ and $s' \in S_{k'}^{92}$ with $E_k^{92} \cap E_{k'}^{92} \neq \emptyset$ (non-overlapping constraint),

d) and there is $\frac{|H|-1}{2}$ pairs of demands edges (k, e) from the odd-hole H denoted by H_{92} (i.e., $v_{k,e} \in H_{92}$ s.t. the demand k selects the edge e for its routing in the solution $\mathcal{S}^{92}$, i.e., $e \in E_k^{92}$ for each node $v_{k,e} \in H_{92}$, and $e' \notin E_{k'}^{92}$ for all $v_{k',e'} \in H \setminus H_{92}$.

Obviously, $\mathcal{S}^{92}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{92}}, z^{\mathcal{S}^{92}})$ is belong to $P(G, K, \mathbb{S})$ and then to $F_{H,C}^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in H} x_e^k + \frac{|H|-1}{2} \sum_{v_{k',e'} \in C} x_{e'}^{k'} = \frac{|H|-1}{2}$. As a result, $F_{H,C}^{\tilde{G}_E^K}$ is not empty (i.e., $F_{H,C}^{\tilde{G}_E^K} \neq \emptyset$). Furthermore, given that $s \in \{w_k, ..., \bar{s}\}$ for each $v_{k,s} \in H$, this means that there exists at least one feasible slot assignment S_k for the demands k in H with $s \notin S_k$ for each $v_{k,s} \in H$. This means that $F_{H,C}^{\tilde{G}_E^K} \neq P(G, K, \mathbb{S})$.
Let denote the inequality $\sum_{v_{k,e} \in H} x_e^k + \frac{|H|-1}{2} \sum_{v_{k',e'} \in C} x_{e'}^{k'} \leq \frac{|H|-1}{2}$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S})$. Suppose that $F_{H,C}^{\tilde{G}_E^K} \subset F = \{(x, z) \in P(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. We show that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K} (w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ as done in the proof of theorem 2.4.15,

b) and $\mu_e^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$ with $v_{k,e} \notin H \cup C$ as done in the proof of theorem 2.4.15,

c) and μ_e^k are equivalent for all $v_{k,e} \in H$ as done in the proof of theorem 2.4.15,

given that the solutions defined in the proof of theorem 2.4.15, their corresponding incidence vector are belong to $P(G, K, \mathbb{S})$ and then to $F_{H,C}^{\tilde{G}_E^K}$ given that they are composed by $\sum_{v_{k,e} \in H} x_e^k + \frac{|H|-1}{2} \sum_{v_{k',e'} \in C} x_{e'}^{k'} = \frac{|H|-1}{2}$.
Let us prove now that $\mu_{e'}^{k'}$ are equivalent for all $v_{k',e'} \in C$. For this, we consider a node $v_{k',e'}$ in C s.t. $e' \notin E_{k'}^{92}$. For that, we consider a solution $\tilde{\mathcal{S}}^{92} = (\tilde{E}^{92}, \tilde{S}^{92})$ in which

a) a feasible path $\tilde{E}_k^{92}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $\tilde{S}_k^{92}$ is assigned to each demand $k \in K$ along each edge $e \in \tilde{E}_k^{92}$ with $|\tilde{S}_k^{92}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}_k^{92}$ and $s" \in \tilde{S}_{k'}^{92}$ with $\tilde{E}_k^{92} \cap \tilde{E}_{k'}^{92} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in \tilde{E}_k^{92}} |\{s' \in \tilde{S}_k^{92}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s \in S_k^{92}$ with $\tilde{E}_k^{92} \cap \tilde{E}_{k'}^{92} \neq \emptyset$,

e) and there is $\frac{|H|-1}{2}$ pairs of demand-edge (k, e) from the odd-hole H denoted by $\tilde{H}_{92}$ (i.e., $v_{k,e} \in \tilde{H}_{92}$ s.t. the demand k selects the edge e for its routing in the solution $\tilde{\mathcal{S}}^{92}$, i.e., $e \in \tilde{E}_k^{92}$ for each node $v_{k,e} \in \tilde{H}_{92}$, and $e" \notin \tilde{E}_{k'}^{92}$ for all $v_{k',e"} \in H \setminus \tilde{H}_{92}$.

$\tilde{\mathcal{S}}^{92}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{92}}, z^{\tilde{\mathcal{S}}^{92}})$ is belong to F and then to $F_{H,C}^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e} \in H} x_e^k + \frac{|H|-1}{2} \sum_{v_{k',e'} \in C} x_{e'}^{k'} = \frac{|H|-1}{2}$. Based on this, we construct a solution $\mathcal{S}'^{93}$ derived from the solution $\tilde{\mathcal{S}}^{92}$ by

a) modifying the path assigned to the demand k' in $\tilde{\mathcal{S}}^{92}$ from $\tilde{E}_{k'}^{92}$ to a path $E'^{93}_{k'}$ passed through the edge e' with $v_{k',e'} \in C$,

b) and modifying the path assigned to each demand k with $v_{k,e} \in H_{92}$ in $\tilde{\mathcal{S}}^{92}$ with $e \in \tilde{E}_k^{92}$ and $v_{k,e} \in H$ from $\tilde{E}_k^{92}$ to a path E'^{93}_k without passing through any edge $e" \in E \setminus (E_0^k \cup E_1^k)$,

c) modifying the last-slots assigned to some demands $\tilde{K} \subset K$ from $\tilde{S}_{\tilde{k}}^{92}$ to $S'^{93}_{\tilde{k}}$ for each $\tilde{k} \in \tilde{K}$ while satisfying non-overlapping constraint.

The paths assigned to the demands $K \setminus (K(H_{92}) \cup \{k'\})$ in $\tilde{\mathcal{S}}^{92}$ remain the same in $\mathcal{S}'^{93}$ (i.e., $E'^{93}_{k"} = \tilde{E}_{k"}^{92}$ for each $k" \in K \setminus \{k, k'\}$), and also without modifying the last-slots assigned to the demands $K \setminus \tilde{K}$ in $\tilde{\mathcal{S}}^{92}$, i.e., $\tilde{S}_k^{92} = S'^{93}_k$ for each demand $k \in K \setminus \tilde{K}$. The solution $\mathcal{S}'^{93}$ is clearly feasible given that

a) a feasible path E'^{93}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S'^{93}_k is assigned to each demand $k \in K$ along each edge $e \in E'^{93}_k$ with $|S'^{93}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S'^{93}_k$ and $s" \in S'^{93}_{k'}$ with $E'^{93}_k \cap E'^{93}_{k'} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E'^{93}_k} |\{s' \in S'^{93}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}'^{93}}, z^{\mathcal{S}'^{93}})$ is belong to F and then to $F_{H,C}^{\tilde{G}_E^K}$ given that it is composed by $\sum_{v_{k,e}\in H} x_e^k + \frac{|H|-1}{2}\sum_{v_{k',e'}\in C} x_{e'}^{k'} = \frac{|H|-1}{2}$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{92}} + \sigma z^{\tilde{\mathcal{S}}^{92}} = \mu x^{\mathcal{S}'^{93}} + \sigma z^{\mathcal{S}'^{93}} = \mu x^{\tilde{\mathcal{S}}^{92}} + \sigma z^{\tilde{\mathcal{S}}^{92}} + \mu_{e'}^{k'} - \sum_{v_{k,e}\in H_{92}} \mu_e^k + \sum_{\tilde{k}\in\tilde{K}}\sum_{s'\in S_{\tilde{k}}'^{93}} \sigma_{s'}^{\tilde{k}} - \sum_{s\in\tilde{S}_{\tilde{k}}^{92}} \sigma_s^{\tilde{k}}$$
$$+ \sum_{e"\in E_{k'}'^{93}\setminus\{e'\}} \mu_{e"}^{k'} - \sum_{e"\in\tilde{E}_{k'}^{92}} \mu_{e"}^{k'} + \sum_{e"\in E_k'^{93}} \mu_{e"}^{k} - \sum_{k\in K(H_{92})}\sum_{e"\in\tilde{E}_k^{92}} \mu_{e"}^{k}.$$

It follows that $\mu_{e'}^{k'} = \sum_{v_{k,e}\in H_{92}} \mu_e^k$ for demand k' and a edge $e'\in E\backslash(E_0^{k'}\cup E_1^{k'})$ with $v_{k',e'}\in C$ given that $\mu_{e"}^k = 0$ for all $k\in K$ and all $e"\in E\setminus(E_0^k\cup E_1^k)$ with $v_{k,e"}\notin H\cup C$, and $\sigma_s^k = 0$ for all $k\in K$ and all $s\in\{w_k,...,\bar{s}\}$. As a result, $\mu_{e'}^{k'} = \rho\frac{|H|-1}{2}$.
Given that the pair $v_{k',e'}$ is chosen arbitrary in the clique C, we iterate the same procedure for all pairs $v_{k',e'}\in C$ s.t. we find

$$\mu_{e'}^{k'} = \rho\frac{|H|-1}{2}, \text{for all pairs } v_{k',e'}\in C.$$

As a result, all $\mu_{e'}^{k'}\in C$ are equivalents s.t.

$$\mu_{e'}^{k'} = \mu_{e"}^{k"} = \rho\frac{|H|-1}{2}, \text{for all pairs } v_{k',e'}, v_{k",e"}\in C$$

On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned}
\mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k'\in K \text{ and all } e'\in E_0^{k'},\\
\mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k'\in K \text{ and all } e'\in E_1^{k'},\\
\sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k'\in K \text{ and all } s'\in\{1,...,w_{k'}-1\}.
\end{aligned}$$

We conclude that for each $k\in K$ and $e\in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e}, \text{ if } e\in E_0^k,\\ \gamma_2^{k,e}, \text{ if } e\in E_1^k,\\ \rho, \text{ if } v_{k,e}\in H,\\ \rho\frac{|H|-1}{2}, \text{ if } v_{k,e}\in C,\\ 0, otherwise, \end{cases}$$

and for each $k\in K$ and $s\in\mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s\in\{1,...,w_k-1\},\\ 0, otherwise. \end{cases}$$

As a result $(\mu,\sigma) = \sum_{v_{k,e}\in H} \rho\alpha_e^k + \sum_{v_{k',e'}\in C} \rho\frac{|H|-1}{2}\alpha_{e'}^{k'} + \gamma Q.$

□

2.4.10 Tranmission-Reach-Cover Inequalities

The inequalities (2.43), (2.44) and (2.45) can be strengthened by defining a minimal cover related to the transmission-reach constraint.

Definition 2.4.8. *Consider a demand $k \in K$. A cover C for the demand k related to the transmission-reach constraint is a subset of edges in $E\setminus(E_0^k \cup E_1^k)$ s.t. $\sum_{e\in C} l_e > \bar{l}_k$, and each pair of edges $(e, e') \in C$ are not non-compatible edges for the demand k. Furthermore, it's said minimal cover for the demand k if and only if for each $e \in C$ we have $\sum_{e'\in C\setminus\{e\}} l_{e'} \leq \bar{l}_k$.*

Based on this, we introduce the following inequalities.

Proposition 2.4.22. *Consider a demand $k \in K$. Let C be a minimal cover related to the tranmission-reach constraint for the demand k. Then, the inequality*

$$\sum_{e\in C} x_e^k \leq |C| - 1, \tag{2.46}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. Consider a demand $k \in K$. If C is minimal cover for k this means that there are at most $|C| - 1$ edges from the set of edges in C that can be used to route the demand k. We strengthen the proof by assuming that the inequality (2.46) is not valid for $P(G, K, \mathbb{S})$. It follows that there exists a C-RSA solution S in which $\tilde{e} \notin E_k$ for an edge $\tilde{e} \in C$ s.t.

$$\sum_{e\in C} x_e^k(S) > |C| - 1.$$

Since $\tilde{e} \notin E_k$ for an edge $\tilde{e} \in C$ this means that $x_{\tilde{e}}^k(S) = 0$, and taking into account that C is minimal cover for the demand k, $x_e^k(S) \leq 1$ for each $e \in C \setminus \{\tilde{e}\}$ and $x_{\tilde{e}}^k(S) \leq 1$, it follows that

$$\sum_{e\in C\setminus\{\tilde{e}\}} x_e^k(S) \leq |C| - 1$$

which contradicts what we supposed before, i.e., $\sum_{e\in C} x_e^k(S) > |C| - 1$.
Hence $\sum_{e\in C} |E_k \cap \{e\}| \leq |C| - 1$.
We conclude at the end that the inequality (2.46) is valid for $P(G, K, \mathbb{S})$. $\square$

Furthermore, the inequality (2.46) induced by the minimal cover C can be lifted by introducing the definition of an extended cover related to the transmission-reach constraint as follows.

Proposition 2.4.23. *Consider a demand $k \in K$. Let C be a minimal cover for the demand k related to the transmission-reach constraint, and $\Xi(C)$ be a subset of edges in $E\setminus(C \cup E_0^k \cup E_1^k)$ where $\Xi = \{e \in E\setminus(C \cup E_1^k \cup E_0^k) : l_e \geq l_{e'} \quad \forall e' \in C$ and e is not non-compatible edges with edges in $C\}$. Then, the inequality*

$$\sum_{e\in C} x_e^k + \sum_{e'\in\Xi(C)} x_{e'}^k \leq |C| - 1, \tag{2.47}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. If C is minimal cover related to the transmission-reach constraint for the demand $k \in K$ this means that there is at most $|C|-1$ edges from the set of edges in $C \cup \Xi(C)$ that can be used to route the demand k. We strengthen the proof by assuming that the inequality (2.47) is not valid for $P(G, K, \mathbb{S})$. It follows that there exists a C-RSA solution S in which $\tilde{e} \notin E_k$ for each edge $\tilde{e} \in \Xi(C)$ s.t.

$$\sum_{e \in C} x_e^k(S) > |C| - 1.$$

Since $\tilde{e} \notin E_k$ for for each edge $\tilde{e} \in \Xi(C)$ this means that $x_{\tilde{e}}^k(S) = 0$, and taking into account that C is minimal cover for the demand k, $x_e^k(S) \leq 1$ for each $e \in C$ and $x_{\tilde{e}}^k(S) \leq 1$, it follows that

$$\sum_{e \in C} x_e^k(S) \leq |C| - 1$$

which contradicts what we supposed before, i.e., $\sum_{e \in C} x_e^k(S) > |C|-1$ and also the inequality (2.46).
Hence $\displaystyle\sum_{e \in C} |E_k \cap \{e\}| + \sum_{\tilde{e} \in \Xi(C)} |E_k \cap \{\tilde{e}\}| \leq |C| - 1.$
We conclude at the end that the inequality (2.47) is valid for $P(G, K, \mathbb{S})$. □

Theorem 2.4.17. *Consider a demand k in K. Let C be a minimal cover in $E \setminus (E_0^k \cup E_1^k)$ for the demand k related to the transmission-reach constraint. Then, the inequality* (2.46) *is facet defining for the polytope $P(G, K, \mathbb{S}, C, k)$ where*

$$P(G, K, \mathbb{S}, C, k) = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{e' \in E \setminus (C \cup E_0^k \cup E_1^k)} x_{e'}^k = 0\}.$$

Proof. Let F_C^k denote the face induced by the inequality (2.46), which is given by

$$F_C^k = \{(x, z) \in P(G, K, \mathbb{S}, C, k) : \sum_{e \in C} x_e^k = |C| - 1\}.$$

In order to prove that inequality $\sum_{e \in C} x_e^k \leq |C| - 1$ is facet defining for $P(G, K, \mathbb{S}, C, k)$, we start checking that F_C^k is a proper face, and $F_C^k \neq P(G, K, \mathbb{S}, C, k)$.
We construct a solution $\mathcal{S}^{96} = (E^{96}, S^{96})$ as below

a) a feasible path E_k^{96} is assigned to each demand $k \in K$ (routing constraint),
b) a set of last-slots S_k^{96} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{96}$ with $|S_k^{96}| \geq 1$ (contiguity and continuity constraints),
c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{96}$ and $s' \in S_{k'}^{96}$ with $E_k^{96} \cap E_{k'}^{96} \neq \emptyset$ (non-overlapping constraint),
d) and there is $|C| - 1$ edges from the cover C that are used to route the demand k in the solution $\mathcal{S}^{96}$ denoted by C_{96} (i.e., if $e \in C_{96}$, this means that the edge e is selected for the routing of the demand k in the solution $\mathcal{S}^{96}$, i.e., $e \in E_k^{96}$ for each demand $e \in C_{96}$, $e' \notin E_k^{96}$ for all $e' \in C \setminus C_{96}$.

Obviously, $\mathcal{S}^{96}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{96}}, z^{\mathcal{S}^{96}})$ is belong to $P(G, K, \mathbb{S}, C, k)$ and then to F_C^k given that it is composed by $\sum_{e \in C} x_e^k = |C| - 1$. As a result, F_C^k is not empty (i.e., $F_C^k \neq \emptyset$). Furthermore, given that $e \in E \setminus (E_0^k \cup E_1^k)$ for each $e \in C$, this means that there exists at least one feasible routing E_k for the demand k in C with $e \notin E_k$. As a result, $F_C^k \neq P(G, K, \mathbb{S}, C, k)$.

We denote the inequality $\sum_{e \in C} x_e^k \leq |C| - 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G, K, \mathbb{S}, C, k)$. Suppose that $F_C^k \subset F = \{(x, z) \in P(G, K, \mathbb{S}, C, k) : \mu x + \sigma z = \tau\}$. We show that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k \in K} |E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k \in K} |E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$,

b) and $\mu_e^{k'} = 0$ for all demands $k' \in K \setminus \{k\}$ and all edges $e \in E \setminus (E_0^{k'} \cup E_1^{k'})$,

c) and $\mu_{e'}^k = 0$ for all edges $e' \in E \setminus (E_0^{k'} \cup E_1^{k'} \cup C)$,

d) and all μ_e^k are equivalents for the set of edges in C of the demand k.

We first show that $\mu_{e'}^{k'} = 0$ for each edge $e' \in E \setminus (E_0^{k'} \cup E_1^{k'})$ of each demand $k' \in K \setminus \{k\}$. For that, we consider a solution $\mathcal{S}'^{96} = (E'^{96}, S'^{96})$ in which

a) a feasible path $E_k'^{96}$ is assigned to each demand $k \in K$ (routing constraint),

b) and a set of last-slots $S_k'^{96}$ is assigned to each demand $k \in K$ along each edge $e' \in E_k'^{96}$ with $|S_k'^{96}| \geq 1$ (contiguity and continuity constraints),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k'^{96}$ and $s" \in S_{k'}'^{96}$ with $E_k'^{96} \cap E_{k'}'^{96} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E_k'^{96}} |\{s' \in S_k'^{96}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) the edge e' is not non-compatible edge with the selected edges $e" \in E_{k'}'^{96}$ of demand k' in the solution $\mathcal{S}'^{96}$, i.e., $\sum_{e" \in E_{k'}'^{96}} l_{e"} + l_{e'} \leq \bar{l}_{k'}$. As a result, $E_k'^{96} \cup \{e'\}$ is a feasible path for the demand k',

e) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s \in S_k'^{96}$ and $s' \in S_{k'}'^{96}$ with $(E_{k'}'^{96} \cup \{e'\}) \cap E_k'^{96} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e' in the set of edges $E_{k'}'^{96}$ selected to route the demand k' in the solution $\mathcal{S}'^{96}$),

f) and there is $|C| - 1$ edges from the cover C that are used to route the demand k in the solution $\mathcal{S}'^{96}$ denoted by C'_{96} (i.e., if $e \in C'_{96}$, this means that the demand k selects the edge e for its routing in the solution $\mathcal{S}'^{96}$, i.e., $e \in E_k'^{96}$ for each edge $e \in C'_{96}$, and $e' \notin E_k'^{96}$ for all $e' \in C \setminus C'_{96}$.

The solution $\mathcal{S}'^{96}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{96}}, z^{\mathcal{S}'^{96}})$ is belong to F and then to F_C^k given that it is composed by $\sum_{e \in C} x_e^k = |C| + 1$. Based on this, we derive a solution $\mathcal{S}^{98}$ obtained from the solution $\mathcal{S}'^{96}$ by adding an unused edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k' in $K \setminus \{k\}$ in the solution $\mathcal{S}^{96}$ which means that $E_{k'}^{98} = E_{k'}'^{96} \cup \{e'\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k'\}$ in $\mathcal{S}'^{96}$ remain the same in the solution $\mathcal{S}^{98}$, i.e., $S_k^{98} = S_k'^{96}$ for each $k \in K$, and $E_{k"}^{98} = E_{k"}'^{96}$ for each $k" \in K \setminus \{k'\}$. $\mathcal{S}^{98}$ is clearly feasible given that

a) and a feasible path E_k^{98} is assigned to each demand $k \in K$ (routing constraint),

b) and a set of last-slots S_k^{98} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{98}$ with $|S_k^{98}| \geq 1$ (contiguity and continuity constraints),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{98}$ and $s" \in S_{k'}^{98}$ with $E_k^{98} \cap E_{k'}^{98} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E_k^{98}} |\{s' \in S_k^{98}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{98}}, z^{\mathcal{S}^{98}})$ is belong to F and then to F^k_C given that it is composed by $\sum_{e\in C} x^k_e = |C|+1$. It follows that

$$\mu x^{\mathcal{S}'^{96}} + \sigma z^{\mathcal{S}'^{96}} = \mu x^{\mathcal{S}^{98}} + \sigma z^{\mathcal{S}^{98}} = \mu x^{\mathcal{S}'^{96}} + \mu^{k'}_{e'} + \sigma z^{\mathcal{S}'^{96}}.$$

As a result, $\mu^{k'}_{e'} = 0$ for demand k' and an edge e'.
As e' is chosen arbitrarily for the demand k' with $e \notin E^{k'}_0 \cup E^{k'}_1$, we iterate the same procedure for all $e" \in E \setminus (E^{k'}_0 \cup E^{k'}_1 \cup \{e'\})$. We conclude that for the demand k'

$$\mu^{k'}_{e'} = 0, \text{ for all } e' \in E \setminus (E^{k'}_0 \cup E^{k'}_1).$$

Moreover, given that k' is chosen arbitrarily in $K \setminus \{k\}$, we iterate the same procedure for all $k" \in K \setminus \{k, k'\}$ and all $e' \in E \setminus (E^{k"}_0 \cup E^{k"}_1)$. We conclude at the end that

$$\mu^{k"}_{e'} = 0, \text{ for all } k" \in K \setminus \{k, k'\} \text{ and all } e' \in E \setminus (E^{k"}_0 \cup E^{k"}_1).$$

We further re-do the same procedure for the demand k and all edges $e' \in E \setminus (E^k_0 \cup E^k_1 \cup C)$. As a result,

$$\mu^k_{e'} = 0, \text{ for all } e' \in E \setminus (E^k_0 \cup E^k_1 \cup C).$$

Let's us show that $\sigma^k_s = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$. For that, we consider a solution $\mathcal{S}"^{96} = (E"^{96}, S"^{96})$ in which

a) a feasible path $E"^{96}_k$ is assigned to each demand $k \in K$ (routing constraint),
b) a set of last-slots $S"^{96}_k$ is assigned to each demand $k \in K$ along each edge $e' \in E"^{96}_k$ with $|S"^{96}_k| \geq 1$ (contiguity and continuity constraints),
c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"^{96}_k$ and $s" \in S"^{96}_{k'}$ with $E"^{96}_k \cap E"^{96}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e'\in E"^{96}_k} |\{s' \in S"^{96}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),
d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"^{96}_{k'}$ with $E"^{96}_k \cap E"^{96}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{96}_k$ assigned to the demand k in the solution $\mathcal{S}"^{96}$),
e) and there is $|C| - 1$ edges from the cover C that are used to route the demand k in the solution $\mathcal{S}"^{96}$ denoted by $C"_{96}$ (i.e., if $e \in C"_{96}$, this means that the demand k selects the edge e for its routing in the solution $\mathcal{S}"^{96}$, i.e., $e \in E"^{96}_k$ for each edge $e \in C"_{96}$, $e" \notin E"^{96}_k$ for all $e" \in C \setminus C"_{96}$.

   $\mathcal{S}"^{96}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{96}}, z^{\mathcal{S}"^{96}})$ is belong to F and then to F^k_C given that it is composed by $\sum_{e\in C} x^k_e = |C|+1$. Based on this, we construct a solution $\mathcal{S}^{99}$ derived from the solution $\mathcal{S}"^{96}$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K \setminus \{k\}$ in $\mathcal{S}"^{96}$ (i.e., $E^{99}_k = E"^{96}_k$ for each $k \in K \setminus \tilde{K}$, and $E^{99}_k \neq E"^{96}_k$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E^{99}_k is assigned to each demand $k \in \tilde{K}$ (routing constraint),
b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"^{96}_k$ and $s" \in S"^{96}_{k'}$ with $E^{99}_k \cap E"^{96}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in\tilde{K}, e'\in E^{99}_k} |\{s' \in S"^{96}_k, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k\in K\setminus\tilde{K}, e'\in E"^{96}_k} |\{s' \in S"^{96}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"^{96}_{k"}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{96}_k$ assigned to the demand k in the solution $\mathcal{S}"^{96}$).

The last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}"^{96}$ remain the same in $\mathcal{S}^{99}$, i.e., $S"^{96}_{k'} = S^{99}_{k'}$ for each demand $k' \in K \setminus \{k\}$, and $S^{99}_k = S"^{96}_k \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{99}$ is clearly feasible given that

a) a feasible path E^{99}_k is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S^{99}_k is assigned to each demand $k \in K$ along each edge $e' \in E^{99}_k$ with $|S^{99}_k| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{99}_k$ and $s" \in S^{99}_{k'}$ with $E^{99}_k \cap E^{99}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E^{99}_k} |\{s' \in S^{99}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{99}}, z^{\mathcal{S}^{99}})$ is belong to F and then to F^k_C given that it is composed by $\sum_{e \in C} x^k_e = |C| + 1$. We have so

$$\mu x^{\mathcal{S}"^{96}} + \sigma z^{\mathcal{S}"^{96}} = \mu x^{\mathcal{S}^{99}} + \sigma z^{\mathcal{S}^{99}} = \mu x^{\mathcal{S}"^{96}} + \sigma z^{\mathcal{S}"^{96}} + \sigma^k_{s'} - \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E"^{96}_k} \mu^{\tilde{k}}_{e'} + \sum_{\tilde{k} \in \tilde{K}} \sum_{e" \in E^{99}_k} \mu^{\tilde{k}}_{e"}.$$

It follows that $\sigma^k_{s'} = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ given that $\mu^{\tilde{k}}_{e'} = 0$ for all the demands $\tilde{k} \in K$ and all edges $e' \in E \setminus (E^{\tilde{k}}_0 \cup E^{\tilde{k}}_1)$ with $e' \notin C$ if $k = \tilde{k}$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k s.t. we find

$$\sigma^k_{s'} = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\}.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma^{k'}_s = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\}.$$

Consequently, we conclude that

$$\sigma^k_s = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\}.$$

Let us prove now that μ^k_e for all $e \in C$ are equivalents. We first consider an edge e' in C s.t. $e \notin E^{96}_{k'}$. For that, we consider a solution $\mathcal{S}^{100} = (E^{100}, S^{100})$ obtained from the solution $\mathcal{S}^{96}$ by

a) selecting an edge e from C_{96} s.t. the demand k used the edge e for its routing in the solution $\mathcal{S}^{96}$,

b) the paths assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{96}$ remain the same in $\mathcal{S}^{100}$ (i.e., $E^{100}_{k"} = E^{96}_{k"}$ for each $k" \in K \setminus \{k\}$),

c) without modifying the last-slots assigned to the demands K in $\mathcal{S}^{96}$, i.e., $S^{96}_k = S^{100}_k$ for each demand $k \in K$,

d) modifying the path assigned to the demand k in $\mathcal{S}^{96}$ from E^{96}_k to a path E^{100}_k passed through the edge e' (i.e., $e' \in E^{100}_k$) with $e' \in C$ s.t. $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and each $s' \in S^{96}_{k'}$ with $E^{96}_k \cap E^{100}_{k'} \neq \emptyset$, and without passing through the edge e, i.e., $e \notin E^{100}_k$.

The solution $\mathcal{S}^{100}$ is feasible given that

a) a feasible path E_k^{100} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{100} is assigned to each demand $k \in K$ along each edge $e \in E_k^{100}$ with $|S_k^{100}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{100}$ and $s" \in S_{k'}^{100}$ with $E_k^{100} \cap E_{k'}^{100} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{100}} |\{s' \in S_k^{100}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{100}}, z^{\mathcal{S}^{100}})$ is belong to F and then to F_C^k given that it is composed by $\sum_{e \in C} x_e^k = |C| - 1$. We then obtain that

$$\mu x^{\mathcal{S}^{96}} + \sigma z^{\mathcal{S}^{96}} = \mu x^{\mathcal{S}^{100}} + \sigma z^{\mathcal{S}^{100}} = \mu x^{\mathcal{S}^{96}} + \sigma z^{\mathcal{S}^{96}} + \mu_{e'}^k - \mu_e^k + \sum_{e" \in E_k^{100} \setminus C} \mu_{e"}^k - \sum_{e" \in E_k^{96} \setminus \{e\} \cup C} \mu_{e"}^k.$$

It follows that $\mu_{e'}^k = \mu_e^k$ for demand k and a edge $e' \in C$ given that $\mu_{e"}^k = 0$ for all $e" \in E \setminus (E_0^k \cup E_1^k)$ with $e" \notin C$. Given that the pair (e, e') are chosen arbitrary in the cover C, we iterate the same procedure for all pairs (e, e') s.t. we find

$$\mu_e^k = \mu_{e'}^k, \text{for all pairs } (e, e') \in C.$$

Consequently, we conclude that

$$\mu_e^k = \rho, \text{ for all } e \in C.$$

On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k' \in K$ and $e' \in E$

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'}, \text{ if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'}, \text{ if } e' \in E_1^{k'}, \\ \rho, \text{ if } e' \in C \text{ and } k' = k, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\} \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{e \in C} \rho \alpha_e^k + \gamma Q$. $\square$

2.4.11 Edge-Capacity-Cover Inequalities

On the other hand, let's us now provide some inequalities related to the capacity constraint.

Proposition 2.4.24. *Consider an edge e in E. Then, the inequality*

$$\sum_{k \in K \setminus K_e} w_k x_e^k \leq \bar{s} - \sum_{k' \in K_e} w_{k'}, \tag{2.48}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. The number of slots allocated in the edge $e \in E$ should be less than the residual capacity of the edge e which is equal to $\bar{s} - \sum_{k' \in K_e} w_{k'}$. □

Based on this, we introduce the following definitions.

Definition 2.4.9. *For an edge $e \in E$, a subset of demands $C \subseteq K$ with $e \notin E_0^k \cap E_1^k$ For each demand $k \in C$, is said a cover for the edge e if $\sum_{k \in C} w_k > \bar{s} - \sum_{k' \in K_e} w_{k'}$.*

Definition 2.4.10. *For an edge e in E, a cover C is said a minimal cover if $C \setminus \{k\}$ is not a cover for all $k \in C$, i.e., $\sum_{k' \in C \setminus \{k\}} w_{k'} \leq \bar{s} - \sum_{k" \in K_e} w_{k"}$.*

Proposition 2.4.25. *Consider an edge e in E. Let C be a minimal cover in K for the edge e. Then, the inequality*

$$\sum_{k \in C} x_e^k \leq |C| - 1, \tag{2.49}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. If C is minimal cover for edge $e \in E$ this means that there are at most $|C|-1$ demands from the set of demands in C that can use the edge e. We strengthen the proof by assuming that the inequality (2.49) is not valid for $P(G, K, \mathbb{S})$. It follows that there exists a C-RSA solution S in which $e \notin E_{k'}$ for a demand $k' \in C$ s.t.

$$\sum_{k \in C} x_e^k(S) > |C| - 1.$$

Since $e \notin E_{k'}$ for a demand $k' \in C$ this means that $x_e^{k'}(S) = 0$, and taking into account that C is minimal cover for the edge e, $x_e^k(S) \leq 1$ for each $k \in C \setminus \{k'\}$ and $x_e^{k'}(S) \leq 1$, it follows that

$$\sum_{k \in C \setminus \{k'\}} x_e^k(S) \leq |C| - 1$$

which contradicts what we supposed before, i.e., $\sum_{k \in C} x_e^k(S) > |C| - 1$.
Hence $\sum_{k \in C} |E_k \cap \{e\}| \leq |C| - 1$.
We conclude at the end that the inequality (2.49) is valid for $P(G, K, \mathbb{S})$. □

We verified that the inequality (2.49) can be easily strengthened by using its extended format which we call extended minimal cover for an edge e as follows.

Proposition 2.4.26. *Consider an edge e in E. Let C be a minimal cover in K for the edge e, and $\Xi(C)$ be a subset of demands in $K \setminus C \cup K_e$ where $\Xi = \{k \in K \setminus C \cup K_e : e \notin E_0^k$ and $w_k \geq w_{k'} \quad \forall k' \in C\}$. Then, the inequality*

$$\sum_{k \in C} x_e^k + \sum_{k' \in \Xi(C)} x_e^{k'} \leq |C| - 1, \tag{2.50}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. If C is minimal cover for edge $e \in E$ this means that there is at most $|C|-1$ demands from the set of demands in $C \cup \Xi(C)$ that can use the edge e. We strengthen the proof by assuming that the inequality (2.50) is not valid for $P(G, K, \mathbb{S})$. It follows that there exists a C-RSA solution S in which $e \notin E_{k'}$ for each demand $k' \in \Xi(C)$ s.t.

$$\sum_{k \in C} x_e^k(S) > |C| - 1.$$

Since $e \notin E_{k'}$ for for each demand $k' \in \Xi(C)$this means that $x_e^{k'}(S) = 0$, and taking into account that C is minimal cover for the edge e, $x_e^k(S) \leq 1$ for each $k \in C$ and $x_e^{k'}(S) \leq 1$, it follows that

$$\sum_{k \in C} x_e^k(S) \leq |C| - 1$$

which contradicts what we supposed before, i.e., $\sum_{k \in C} x_e^k(S) > |C|-1$ and also the inequality (2.49).
Hence $\sum_{k \in C} |E_k \cap \{e\}| + \sum_{k' \in \Xi(C)} |E_{k'} \cap \{e\}| \leq |C| - 1$.
We conclude at the end that the inequality (2.49) is valid for $P(G, K, \mathbb{S})$. □

Furthermore, the inequality (2.49) can have more generalized strengthening format using lifting procedures proposed by Nemhauser and Wolsey in [109].

Theorem 2.4.18. *Consider an edge e in E. Let C be a minimal cover in K for the edge e. Then, the inequality* (2.49) *is facet defining for the polytope $P(G, K, \mathbb{S}, C, e)$ where*

$$P(G, K, \mathbb{S}, C, e) = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{k' \in K \setminus (C \cup K_e)} x_e^{k'} = 0\}.$$

Proof. Let F_C^e denote the face induced by the inequality (2.49), which is given by

$$F_C^e = \{(x, z) \in P(G, K, \mathbb{S}, C, e) : \sum_{k \in C} x_e^k = |C| - 1\}.$$

In order to prove that inequality $\sum_{k \in C} x_e^k \leq |C| - 1$ is facet defining for $P(G, K, \mathbb{S}, C, e)$, we start checking that F_C^e is a proper face, and $F_C^e \neq P(G, K, \mathbb{S}, C, e)$.
We construct a solution $\mathcal{S}^{101} = (E^{101}, S^{101})$ as below

a) a feasible path E_k^{101} is assigned to each demand $k \in K$ (routing constraint),
b) a set of last-slots S_k^{101} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{101}$ with $|S_k^{101}| \geq 1$ (contiguity and continuity constraints),

c) $\{s-w_k+1,...,s\} \cap \{s'-w_{k'}+1,...,s'\} = \emptyset$ for each $k,k' \in K$ and each $s \in S_k^{101}$ and $s' \in S_{k'}^{101}$ with $E_k^{101} \cap E_{k'}^{101} \neq \emptyset$ (non-overlapping constraint),

d) and there is $|C|-1$ demands from the cover C which pass through the edge e in the solution $\mathcal{S}^{101}$ denoted by C_{101} (i.e., if $k \in C_{101}$, this means that the demand k selects the edge e for its routing in the solution $\mathcal{S}^{101}$, i.e., $e \in E_k^{101}$ for each demand $k \in C_{101}$, $e' \notin E_{k'}^{101}$ for all $k' \in C \setminus C_{101}$.

Obviously, $\mathcal{S}^{101}$ is a feasible solution for the problem given that it satisfies all the constraints of our cut formulation (2.2)-(2.10). Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{101}}, z^{\mathcal{S}^{101}})$ is belong to $P(G,K,\mathbb{S},C,e)$ and then to F_C^e given that it is composed by $\sum_{k\in C} x_e^k = |C|-1$. As a result, F_C^e is not empty (i.e., $F_C^e \neq \emptyset$). Furthermore, given that $e \in E \setminus (E_0^k \cup E_1^k)$ for each $k \in C$, this means that there exists at least one feasible routing E_k for each demand k in C with $e \notin E_k$. This means that $F_C^e \neq P(G,K,\mathbb{S},C,\Xi(C),e)$.
We denote the inequality $\sum_{k\in C} x_e^k \leq |C|-1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P(G,K,\mathbb{S},C,e)$. Suppose that $F_C^e \subset F = \{(x,z) \in P(G,K,\mathbb{S},C,e) : \mu x + \sigma z = \tau\}$. We show that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (s.t. $\gamma_1 \in \mathbb{R}^{\sum_{k\in K}|E_0^k|}, \gamma_2 \in \mathbb{R}^{\sum_{k\in K}|E_1^k|}, \gamma_3 \in \mathbb{R}^{\sum_{k\in K}(w_k-1)}$) s.t. $(\mu,\sigma) = \rho(\alpha,\beta) + \gamma Q$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$,

b) and $\mu_{e'}^k = 0$ for all demands $k \in K$ and all edges $e \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e'$ if $k \in C$,

c) and all μ_e^k are equivalents for the set of demands in C.

We first show that $\mu_{e'}^k = 0$ for each edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$ with $e \neq e'$ if $k \in C$. For that, we consider a solution $\mathcal{S}'^{101} = (E'^{101}, S'^{101})$ in which

a) a feasible path $E_k'^{101}$ is assigned to each demand $k \in K$ (routing constraint),

b) and a set of last-slots $S_k'^{101}$ is assigned to each demand $k \in K$ along each edge $e' \in E_k'^{101}$ with $|S_k'^{101}| \geq 1$ (contiguity and continuity constraints),

c) and $\{s'-w_k+1,...,s'\} \cap \{s"-w_{k'}+1,...,s"\} = \emptyset$ for each $k,k' \in K$ and each $s' \in S_k'^{101}$ and $s" \in S_{k'}'^{101}$ with $E_k'^{101} \cap E_{k'}'^{101} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in K, e'\in E_k'^{101}} |\{s' \in S_k'^{101}, s" \in \{s'-w_k+1,...,s'\}| \leq 1$ (non-overlapping constraint),

d) the edge e' is not non-compatible edge with the selected edges $e" \in E_k'^{101}$ of demand k in the solution $\mathcal{S}'^{101}$, i.e., $\sum_{e"\in E_k'^{101}} l_{e"} + l_{e'} \leq \bar{l}_k$. As a result, $E_k'^{101} \cup \{e'\}$ is a feasible path for the demand k,

e) $\{s-w_k+1,...,s\} \cap \{s'-w_{k'}+1,...,s'\} = \emptyset$ for each $k' \in K$ and $s \in S_k'^{101}$ and $s' \in S_{k'}'^{101}$ with $(E_k'^{101} \cup \{e'\}) \cap E_{k'}'^{101} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the edge e' in the set of edges $E_k'^{101}$ selected to route the demand k in the solution $\mathcal{S}'^{101}$),

f) and there is $|C|-1$ demands from the cover C which pass through the edge e in the solution $\mathcal{S}'^{101}$ denoted by C'_{101} (i.e., if $k \in C'_{101}$, this means that the demand k selects the edge e for its routing in the solution $\mathcal{S}'^{101}$, i.e., $e \in E_k'^{101}$ for each demand $k \in C'_{101}$, $e' \notin E_{k'}'^{101}$ for all $k' \in C \setminus C'_{101}$.

$\mathcal{S}'^{101}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{101}}, z^{\mathcal{S}'^{101}})$ is belong to F and then to F_C^e given that it is composed by $\sum_{k\in C} x_e^k = |C|+1$. Based on this, we derive a solution $\mathcal{S}^{102}$ obtained from the solution $\mathcal{S}'^{101}$ by adding an unused edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in K in the solution $\mathcal{S}^{101}$ which means that $E_k^{102} = E_k'^{101} \cup \{e'\}$. The last-slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^{101}$

remain the same in the solution $\mathcal{S}^{102}$, i.e., $S_k^{102} = S_k'^{101}$ for each $k \in K$, and $E_{k'}^{102} = E_{k'}'^{101}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{102}$ is clearly feasible given that

a) and a feasible path E_k^{102} is assigned to each demand $k \in K$ (routing constraint),

b) and a set of last-slots S_k^{102} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{102}$ with $|S_k^{102}| \geq 1$ (contiguity and continuity constraints),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{102}$ and $s" \in S_{k'}^{102}$ with $E_k^{102} \cap E_{k'}^{102} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E_k^{102}} |\{s' \in S_k^{102}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{102}}, z^{\mathcal{S}^{102}})$ is belong to F and then to F_C^e given that it is composed by $\sum_{k \in C} x_e^k = |C| + 1$. It follows that

$$\mu x^{\mathcal{S}'^{101}} + \sigma z^{\mathcal{S}'^{101}} = \mu x^{\mathcal{S}^{102}} + \sigma z^{\mathcal{S}^{102}} = \mu x^{\mathcal{S}'^{101}} + \mu_{e'}^k + \sigma z^{\mathcal{S}'^{101}}.$$

As a result, $\mu_{e'}^k = 0$ for demand k and an edge e'.
As e' is chosen arbitrarily for the demand k with $e \notin E_0^k \cup E_1^k$ and $e \neq e'$ if $k \in C$, we iterate the same procedure for all $e" \in E \setminus (E_0^k \cup E_1^k \cup \{e'\})$ with $e \neq e"$ if $k \in C$. We conclude that for the demand k

$$\mu_{e'}^k = 0, \text{ for all } e' \in E \setminus (E_0^k \cup E_1^k) \text{ with } e \neq e' \text{ if } k \in C.$$

Moreover, given that k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$ and all $e' \in E \setminus (E_0^k \cup E_1^k)$. We conclude at the end that

$$\mu_{e'}^k = 0, \text{ for all } k \in K \text{ and all } e' \in E \setminus (E_0^k \cup E_1^k) \text{ with } e \neq e' \text{ if } k \in C.$$

Let's us show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$. For that, we consider a solution $\mathcal{S}"^{101} = (E"^{101}, S"^{101})$ in which

a) a feasible path $E"_k^{101}$ is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots $S"_k^{101}$ is assigned to each demand $k \in K$ along each edge $e' \in E"_k^{101}$ with $|S"_k^{101}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S"_k^{101}$ and $s" \in S"_{k'}^{101}$ with $E"_k^{101} \cap E"_{k'}^{101} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E"_k^{101}} |\{s' \in S"_k^{101}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S"_{k'}^{101}$ with $E"_k^{101} \cap E"_{k'}^{101} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"_k^{101}$ assigned to the demand k in the solution $\mathcal{S}"^{101}$),

e) and there is $|C| - 1$ demands from the cover C which pass through the edge e in the solution $\mathcal{S}"^{101}$ denoted by $C"_{101}$ (i.e., if $k \in C"_{101}$, this means that the demand k selects the edge e for its routing in the solution $\mathcal{S}"^{101}$, i.e., $e \in E"_k^{101}$ for each demand $k \in C"_{101}$, $e" \notin E"_{k"}^{101}$ for all $k" \in C \setminus C"_{101}$.

$\mathcal{S}"^{101}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\mathcal{S}"^{101}}, z^{\mathcal{S}"^{101}})$ is belong to F and then to F_C^e given that it is composed by $\sum_{k \in C} x_e^k = |C| + 1$. Based on this, we construct a solution $\mathcal{S}^{103}$ derived from the solution $\mathcal{S}"^{101}$ by adding the slot s' as last-slot to the demand k with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}"^{101}$ (i.e., $E_k^{103} = E"_k^{101}$ for each $k \in K \setminus \tilde{K}$, and $E_k^{103} \neq E"_k^{101}$ for each $k \in \tilde{K}$) s.t.

a) a new feasible path E_k^{103} is assigned to each demand $k \in \tilde{K}$ (routing constraint),

b) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k \in \tilde{K}$ and $k' \in K \setminus \tilde{K}$ and each $s' \in S"^{101}_k$ and $s" \in S"^{101}_{k'}$ with $E_k^{103} \cap E"^{101}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}, e' \in E_k^{103}} |\{s' \in S"^{101}_k, s" \in \{s' - w_k + 1, ..., s'\}| + \sum_{k \in K \setminus \tilde{K}, e' \in E"^{101}_k} |\{s' \in S"^{101}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in \tilde{K}$ and $s" \in S"^{101}_{k"}$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S"^{101}_k$ assigned to the demand k in the solution $\mathcal{S}"^{101}$).

The last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}"^{101}$ remain the same in $\mathcal{S}^{103}$, i.e., $S"^{101}_{k'} = S^{103}_{k'}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{103} = S"^{101}_k \cup \{s\}$ for the demand k. The solution $\mathcal{S}^{103}$ is clearly feasible given that

a) a feasible path E_k^{103} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{103} is assigned to each demand $k \in K$ along each edge $e' \in E_k^{103}$ with $|S_k^{103}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{103}$ and $s" \in S_{k'}^{103}$ with $E_k^{103} \cap E_{k'}^{103} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E_k^{103}} |\{s' \in S_k^{103}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{103}}, z^{\mathcal{S}^{103}})$ is belong to F and then to F_C^e given that it is composed by $\sum_{k \in C} x_e^k = |C| + 1$. We have so

$$\mu x^{\mathcal{S}"^{101}} + \sigma z^{\mathcal{S}"^{101}} = \mu x^{\mathcal{S}^{103}} + \sigma z^{\mathcal{S}^{103}} = \mu x^{\mathcal{S}"^{101}} + \sigma z^{\mathcal{S}"^{101}} + \sigma_{s'}^k - \sum_{\tilde{k} \in \tilde{K}} \sum_{e' \in E"^{101}_k} \mu_{e'}^{\tilde{k}} + \sum_{\tilde{k} \in \tilde{K}} \sum_{e" \in E_k^{103}} \mu_{e"}^{\tilde{k}}.$$

It follows that $\sigma_{s'}^k = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ given that $\mu_{e'}^k = 0$ for all the demands $k \in K$ and all edges $e' \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e'$ if $k \in C$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k s.t. we find

$$\sigma_{s'}^k = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\}.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\}.$$

Consequently, we conclude that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\}.$$

Let us prove now that μ_e^k for all $k \in K$ with $k \in C$ are equivalents. For that, we consider a demand k' in C s.t. $e \notin E_{k'}^{101}$. For that, we consider a solution $\mathcal{S}^{104} = (E^{104}, S^{104})$ from the solution $\mathcal{S}^{101}$ by

a) selecting a demand k from C_{101} s.t. the demand k used the edge e for its routing in the solution $\mathcal{S}^{101}$,

b) the paths assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{101}$ remain the same in $\mathcal{S}^{104}$ (i.e., $E_{k"}^{104} = E_{k"}^{101}$ for each $k" \in K \setminus \{k, k'\}$),

c) without modifying the last-slots assigned to the demands K in $\mathcal{S}^{101}$, i.e., $S_k^{101} = S_k^{104}$ for each demand $k \in K$,

d) modifying the path assigned to the demand k' in $\mathcal{S}^{101}$ from $E_{k'}^{101}$ to a path $E_{k'}^{104}$ passed through the edge e (i.e., $e \in E_{k'}^{104}$) with $k' \in C$ s.t. $\{s-w_k+1,...,s\} \cap \{s'-w_{k'}+1,...,s'\} = \emptyset$ for each $k \in K$ and each $s' \in S_{k'}^{101}$ and each $s \in S_k^{101}$ with $E_k^{101} \cap E_{k'}^{104} \neq \emptyset$,

e) modifying the path assigned to the demand k in $\mathcal{S}^{101}$ with $e \in E_k^{101}$ and $k \in C$ from E_k^{101} to a path E_k^{104} without passing through the edge e (i.e., $e \notin E_k^{104}$) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k"} + 1, ..., s'\} = \emptyset$ for each $k" \in K \setminus \{k, k'\}$ and each $s \in S_k^{101}$ and each $s' \in S_{k"}^{101}$ with $E_{k"}^{101} \cap E_k^{104} \neq \emptyset$, and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $s \in S_k^{101}$ and each $s' \in S_{k'}^{101}$ with $E_{k"}^{104} \cap E_k^{104} \neq \emptyset$.

The solution $\mathcal{S}^{104}$ is feasible given that

a) a feasible path E_k^{104} is assigned to each demand $k \in K$ (routing constraint),

b) a set of last-slots S_k^{104} is assigned to each demand $k \in K$ along each edge $e \in E_k^{104}$ with $|S_k^{104}| \geq 1$ (contiguity and continuity constraints),

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{104}$ and $s" \in S_{k'}^{104}$ with $E_k^{104} \cap E_{k'}^{104} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{104}} |\{s' \in S_k^{104}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(x^{\mathcal{S}^{104}}, z^{\mathcal{S}^{104}})$ is belong to F and then to F_C^e given that it is composed by $\sum_{k \in C} x_e^k = |C| - 1$. We then obtain that

$$\begin{aligned} \mu x^{\mathcal{S}^{101}} + \sigma z^{\mathcal{S}^{101}} = \mu x^{\mathcal{S}^{104}} + \sigma z^{\mathcal{S}^{104}} &= \mu x^{\mathcal{S}^{101}} + \sigma z^{\mathcal{S}^{101}} + \mu_e^{k'} - \mu_e^k \\ &+ \sum_{e" \in E_{k'}^{104} \setminus \{e\}} \mu_{e"}^{k'} - \sum_{e" \in E_{k'}^{101}} \mu_{e"}^{k'} + \sum_{e" \in E_k^{104}} \mu_{e"}^k - \sum_{e" \in E_k^{101} \setminus \{e\}} \mu_{e"}^k. \end{aligned}$$

It follows that $\mu_e^{k'} = \mu_e^k$ for demand k' and a edge $e' \in E \setminus (E_0^{k'} \cup E_1^{k'})$ with $v_{k'} \in C$ given that $\mu_{e"}^k = 0$ for all $k \in K$ and all $e" \in E \setminus (E_0^k \cup E_1^k)$ with $k \notin C$.
Given that the pair (k, k') are chosen arbitrary in the cover C, we iterate the same procedure for all pairs (k, k') s.t. we find

$$\mu_e^k = \mu_e^{k'}, \text{for all pairs } (k, k') \in C.$$

Consequently, we conclude that

$$\mu_e^k = \rho, \text{ for all } k \in C.$$

On the other hand, we use the same technique applied in the polyhedron dimension proof 2.3.1 to prove that

$$\begin{aligned} \mu_{e'}^{k'} &= \gamma_1^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} &= \gamma_2^{k',e'}, \text{ for all } k' \in K \text{ and all } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} &= \gamma_3^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}. \end{aligned}$$

We conclude that for each $k' \in K$ and $e' \in E$

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'}, \text{ if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'}, \text{ if } e' \in E_1^{k'}, \\ \rho, \text{ if } k' \in C \text{ and } e' = e, \\ 0, otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s}, \text{if } s \in \{1, ..., w_k - 1\} \\ \qquad\qquad 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum_{k \in C} \rho \alpha_e^k + \gamma Q$.

□

2.5 Symmetry-Breaking Inequalities

We have noticed that several symmetrical solutions may appear given that there exist several feasible solutions that have the same value of the solution (called equivalents solutions), and they can be found by doing some permutations between the slots assigned to some demands without changing the selected paths (routing) while satisfying the C-RSA constraints. There exists several methods to break the symmetry. See, for example, perturbation method proposed by Margot in [101], isomorphism pruning method by Margot et al. in [102] and [103], orbital branching method by Ostrowski et al. in [113] and [114], orbital fixing method by Kaibel et al. in [121], and symmetry-breaking constraints by Kaibel and Pfetsch in [120] which is applied in our study. We aim to introduce breaking-symmetry inequalities to remove the Sub-problems in the enumeration tree that are equivalent due to the equivalency of their associated solutions. To do so, we derive the following inequalities.

Proposition 2.5.1. *Consider a demand k, slot $s \in \{1, ..., \bar{s} - 1\}$. Let s' be a slot in $\{s, ..., \bar{s}\}$*

$$\sum_{s"=s'}^{\min(s'+w_k-1, \bar{s})} z_{s"}^k - \sum_{k' \in K} \sum_{s"=s}^{\min(s+w_{k'}-1, \bar{s})} z_{s"}^{k'} \leq 0. \tag{2.51}$$

This ensures that the slot s' can be assigned to the demand k iff the slot s (which precedes the slot s') is already assigned to at least one demand k' in K. A similar idea was proposed by Mendez-Diaz and Zabala in [105] to break the symmetry for the vertex coloring problem. Note that the inequalities (5.19) are not valid for the polytope $P(G, K, \mathbb{S})$ given that they cut off some feasible regions in the polytope $P(G, K, \mathbb{S})$. In any case, we ensure that there exists at least one optimal solution from our original problem that remains feasible and belongs to the convex hull of non-symmetric solutions of the C-RSA problem.

2.6 Lower Bounds

In this section, we derive some lower bounds for the C-RSA. Let p_k^* denote the minimum-cost path between origin node o_k and destination node d_k for the demand k with total length smaller than the transmission-reach $\bar{l}_k$. We know in advance that the optimal path that will be selected for the demand k in the optimal solution, its total cost is at least equal to the total cost of the minimum-cost path p_k^*. Based on this, we introduce the following inequalities.

Proposition 2.6.1. *Consider a demand $k \in K$. Then, the inequality*

$$\sum_{e \in E} c_e x_e^k \geq \sum_{e \in E(p_k^*)} c_e, \tag{2.52}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It's trivial given that in any feasible solution S in $P(G, K, \mathbb{S})$, the total cost of the path selected to the demand k is greatest than or equal to the total cost of the minimum-cost path p_k^*. □

The inequality (2.52) is then used to derive a lower bounds for the C-RSA as follows.

Proposition 2.6.2. *The inequality*

$$\sum_{k \in K} \sum_{e \in E} c_e x_e^k \geq \sum_{k \in K} \sum_{e \in E(p_k^*)} c_e, \tag{2.53}$$

is valid for $P(G, K, \mathbb{S})$.

Proof. It's trivial given the optimal value is at least equal to the sum of the total cost of minimum-cost path over all the demands in K.

□

The separation problem associated with inequality (2.53) is equivalent to solving the Resource Constrained Shortest Path (RCSP) Problem for each demand k. The RCSP is well known to be a NP-hard problem [46]. To do so, we propose a pseudo-polynomial time algorithm using dynamic programming [47] to compute the minimum-cost path for each demand k while satisfying the transmission-reach constraint. For each demand $k \in K$, we associate to each node $v \in V \setminus V_0^k$ in the graph G a set of labels L^v s.t. each label corresponds to differents paths from th origin node o_k to the node v, and each label p is specified by a cost equals to $\sum_{e \in E(p)} c_e$, and a weight equals to $\sum_{e \in E(p)} l_e$. We denote by T_v the set of labels on node $v \in V$. For each demand k and slot $s \in \{w_k, ..., \bar{s}\}$, the complexity of the algorithm is bounded by $\mathcal{O}(|E \setminus E_0^k| \bar{l}_k)$ [47]. Algorithm 1 summarizes the different steps of the dynamic programming algorithm.

Algorithm 1: Dynamic Programming Algorithm for the Computation of Lower Bounds for the C-RSA

Data: An undirected, loopless, and connected graph $G = (V, E)$, a spectrum $\mathbb{S}$, a demand k
Result: constrained minimum-cost path p_k^* for the demand k
Set $L^{o_k} = \{(0, 0)\}$ and $L^v = \emptyset$ for each node $v \in V \setminus (V_0^k \cup \{o_k\})$;
Set $T^v = \emptyset$ for each node $v \in V \setminus V_0^k$;
STOP= FALSE;
while $\cup_{v \in V}(L_v \setminus T_v) \neq \emptyset$ **do**
  Select a node $i \in V \setminus V_0^k$ and a label $p \in L^i \setminus T^i$ having the smallest value of $\sum_{e \in E(p)} c_e$;
  for *each* $e = ij \in \delta(i) \setminus E_0^k$ *s.t.* $\sum_{e' \in E(p)} l_{e'} + l_e \leq \bar{l}_k$ **do**
    if $j \notin V(p)$ **then**
      Set $p' = p \cup \{e\}$;
      Update the set of label $L^j = L^i \cup \{p'\}$;
    end
  end
  Set $T^i = T^i \cup \{p\}$;
end
We select one label p from the labels L^{d_k} of destination node d_k and set $p^* = p$;
return constrained minimum-cost path p_k^* for the demand k;

2.7 Concluding Remarks

In this chapter, we focused on a complex variant of the Routing and Spectrum Assignment (RSA) problem, called the Constrained-Routing and Spectrum Assignment (C-RSA). We first proposed a new integer linear programming formulation based on the so-called cut formulation for the C-RSA. We investigate the facial structure of the associated polyhedron by showing that some basic inequalities are facet-defining under certain conditions. We further identified several families of valid inequalities to obtain tighter LP bounds. Moreover, we studied the facial structure of these valid inequalities, and shown that are facet defining for the associated polyhedron under certain necessary and sufficient conditions. We end our chapter by introducing some symmetry-breaking inequalities to well manage the so-called equivalents Sub-problems.

Chapter 3

Branch-and-Cut Algorithm for the C-RSA Problem

Based on the theoretical results presented in the chapter (2), we devise a Branch-and-Cut algorithm to solve the C-RSA problem. Our aim is to study the effectiveness of the algorithm, and assess the impact of each valid inequality on the effectiveness of the Branch-and-Cut algorithm. First, we give an overview of the algorithm. Then, we describe the separation procedure used for each valid inequality based on exact algorithms, greedy-algorithms, and heuristics. At the end, we provide a detailed behavior study of the Branch-and-Cut algorithm.

3.1 Branch-and-Cut Algorithm

3.1.1 Description

In what follows, we describe the Branch-and-Cut algorithm. Consider an undirected, loopless, and connected graph $G = (V, E)$, which is specified by a set of nodes V, and a multiset E of links. Each link $e = ij \in E$ is associated with a length $\ell_e \in \mathbb{R}_+$ (in kms), a cost $c_e \in \mathbb{R}_+$ s.t. each link $e \in E$ is divided into $\bar{s} \in \mathbb{N}_+$ slots. Let $\mathbb{S} = \{1, \ldots, \bar{s}\}$ be an optical spectrum of available frequency slots with $\bar{s} \leq 320$, and K be a multiset of demands s.t. each demand $k \in K$ is specified by an origin node $o_k \in V$, a destination node $d_k \in V \setminus \{o_k\}$, a slot-width $w_k \in \mathbb{Z}_+$, and a transmission-reach $\bar{\ell}_k \in \mathbb{R}_+$ (in kms). We first consider a restricted linear problem denoted by LP_0 given by the inequalities (2.3)-(2.5) and (2.7)-(2.10) s.t. the cut inequalities (2.2) and non-overlapping inequalities (2.6) are not included in LP_0. LP_0 is so equivalent to

$$\min \sum_{k \in K} \sum_{e \in E} l_e x_e^k$$

$$\sum_{e \in E} l_e x_e^k \leq \bar{\ell}_k, \forall k \in K,$$

$$x_e^k = 0, \forall k \in K, \forall e \in E_0^k,$$

$$x_e^k = 1, \forall k \in K, \forall e \in E_1^k,$$

$$z_s^k = 0, \forall k \in K, \forall s \in \{1, ..., w_k - 1\},$$

$$\sum_{s=w_k}^{\bar{s}} z_s^k = 1, \forall k \in K,$$

$$0 \leq x_e^k \leq 1, \forall k \in K, \forall e \in E,$$

$$0 \leq z_s^k \leq 1, \forall k \in K, \forall s \in \mathbb{S}.$$

3.1.2 Test of Feasibility

Given an optimal solution $(\bar{x}, \bar{z})$ for the relaxation of LP_0. It is feasible for the C-RSA problem iff $(\bar{x}, \bar{z})$ is integral and it satisfies the cut inequalities (2.2) and non-overlapping inequalities (2.6). Usually, $(\bar{x}, \bar{z})$ does not satisfy the inequalities (2.2) and (2.6). As a result, $(\bar{x}, \bar{z})$ is not feasible for the C-RSA problem. For that, we generate several valid inequalities violated by a solution $(\bar{x}, \bar{z})$ at each iteration of the Branch-and-Cut algorithm. This is known under the name "Separation Problem" which consists in identifying for a given class of valid inequalities the existence of one or more inequalities of this class that are violated by the current solution. We repeat this procedure in each iteration of the algorithm until non violated inequality is identified. As a result, the final solution is optimal for the linear relaxation of our cut formulation. Furthermore, if it is integral, then it is optimal for the C-RSA problem. Otherwise, we create two subproblems called childs by branching on a fractional variable (variable branching rule) or on some constraints using the Ryan & Foster branching rule (constraint branching rule). Based on this, we devise a basic Branch-and-Cut algorithm by combining cutting-plane algorithm based on the separation of the cut inequalities (2.2) and non-overlapping inequalities (2.6), and a Branch-and-Bound algorithm.
On the other hand, to accelerate the Branch-and-Cut algorithm, we already introduced several classes of valid inequalities used to obtain tighter LP bounds. Based on this, and at each iteration in a certain level of the Branch-and-Cut algorithm, one can identify one or more than one violated inequality by the current fractional solution for a given class of valid inequalities. Algorithm 2 summarizes the different steps of Branch-and-Cut algorithm taking into account additional valid inequalities for a given class of valid inequalities.
To do so, we study the separation problem of each valid inequality as follows.

3.1.3 Separation of Non-Overlapping Inequalities

Consider a fractional solution $(\bar{x}, \bar{z})$, and an edge $e \in E$ and a slot $s \in \mathbb{S}$. The separation problem associated with the inequality (2.6) consists in identifying all pairs of demands $k, k' \in K$ s.t.

$$\bar{x}_e^k + \bar{x}_e^{k'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} \bar{z}_{s'}^k + \sum_{s"=s}^{\min(s+w_{k'}-1,\bar{s})} \bar{z}_{s"}^{k'} > 3.$$

To do so, we propose an exact algorithm in $\mathcal{O}(|E| * \bar{s} * |K| * \log(|K|))$ which works as follows. We select each pair of demands $k, k' \in K$ with $x_e^k > 0$, $\sum_{s'=s}^{min(s+w_k-1,\bar{s})} z_{s'}^k > 0$, $\bar{x}_e^{k'} > 0$ and $\sum_{s"=s}^{\min(s+w_{k'}-1,\bar{s})} \bar{z}_{s"}^{k'} > 0$. We then add the inequality (2.6) induced by each selected pair of demands k, k' for the slot s ove edge e to the current LP if it is violated, i.e.,

$$x_e^k + x_e^{k'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s"=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s"}^{k'} \leq 3.$$

Otherwise, we conclude that such inequality does not exist for the current solution $(\bar{x}, \bar{z})$. On the other hand, given that the inequalities (2.5) are taken in format of equations when implementing the B&C algorithm (i.e., $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$ for all $k \in K$). Based on this, and taking into account the non-overlapping inequalities (2.6), we propose a new non-overlapping inequality (3.1) more efficient compared to the ones of (2.6).

Proposition 3.1.1. *Consider an edge e, and a pair of demands $k, k' \in K$ with $e \notin E_0^k \cup E_0^{k'}$.*

Algorithm 2: Branch-and-Cut Algorithm for the C-RSA

Data: An undirected, loopless, and connected graph $G = (V, E)$, a spectrum $\mathbb{S}$, a multi-set K of demands, and a given class of valid inequality

Result: Optimal solution for the C-RSA problem

1 LP$\longleftarrow$ LP_0;
2 Stop= FALSE;
3 **while** *STOP==FALSE* **do**
4   Solve the linear program LP;
5   Let (x^*, z^*) be the optimal solution of LP;
6   **if** *there exist inequalities from the cut inequalities* (2.2)*, non-overlapping inequalities* (2.6)*, and those of the given class that are violated by the current solution* (x^*, z^*) **then**
7   Add them to LP ;
8   **end**
9   **else**
10   STOP = TRUE;
11   **end**
12 **end**
13 Consider the optimal solution (x^*, z^*) of LP ;
14 **if** (x^*, z^*) *is integer for the C-RSA* **then**
15   (x^*, z^*) is an optimal solution for the C-RSA;
16   End of the Branch-and-Cut algorithm ;
17 **end**
18 **else**
19   Create two Sub-problems by branching one some variables or constraints ;
20 **end**
21 **for** *each Sub-problem not yet solved* **do**
22   go to 3 ;
23 **end**
24 **return** the best optimal solution (x^*, z^*) for the Pricer;

Let s be a slot in $\{w_k, ..., \bar{s}\}$. Then, the inequality

$$x_e^k + x_e^{k'} + z_s^k + \sum_{s"=s-w_k+1}^{\min(s+w_{k'}-1,\bar{s})} z_{s"}^{k'} \leq 3, \tag{3.1}$$

is valid for $P'(G,K,\mathbb{S}) = \{(x,z) \in P(G,K,\mathbb{S}) : \sum_{s=w_k}^{\bar{s}} z_s^k = 1$ *for all* $k \in K\}$.

The separation problem associated with the inequality (3.1) consists in identifying for each demand $k \in K$, a demands $k' \in K$ s.t.

$$\bar{x}_e^k + \bar{x}_e^{k'} + \bar{z}_s^k + \sum_{s"=s-w_k+1}^{\min(s+w_{k'}-1,\bar{s})} \bar{z}_{s"}^{k'} > 3.$$

To do so, we propose an exact algorithm in $\mathcal{O}(|E| * \bar{s} * |K| * (|K|-1))$ which works as follows. For each demand k and each slot $s \in \{w_k, ..., \bar{s}\}$ over edge e with $x_e^k > 0$, $z_s^k > 0$, we select each demand $k' \in K$ with $\bar{x}_e^{k'} > 0$ and $\sum_{s"=s-w_k+1}^{\min(s+w_{k'}-1,\bar{s})} \bar{z}_{s"}^{k'} > 0$. We then add the following inequality to the current LP if it is violated, i.e.,

$$x_e^k + x_e^{k'} + z_s^k + \sum_{s"=s-w_k+1}^{\min(s+w_{k'}-1,\bar{s})} z_{s"}^{k'} \leq 3.$$

Otherwise, we conclude that there does not exist an inequality from the non-overlapping inequalities (3.1) violated current solution $(\bar{x}, \bar{z})$. Note that, from an efficiency point of view, the inequalities (3.1) replace the inequalities (2.6) in the B&C algorithm.

3.1.4 Separation of Cut Inequalities

In this section we discuss the separation problem of our cut inequalities (2.2). Its associated separation problem consists in identifying a cut inequalities (2.2) that is violated by a given fractional solution $(\bar{x}, \bar{z})$. For each demand $k \in K$, this can be done in polynomial time [55] as shown in the theorem of Ford and Fulkerson by finding a minimum cut separating the origin-node o_k and destination-node d_k. As a result, this can be done exactly [55] and very effectively in $\mathcal{O}(|V \setminus V_0^k|^2 * \sqrt{|E \setminus E_0^k|})$ using an efficient implementation of minimum cut algorithm based on the so-called preflow push-relabel algorithm of Goldberg and Tarjan [62] to compute maximum flow/minimum cut in the proper graph G_k of demand k by assigning a positif weight $\bar{x}_e^k$ for each edge e in the graph G_k. For that, we use a C++ library proposed by the LEMON GRAPH library [86] which calls the algorithm of Goldberg and Tarjan for the minimum cut computation. Based on this, we conclude that the separation of the cut inequalities (2.2) can be done in $\mathcal{O}(|V|^2 * \sqrt{|E|} * |K|)$ in the worst case.

3.1.5 Separation of Edge-Slot-Assignment Inequalities

Consider a fractional solution $(\bar{x}, \bar{z})$, and an edge $e \in E$ and a slot $s \in \mathbb{S}$. The separation problem associated with the inequality (2.25) consists in identifying a subset of demands $\tilde{K}^* \subset K$ s.t.

$$\sum_{k \in \tilde{K}^*} \bar{x}_e^k + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} \bar{z}_{s'}^k > |\tilde{K}^*| + 1.$$

To do so, we propose an exact algorithm in $\mathcal{O}(|K| * |E| * \bar{s})$ which works as follows. The main idea is to iteratively add each demand $k \in K$ to $\tilde{K}^*$ iff $x_e^k > 0$ and $\sum_{s'=s}^{min(s+w_k-1,\bar{s})} z_{s'}^k > 0$. We then add the inequality (2.25) induced by $\tilde{K}^*$ to the current LP if it is violated, i.e.,

$$\sum_{k \in \tilde{K}^*} x_e^k + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq |\tilde{K}^*| + 1.$$

Otherwise, we conclude that such inequality does not exist for the current solution $(\bar{x}, \bar{z})$. Moreover, if such violated inequality is identified, it can be easily lifted introducing the inequality (2.27) induced by $\tilde{K}^*$ and a subset of demands $K_e \setminus \tilde{K}^*$ as follows

$$\sum_{k \in \tilde{K}^*} x_e^k + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{k' \in K_e \setminus \tilde{K}^*} \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} \leq |\tilde{K}^*| + 1.$$

On the other hand, given that the inequalities (2.5) are taken in format of equations when implementing the B&C algorithm. Based on this, and taking into account the non-overlapping inequalities (2.6), we define another conflict graph totally different compared with the conflict graphs introduced previously.

Definition 3.1.1. *Let $\tilde{G}_S^e$ be a conflict graph defined as follows. For each slot $s \in \{w_k, ..., \bar{s}\}$ and demand $k \in K$ with $e \notin E_0^k$, consider a node $v_{k,s}$ in $\tilde{G}_S^e$. Two nodes $v_{k,s}$ and $v_{k',s'}$ are linked by an edge in $\tilde{G}_S^e$ if and only if*

a) $k = k'$,

b) *or* $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ *if* $k \neq k'$ *and* $(k, k') \notin K_c^e$.

The conflict graph $\tilde{G}_S^e$ is not a perfect graph given that some nodes $v_{k,s}$ and $v_{k',s'}$ are linked even if the $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$, i.e., when $k = k'$.

Proposition 3.1.2. *Consider an edge $e \in E$. Let C be a clique in the conflict graph $\tilde{G}_S^e$ with $|C| \geq 3$, and $\sum_{k \in C} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus C} w_{k'}$. Then, the inequality*

$$\sum_{v_{k,s} \in C} (x_e^k + z_s^k) \leq |C| + 1, \tag{3.2}$$

is valid for $P'(G, K, \mathbb{S}) = \{(x, z) \in P(G, K, \mathbb{S}) : \sum_{s=w_k}^{\bar{s}} z_s^k = 1 \text{ for all } k \in K\}$.

Proof. It is trivial given the definition of a clique set in the conflict graph $\tilde{G}_S^e$ s.t. for each two linked nodes $v_{k,s}$ and $v_{k',s'}$ in $\tilde{G}_S^e$, we have

$$x_e^k + x_e^{k'} + z_s^k + z_{s'}^{k'} \leq 3.$$

This can be generalized for a triplet of linked nodes $v_{k,s}$ and $v_{k',s'}$ and $v_{k'',s''}$ with $w_k + w_{k'} + w_{k"} \leq \bar{s} - \sum_{\tilde{k} \in K_e \setminus \{k,k',k"\}} w_{\tilde{k}}$, such that for each linked nodes $(v_{k,s}, v_{k',s'})$ and $(v_{k,s}, v_{k'',s''})$ and $(v_{k',s'}, v_{k'',s''})$, we have

$$x_e^k + x_e^{k'} + z_s^k + z_{s'}^{k'} \leq 3,$$
$$x_e^k + x_e^{k"} + z_s^k + z_{s"}^{k"} \leq 3,$$
$$x_e^{k'} + x_e^{k"} + z_{s'}^{k'} + z_{s"}^{k"} \leq 3.$$

By adding the three previous inequalities, we get the following inequality using the chvatal gomory procedure

$$2x_e^k + 2x_e^{k'} + 2x_e^{k"} + 2z_s^k + 2z_{s'}^{k'} + 2z_{s"}^{k"} \leq 9$$
$$\Rightarrow x_e^k + x_e^{k'} + x_e^{k"} + z_s^k + z_{s'}^{k'} + z_{s"}^{k"} \leq 4 \text{ given that } \left\lfloor \frac{9}{2} \right\rfloor = 4.$$

This can be generalized for each clique C with $|C| \geq 4$ while showing that the inequality (3.2) can be seen as Chvatal-Gomory cuts. For that, and using the Chvatal-Gomory and recurrence procedures, we get that for all $C' \subset C$ with $|C'| = |C| - 1$ and $|C'| \geq 3$

$$\sum_{v_{k,s} \in C'} x_e^k + z_s^k \leq |C'| + 1.$$

By adding the previous inequalities for all $C' \subset C$ with $|C'| = |C| - 1$, and doing then some simplification, we get at the end that

$$\sum_{v_{k,s} \in C} x_e^k \ + \ z_s^k \quad \leq \quad \left\lfloor |C| + \frac{|C|}{|C|-1} \right\rfloor \quad \Rightarrow \quad \sum_{v_{k,s} \in C} x_e^k \ + \ z_s^k \quad \leq \quad |C| \ + \ 1$$

given that $\left\lfloor \dfrac{|C|}{|C|-1} \right\rfloor = 1$. We conclude that the inequality (3.2) is valid for $P(G, K, \mathbb{S})$. $\square$

Remark 3.1.1. *The inequality* (3.1) *is a particular case of inequality* (3.2) *for a clique* $C = \{v_{k,s}\} \cup \{v_{k',s'} \in \tilde{G}_c^e \text{ s.t. } \{s' - w_k' + 1, ..., s'\} \cap \{s - w_k + 1, ..., s\} \neq \emptyset\}$.

Remark 3.1.2. *The inequality* (3.2) *associated with a clique C over edge e, it is dominated by the inequality* (2.32) *associated with an interval $I = [s_i, s_j]$ and the subset of demands $\tilde{K}$ over edge e iff*

a) $\tilde{s} \in \{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, s'\}$ *for each pair of nodes* $(v_{k,s}, v_{k',s'})$ *in* C,

b) and $[\min_{v_{k,s} \in C}(s - w_k + 1), \max_{v_{k,s} \in C} s] \subset I$.

Proof. Consider an edge e and an interval of contiguous slots $I = [s_i, s_j] \subseteq [1, \bar{s}]$. Let C be a clique in the conflict graph $\tilde{G}_S^e$, and $\tilde{K} = \{k \in K \text{ s.t. } v_{k,s} \in C\}$ be a subset of demands in K with $\tilde{K}$ is a clique in the conflict graph $\tilde{G}_I^e$ for the interval $I = [s_i, s_j]$.

Necessity: First, assume that

a) $\tilde{s} \in \{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, s'\}$ for each pair of nodes $(v_{k,s}, v_{k',s'})$ in C,

b) and $[\min_{v_{k,s} \in C}(s - w_k + 1), \max_{v_{k,s} \in C} s] \subset I$.

Given that $s - w_k + 1 \geq \min_{v_{k',s'} \in C}(s' - w_{k'} + 1)$ and $s \leq \max_{v_{k',s'} \in C} s'$ for each $v_{k,s} \in C$, and that $|\{s - w_k + 1, ..., s\}| = w_k$ for each $v_{k,s} \in C$, it follows that $s \in \{s_i + w_k - 1, ..., s_j\}$ for each $v_{k,s} \in C$ of demand $k \in \tilde{K}$. As a result, we get that

$$\sum_{k \in \tilde{K}} x_e^k + \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k = \sum_{k \in \tilde{K}} x_e^k + \sum_{k \in \tilde{K}} z_s^k + \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\} \setminus \{s\}} z_{s'}^k \quad (3.3)$$
$$\Longrightarrow \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k = \sum_{k \in \tilde{K}} z_s^k + \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\} \setminus \{s\}} z_{s'}^k. \quad (3.4)$$

Taking into account that $\tilde{K} = \{k \in K \text{ s.t. } v_{k,s} \in C\}$, this means that

$$\sum_{k \in \tilde{K}} z_s^k = \sum_{v_{k,s} \in C} z_s^k.$$

This implies that

$$\sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k = \sum_{v_{k,s} \in C} z_s^k + \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\} \setminus \{s\}} z_{s'}^k$$
$$\implies \sum_{v_{k,s} \in C} z_s^k \preceq \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k.$$

Given that the demands are independants, it follows that

$$z_s^k \preceq \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k \text{ for each } v_{k,s} \in C.$$

Hence, the inequality (3.2) is dominated by the inequality (2.32).
Sufficiency: Assume that the inequality (3.2) is dominated by the inequality (2.32). It follows that

$$\sum_{v_{k,s} \in C} x_e^k + z_s^k \preceq \sum_{k \in \tilde{K}} x_e^k + \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k \implies \sum_{v_{k,s} \in C} z_s^k \preceq \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k$$
$$\implies \sum_{k \in \tilde{K}} z_s^k \preceq \sum_{k \in \tilde{K}} \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k \implies z_s^k \preceq \sum_{s' \in \{s_i + w_k - 1, ..., s_j\}} z_{s'}^k \text{ for each } k \in \tilde{K}$$
$$\implies s \in \{s_i + w_k - 1, ..., s_j\} \text{ for each } k \in \tilde{K} \implies s \in \{s_i + w_k - 1, ..., s_j\} \text{ for each node } v_{k,s} \in C$$
$$\implies s - w_k + 1 \in I \text{ for each node } v_{k,s} \in C \implies \min_{v_{k,s} \in C} (s - w_k + 1) \in I$$
$$\text{and } \max_{v_{k,s} \in C} s \in I \text{ for each node } v_{k,s} \in C \implies [\min_{v_{k,s} \in C} (s - w_k + 1), \max_{v_{k,s} \in C} s] \subseteq I.$$

Furthermore, and given that $w_k + w_{k'} > |I|$ for each pair of demands $k, k' \in \tilde{K}$, it follows that $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s\} \neq \emptyset$ for each $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ of each pair of demands $k, k' \in \tilde{K}$. Hence, $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s\} \neq \emptyset$ for each pair $(v_{k,s}, v_{k',s'}) \in C$ since $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. We conclude at the end that

a) $\tilde{s} \in \{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, s'\}$ for each pair of nodes $(v_{k,s}, v_{k',s'})$ in C,
b) and $[\min_{v_{k,s} \in C} (s - w_k + 1), \max_{v_{k,s} \in C} s] \subset I$,

which ends the proof. □

3.1.6 Separation of Edge-Slot-Assignment-Clique Inequalities

Consider an edge $e \in E$, and a fractional solution $(\bar{x}, \bar{z})$. The separation algorithm for the inequality (3.2) consists in identifying a maximal clique C^* in the conflict graph $\tilde{G}_S^e$ s.t.

$$\sum_{v_{k,s} \in C^*} \bar{x}_e^k + \bar{z}_s^k > |C| + 1.$$

To do this, we use the greedy algorithm introduced by Nemhauser and Sigismondi in [110] to identify a maximal clique C^* in the conflict graph $\tilde{G}_S^e$ given that computing a maximal clique in such a graph is also NP-hard problem [123]. Based on this, we first assign a positive weight $\bar{z}_s^k * \bar{x}_e^k$ to each node $v_{k,s}$ in the conflict graph $\tilde{G}_S^e$. We then select a node $v_{k,s}$ in the conflict graph $\tilde{G}_S^e$ having the largest weight compared with the other nodes in $\tilde{G}_S^e$, and set $C^* = \{v_{k,s}\}$. After that, we iteratively add each node $v_{k',s'}$ to the current C^* if it is linked with all the nodes $v_{k,s}$ already assigned to the current clique C^* and $\bar{z}_{s'}^{k'} > 0$ and $\bar{x}_e^{k'} > 0$. At the end, we add the inequality (3.2) induced by the clique C^* for edge e to the current LP if it is violated, i.e., we add the following inequality

$$\sum_{v_{k,s} \in C^*} x_e^k + z_s^k \leq |C| + 1.$$

Furthermore, it can be lifted by identifying a maximal clique N^* s.t. each $v_{k',s'} \in N^*$ is linked with all the nodes $v_{k,s} \in C^* \cup (N^* \setminus \{v_{k',s'}\})$ in $\tilde{G}_S^e$. For that, we use also the greedy algorithm introduced by Nemhauser and Sigismondi in [110] to identify the clique N^* as follows. We first set $N^* = \{v_{k',s'}\}$ with $v_{k',s'} \notin C^*$ a node in $\tilde{G}_S^e$ having the largest value of node-degree (i.e., $|\delta(v_{k',s'})|$) in $\tilde{G}_S^e$ and $v_{k',s'}$ is linked with all the nodes $v_{k,s} \in C^*$ in $\tilde{G}_S^e$ and $k' \in K_e$. Afterwards, we iteratively add each node $v_{k",s"} \notin C^* \cup N^*$ to the current N^* if it is linked in $\tilde{G}_S^e$ with all the nodes already assigned to C^* and N^* and $k" \in K_e$. At the end, we add the inequality (3.2) induced by the clique $C^* \cup N^*$ to the current LP, i.e.,

$$\sum_{v_{k,s} \in C^*} (x_e^k + z_s^k) + \sum_{v_{k',s'} \in N^*} z_{s'}^{k'} \leq 1.$$

3.1.7 Separation of Edge-Interval-Capacity-Cover Inequalities

Let's discuss the separation problem of the inequality (2.30). Given a fractional solution $(\bar{x}, \bar{z})$, and an edge $e \in E$. We first construct a set of intervals of contiguous slots $I \in I_e$ s.t. each interval of contiguous slots I_e is identified by generating two slots s_i and s_j randomly in $\mathbb{S}$ with $s_j \geq s_i + 2\max_{k \in K \setminus \bar{K}_e} w_k$. Consider now an interval of contiguous slots $I = [s_i, s_j] \in I_e$ over an edge e. The separation problem associated with the inequality (2.30) is NP-hard [125] given that it consists in identifying a cover $\tilde{K}^*$ for the interval $I = [s_i, s_j]$ over the edge e, s.t.

$$\sum_{k \in \tilde{K}^*} \bar{x}_e^k + \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > 2|\tilde{K}^*| - 1.$$

For that, we use a greedy algorithm introduced by Nemhauser and Sigismondi in [110] as follows. We first select a demand $k \in K$ having the largest number of requested slot w_k with $\bar{x}_e^k > 0$ and $\sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > 0$, and then set $\tilde{K}^*$ to $\tilde{K}^* = \{k\}$. After that, we iteratively add each demand $k' \in K \setminus \tilde{K}^*$ to $\tilde{K}^*$ with $\bar{x}_e^{k'} > 0$ and $\sum_{s'=s_i+w_{k'}-1}^{s_j} \bar{z}_{s'}^{k'} > 0$, until a cover $\tilde{K}^*$ is obtained for the interval I over the edge e with $\sum_{k \in \tilde{K}^*} w_k > |I|$. We further derive a minimal cover from the cover $\tilde{K}^*$ by deleting each demand $k \in \tilde{K}^*$ if $\sum_{k' \in \tilde{K}^* \setminus \{k\}} w_{k'} \leq |I|$. We then add the inequality (2.30) induced by the minimal cover $\tilde{K}^*$ for the interval I and edge e if it is violated, i.e., we add the following valid inequality to the current LP

$$\sum_{k \in \tilde{K}^*} x_e^k + \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k \leq 2|\tilde{K}^*| - 1.$$

Furthermore, the inequality (2.30) induced by the minimal cover $\tilde{K}^*$ can be lifted in polynomal time $\mathcal{O}(K_e \setminus \tilde{K})$ by introducing an extended cover inequality (2.31) as follows

$$\sum_{k\in\tilde{K}^*} x_e^k + \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + \sum_{k'\in\tilde{K}_e^*}\sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq 2|\tilde{K}^*| - 1,$$

where $w_{k'} \geq w_k$ for each $k \in \tilde{K}^*$ and each $k' \in \tilde{K}_e^*$.

3.1.8 Separation of Edge-Interval-Clique Inequalities

The separation problem related to the inequality (2.32) is NP-hard [116][123] given that it consists in identifying a maximal clique C^* in the conflict graph $\tilde{G}_I^e$ for a given edge e and a given interval $I = [s_i, s_j]$ s.t.

$$\sum_{k\in C^*} \bar{x}_e^k + \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > |C^*| + 1,$$

for a given fractional solution $(\bar{x}, \bar{z})$ of the current LP.
We start our procedure of separation by constructing a set of intervals of contiguous slots $I = [s_i, s_j] \in I_e$ for a given edge $e \in E$ s.t. each interval of contiguous slots $I = [s_i, s_j] \in I_e$ is identified for each slot $s_i \in \mathbb{S}$ and slot s_j with $s_j \in \{s_i + \max_{k\in K\setminus\bar{K}_e} w_k, ..., \min(\bar{s}, s_i + 2\max_{k\in K\setminus\bar{K}_e} w_k)\}$. Consider now an interval of contiguous slots $I = [s_i, s_j] \in I_e$ over an edge e, and its associated conflict graph $\tilde{G}_I^e$. We then use a greedy algorithm introduced by Nemhauser and Sigismondi in [110] to identify a maximal clique in the conflict graph $\tilde{G}_I^e$ as follows. We first associate a positive weight for each node v_k in $\tilde{G}_I^e$ equals to $\bar{x}_e^k * \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k$. We then set $C^* = \{k\}$ s.t. k is a demand in K having the largest number of slots w_k and weight $\bar{x}_e^k * \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k$. After that, we iteratively add each demand k' having $\bar{x}_e^{k'} > 0$ and $\sum_{s'=s_i+w_{k'}-1}^{s_j} \bar{z}_{s'}^{k'} > 0$ s.t. its corresponding node $v_{k'}$ is linked with all the nodes v_k with k already assigned to the current C^*. After that, we check if the inequality (2.32) induced by the maximal clique C^* for the interval I and edge e is violated or not. If so, we add the inequality (2.32) induced by the maximal clique C^* to the current LP, i.e.,

$$\sum_{k\in C^*} x_e^k + \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k \leq |C^*| + 1.$$

One can strengthen this additional inequality by adding the inequality (2.33) induced by the maximal clique C^* and $C_e^* \subset K_e \setminus C^*$, i.e.,

$$\sum_{k\in C^*} x_e^k + \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + \sum_{k'\in C_e^*}\sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq |C^*| + 1,$$

s.t.

a) $w_{k'} + w_k \geq |I| + 1$ for each $k \in C^*$ and $k' \in C_e^*$,

b) $w_{k'} + w_{k"} \geq |I| + 1$ for each $k' \in C_e^*$ and $k" \in C_e^*$,

c) $w_{k'} \leq |I|$ and $2w_{k'} \geq |I| + 1$ for each $k' \in C_e^*$.

3.1.9 Separation of Interval-Clique Inequalities

Given a fractional solution $(\bar{x}, \bar{z})$, and an interval of contiguous slots $I = [si, s_j]$. Our separation algorithm for the inequality (2.35) consists in identifying a maximal clique C^* in the conflict graph $\tilde{G}_I^E$ s.t.

$$\sum_{k \in C^*} \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > 1.$$

As result, its associated separation problem is NP-hard given that computing a maximal clique in a given graph is known to be a NP-hard problem [123]. For that, we also use the greedy algorithm introduced by Nemhauser and Sigismondi in [110] to identify a maximal clique in the conflict graph $\tilde{G}_I^E$ as follows. We first generate a set of intervals of contiguous slots denoted by I_E s.t. each interval of contiguous slots $I = [s_i, s_j] \in I_E$ is given for each slot $s_i \in \mathbb{S}$ and slot s_j with $s_j \in \{s_i + \max\limits_{\substack{k \in K, \\ |E_1^k| \geq 1}} w_k, ..., \min(\bar{s}, s_i + 2 \max\limits_{\substack{k \in K, \\ |E_1^k| \geq 1}} w_k)\}$. We then consider an interval of contiguous slots $I = [s_i, s_j] \in I_E$ and its associated conflict graph $\tilde{G}_I^E$. We associate a positive weight $\sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k$ for each node v_k in $\tilde{G}_I^E$. We select a demand k s.t. k is a demand in K having the largest number of slots w_k and weight $\sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k$, and then set $C^* = \{k\}$. After that, we iteratively add each demand k' having $\sum_{s'=s_i+w_{k'}-1}^{s_j} \bar{z}_{s'}^{k'} > 0$ s.t. its corresponding node $v_{k'}$ is linked with all the nodes v_k with $k \in C^*$. At the end, we add the inequality (2.35) induced by the maximal clique C^* if it is violated, i.e., by adding the following inequality to the current LP

$$\sum_{k \in C^*} \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k \leq 1.$$

Moreover, this additional inequality can be strengthened as follows

$$\sum_{k \in C^*} \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + \sum_{k' \in C_e^*} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq 1,$$

where $C_E^* \subset K \setminus C^*$ s.t.

a) $w_{k'} + w_k \geq |I| + 1$ and $E_1^k \cap E_1^{k'} \neq \emptyset$ for each $k \in C^*$ and $k' \in C_E^*$,

b) $w_{k'} + w_{k"} \geq |I| + 1$ and $E_1^{k'} \cap E_1^{k"} \neq \emptyset$ for each $k' \in C_E^*$ and $k" \in C_E^*$,

c) $w_{k'} \leq |I|$ and $2w_{k'} \geq |I| + 1$ for each $k' \in C_E^*$.

3.1.10 Separation of Interval-Odd-Hole Inequalities

For the inequality (2.36), we propose a separation algorithm that consists in identifying an odd-hole H^* in the conflict graph $\tilde{G}_I^E$ for a given Interval I and a fractional solution $(\bar{x}, \bar{z})$ s.t.

$$\sum_{k \in H^*} \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > \frac{|H^*| - 1}{2}.$$

This can be done in polynomial time as shown by Rebennack et al. in [139] and [140]. Based on this, we use the exact algorithm proposed by the same authors which consists of finding a minimum weighted odd-cycle in a graph. For that, we should first generate a set of intervals of contiguous slots I_E as we did before in the section 3.1.9. We then consider a conflict graph $\tilde{G}_I^E$ associated with a given interval of contiguous slots $I \in I_E$. We construct an auxiliary conflict graph $\tilde{G}_I'^E$ which can be seen as a bipartite graph by duplicating each node v_k in $\tilde{G}_I^E$ (i.e., v_k and v_k') and each two nodes are linked in $\tilde{G}_I'^E$ if their original nodes are linked in $\tilde{G}_I^E$. We assign to each link (v_a, v_b) in $\tilde{G}_I'^E$ a weight equals to $\frac{1-\sum_{s'=s_i+w_a-1}^{s_j} \bar{z}_{s'}^a - \sum_{s'=s_i+w_b-1}^{s_j} \bar{z}_{s'}^b}{2}$. We then compute for each node v_k in $\tilde{G}_I^E$, the shortest path between v_k and its copy in the auxiliary conflict graph $\tilde{G}_I'^E$ denoted by $p_{v_k,v_k'}$. After that, we check if the total sum of weight over edges belong this path is smallest than $\frac{1}{2}$,

$$\sum_{(v_a,v_b)\in E(p_{v_k,v_k'})} \frac{1-\sum_{s'=s_i+w_a-1}^{s_j} \bar{z}_{s'}^a - \sum_{s'=s_i+w_b-1}^{s_j} \bar{z}_{s'}^b}{2} < \frac{1}{2}.$$

If so, the odd-hole H^* is composed by all the original nodes of nodes belong the computed shortest path $p_{v_k,v_k'}$, i.e., $V(p_{v_k,v_k'}) \setminus \{v_k'\}$. We then add the inequality (2.36) induced by the odd-hole H^* to the current LP, i.e.,

$$\sum_{k\in H^*} \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k \leq \frac{|H^*|-1}{2}.$$

It can be lifted using the greedy algorithm introduced by Nemhauser and Sigismondi in [110] to identify a maximal clique C^* in the conflict graph $\tilde{G}_I^E$ s.t.

a) $w_{k'} + w_k \geq |I| + 1$ and $E_1^k \cap E_1^{k'} \neq \emptyset$ for each $k \in H^*$ and $k' \in C^*$,

b) $w_{k'} + w_{k"} \geq |I| + 1$ and $E_1^{k'} \cap E_1^{k"} \neq \emptyset$ for each $k' \in C^*$ and $k" \in C^*$,

c) $w_{k'} \leq |I|$ and $2w_{k'} \geq |I| + 1$ for each $k' \in C^*$.

For that, we first assign a positive weight equals to the number of slots request $w_{k'}$ by the demand k' for each node $v_{k'}$ linked with all the nodes $v_k \in H^*$ in the conflict graph $\tilde{G}_I^E$. We then select the node $v_{k'}$ linked with all the nodes $v_k \in H^*$ in the conflict graph $\tilde{G}_I^E$ having the largest weight, and set C^* to $\{k'\}$. After that, we iteratively add each demand $k"$ to the current clique C^* if its associated node $v_{k"}$ is linked with all the nodes $v_k \in H^*$ and nodes $v_{k'} \in C^*$. As a result, we add the inequality (2.37) induced by the odd-hole H^* and clique C^* to the current LP, i.e.,

$$\sum_{k\in H^*} \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + \frac{|H^*|-1}{2} \sum_{k'\in C^*} \sum_{s"=s_i+w_{k'}-1}^{s_j} z_{s"}^{k'} \leq \frac{|H^*|-1}{2}.$$

3.1.11 Separation of Slot-Assignment-Clique Inequalities

Now, we describe the separation algorithm for the inequality (2.38). It consists in identifying a maximal clique C^* in the conflict graph $\tilde{G}_S^E$ s.t.

$$\sum_{v_{k,s}\in C^*} \bar{z}_s^k > 1,$$

for a given fractional solution $(\bar{x}, \bar{z})$ of the current LP.
To do so, we use the greedy algorithm introduced by Nemhauser and Sigismondi in [110] to

identify a maximal clique C^* in the conflict graph $\tilde{G}_S^E$ given that computing a maximal clique in such a graph is also NP-hard problem [123]. Based on this, we first assign a positive weight $\bar{z}_s^k$ to each node $v_{k,s}$ in the conflict graph $\tilde{G}_S^E$. We then select a node $v_{k,s}$ in the conflict graph $\tilde{G}_S^E$ having the largest weight compared with the other nodes in $\tilde{G}_S^E$, and set $C^* = \{v_{k,s}\}$. After that, we iteratively add each node $v_{k',s'}$ to the current C^* if it is linked with all the nodes $v_{k,s}$ already assigned to the current clique C^* and $\bar{z}_{s'}^{k'} > 0$. At the end, we add the inequality (2.38) induced by the clique C^* to the current LP if it is violated, i.e., we add the following inequality

$$\sum_{v_{k,s} \in C^*} z_s^k \leq 1.$$

Furthermore, it can be lifted by identifying a maximal clique N^* s.t. each $v_{k',s'} \in N^*$ is linked with all the nodes $v_{k,s} \in C^* \cup (N^* \setminus \{v_{k',s'}\})$ in $\tilde{G}_S^E$. For that, we use also the greedy algorithm introduced by Nemhauser and Sigismondi in [110] to identify the clique N^* as follows. We first set $N^* = \{v_{k',s'}\}$ with $v_{k',s'} \notin C^*$ a node in $\tilde{G}_S^E$ having the largest value of node-degree (i.e., $|\delta(v_{k',s'})|$) in $\tilde{G}_S^E$ and $v_{k',s'}$ is linked with all the nodes $v_{k,s} \in C^*$ in $\tilde{G}_S^E$. Afterwards, we iteratively add each node $v_{k',s'} \notin C^* \cup N^*$ to the current N^* if it is linked in $\tilde{G}_S^E$ with all the nodes already assigned to C^* and N^*. At the end, we add the inequality (2.38) induced by the clique $C^* \cup N^*$ to the current LP, i.e.,

$$\sum_{v_{k,s} \in C^*} z_s^k + \sum_{v_{k',s'} \in N^*} z_{s'}^{k'} \leq 1.$$

3.1.12 Separation of Slot-Assignment-Odd-Hole Inequalities

The separation algorithm of the inequality (2.40) can be performed by identifying an odd-hole H^* in the conflict graph $\tilde{G}_S^E$ for a given fractional solution $(\bar{x}, \bar{z})$ s.t.

$$\sum_{v_{k,s} \in H^*} \bar{z}_s^k > \frac{|H^*| - 1}{2}.$$

This can be done in polynomial time as shown by Rebennack et al. in [139] and [140] by finding a minimum weighted odd-cycle in the conflict graph $\tilde{G}_S^E$. To do so, we first construct an auxiliary conflict graph $\tilde{G}_S'^E$ which can be seen also as a bipartite graph by duplicating each node $v_{k,s}$ in $\tilde{G}_S^E$ (i.e., $v_{k,s}$ and $v'_{k,s}$) s.t. each two nodes are linked in $\tilde{G}_S'^E$ if their original nodes are linked in $\tilde{G}_S^E$. We assign to each link $(\tilde{v}_{k,s}, \tilde{v}_{k',s'})$ in $\tilde{G}_S'^E$ a weight equals to $\frac{1-\bar{z}_s^k-\bar{z}_{s'}^{k'}}{2}$. We then compute for each node $v_{k,s}$ in $\tilde{G}_S^E$, the shortest path between $v_{k,s}$ and its copy $v'_{k,s}$ in the auxiliary conflict graph $\tilde{G}_S'^E$ denoted by $p_{v_{k,s},v'_{k,s}}$. After that, we check if the total sum of weight over edges belonging to this path is smaller than $\frac{1}{2}$. If so, the odd-hole H^* is composed by all the original nodes of nodes belong the computed shortest path $p_{v_{k,s},v'_{k,s}}$, i.e., $V(p_{v_{k,s},v'_{k,s}}) \setminus \{v'_{k,s}\}$. As a result, the following inequality (2.40) induced by the odd-hole H^*

$$\sum_{v_{k,s} \in H^*} z_s^k \leq \frac{|H^*| - 1}{2},$$

should be added to the current LP. Moreover, one can strengthen the inequality (2.40) induced by the odd-hole H^* using the greedy algorithm introduced by Nemhauser and Sigismondi in [110] to identify a maximal clique C^* in the conflict graph $\tilde{G}_S^E$ s.t. each node $v_{k',s'} \in C^*$ should have a link with all the nodes $v_{k,s} \in H^*$, and all the nodes $v_{k",s"} \in C^* \setminus \{v_{k',s'}\}$ in the conflict graph $\tilde{G}_S^E$. For that, we first assign a node $v_{k',s'} \notin H^*$ to the clique C^* (i.e., $C^* = \{v_{k',s'}\}$) s.t. $v_{k',s'}$ has the largest value of node-degree (i.e., $|\delta(v_{k',s'})|$) in $\tilde{G}_S^E$ and $v_{k',s'}$ is linked with all the nodes $v_{k,s} \in H^*$ in $\tilde{G}_S^E$. After that, we iteratively add each node $v_{k',s'} \notin H^* \cup C^*$ to the current clique C^* if it is linked in $\tilde{G}_S^E$ with all the nodes already assigned to the odd-hole H^* and the clique C^*. We then add the inequality (2.42) induced by the odd-hole H^* and clique C^*

$$\sum_{v_{k,s} \in H^*} z_s^k + \frac{|H^*|-1}{2} \sum_{v_{k',s'} \in C^*} z_{s'}^{k'} \leq \frac{|H^*|-1}{2},$$

3.1.13 Separation of Incompatibility-Clique Inequalities

Consider now the inequality (2.43), and a fractional solution $(\bar{x}, \bar{z})$. Its associated separation algorithm consists in identifying a maximal clique C^* in the conflict graph $\tilde{G}_E^K$ s.t.

$$\sum_{v_{k,e} \in C^*} \bar{x}_e^k > 1.$$

The separation problem related to this inequality is NP-hard given that computing a maximal clique in the conflict graph $\tilde{G}_E^K$ is NP-hard problem [123]. For that, we also use the greedy algorithm introduced by Nemhauser and Sigismondi in [110] to identify a maximal clique in the conflict graph $\tilde{G}_E^K$ taking into account the fractional solution $(\bar{x}, \bar{z})$ as follows. We first assign a positive weight $\bar{x}_e^k$ to each node $v_{k,e}$ in the conflict graph $\tilde{G}_E^K$. We then select a node $v_{k,e}$ in the conflict graph $\tilde{G}_E^K$ having the largest weight $\bar{x}_e^k$, and set $C^* = \{v_{k,e}\}$. After that, we iteratively add each node $v_{k',e'}$ to the current C^* if it is linked with all the nodes $v_{k,e} \in C^*$ and $\bar{x}_{e'}^{k'} > 0$. At the end, the following inequality (2.43) induced by the clique C^*

$$\sum_{v_{k,e} \in C^*} x_e^k \leq 1,$$

should be added to the current LP if it is violated. Furthermore, one can strengthen the additional inequality (2.43) by identifying a maximal clique N^* s.t. each $v_{k',e'} \in N^*$ is linked with all the nodes $v_{k,e} \in C^* \cup (N^* \setminus \{v_{k',e'}\})$ in $\tilde{G}_E^K$. For that, we use also the greedy algorithm introduced by Nemhauser and Sigismondi in [110] to identify the clique N^* as follows. We first set $N^* = \{v_{k',e'}\}$ with $v_{k',e'} \notin C^*$ a node in $\tilde{G}_E^K$ having the largest degree $|\delta(v_{k',e'})|$ in $\tilde{G}_E^K$ and should be also linked with all the nodes $v_{k,e} \in C^*$ in $\tilde{G}_E^K$. We then iteratively add each node $v_{k',e'} \notin C^* \cup N^*$ to the current N^* if it is linked in $\tilde{G}_E^K$ with all the nodes already assigned to C^* and N^*. At the end, we add the inequality (2.43) induced by the clique $C^* \cup N^*$ to the current LP, i.e.,

$$\sum_{v_{k,e} \in C^*} x_e^k + \sum_{v_{k',e'} \in N^*} x_{e'}^{k'} \leq 1.$$

3.1.14 Separation of Incompatibility-Odd-Hole Inequalities

The separation algorithm related to the inequality (2.44) can be done in polynomial time by finding a minimum weighted odd-cycle in the conflict graph $\tilde{G}_E^K$ as shown by Rebennack et al. in [139] and [140]. For that, our aims is to identify an odd-hole H^* in the conflict graph $\tilde{G}_E^K$ s.t.

$$\sum_{v_{k,e} \in H^*} \bar{x}_e^k > \frac{|H^*|-1}{2},$$

for a given fractional solution $(\bar{x}, \bar{z})$ of the current LP.
We start its procedure of separation by constructing an auxiliary conflict graph $\tilde{G}'^{K}_{E}$ by duplicating each node $v_{k,e}$ in $\tilde{G}^{K}_{E}$ (i.e., $v_{k,e}$ and $v'_{k,e}$) s.t. each two nodes are linked in $\tilde{G}'^{K}_{E}$ if their original nodes are linked in $\tilde{G}^{K}_{E}$. We assign to each link $(\tilde{v}_{k,e}, \tilde{v}_{k',e'})$ in $\tilde{G}'^{K}_{E}$ a weight $\frac{1-\bar{x}^{k}_{e}-\bar{x}^{k'}_{e'}}{2}$. After that, we compute for each node $v_{k,e}$ in $\tilde{G}^{K}_{E}$, the shortest path between $v_{k,e}$ and its copy $v'_{k,e}$. We denote this shortest path by $p_{v_{k,e},v'_{k,e}}$. Note that if the total sum of weight over edges belonging to this path is smaller than $\frac{1}{2}$, this means that there exists odd-hole H^* composed by all the original nodes of nodes belong the computed shortest path $p_{v_{k,e},v'_{k,s}}$, i.e., $V(p_{v_{k,e},v'_{k,s}}) \setminus \{v'_{k,s}\}$, s.t. its associated inequality (2.44) is violated by the current fractional solution $(\bar{x}, \bar{z})$ to the current LP. As a result, we add following inequality (2.44) induced by the odd-hole H^*

$$\sum_{v_{k,e} \in H^*} x^{k}_{e} \leq \frac{|H^*| - 1}{2}.$$

Moreover, the inequality (2.44) induced by the odd-hole H^* can be lifted using the greedy algorithm introduced by Nemhauser and Sigismondi in [110] by identifying a maximal clique C^* in the conflict graph $\tilde{G}^{K}_{E}$ s.t. each node $v_{k',e'} \in C^*$ should have a link with all the nodes $v_{k,e} \in H^*$, and all the nodes $v_{k'",e"} \in C^* \setminus \{v_{k',e'}\}$ in the conflict graph $\tilde{G}^{K}_{E}$. To do so, we first assign a node $v_{k',e'} \notin H^*$ to the clique C^* (i.e., $C^* = \{v_{k',e'}\}$) having the largest degree $|\delta(v_{k',e'})|$ in $\tilde{G}^{K}_{E}$, and $v_{k',e'}$ should be linked with all the nodes $v_{k,e} \in H^*$ in $\tilde{G}^{K}_{E}$. After that, we iteratively add each node $v_{k',e'} \notin H^* \cup C^*$ to the current clique C^* if it is linked in $\tilde{G}^{K}_{E}$ with all the nodes already assigned to $H^* \cup C^*$. We then add the inequality (2.45) induced by the odd-hole H^* and the clique C^*

$$\sum_{v_{k,e} \in H^*} x^{k}_{e} + \frac{|H^*| - 1}{2} \sum_{v_{k',e'} \in C^*} x^{k'}_{e'} \leq \frac{|H^*| - 1}{2}.$$

3.1.15 Separation of Transmission-Reach-Cover Inequalities

In this section, we study the separation problem of the inequality (2.46). Given a fractional solution $(\bar{x}, \bar{z})$, and a demand $k \in K$. The separation problem associated with the inequality (2.46) is NP-hard [125] given that it consists in identifying a cover C^* related to the transmission-reach constraint of the demand k, s.t.

$$\sum_{e \in c^*} \bar{x}^{k}_{e} > |C^*| - 1.$$

For this, we propose a separation algorithm based on a greedy algorithm introduced by Nemhauser and Sigismondi in [110]. We first select an edge $e \in E \setminus (E^{k}_{0} \cup E^{k}_{1})$ having the largest length l_e with $\bar{x}^{k}_{e} > 0$, and set C^* to $C^* = \{e\}$. After that, we iteratively add each edge $e' \in E \setminus (E^{k}_{0} \cup E^{k}_{1} \cup C^*)$ to C^* while $\sum_{e \in C^*} l_e \leq \bar{l}_k$ and e' is not non-compatible edge with the edges already added to the cover C^*, i.e., until a cover C^* is obtained for the the demand k with $\sum_{e \in C^*} l_e > \bar{l}_k$. We further derive a minimal cover from the cover C^* by deleting each edge $e \in C^*$ if $\sum_{e' \in C^* \setminus \{e\}} l_{e'} \leq \bar{l}_k$. We then add the inequality (2.46) induced by the minimal cover C^* for the demand k to the current LP if it is violated, i.e.,

$$\sum_{e \in C^*} x^{k}_{e} \leq |C^*| - 1.$$

Furthermore, the inequality (2.46) induced by the minimal cover C^* can be lifted by introducing an extended cover inequality (2.47) as follows

$$\sum_{e \in C^*} x_e^k + \sum_{e' \in E(C^*)} x_{e'}^k \leq |C^*| - 1,$$

where $l_{e'} \geq l_e$ for each $e \in C^*$ and each $e' \in E(C^*)$ with $e' \notin E_0^k \cup E_1^k$ and e' is not a non-compatible edge with each edge $e \in C^*$.

3.1.16 Separation of Edge-Capacity-Cover Inequalities

Let's now study the separation problem of the inequality (2.49). Given a fractional solution $(\bar{x}, \bar{z})$, and an edge $e \in E$. The separation problem associated with the inequality (2.49) is NP-hard [125] given that it consists in identifying a cover $\tilde{K}^*$ the edge e, s.t.

$$\sum_{k \in \tilde{K}^*} \bar{x}_e^k > |\tilde{K}^*| - 1.$$

To do so, we propose a separation algorithm based on a greedy algorithm introduced by Nemhauser and Sigismondi in [110]. We first select a demand $k \in K \setminus K_e$ having largest number of requested slot w_k with $\bar{x}_e^k > 0$, and set $\tilde{K}^*$ to $\tilde{K}^* = \{k\}$. After that, we iteratively add each demand $k' \in K \setminus (K_e \cup \tilde{K}^*)$ to $\tilde{K}^*$ while $\sum_{k \in \tilde{K}^*} w_k \leq \bar{s} - \sum_{\tilde{k} \in K_e} w_{\tilde{k}}$, i.e., until a cover $\tilde{K}^*$ is obtained for the the edge e with $\sum_{k \in \tilde{K}^*} w_k > \bar{s} - \sum_{\tilde{k} \in K_e} w_{\tilde{k}}$. We further derive a minimal cover from the cover $\tilde{K}^*$ by deleting each demand $k \in \tilde{K}^*$ if $\sum_{k' \in \tilde{K}^* \setminus \{k\}} w_{k'} \leq \bar{s} - \sum_{\tilde{k} \in K_e} w_{\tilde{k}}$. We then add the inequality (2.49) induced by the minimal cover $\tilde{K}^*$ for the edge e to the current LP if it is violated, i.e.,

$$\sum_{k \in \tilde{K}^*} x_e^k \leq |\tilde{K}^*| - 1.$$

Furthermore, the inequality (2.49) induced by the minimal cover $\tilde{K}^*$ can be lifted by introducing an extended cover inequality (2.50) as follows

$$\sum_{k \in \tilde{K}^*} x_e^k + \sum_{k' \in \tilde{K}_e^*} x_e^{k'} \leq |\tilde{K}^*| - 1,$$

where $w_{k'} \geq w_k$ for each $k \in \tilde{K}^*$ and each $k' \in \tilde{K}_e^*$ with $k' \notin K_e$.

3.1.17 Primal Heuristic

Here, we propose a primal heuristic to boost the performance of the Branch-and-Cut algorithm. It is based on a hybrid method between a local search algorithm and a greedy-algorithm. Given an optimal fractional solution $(\bar{x}, \bar{z})$ in a certain node of the B&C tree, our primal heuristic consists in constructing an integral "feasible" solution from this fractional solution. To do so, we first construct several paths R_k for each demand $k \in K$ based on the fractional values $\bar{x}_e^k$ using network flow algorithms s.t. each path $p \in R_k$ satisfies the cut inequalities (2.2). We then use a local search algorithm which consists in generating at each iteration a sequence of demands L (order) numeroted with $L = 1', 2', ..., |K|' - 1, |K|'$. Based on this sequence of demands, our greedy algorithm selects a path p from R_k and a slot s for each demand $k' \in L$ with $\bar{z}_s^{k'} \neq 0$ and $\bar{x}_e^k \neq 0$ for each $e \in E(p)$, while respecting the non-overlapping constraint with the set of demands that precede the demand k' in the list L

(i.e., the demands $1', 2, ..., k'-1$). However, if there does not exist such pair of path p and slot s for the demand k', we then select a path p and a slot s for the demand $k' \in L$ with $\bar{z}_s^{k'} = 0$ with $s \in \{w_{k'}, ..., \bar{s}\}$ and $\bar{x}_e^k \neq 0$ for each $e \in E(p)$ while respecting the non-overlapping constraint with the set of demands that precede the demand k' in the list L.
After that, we compute the associated total length of the paths selected for the set of demands K in the final solution $\mathcal{S}$ given by the greedy-algorithm (i.e., $\sum_{k\in K}\sum_{e\in E_k} l_e$). Our local search algorithm generates a new sequence by doing some permutation of demands in the last sequence of demands, if the value of the solution given by greedy-algorithm is smaller than the value of the best solution found until the current iteration. Otherwise, we stop the algorithm, and we give in output the best solution found during our primal heuristic induced by the best sequence of demands having the smallest value of total length of the selected path compared with the others generated sequences.

3.2 Computational Study

3.2.1 Implementation's Feature

We have used C++ programming language to implement the B&C algorithm under Linux using three framworks, Cplex 12.9 [37], Gurobi 9.0 [68], and "Solving Constraint Integer Programs" (SCIP 7.0) [149] framework using Cplex 12.9 as LP solver. It has been tested on LIMOS high performance servers with a memory size limited to 64 gb while benefiting from parallelism by activating 8 threads using Gurobi or SCIP (which is not possible using Cplex when using cutting-plane based method), and with a CPU time limited to 5 hours (18000 s).

3.2.2 Description of Instances

We further proposed a deep study of the behavior of the algorithm using two types of instances: random and real, and 14 graphs (topologies). They are composed of two types of graphs: real, and other realistics from SND-Lib [112] with a number of links $21 \leq |E| \leq 166$, and a number of nodes $14 \leq |V| \leq 161$ as shown in the table of Figure 3.2.2. Note that we tested 4 instances for each triplet $(G, K, \bar{s})$ with $|K| \in \{10, 20, 30, 40, 50, 100, 150, 200, 250, 300\}$, and $\bar{s}$ up to 320 slots.

Topology (Graph)		Number of nodes	Number of links	Max Node Degree	Min Node Degree	Average Node Degree
Real Topology	German	17	25	5	2	2,94
	Nsfnet	14	21	4	2	3
	Spain	30	56	6	2	3,73
	Conus75	75	99	5	2	2,64
	Coronet100	100	136	5	2	2,72
Realistic Topology (SND-LIB)	Europe	28	41	5	2	2,92
	France	25	45	10	2	3,6
	German50	50	88	5	2	3,52
	Brain	161	166	37	1	2,06
	Giul39	39	86	8	3	4,41
	India35	35	80	9	2	4,57
	Pioro40	40	89	5	4	4,45
	Ta65	65	108	10	1	3,32
	Zib54	54	80	10	1	2,96

Figure 3.1: Characteristics of different topologies used for our experiments.

3.2.3 Computational Results

Based on some preliminary results, the cover-based inequalities (2.49) and (2.30) are shown to be efficient than the clique-based inequalities (2.38), (3.2) and (2.32). In fact, the B&C algorithm is very efficient using SCIP and Gurobi when adding the cover-based inequalities (2.49) and (2.30). We notice that adding these families of valid inequalities allows solving to optimality some instances that are not solved to optimality using B&C_Cplex, B&C_Gurobi and B&C_SCIP. Furthermore, they allow reducing the average gap, average number of nodes, and the average cpu time. On the other hand, we observed that the valid inequalities do not work well when using Cplex. This is due to deactivating the inequalities of the proper Cplex cut generation, and Cplex does not work well without its proper cut generation even if the valid inequalities shown to be efficient using Gurobi and Cplex for the instances tested. The results show also that several inequalities of the cover-based inequalities (2.49) and (2.30), and clique-based inequalities (2.38), (3.2) and (2.32), they are generated along the B&C algorithm. However, the number of clique-based inequalities (2.38) generated is very less compared with other inequalities. Based on these results, we conclude that the valid inequalities are very useful to obtain tighter LP bounds using Gurobi and SCIP. As a result, we combine these families of valid inequalities s.t. their separation is performed along the B&C algorithm (using Cplex, Gurobi and SCIP) in the following order

a) edge-capacity-cover inequalities (2.49),

b) edge-Interval-Capacity-Cover inequalities (2.30),

c) edge-slot-assignment-clique inequalities (3.2),

d) edge-interval-clique inequalities (2.32),

e) slot-assignment-clique inequalities (2.38).

Using this, we provide a comparative study between Cplex, Gurobi and SCIP using the B&C (without additional valid inequalities) algorithm. To do so, we evaluate the impact of the valid inequalities used within the B&C algorithm. For this, we present some computational results using several instances with a number of demand ranges in $\{10, 20, 30, 40, 50, 100, 150, 200, 250, 300\}$ and $\bar{s}$ up to 320 slots. We classify instances in two classes: small-sized instances with number of demands $\{10, 20, 30, 40, 50\}$ and $\bar{s}$ up to 180, and ones of large-sized instances with number of demands ranges in $\{100, 150, 200, 250, 300\}$ and $\bar{s}$ up to 320. We use two types of topologies: real, and realistic ones from SND-LIB already described in the Table 3.2.2. Our first series of computational results presented in Tables 3.1, it concerns a comparaison between the results obtained for the B&C algorithm using Cplex and SCIP (without or with additional valid inequalities). On the other hand, in the second series of computational results are shown in the Tables 3.2, we present the results found for the B&C algorithm using Gurobi and SCIP (without or with additional valid inequalities). In the third series shown in the Tables 3.3, we compare the results found by the B&C algorithm under using Cplex (without or with additional valid inequalities) with those that are found using SCIP (without or with additional valid inequalities). We consider 4 criteria in the different tables, average number of nodes in the enumeration tree (Nb_Nd), average gap (Gap) which represents the relative error between the lower bound gotten at the end of the resolution and best upper bound, average Cpu time computation (T_Cpu), average number of violated inequalities added (Nbr_Cuts). For each instance, we use Cplex with benefiting of its automatic cut generation and without our additional valid inequalities (denoted by B&C_Cplex in the different tables), Cplex using our valid inequalities and disabling its proper

cut generation (denoted by B&C_CPX_With_Additional_Valid_Ineq), Gurobi with benefiting of its automatic cut generation and without our additional valid inequalities (denoted by B&C_Gurobi), Gurobi using our valid inequalities and disabling the Gurobi proper cut generation (denoted by B&C_GRB_With_Additional_Valid_Ineq), SCIP with benefiting of its automatic cut generation and without our additional valid inequalities (denoted by B&C_SCIP), SCIP using our valid inequalities and disabling the SCIP proper cut generation (denoted by B&C_SCIP_With_Additional_Valid_Ineq). To make the results and the comparison more readable, we just present the results already found for a subset of instances based on 2 real topologies: German, Nsfnet, and 2 realistic topologies: India35 and Pioro40.
The results show that adding the valid inequalities is very efficient. They improve the effectiveness of the B&C algorithm compared with the last approach when adding just one family of valid inequalities within the B&C algorithm. In fact, we first notice that introducing valid inequalities allows solving several instances to optimality that are not solved to optimality using B&C_Cplex, B&C_Gurobi and B&C_SCIP. Furthermore, they enabled reducing the average number of nodes in the B&C tree, and also the average Cpu time for several instances. On the other hand, and when the optimality is not guaranteed, adding valid inequalities decreases the average gap for several instances. However, there exists few instances in which adding valid inequalities does not improve the results of B&C algorithm. We further observe that using the valid inequalities within Gurobi (i.e., B&C_SCIP_With_Additional_Valid_Ineq) is shown to be very efficient compared with Cplex and Gurobi (see for example the Tables 3.1 and 3.2). However, and looking at the instances that are solved to optimality introducing the valid inequalities using Gurobi and SCIP, we notice that we have less number of nodes and time cpu using SCIP compared with Gurobi (see for example the Tables 3.2). Furthermore, introducing the valid inequalities using SCIP works much betther than SCIP, Cplex and Gurobi even with using their proper cuts s.t. B&C_SCIP_With_Additional_Valid_Ineq is able to solve several instances to optimality that are not solved using B&C_CPX, B&C_GRB and B&C_SCIP. This means that we are able to beat Cplex, Gurobi and SCIP introducing the valid inequalities using SCIP. On the other hand, and considering large-sized instances with $|K| \geq 200$, we noticed that adding valid inequalities does not improve the effectiveness of the B&C algorithm s.t. there exist some instances that are solved to optimality using B&C_Cplex and B&C_Gurobi that are not solved to optimality with B&C_CPX_With_Additional_Valid_Ineq, B&C_GRB_With_Additional_Valid_Ineq and B&C_SCIP_With_Additional_Valid_Ineq. Based on these results, we conclude that using the valid inequalities allows obtaining tighter LP bound much more for instances with number of demands up to 150 than for large-sized instances with $|K| \geq 200$.

3.3 Concluding Remarks

Based on the theoretical results obtained previously in the chapter 2, we developed a Branch-and-Cut algorithm to solve the problem. The valid inequalities are shown to be efficient and allow improving the effectiveness of the B&C algorithm.

Instances			B&C_CPX			B&C_CPX_With_Add_Valid_Ineq				B&C_SCIP			B&C_SCIP_With_Add_Valid_Ineq			
Topology	$\|K\|$	$\|\mathbb{S}\|$	**Nbr_Nd**	**Gap**	**T_Cpu**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**T_Cpu**	**Nbr_Nd**	**Gap**	**T_Cpu**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**T_Cpu**
German	10	15	5534,75	0,00	42,37	9071,75	0,00	1524,25	101,76	1310,25	0,00	14,35	59	0,00	429,75	0,83
German	20	45	109616,75	0,27	4382,25	68601,50	4,46	46110,75	18000	185956	0,27	3895,50	141	0,00	2403,50	3,89
German	30	45	71995	0,38	8845,19	16945,75	18,73	53497,50	18000	401335,75	1,60	11740,04	160376,50	1,46	129867,25	8334,95
German	40	45	75469	3,82	17778,28	363,33	60,47	41306,33	18000	315993,66	8,33	16206,36	383058,66	3,70	224642,33	16624,87
German	50	55	44143	5,32	17823,61	470,75	61,19	52946	18000	246146,50	9,62	16675,88	251152,50	13,73	305309,75	17074,95
German	100	140	1	0,00	50,15	33,75	60,75	20802	18000	1158,50	0,00	340,10	3014,25	0,00	17668,50	617,80
German	150	210	1	0,00	327,54	150,50	62,40	39133,25	18000	12759	0,01	7329,06	3609	0,00	24782,25	3057,79
Nsfnet	10	15	72616,25	0,00	644,88	140046,50	0,00	3760,75	2640,70	13462	0,00	113,64	1	0,00	95,75	0,15
Nsfnet	20	20	121350,25	32,76	18000	27987,75	41,38	53257,75	18000	699646	9,51	14242,82	21586	0,00	24587	192,27
Nsfnet	30	30	67751	17,13	17795,60	6322,75	49,13	66315,50	18000	272065	40,99	16558,66	281569,66	3,29	340177	11048,71
Nsfnet	40	35	32491	21,46	17879,34	1687,25	59,53	62749,25	18000	225696,67	46,74	16813,88	119841,66	1,17	163519,33	5673,46
Nsfnet	50	50	21256,75	19,80	17873,32	2216,25	59,73	53734,50	18000	247873,25	43,09	16914,53	148476,50	5,91	340399,25	17405,09
Nsfnet	100	120	1064,50	20,28	17885,46	99	66,23	35031,75	18000	56598,50	57,19	17779,07	1	0,00	464,25	40,87
Nsfnet	150	160	145	24,53	17893,17	1	66,94	142594,25	18000	12663	58,50	17927,20	1	0,00	496,25	136,02
India35	10	40	968,50	0,00	40,87	106140	23,94	48081	18000	1907,25	0,00	87,60	1	0,00	779,75	1,80
India35	20	40	1	0,00	12,23	38307,25	41,43	33858	18000	9	0,00	4	7	0,00	2437,75	5,92
India35	30	40	44170,25	0,00	4149,40	19389,50	49,99	80759,50	18000	91798	0,00	7821,50	32156,75	0,00	10917,50	2309,66
India35	40	40	32029	2,80	17945,92	9583,50	53,19	94366,50	18000	161514	2,42	17486,08	191812	0,18	27270	17333,53
India35	50	80	1	0,00	60,98	4763	58,61	35364,75	18000	34	0,00	22,13	69,25	0,00	11105,75	112,19
India35	100	120	6217,50	0,71	9493,81	230	67,42	28590,25	18000	24797	0,32	9137,26	23403,75	0,44	48702,75	9494,52
India35	150	200	4948,25	0,28	13807,72	287,75	67,65	19196,75	18000	16809	0,21	13739,65	1026	0,00	19898	4101,80
Pioro40	10	40	1	0,00	6,23	123725,75	14,29	1705,75	18000	1	0,00	1,49	1	0,00	202,25	1,69
Pioro40	20	40	1	0,00	15,77	52857,25	29,75	17720,25	18000	1,50	0,00	3,44	1	0,00	1255,75	4,88
Pioro40	30	40	1	0,00	26,32	27075,50	44,66	47926,50	18000	1,50	0,00	5,72	6,25	0,00	2972,50	10,54
Pioro40	40	40	32219,75	0,19	8966,91	17658,50	52,21	58468,25	18000	83597	0,20	8692,50	67151	0,12	53544,50	8711,30
Pioro40	50	80	1	0,00	84,98	7344,75	59,79	6049,25	18000	14	0,00	15,93	4	0,00	8606	54,39
Pioro40	100	80	12576	0,05	13517,75	1555,50	66,37	54491,75	18000	21281,75	0,04	9087,52	23785,75	0,04	57123,75	9916,63
Pioro40	150	160	1	0,00	626,31	765,75	68,03	11064,75	18000	823,50	0,00	816,89	124,50	0,00	22814	1509,87

Table 3.1: Table of comparison for the B&C Algorithm: Cplex (Without or With Additional Valid Inequalities) Vs SCIP (Without or With Additional Valid Inequalities).

Instances			B&C_GRB			B&C_GRB_With_Add_Valid_Ineq				B&C_SCIP			B&C_SCIP_With_Add_Valid_Ineq			
Topology	$\lvert K\rvert$	$\lvert\mathbb{S}\rvert$	Nbr_Nd	Gap	T_Cpu	Nbr_Nd	Gap	Nbr_Cuts	T_Cpu	Nbr_Nd	Gap	T_Cpu	Nbr_Nd	Gap	Nbr_Cuts	T_Cpu
German	10	15	4906,25	0,00	981,48	3460,25	0,00	1866,50	149,13	1310,25	0,00	14,35	59	0,00	429,75	0,83
German	20	45	20752	0,27	4889,90	28093,75	0,27	33413	4962,48	185956	0,27	3895,50	141	0,00	2403,50	3,89
German	30	45	24321,50	0,39	9279,54	3614,50	0,00	19141,50	882,68	401335,75	1,60	11740,04	160376,50	1,46	129867,25	8334,95
German	40	45	35451,67	5,52	18000	9920	1,82	216792,33	15453,54	315993,66	8,33	16206,36	383058,66	3,70	224642,33	16624,87
German	50	55	18901,50	6,15	18000	6679	17,55	263086,50	18000	246146,50	9,62	16675,88	251152,50	13,73	305309,75	17074,95
German	100	140	1	0,00	1634,37	1746,50	0,00	200920	15425,53	1158,50	0,00	340,10	3014,25	0,00	17668,50	617,80
German	150	210	64,75	0,00	3184,56	979,75	73,06	44188,25	18000	12759	0,01	7329,06	3609	0,00	24782,25	3057,79
Nsfnet	10	15	15222,75	0,00	2087,33	4385	0,00	4095,25	216,83	13462	0,00	113,64	1	0,00	95,75	0,15
Nsfnet	20	20	51525	12,77	18000	181730	2,58	99411	18000	699646	9,51	14242,82	21586	0,00	24587	192,27
Nsfnet	30	30	27735	22,41	18000	19702,67	13,20	263905,66	18000	272065	40,99	16558,66	281569,66	3,29	340177	11048,71
Nsfnet	40	35	12631	34,18	18000	6239,33	24,31	307366	18000	225696,67	46,74	16813,88	119841,66	1,17	163519,33	5673,46
Nsfnet	50	50	8733,50	29,35	18000	5265,75	47,22	347095,75	18000	247873,25	43,09	16914,53	148476,50	5,91	340399,25	17405,09
Nsfnet	100	120	7790,50	29,60	18000	2253,25	41,60	326605	18000	56598,50	57,19	17779,07	1	0,00	464,25	40,87
Nsfnet	150	160	4255,25	33,05	18000	871	100000	80789	18000	12663	58,50	17927,20	1	0,00	496,25	136,02
India35	10	40	2139,50	0,00	481,60	1658,25	0,00	1902,50	281,53	1907,25	0,00	87,60	1	0,00	779,75	1,80
India35	20	40	103	0,00	541,13	1722,25	0,00	4866	572,66	9	0,00	4	7	0,00	2437,75	5,92
India35	30	40	19437,25	0,00	5132,47	1879,50	0,00	11196,25	914,49	91798	0,00	7821,50	32156,75	0,00	10917,50	2309,66
India35	40	40	28219	1,40	18000	9945	0,00	58588,25	7309,79	161514	2,42	17486,08	191812	0,18	27270	17333,53
India35	50	80	386,50	0,00	1221,34	2174,25	0,00	18163	3970,39	34	0,00	22,13	69,25	0,00	11105,75	112,19
India35	100	120	6036,25	0,75	10772,47	2399	95,14	30646,50	18000	24797	0,32	9137,26	23403,75	0,44	48702,75	9494,52
India35	150	200	4164	0,35	11292,87	726,50	98,71	7760,25	18000	16809	0,21	13739,65	1026	0,00	19898	4101,80
Pioro40	10	40	905,75	0,00	275,37	1756	0,00	578,50	232,46	1	0,00	1,49	1	0,00	202,25	1,69
Pioro40	20	40	544,75	0,00	527,67	1613,75	0,00	2330,50	512,12	1,50	0,00	3,44	1	0,00	1255,75	4,88
Pioro40	30	40	2	0,00	539,60	1669,25	0,00	5362,25	863,04	1,50	0,00	5,72	6,25	0,00	2972,50	10,54
Pioro40	40	40	12237,75	0,19	9290,28	10971	0,01	41439,25	5815,90	83597	0,20	8692,50	67151	0,12	53544,50	8711,30
Pioro40	50	80	10,25	0,00	773,69	1800	0,00	9409,50	2780,11	14	0,00	15,93	4	0,00	8606	54,39
Pioro40	100	80	2785,50	0,54	5680,84	4749	1,76	72888,25	17854,91	21281,75	0,04	9087,52	23785,75	0,04	57123,75	9916,63
Pioro40	150	160	2,25	0,00	4805,69	1068,25	99,51	8076	18000	823,50	0,00	816,89	124,50	0,00	22814	1509,87

Table 3.2: Table of comparison for the B&C Algorithm: Gurobi (Without or With Additional Valid Inequalities) Vs SCIP (Without or With Additional Valid Inequalities).

Instances			B&C_CPX			B&C_CPX_With_Add_Valid_Ineq				B&C_GRB			B&C_GRB_With_Add_Valid_Ineq			
Topology	$\|K\|$	$\|\mathbb{S}\|$	**Nbr_Nd**	**Gap**	**T_Cpu**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**T_Cpu**	**Nbr_Nd**	**Gap**	**T_Cpu**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**T_Cpu**
German	10	15	5534,75	0,00	42,37	9071,75	0,00	1524,25	101,76	4906,25	0,00	981,48	3460,25	0,00	1866,50	149,13
German	20	45	109616,75	0,27	4382,25	68601,50	4,46	46110,75	18000	20752	0,27	4889,90	28093,75	0,27	33413	4962,48
German	30	45	71995	0,38	8845,19	16945,75	18,73	53497,50	18000	24321,50	0,39	9279,54	3614,50	0,00	19141,50	882,68
German	40	45	75469	3,82	17778,28	363,33	60,47	41306,33	18000	35451,67	5,52	18000	9920	1,82	216792,33	15453,54
German	50	55	44143	5,32	17823,61	470,75	61,19	52946	18000	18901,50	6,15	18000	6679	17,55	263086,50	18000
German	100	140	1	0,00	50,15	33,75	60,75	20802	18000	1	0,00	1634,37	1746,50	0,00	200920	15425,53
German	150	210	1	0,00	327,54	150,50	62,40	39133,25	18000	64,75	0,00	3184,56	979,75	73,06	44188,25	18000
Nsfnet	10	15	72616,25	0,00	644,88	140046,50	0,00	3760,75	2640,70	15222,75	0,00	2087,33	4385	0,00	4095,25	216,83
Nsfnet	20	20	121350,25	32,76	18000	27987,75	41,38	53257,75	18000	51525	12,77	18000	181730	2,58	99411	18000
Nsfnet	30	30	67751	17,13	17795,60	6322,75	49,13	66315,50	18000	27735	22,41	18000	19702,67	13,20	263905,66	18000
Nsfnet	40	35	32491	21,46	17879,34	1687,25	59,53	62749,25	18000	12631	34,18	18000	6239,33	24,31	307366	18000
Nsfnet	50	50	21256,75	19,80	17873,32	2216,25	59,73	53734,50	18000	8733,50	29,35	18000	5265,75	47,22	347095,75	18000
Nsfnet	100	120	1064,50	20,28	17885,46	99	66,23	35031,75	18000	7790,50	29,60	18000	2253,25	41,60	326605	18000
Nsfnet	150	160	145	24,53	17893,17	1	66,94	142594,25	18000	4255,25	33,05	18000	871	100000	80789	18000
India35	10	40	968,50	0,00	40,87	106140	23,94	48081	18000	2139,50	0,00	481,60	1658,25	0,00	1902,50	281,53
India35	20	40	1	0,00	12,23	38307,25	41,43	33858	18000	103	0,00	541,13	1722,25	0,00	4866	572,66
India35	30	40	44170,25	0,00	4149,40	19389,50	49,99	80759,50	18000	19437,25	0,00	5132,47	1879,50	0,00	11196,25	914,49
India35	40	40	32029	2,80	17945,92	9583,50	53,19	94366,50	18000	28219	1,40	18000	9945	0,00	58588,25	7309,79
India35	50	80	1	0,00	60,98	4763	58,61	35364,75	18000	386,50	0,00	1221,34	2174,25	0,00	18163	3970,39
India35	100	120	6217,50	0,71	9493,81	230	67,42	28590,25	18000	6036,25	0,75	10772,47	2399	95,14	30646,50	18000
India35	150	200	4948,25	0,28	13807,72	287,75	67,65	19196,75	18000	4164	0,35	11292,87	726,50	98,71	7760,25	18000
Pioro40	10	40	1	0,00	6,23	123725,75	14,29	1705,75	18000	905,75	0,00	275,37	1756	0,00	578,50	232,46
Pioro40	20	40	1	0,00	15,77	52857,25	29,75	17720,25	18000	544,75	0,00	527,67	1613,75	0,00	2330,50	512,12
Pioro40	30	40	1	0,00	26,32	27075,50	44,66	47926,50	18000	2	0,00	539,60	1669,25	0,00	5362,25	863,04
Pioro40	40	40	32219,75	0,19	8966,91	17658,50	52,21	58468,25	18000	12237,75	0,19	9290,28	10971	0,01	41439,25	5815,90
Pioro40	50	80	1	0,00	84,98	7344,75	59,79	6049,25	18000	10,25	0,00	773,69	1800	0,00	9409,50	2780,11
Pioro40	100	80	12576	0,05	13517,75	1555,50	66,37	54491,75	18000	2785,50	0,54	5680,84	4749	1,76	72888,25	17854,91
Pioro40	150	160	1	0,00	626,31	765,75	68,03	11064,75	18000	2,25	0,00	4805,69	1068,25	99,51	8076	18000

Table 3.3: Table of comparison for the B&C Algorithm: Cplex (Without or With Additional Valid Inequalities) Vs Gurobi (Without or With Additional Valid Inequalities).

Chapter 4

Path Formulation and Branch-and-Cut-and-Price Algorithm for the C-RSA Problem

In this chapter, we first introduce an extended integer linear programming model based on the so-called *path formulation* using the path variables. Using this and several classes of valid inequalities previously introduced, we derive two exact algorithms: Branch-and-Price and Branch-and-Cut-and-Price to solve the C-RSA problem. In this section, we describe the framework of these algorithms. First, we give an overview of our column generation algorithm. Then, we discuss the pricing problem. We further present the different separation procedures associated with the different classes of valid inequalities useful to boost the performance of the algorithms. We give at the end some computational results and a comparative study between Branch-and-Price and Branch-and-Cut-and-Price algorithms. We close our chapter with some concluding remarks

4.1 Path Formulation

Let P^k denote the set of all feasible (o_k,d_k) paths in G s.t. for each demand $k \in K$, we have

$$\sum_{e \in E(p_k)} l_e \leq \bar{l}_k, \text{ for all } p_k \in P^k.$$

The basic *path formulation* is based on the cut formulation's variables (i.e., the variables x and z), and a new family of variables defined as follows. We consider for $k \in K$ and $p \in P^k$ and $s \in S$, a variable $y^k_{p,s}$ which takes 1 if slot s is the last slot allocated along the path p for the routing of demand k and 0 if not, s.t. s represents the last slot of the interval of contiguous slots of width w_k allocated by the demand $k \in K$, with $s \in S$ and $p \in P^k$. Note that all the slots $s' \in \{s - w_k + 1, ..., s\}$ should be assigned to demand k along the path p whenever $y^k_{p,s} = 1$. Let $P^k(e)$ denote set of all admissible (o_k,d_k) paths going through the edge e in G for the demand k.
In this case, the C-RSA is also equivalent to the following integer linear programming

$$\min \sum_{k \in K} \sum_{p \in P^k} \sum_{e \in E(p)} \sum_{s=w_k}^{\bar{s}} c_e y^k_{p,s}, \tag{4.1}$$

subject to

$$\sum_{p \in P^k} \sum_{s=1}^{w_k - 1} y^k_{p,s} = 0, \forall k \in K, \tag{4.2}$$

$$\sum_{p \in P^k} \sum_{s=w_k}^{\bar{s}} y^k_{p,s} = 1, \forall k \in K, \tag{4.3}$$

$$\sum_{k \in K} \sum_{p \in P^k(e)} \sum_{s'=s}^{s+w_k-1} y^k_{p,s'} \leq 1, \forall e \in E, \forall s \in S, \tag{4.4}$$

$$y^k_{p,s} \geq 0, \forall k \in K, \forall p \in P^k, \forall s \in S, \tag{4.5}$$

$$y^k_{p,s} \in \{0,1\}, \forall k \in K, \forall p \in P^k, \forall s \in S. \tag{4.6}$$

Inequalities (4.2) express the fact that a demand $k \in K$ cannot occupy a slot s as the last slot before her slot-width w_k. Inequalities (4.3) express the routing and spectrum constraints at the same time s.t. they ensure that exactly one slot $s \in \{w_k, \ldots, \bar{s}\}$ is assigned as last slot for the routing of demand k, and exactly one single path from P^k is allocated by each demand $k \in K$. Note that a slot $s \in S$ is said an allocated slot by the demand k iff $\sum_{p \in P^k} \sum_{s'=s}^{s+w_k-1} y^k_{p,s'} = 1$ which means that s is covered by the interval of contiguous slots allocated by demand k. Inequalities (4.4) ensure that a slot s over the edge e cannot be allocated to at most by one demand $k \in K$. Inequalities (4.5) are trivial inequalities, and constraints (4.6) are the integrality constraints.
To benefit from some theoretical results done in the chapter 2, we introduce the two variables x^k_e and z^k_s used in the cut formulation presented in the chapter 2. As a result, all the valid inequalities for the polytope associated with the cut formulation, they still valid for the polytope associated with the path formulation following the addition of these two variables and the two following constraint

$$x^k_e - \sum_{p \in B^k(e)} \sum_{s=w_k}^{\bar{s}} y^k_{p,s} = 0, \forall k \in K, \forall e \in E, \tag{4.7}$$

and

$$z^k_s - \sum_{p \in B^k} y^k_{p,s} = 0, \forall k \in K, \forall s \in \mathbb{S}. \tag{4.8}$$

Therefore, the C-RSA is then equivalent to the extended formulation based on the following integer linear programming

$$\min \sum_{k \in K} \sum_{e \in E} c_e x_e^k, \tag{4.9}$$

$$\sum_{p \in P^k} \sum_{s=1}^{w_k - 1} y_{p,s}^k = 0, \forall k \in K, \tag{4.10}$$

$$\sum_{p \in P^k} \sum_{s=w_k}^{\bar{s}} y_{p,s}^k = 1, \forall k \in K, \tag{4.11}$$

$$x_e^k - \sum_{p \in P^k(e)} \sum_{s=w_k}^{\bar{s}} y_{p,s}^k = 0, \forall k \in K, \forall e \in E, \tag{4.12}$$

$$z_s^k - \sum_{p \in P^k} y_{p,s}^k = 0, \forall k \in K, \forall s \in \mathbb{S}, \tag{4.13}$$

$$\sum_{k \in K} \sum_{p \in P^k(e)} \sum_{s'=s}^{s+w_k-1} y_{p,s'}^k \leq 1, \forall e \in E, \forall s \in S, \tag{4.14}$$

$$y_{p,s}^k \geq 0, \forall k \in K, \forall p \in P^k, \forall s \in S, \tag{4.15}$$

$$x_e^k \geq 0, \forall k \in K, \forall e \in E, \tag{4.16}$$

$$z_s^k \geq 0, \forall k \in K, \forall s \in \mathbb{S}, \tag{4.17}$$

$$y_{p,s}^k \in \{0,1\}, \forall k \in K, \forall p \in P^k, \forall s \in S, \tag{4.18}$$

$$x_e^k \in \{0,1\}, \forall k \in K, \forall e \in E, \tag{4.19}$$

$$z_s^k \in \{0,1\}, \forall k \in K, \forall s \in \mathbb{S}. \tag{4.20}$$

4.2 Column Generation Algorithm

As it has been mentioned previously, our path formulation contains a huge number of variables which can be exponentiel in the worst case due to the number of all feasible paths for each traffic demand. To manage that, we use a column generation algorithm to solve its linear relaxation. To do so, we begin the algorithm with a restricted linear program of our path formulation by considering a feasible subset of variables (columns). For that, we first generate a subset of feasible paths for each demand $k \in K$ denoted by $B^k \subset P^k$ s.t. the variables $y_{p,s}^k$ for each $k \in K$, $p \in B^k$ and $s \in \mathbb{S}$ induce a feasible basis for the restricted linear program. This means that there exists at least one feasible solution for the restricted linear program. Based on this, we derive the so-called restricted master problem (RMP) as follows

$$\min \sum_{k \in K} \sum_{e \in E} c_e x_e^k,$$

subject to

$$\sum_{p\in B^k}\sum_{s=1}^{w_k-1} y^k_{p,s} = 0, \forall k \in K,$$

$$\sum_{p\in B^k}\sum_{s=w_k}^{\bar{s}} y^k_{p,s} = 1, \forall k \in K,$$

$$x^k_e - \sum_{p\in B^k(e)}\sum_{s=w_k}^{\bar{s}} y^k_{p,s} = 0, \forall k \in K, \forall e \in E,$$

$$z^k_s - \sum_{p\in B^k} y^k_{p,s} = 0, \forall k \in K, \forall s \in \mathbb{S},$$

$$\sum_{k\in K}\sum_{p\in B^k(e)}\sum_{s'=s}^{s+w_k-1} y^k_{p,s'} \leq 1, \forall e \in E, \forall s \in S,$$

$$y^k_{p,s} \geq 0, \forall k \in K, \forall p \in B^k, \forall s \in S,$$

$$x^k_e \geq 0, \forall k \in K, \forall e \in E,$$

$$z^k_s \geq 0, \forall k \in K, \forall s \in \mathbb{S}.$$

At each iteration, our column generation algorithm check if there exists a variable $y^k_{p,s}$ with $p \notin B^k$ for a demand k and slot s having a negative reduced cost using the solution of the dual problem, and add it to B^k. This procedure is based on the so-called "Pricing Problem".

4.2.1 Pricing Problem

As noted later, we consider an initial restricted master problem denoted by RMP_0 which is based on an initial subset of variables induced by a subset of feasible path $B^k \subset P^k$ for each demand $k \in K$. The pricing problem consists in finding a feasible path p for a demand k and slot s having a negative reduced cost using the optimal solution of the dual problem. To do so, we consider the following dual variables

a) α associated with the equations (4.10) such that $\alpha_k \in \mathbb{R}$ for all $k \in K$,

b) β associated with the equations (4.11) such that $\beta^k \in \mathbb{R}$ for all $k \in K$,

c) μ associated with the inequalities (4.14) such that $\mu^e_s \leq 0$ for all $e \in E$ and $s \in \mathbb{S}$,

d) λ associated with the equations (4.12) such that $\lambda^k_e \in \mathbb{R}$ for all $k \in K$ and $e \in E$,

e) ρ associated with the equations (4.13) such that $\rho^k_s \in \mathbb{R}$ for all $k \in K$ and $s \in \mathbb{S}$,

Th dual problem is then equivalent to

$$\max -\sum_{k\in K}\beta^k + \sum_{e\in E}\sum_{s\in\mathbb{S}}\mu^e_s, \tag{4.21}$$

subject to

$$\beta^k - \sum_{e\in E(p)}(\lambda^k_e + \sum_{s'=s-w_k+1}^{s}\mu^e_{s'}) - \rho^k_s \geq 0, \forall k \in K, \forall p \in P^k, \forall s \in \{w_k, ..., \bar{s}\}, \tag{4.22}$$

$$c_e + \lambda^k_e \geq 0, \forall k \in K, \forall e \in E, \tag{4.23}$$

$$\alpha^k + \rho^k_s \geq 0, \forall k \in K, \forall s \in \mathbb{S}, \tag{4.24}$$

$$\mu^e_s \leq 0, \forall e \in E, \forall s \in \mathbb{S}. \tag{4.25}$$

As a result, we obtain that for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$

$$rc_s^k = \beta^k - \rho_s^k + \min_{p \in P^k \setminus B^k} [\sum_{e \in E(p)} -\lambda_e^k - \sum_{s'=s-w_k+1}^{s} \mu_{s'}^e], \tag{4.26}$$

which defines the reduced-cost associated with each demand k and slot s. This is equivalent to the separation problem associated with the dual constraint (4.22). It consists in identifying a path p for a demand k and slot s s.t.

$$\beta^k - \rho_s^k + \sum_{e \in E(p)} (-\lambda_e^k - \sum_{s'=s-w_k+1}^{s} \mu_{s'}^e) < 0.$$

Based on this, and taking into account the tranmission-reach constraint, the pricing problem consists in solving a Resource Constrained Shortest Path (RCSP) Problem, also called Weight Constrained Shortest Path (WCSP) Problem. This problem is well known to be a NP-hard problem [46]. Several algorithms have been proposed in the literature to solve this problem based on dynamic programming algorithms, heuristics and some techniques related to the lagrangian decomposition. As background references we mention [19, 47, 50, 84, 97]. In this work, we have developed an efficient algorithm based on dynamic programming algorithm which allows us to add a path p with a negative reduced cost for each pair of demand k and slot s if it exists while respecting that the length of this path p must be less than $\bar{l}_k$. We repeat this procedure in each iteration of our column generation until no new column is found (i.e., $rc_s^k \geq 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$. As a result, the final solution is optimal for the linear relaxation of our path formulation. Furthermore, if it is integral, then it is optimal for the C-RSA problem. Otherwise, we create two subproblems called childs by branching on fractional variables (variable branching rule) or on some constraints using the Ryan & Foster branching rule [145] (constraint branching rule).

4.2.2 Dynamic Programming Algorithm for the Pricer

In this section, we propose a pseudo-polynomial time algorithm using dynamic programming adapated to our C-RSA problem taking into account the transmission-reach constraint to identify a feasible path for a given pair of demand k and slot s. It is based on the dynamic programming algorithm proposed by Dumitrescu et al. in [47] to solve the RCSP problem. For each demand $k \in K$ and slot s, we associate to each node $v \in V$ in the graph G a set of labels L^v s.t. each label corresponds to differents paths from th origin node o_k to the node v, and each label p is specified by a cost equals to $\sum_{e \in E(p)} (-\lambda_e^k - \sum_{s'=s-w_k+1}^{s} \mu_{s'}^e)$, and a weight equals to $\sum_{e \in E(p)} l_e$. We denote by T_v the set of labels on node $v \in V$. For each demand k and slot $s \in \{w_k, ..., \bar{s}\}$, the complexity of the algorithm is bounded by $\mathcal{O}(|E \setminus E_0^k|\bar{l}_k)$ [47].

Algorithm 3 summarizes the different steps of the dynamic programming algorithm.

4.2.3 Basic Columns

The basic sub-set of paths used to define the restricted master problem, they are generated using a brute-force search algorithm which creates a search tree that covers all the feasible paths P^k for each demand k. It is then used to pre-compute an initial subset B^k of feasible paths for each demand $k \in K$ taking into account the transmission-reach constraint which allows us to prune some non intersecting nodes in our search tree of this algorithm.

Algorithm 3: Dynamic Programming Algorithm

Data: An undirected, loopless, and connected graph $G = (V, E)$, a spectrum $\mathbb{S}$, a multi-set K of demands, a linear program LP, a demand k and a slot $s \in \{w_k, ..., \bar{s}\}$, a set B^k of feasible paths already exists in the current LP for the demand $k \in K$ and slot s, and the optimal values of the duals variables $(\alpha, \beta, \mu, \lambda, \rho)$

Result: Optimal path $p*$ for the demand k and slot s

Set $L^{o_k} = \{(0, 0)\}$ and $L^v = \emptyset$ for each node $v \in V \setminus (V_0^k \cup \{o_k\})$;
Set $T^v = \emptyset$ for each node $v \in V \setminus V_0^k$;
STOP= FALSE;
while *STOP==FALSE* **do**
    if $\cup_{v \in V}(L_v \setminus T_v) = \emptyset$ **then**
        STOP= TRUE;
        Set $p^* = \emptyset$;
        We select one label p from the labels L^{d_k} of destination node d_k s.t. $p \notin B^k$ with $\beta^k - \rho_s^k + \sum_{e \in E(p)}(-\lambda_e^k - \sum_{s'=s-w_k+1}^{s} \mu_{s'}^e) < 0$;
        if *such label exists* **then**
            Set $p^* = p$;
        end
    end
    if $\cup_{v \in V}(L_v \setminus T_v) \neq \emptyset$ **then**
        Select a node $i \in V \setminus V_0^k$ and a label $p \in L^i \setminus T^i$ having the smallest value of $\sum_{e \in E(p)} l_e$;
        for *each* $e = ij \in \delta(i) \setminus E_0^k$ *s.t.* $\sum_{e' \in E(p)} l_{e'} + l_e \leq \bar{l}_k$ **do**
            if $j \notin V(p)$ **then**
                Set $p' = p \cup \{e\}$;
                Update the set of label $L^j = L^i \cup \{p'\}$;
            end
        end
        Set $T^i = T^i \cup \{p\}$;
    end
end
return the best optimal path p^* for the demand k and slot s;

4.3 Branch-and-Price and Branch-and-Cut-and-Price Algorithms

Based on these features, we derive a Branch-and-Price algorithm by combining a column generation algorithm with a Branch-and-Bound algorithm.

4.3.1 Description

The main purpose of this algorithm is to solve a sequence of linear programs using the column generation algorithm at each node of a Branch-and-Bound algorithm. At each iteration of a certain level of the algorithm, we solve our pricing problem by identifying one or more than one new column by solving a RCSP problem for each demand k and slot $s \in \{w_k, ..., \bar{s}\}$ using the dynamic programming algorithm. Algorithm 4 summarizes the different steps of the Branch-and-Price algorithm.

Let us describe the Branch-and-Cut-and-Price based on the Branch-and-Price algorithm combined with a cutting-plane based algorithm by adding several valid inequalities useful to obtain tighter bounds. Consider a fractional solution $\bar{y}$. At each iteration of the Branch-and-Price algorithm, our aim is to identify for a given class of valid inequalities the existence of one or more than one inequalities of this class that are violated by the current solution. We repeat this procedure in each iteration of the algorithm until non violated inequality is identified. Algorithm 5 summarizes the different steps of the Branch-and-Cut-and-Price algorithm for a given class of valid inequalities.

As mentioned before, the Branch-and-Cut-and-Price is based on the path formulation and all the classes of valid inequalities already identified in the chapter (2).

4.3.2 Primal Heuristic

Here, we propose a primal heuristic based on a hybrid method between local search algorithm and a greedy-algorithm. It is necessary to boost the performance of the algorithms, obtain tighter bounds, accelerate the algorithm, and reduce the memory consumed by the tree of B&P and B&C&P by pruning certain nodes that are not interesting. Given a feasible fractional solution $\bar{y}$, our primal heuristic consists in constructing an integral "feasible" solution from this fractional solution. To do so, we propose a local search algorithm which consists in generating at each iteration a sequence of demands L (order) numeroted with $L = 1', 2', ..., |K|' - 1, |K|'$. Based on this sequence of demands, our greedy algorithm selects a path p and a slot s for each demand $k' \in L$ with $y^{k'}_{p,s} \neq 0$ while respecting the non-overlapping constraint with the set of demands that precede the demand k' in the list L (i.e., the demands $1', 2, ..., k' - 1$). However, if there does not exist such pair of path p and slot s for the demand k', we then select a path p and a slot s for the demand $k' \in L$ with $y^{k'}_{p,s} = 0$ and $s \in \{w_{k'}, ..., \bar{s}\}$ while respecting the non-overlapping constraint with the set of demands that precede the demand k' in the list L. After that, we compute the associated total length of the paths selected for the set of demands K in the final solution $\mathcal{S}$ given by the greedy-algorithm. Our local search algorithm generates a new sequence by doing some permutation of demands in the last sequence of demands, if the value of the solution given by greedy-algorithm is smaller than the value of the best solution found until the current iteration. Otherwise, we stop the algorithm, and we give in output the best solution found during our primal heuristic induced by the best sequence of demands having the smallest value of total length of the selected path compared with the others generated sequences.

Algorithm 4: Branch-And-Price Algorithm for the C-RSA

Data: An undirected, loopless, and connected graph $G = (V, E)$, a spectrum $\mathbb{S}$, a multi-set K of demands, a set B^k of precomputed feasible paths for each demand $k \in K$

Result: Optimal solution for the C-RSA problem

LP$\longleftarrow$ RMP_0;
Stop= FALSE;
while *STOP==FALSE* **do**
    Solve the linear program LP;
    Let (y^*, x^*, z^*) be the optimal solution of LP;
    Consider the optimal values of the duals variables $(\alpha^*, \beta^*, \mu^*, \lambda^*, \rho^*)$;
    ADD = FALSE;
    for *each demand* $k \in K$ **do**
        for *each slot* $s \in \{w_k, ..., \bar{s}\}$ **do**
            Compute its associated reduced cost rc_s^k;
            if $rc_s^k < 0$ **then**
                Consider the optimal path p^* for the demand k and slot s with $rc_s^k(p) < 0$;
                Add the new variable (column) $y^k_{p*,s}$ to the current LP;
                ADD= TRUE ;
            end
        end
    end
    if *ADD==FALSE* **then**
        STOP = TRUE;
    end
end
Consider the optimal solution y^* of LP ;
if y^* *is integer for the C-RSA* **then**
    y^* is an optimal solution for the C-RSA;
    End of the Branch-and-Price algorithm ;
end
else
    Create two Sub-problems by branching one some variables or constraints ;
end
for *each Sub-problem not yet solved* **do**
    go to 3 ;
end
return the best optimal solution (y^*, x^*, z^*) for the C-RSA;

Algorithm 5: Branch-and-Cut-and-Price Algorithm for the C-RSA

Data: An undirected, loopless, and connected graph $G = (V, E)$, a spectrum $\mathbb{S}$, a multi-set K of demands, a set B^k of precomputed feasible paths for each demand $k \in K$, and a given class of valid inequality

Result: Optimal solution for the C-RSA problem

1 LP$\longleftarrow$ RMP_0;
2 Stop= FALSE;
3 **while** *STOP==FALSE* **do**
4   Solve the linear program LP;
5   Let (y^*, x^*, z^*) be the optimal solution of LP;
6   Consider the optimal values of the duals variables $(\alpha^*, \beta^*, \mu^*, \lambda^*, \rho^*)$;
7   ADD = FALSE;
8   **for** *each demand* $k \in K$ **do**
9   **for** *each slot* $s \in \{w_k, ..., \bar{s}\}$ **do**
10   Compute its associated reduced cost rc_s^k;
11   **if** $rc_s^k < 0$ **then**
12   Consider the optimal path p^* for the demand k and slot s with $rc_s^k(p) < 0$;
13   Add the new variable (column) $y_{p*,s}^k$ to the current LP;
14   ADD= TRUE ;
15   **end**
16   **end**
17   **end**
18   **if** *ADD==FALSE* **then**
19   **if** *there exist inequalities from the given class that are violated by the current solution* y^* **then**
20   Add them to LP ;
21   **end**
22   **else**
23   STOP = TRUE;
24   **end**
25   **end**
26 **end**
27 Consider the optimal solution y^* of LP ;
28 **if** y^* *is integer for the C-RSA* **then**
29   y^* is an optimal solution for the C-RSA;
30   End of the Branch-and-Cut-and-Price algorithm ;
31 **end**
32 **else**
33   Create two Sub-problems by branching one some variables or constraints ;
34 **end**
35 **for** *each Sub-problem not yet solved* **do**
36   go to 3 ;
37 **end**
38 **return** the best optimal solution (y^*, x^*, z^*) for the C-RSA;

4.4 Computational Study

4.4.1 Implementation's Feature

the B&P and B&C&P algorithms described in the current chapter have been implemented in C++ under Linux using the "Solving Constraint Integer Programs" (SCIP 6.0.2) framework. For the resolution of the linear relaxation at each node in the B&P and B&C&P trees, SCIP uses Cplex 12.9. These have been tested on LIMOS high-performance servers with a memory size limited to 64 Gb while benefiting from parallelism by activating 8 threads, and with a CPU time limited to 5 hours (18000 s). In addition, we have proposed a deep comparative study between the two algorithms using the same topologies presented in the Figure 3.2.2, and the same instances used in the section 3.2.2.

4.4.2 Computational Results

Preliminary results show that introducing each family of valid inequalities improves the effectiveness of the B&P algorithm. In fact, the results first show that introducing each family of valid inequalities enables reducing the average number of nodes in the B&C&P tree, and also the average CPU time for several instances. Furthermore, we observe that adding valid inequalities decreases the average number of added columns for several instances. On the other hand, the results show that the cover-based inequalities (2.49) and (2.30) are efficient compared with those of clique-based inequalities (2.38), (3.2) and (2.32). In fact, the B&C&P algorithm is very efficient when adding the cover-based inequalities (2.49) and (2.30). We notice that adding these families of valid inequalities reduces the average gap, average number of nodes, average CPU time, and also the number of generated columns. Moreover, the results show also that several inequalities of the cover-based inequalities (2.49) and (2.30), and clique-based inequalities (2.38), (3.2) and (2.32), they are generated along the B&C&P algorithm. However, the number of clique-based inequalities (2.38) generated is very less high for the instances tested s.t. they have not generated for several instances. Based on these results, we conclude that the valid inequalities are very interesting to obtain tighter bounds and strengthen the linear relaxation of our path formulation. As a result, we combine these families of valid inequalities s.t. their separation is performed along with the B&C&P algorithm in the following order

a) edge-capacity-cover inequalities (2.49),

b) edge-Interval-Capacity-Cover inequalities (2.30),

c) edge-slot-assignment-clique inequalities (3.2),

d) edge-interval-clique inequalities (2.32),

e) slot-assignment-clique inequalities (2.38).

Based on this, we provide a comparative study between B&P (without additional valid inequalities) and B&C&P (with additional valid inequalities) algorithms. To do so, we evaluate the impact of valid inequalities used within the B&C&P algorithm. We present some computational results using several instances with a number of demand ranges in $\{10, 20, 30, 40, 50, 100, 150, 200, 250, 300\}$ and $\bar{s}$ up to 320 slots. We distinguish two types of instances: small-sized instances with number of demands $\{10, 20, 30, 40, 50\}$ and $\bar{s}$ up to 180, and ones of large-sized instances with number of demands ranges in $\{100, 150, 200, 250, 300\}$ and $\bar{s}$ up to 320. We use two types of topologies: real, and realistic ones from SND-LIB already described in Table 3.2.2. To make the results more readable, we just present a subset

of instances for 2 real topologies: German, Nsfnet,, Spain, and 2 realistic topologies: India35 and Pioro40 with number of demands $|K|$ up to 150 and $\bar{s}$ up to 200 slots.
We consider 5 criteria in the different tables, average number of nodes in the enumeration tree (Nb_Nd), average gap (Gap) which represents the relative error between the lower bound gotten at the end of the resolution and best upper bound, average number of generated columns (Nbr_Cols), average number of violated inequalities added (Nbr_Cuts), average Cpu time computation (T_Cpu). For each instance, we use SCIP without our additional valid inequalities (denoted by B&P_SCIP in the different tables), and SCIP using our valid inequalities and disabling its proper cut generation (denoted by B&C&P_SCIP). The results show that adding the valid inequalities improves the effectiveness of the B&C&P algorithm compared with the last approach described in the last subsequent when adding just one family of valid inequalities. In fact, we first notice that introducing valid inequalities allows solving several instances to optimality that are not solved to optimality using the B&P algorithm. Furthermore, they enabled reducing the average number of nodes in the B&C&P tree, and also the average Cpu time for several instances. On the other hand, and when the optimality is not guaranteed, adding valid inequalities decreases the average gap for several instances. However, there exist few instances very rare in which adding valid inequalities does not improve the results of the B&P algorithm. Based on these results, we ensure that using the valid inequalities strengthens the linear relaxation the path formulation.

Instances			**B&P_SCIP**				**B&C&P_SCIP**				
Topology	$\lvert K\rvert$	$\bar{s}$	**Nbr_Nd**	**Gap**	**Nbr_Cols**	**T_Cpu**	**Nbr_Nd**	**Gap**	**Nbr_Cols**	**Nbr_Cuts**	**T_Cpu**
German	10	15	28	0,00	13,50	0,88	1	0,00	3,25	6,25	0,07
German	20	45	39	0,00	0	6,31	1	0,00	0	7,75	0,25
German	30	45	1	0,00	0	0,20	1	0,00	0	0	0,31
German	40	45	1489,67	0,33	324,67	6000,12	1557,67	0,13	309,67	339	5998,03
German	50	55	3550,75	0,18	412,50	13506,57	1513	0,14	371	385	9020,19
German	100	140	1	0,00	0	9,86	2	0,00	0	6,25	64,73
German	150	210	34	0,00	0	417,78	51	0,00	0	24,75	932,25
Nsfnet	10	15	11	0,00	41,50	0,37	1	0,00	0	0	0,02
Nsfnet	20	20	190,5	0,00	168.5	34.66	1	0,00	165267,25	26.5	4487,61
Nsfnet	30	30	4373,67	1,57	347	12041,42	1	0,00	99961	9,33	11902,57
Nsfnet	40	35	4817,67	0,50	331	17990,15	1	0,00	0	16	3,62
Nsfnet	50	50	2218	0,54	566	13506,61	1	0,00	108442,50	18,25	8932,48
Nsfnet	100	120	2029	2,01	1849	18000	1	0,00	0	0	6,36
Nsfnet	150	160	321,50	11,66	1847,25	17996,69	1	0,00	0	0	32,48
India35	10	40	2	0,00	0	0,56	1	0,00	0	8,50	0,28
India35	20	40	1	0,00	36	0,66	1	0,00	36	0	0,57
India35	30	40	71,50	0,00	109	49,93	9,50	0,00	34,50	43	9,51
India35	40	40	3975,50	0,38	2046,25	17958,21	2754,50	0,11	17896,50	737,50	13542,45
India35	50	80	1	0,00	69,50	3,87	1	0,00	69,50	24	6,37
India35	100	120	496	0,01	50,50	9072,59	353,50	0,00	98	356,25	4820,46
India35	150	200	292	0,01	96,50	9831,30	100,50	0,10	96,50	389,25	8155,83
Pioro40	10	40	1	0,00	27,25	0,26	1	0,00	27,25	0	0,20
Pioro40	20	40	1	0,00	73,75	0,52	1	0,00	73,75	0	0,50
Pioro40	30	40	1	0,00	100,50	0,81	1	0,00	100,50	0	0,68
Pioro40	40	40	101,50	0,00	563,25	116,31	7,50	0,00	261,75	19,50	9,75
Pioro40	50	80	1	0,00	466,25	5,27	1	0,00	466,25	0	4,16
Pioro40	100	80	789,50	0,01	1012,75	9062,39	663	0,01	929	282	4820,38
Pioro40	150	160	1	0,00	1279,50	150,54	1	0,00	1276	0	176,69

Table 4.1: Table of comparison between B&P and B&C&P: With SCIP Cuts Vs Without SCIP Cuts and with additional valid inequalities.

4.4.3 Comparative Study Between Branch-and-Cut and Branch-and-Cut-and-Price Algorithms

Based on the Branch-and-Cut, Branch-and-Price and Branch-and-Cut-and-Price algorithms already devised in the previous sections, we present a comparison between theses algorithms

using several instances with number of demands ranges in $\{10, 20, 30, 40, 50, 100, 150, 200, 250, 300\}$ and $\bar{s}$ up to 320 slots. Our first series of computational results presented in Tables 4.2, 4.3, and 4.2. They concern the results obtained for the Branch-and-Cut algorithm using Cplex (without or with additional valid inequalities) compared with those of Branch-and-Price and Branch-and-Cut-and-Price using SCIP. We denote by B&C_CPX when using Cplex with benefiting of its automatic cut generation and without our additional valid inequalities, and by B&C_CPX_With_Additional_Valid_Ineq when using Cplex with our additional valid inequalities and disabling its proper cut generation. On the other hand, in the second series of computational results are shown in the Tables 4.3, we present the results found for the Branch-and-Cut algorithm using Gurobi (without or with additional valid inequalities) compared with those of Branch-and-Price and Branch-and-Cut-and-Price using SCIP. We denote by B&C_GRB when using Gurobi with benefiting of its automatic cut generation and without our additional valid inequalities, and by B&C_GRB_With_Additional_Valid_Ineq when using Gurobi with our additional valid inequalities and disabling its proper cut generation. Results obtained by the Branch-and-Cut algorithm using SCIP compared with those those of Branch-and-Price and Branch-and-Cut-and-Price using SCIP, they are shown in the Tables 4.4. Let denote by B&C_SCIP when using SCIP with benefiting of its automatic cut generation and without our additional valid inequalities, and by B&C_SCIP_With_Additional_Valid_Ineq when using our additional valid inequalities and disabling its proper cut generation.
Based on the reported results, we notice that the B&C&P algorithm seems to be very efficient compared with B&C algorithm s.t. it is able to provide optimal solutions for several instances, which is not the case for the B&C algorithm (without or with additional valid inequalities) using Cplex, Gurobi, and SCIP within the CPU time limit (5 hours). Furthermore, several instances solved to optimality by B&C algorithm using Cplex, Gurobi, and SCIP could also be solved to optimality within the B&C&P algorithm. The average number of explored nodes using the B&C&P algorithm is greatly reduced for several instances compared with the B&C algorithm using Cplex, Gurobi, and SCIP. Moreover, the average CPU time is significantly reduced using the B&C&P algorithm compared with the B&C algorithm using Cplex, Gurobi, and SCIP (without or with additional valid inequalities). On the other hand, and when using B&P_SCIP algorithm, we notice that we are able to beat B&C_SCIP_With_Add_Valid_Ineq such that B&P_SCIP is able to provide optimal solutions for several instances that are not solved to the optimum by the B&C algorithm using Cplex (see Table 4.2), and Gurobi (see Table 4.3). Furthermore, we noticed that the average number of explored nodes and the average CPU time using the B&P algorithm are greatly reduced for several instances compared with the B&C algorithm using Cplex and Gurobi. However, the B&C algorithm using SCIP with additional valid inequalities is able to beat the B&P algorithm. The results in Table 4.4 show that B&C_SCIP_With_Add_Valid_Ineq provide optimal solutions for several instances, which is not the case for the B&P algorithm. But when the optimality is verified by these two algorithms, we found that using the B&P algorithm reduce the average number of explored nodes and the average CPU time for several instances compared with B&C_SCIP_With_Add_Valid_Ineq.

4.5 Concluding Remarks

The different classes of valid inequalities for the path formulation are shown to be efficient within the Branch-and-Cut-and-Price algorithm. As a result, we notice that the B&C&P algorithm was very efficient compared with the B&P algorithm using several instances. Furthermore, the B&C&P algorithm is able to beat the B&C algorithm. Some instances are still difficult to solve with both B&P and B&C&P algorithms.

Instances			B&C_CPX			B&C_CPX_With_Add_Valid_Ineq				B&P_SCIP				B&C&P_SCIP_With_Add_Valid_Ineq				
Topology	$\lvert K\rvert$	$\lvert \mathbb{S}\rvert$	Nbr_Nd	Gap	T_Cpu	Nbr_Nd	Gap	Nbr_Cuts	T_Cpu	Nbr_Nd	Gap	Nbr_Cols	T_Cpu	Nbr_Nd	Gap	Nbr_Cols	Nbr_Cuts	T_Cpu
German	10	15	5534,75	0,00	42,37	9071,75	0,00	1524,25	101,76	28	0,00	13,50	0,88	1	0,00	3,25	6,25	0,07
German	20	45	109616,75	0,27	4382,25	68601,50	4,46	46110,75	18000	39	0,00	0	6,31	1	0,00	0	7,75	0,25
German	30	45	71995	0,38	8845,19	16945,75	18,73	53497,50	18000	1	0,00	0	0,20	1	0,00	0	0	0,31
German	40	45	75469	3,82	17778,28	363,33	60,47	41306,33	18000	1489,67	0,33	324,67	6000,12	1557,67	0,13	309,67	339	5998,03
German	50	55	44143	5,32	17823,61	470,75	61,19	52946	18000	3550,75	0,18	412,50	13506,57	1513	0,14	371	385	9020,19
German	100	140	1	0,00	50,15	33,75	60,75	20802	18000	1	0,00	0	9,86	2	0,00	0	6,25	64,73
German	150	210	1	0,00	327,54	150,50	62,40	39133,25	18000	34	0,00	0	417,78	51	0,00	0	24,75	932,25
Nsfnet	10	15	72616,25	0,00	644,88	140046,50	0,00	3760,75	2640,70	11	0,00	41,50	0,37	1	0,00	0	0	0,02
Nsfnet	20	20	121350,25	32,76	18000	27987,75	41,38	53257,75	18000	145	0,00	206	7,79	1	0,00	0	11	0,19
Nsfnet	30	30	67751	17,13	17795,60	6322,75	49,13	66315,50	18000	4373,67	1,57	347	12041,42	1	0,00	99961	9,33	11902,57
Nsfnet	40	35	32491	21,46	17879,34	1687,25	59,53	62749,25	18000	4817,67	0,50	331	17990,15	1	0,00	0	16	3,62
Nsfnet	50	50	21256,75	19,80	17873,32	2216,25	59,73	53734,50	18000	2218	0,54	566	13506,61	1	0,00	108442,50	18,25	8932,48
Nsfnet	100	120	1064,50	20,28	17885,46	99	66,23	35031,75	18000	2029	2,01	1849	18000	1	0,00	0	0	6,36
Nsfnet	150	160	145	24,53	17893,17	1	66,94	142594,25	18000	321,50	11,66	1847,25	17996,69	1	0,00	0	0	32,48
India35	10	40	968,50	0,00	40,87	106140	23,94	48081	18000	2	0,00	0	0,56	1	0,00	0	8,50	0,28
India35	20	40	1	0,00	12,23	38307,25	41,43	33858	18000	1	0,00	36	0,66	1	0,00	36	0	0,57
India35	30	40	44170,25	0,00	4149,40	19389,50	49,99	80759,50	18000	71,50	0,00	109	49,93	9,50	0,00	34,50	43	9,51
India35	40	40	32029	2,80	17945,92	9583,50	53,19	94366,50	18000	3975,50	0,38	2046,25	17958,21	2754,50	0,11	17896,50	737,50	13542,45
India35	50	80	1	0,00	60,98	4763	58,61	35364,75	18000	1	0,00	69,50	3,87	1	0,00	69,50	24	6,37
India35	100	120	6217,50	0,71	9493,81	230	67,42	28590,25	18000	496	0,01	50,50	9072,59	353,50	0,00	98	356,25	4820,46
India35	150	200	4948,25	0,28	13807,72	287,75	67,65	19196,75	18000	292	0,01	96,50	9831,30	100,50	0,10	96,50	389,25	8155,83
Pioro40	10	40	1	0,00	6,23	123725,75	14,29	1705,75	18000	1	0,00	27,25	0,26	1	0,00	27,25	0	0,20
Pioro40	20	40	1	0,00	15,77	52857,25	29,75	17720,25	18000	1	0,00	73,75	0,52	1	0,00	73,75	0	0,50
Pioro40	30	40	1	0,00	26,32	27075,50	44,66	47926,50	18000	1	0,00	100,50	0,81	1	0,00	100,50	0	0,68
Pioro40	40	40	32219,75	0,19	8966,91	17658,50	52,21	58468,25	18000	101,50	0,00	563,25	116,31	7,50	0,00	261,75	19,50	9,75
Pioro40	50	80	1	0,00	84,98	7344,75	59,79	6049,25	18000	1	0,00	466,25	5,27	1	0,00	466,25	0	4,16
Pioro40	100	80	12576	0,05	13517,75	1555,50	66,37	54491,75	18000	789,50	0,01	1012,75	9062,39	663	0,01	929	282	4820,38
Pioro40	150	160	1	0,00	626,31	765,75	68,03	11064,75	18000	1	0,00	1279,50	150,54	1	0,00	1276	0	176,69

Table 4.2: Table of comparison between B&C, B&P and B&C&P Algorithms: Cplex (Without or With Additional Valid Inequalities) Vs SCIP (Without or With Additional Valid Inequalities).

Instances			B&C_GRB			B&C_GRB_With_Add_Val_Ineq				B&P_SCIP				B&C&P_SCIP_With_Add_Val_Ineq				
Topology	$\lvert K\rvert$	$\lvert\mathbb{S}\rvert$	**Nbr_Nd**	**Gap**	**T_Cpu**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**T_Cpu**	**Nbr_Nd**	**Gap**	**Nbr_Cols**	**T_Cpu**	**Nbr_Nd**	**Gap**	**Nbr_Cols**	**Nbr_Cuts**	**T_Cpu**
German	10	15	4906,25	0,00	981,48	3460,25	0,00	1866,50	149,13	28	0,00	13,50	0,88	1	0,00	3,25	6,25	0,07
German	20	45	20752	0,27	4889,90	28093,75	0,27	33413	4962,48	39	0,00	0	6,31	1	0,00	0	7,75	0,25
German	30	45	24321,50	0,39	9279,54	3614,50	0,00	19141,50	882,68	1	0,00	0	0,20	1	0,00	0	0	0,31
German	40	45	35451,67	5,52	18000	9920	1,82	216792,33	15453,54	1489,67	0,33	324,67	6000,12	1557,67	0,13	309,67	339	5998,03
German	50	55	18901,50	6,15	18000	6679	17,55	263086,50	18000	3550,75	0,18	412,50	13506,57	1513	0,14	371	385	9020,19
German	100	140	1	0,00	1634,37	1746,50	0,00	200920	15425,53	1	0,00	0	9,86	2	0,00	0	6,25	64,73
German	150	210	64,75	0,00	3184,56	979,75	73,06	44188,25	18000	34	0,00	0	417,78	51	0,00	0	24,75	932,25
Nsfnet	10	15	15222,75	0,00	2087,33	4385	0,00	4095,25	216,83	11	0,00	41,50	0,37	1	0,00	0	0	0,02
Nsfnet	20	20	51525	12,77	18000	181730	2,58	99411	18000	145	0,00	206	7,79	1	0,00	0	11	0,19
Nsfnet	30	30	27735	22,41	18000	19702,67	13,20	263905,66	18000	4373,67	1,57	347	12041,42	1	0,00	99961	9,33	11902,57
Nsfnet	40	35	12631	34,18	18000	6239,33	24,31	307366	18000	4817,67	0,50	331	17990,15	1	0,00	0	16	3,62
Nsfnet	50	50	8733,50	29,35	18000	5265,75	47,22	347095,75	18000	2218	0,54	566	13506,61	1	0,00	108442,50	18,25	8932,48
Nsfnet	100	120	7790,50	29,60	18000	2253,25	41,60	326605	18000	2029	2,01	1849	18000	1	0,00	0	0	6,36
Nsfnet	150	160	4255,25	33,05	18000	871	100000	80789	18000	321,50	11,66	1847,25	17996,69	1	0,00	0	0	32,48
India35	10	40	2139,50	0,00	481,60	1658,25	0,00	1902,50	281,53	2	0,00	0	0,56	1	0,00	0	8,50	0,28
India35	20	40	103	0,00	541,13	1722,25	0,00	4866	572,66	1	0,00	36	0,66	1	0,00	36	0	0,57
India35	30	40	19437,25	0,00	5132,47	1879,50	0,00	11196,25	914,49	71,50	0,00	109	49,93	9,50	0,00	34,50	43	9,51
India35	40	40	28219	1,40	18000	9945	0,00	58588,25	7309,79	3975,50	0,38	2046,25	17958,21	2754,50	0,11	17896,50	737,50	13542,45
India35	50	80	386,50	0,00	1221,34	2174,25	0,00	18163	3970,39	1	0,00	69,50	3,87	1	0,00	69,50	24	6,37
India35	100	120	6036,25	0,75	10772,47	2399	95,14	30646,50	18000	496	0,01	50,50	9072,59	353,50	0,00	98	356,25	4820,46
India35	150	200	4164	0,35	11292,87	726,50	98,71	7760,25	18000	292	0,01	96,50	9831,30	100,50	0,10	96,50	389,25	8155,83
Pioro40	10	40	905,75	0,00	275,37	1756	0,00	578,50	232,46	1	0,00	27,25	0,26	1	0,00	27,25	0	0,20
Pioro40	20	40	544,75	0,00	527,67	1613,75	0,00	2330,50	512,12	1	0,00	73,75	0,52	1	0,00	73,75	0	0,50
Pioro40	30	40	2	0,00	539,60	1669,25	0,00	5362,25	863,04	1	0,00	100,50	0,81	1	0,00	100,50	0	0,68
Pioro40	40	40	12237,75	0,19	9290,28	10971	0,01	41439,25	5815,90	101,50	0,00	563,25	116,31	7,50	0,00	261,75	19,50	9,75
Pioro40	50	80	10,25	0,00	773,69	1800	0,00	9409,50	2780,11	1	0,00	466,25	5,27	1	0,00	466,25	0	4,16
Pioro40	100	80	2785,50	0,54	5680,84	4749	1,76	72888,25	17854,91	789,50	0,01	1012,75	9062,39	663	0,01	929	282	4820,38
Pioro40	150	160	2,25	0,00	4805,69	1068,25	99,51	8076	18000	1	0,00	1279,50	150,54	1	0,00	1276	0	176,69

Table 4.3: Table of comparison between B&C, B&P and B&C&P Algorithms: Gurobi (Without or With Additional Valid Inequalities) Vs SCIP (Without or With Additional Valid Inequalities).

Instances			B&C_SCIP			B&C_SCIP_With_Add_Val_Ineq				B&P_SCIP				B&C&P_SCIP_With_Add_Val_Ineq				
Topology	$\lvert K\rvert$	$\lvert\mathbb{S}\rvert$	**Nbr_Nd**	**Gap**	**T_Cpu**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**T_Cpu**	**Nbr_Nd**	**Gap**	**Nbr_Cols**	**T_Cpu**	**Nbr_Nd**	**Gap**	**Nbr_Cols**	**Nbr_Cuts**	**T_Cpu**
German	10	15	1310,25	0,00	14,35	59	0,00	429,75	0,83	28	0,00	13,50	0,88	1	0,00	3,25	6,25	0,07
German	20	45	185956	0,27	3895,50	141	0,00	2403,50	3,89	39	0,00	0	6,31	1	0,00	0	7,75	0,25
German	30	45	401335,75	1,60	11740,04	160376,50	1,46	129867,25	8334,95	1	0,00	0	0,20	1	0,00	0	0	0,31
German	40	45	315993,66	8,33	16206,36	383058,66	3,70	224642,33	16624,87	1489,67	0,33	324,67	6000,12	1557,67	0,13	309,67	339	5998,03
German	50	55	246146,50	9,62	16675,88	251152,50	13,73	305309,75	17074,95	3550,75	0,18	412,50	13506,57	1513	0,14	371	385	9020,19
German	100	140	1158,50	0,00	340,10	3014,25	0,00	17668,50	617,80	1	0,00	0	9,86	2	0,00	0	6,25	64,73
German	150	210	12759	0,01	7329,06	3609	0,00	24782,25	3057,79	34	0,00	0	417,78	51	0,00	0	24,75	932,25
Nsfnet	10	15	13462	0,00	113,64	1	0,00	95,75	0,15	11	0,00	41,50	0,37	1	0,00	0	0	0,02
Nsfnet	20	20	699646	9,51	14242,82	21586	0,00	24587	192,27	145	0,00	206	7,79	1	0,00	0	11	0,19
Nsfnet	30	30	272065	40,99	16558,66	281569,66	3,29	340177	11048,71	4373,67	1,57	347	12041,42	1	0,00	99961	9,33	11902,57
Nsfnet	40	35	225696,67	46,74	16813,88	119841,66	1,17	163519,33	5673,46	4817,67	0,50	331	17990,15	1	0,00	0	16	3,62
Nsfnet	50	50	247873,25	43,09	16914,53	148476,50	5,91	340399,25	17405,09	2218	0,54	566	13506,61	1	0,00	108442,50	18,25	8932,48
Nsfnet	100	120	56598,50	57,19	17779,07	1	0,00	464,25	40,87	2029	2,01	1849	18000	1	0,00	0	0	6,36
Nsfnet	150	160	12663	58,50	17927,20	1	0,00	496,25	136,02	321,50	11,66	1847,25	17996,69	1	0,00	0	0	32,48
India35	10	40	1907,25	0,00	87,60	1	0,00	779,75	1,80	2	0,00	0	0,56	1	0,00	0	8,50	0,28
India35	20	40	9	0,00	4	7	0,00	2437,75	5,92	1	0,00	36	0,66	1	0,00	36	0	0,57
India35	30	40	91798	0,00	7821,50	32156,75	0,00	10917,50	2309,66	71,50	0,00	109	49,93	9,50	0,00	34,50	43	9,51
India35	40	40	161514	2,42	17486,08	191812	0,18	27270	17333,53	3975,50	0,38	2046,25	17958,21	2754,50	0,11	17896,50	737,50	13542,45
India35	50	80	34	0,00	22,13	69,25	0,00	11105,75	112,19	1	0,00	69,50	3,87	1	0,00	69,50	24	6,37
India35	100	120	24797	0,32	9137,26	23403,75	0,44	48702,75	9494,52	496	0,01	50,50	9072,59	353,50	0,00	98	356,25	4820,46
India35	150	200	16809	0,21	13739,65	1026	0,00	19898	4101,80	292	0,01	96,50	9831,30	100,50	0,10	96,50	389,25	8155,83
Pioro40	10	40	1	0,00	1,49	1	0,00	202,25	1,69	1	0,00	27,25	0,26	1	0,00	27,25	0	0,20
Pioro40	20	40	1,50	0,00	3,44	1	0,00	1255,75	4,88	1	0,00	73,75	0,52	1	0,00	73,75	0	0,50
Pioro40	30	40	1,50	0,00	5,72	6,25	0,00	2972,50	10,54	1	0,00	100,50	0,81	1	0,00	100,50	0	0,68
Pioro40	40	40	83597	0,20	8692,50	67151	0,12	53544,50	8711,30	101,50	0,00	563,25	116,31	7,50	0,00	261,75	19,50	9,75
Pioro40	50	80	14	0,00	15,93	4	0,00	8606	54,39	1	0,00	466,25	5,27	1	0,00	466,25	0	4,16
Pioro40	100	80	21281,75	0,04	9087,52	23785,75	0,04	57123,75	9916,63	789,50	0,01	1012,75	9062,39	663	0,01	929	282	4820,38
Pioro40	150	160	823,50	0,00	816,89	124,50	0,00	22814	1509,87	1	0,00	1279,50	150,54	1	0,00	1276	0	176,69

Table 4.4: Table of comparison between B&C, B&P and B&C&P Algorithms: SCIP (Without or With Additional Valid Inequalities) Vs SCIP (Without or With Additional Valid Inequalities).

Chapter 5

Compact Formulation and Polyhedra for the SA Sub-problem

In this chapter, we focus on the Spectrum Assignment (SA) sub-problem. First, we propose an integer linear programming compact formulation and investigate the facial structure of the associated polytope. Moreover, we identify several classes of valid inequalities for the polytope s.t. some of them come from those that are already proposed for the C-RSA. We further prove that these inequalities are facet-defining, and discuss their separation problems. Based on these results, we devise a Branch-and-Cut (B&C) algorithm for the SA problem.

5.1 The Spectrum Assignment Sub-problem

The SA can be stated as follows. Consider a SFON as an undirected, loopless, and connected graph $G = (V, E)$, and an optical spectrum $\mathbb{S} = \{1, \dots, \bar{s}\}$ of available frequency slots. Let K be a multiset of demands such that each demand k is specified by an origin node $o_k \in V$, a destination node $d_k \in V \setminus \{o_k\}$, a slot-width $w_k \in \mathbb{Z}_+$, and a routing path p_k from its origin o_k to its destination d_k through G. The SA consists of determining for each demand $k \in K$ an interval of contiguous frequency slots $S_k \subset \mathbb{S}$ of width equal to w_k (continuity and contiguity constraints) such that $S_k \cap S_{k'} = \emptyset$ for each pair of demands $k, k' \in K$ $(k \neq k')$ with paths sharing an edge , i.e., $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint), while optimizing the number of slots allocated in $\mathbb{S}$.
The SA is well known to be NP-hard problem [13]. It is equivalent to the problems of wavelength assignment, interval coloring, and dynamic storage allocation [13] that are well known to be NP-hard.

5.2 Compact Formulation

Here we introduce an integer linear programming compact formulation for the SA problem. For $s \in \mathbb{S}$, let u_s be a variable which takes 1 if the slot s is used and 0 if not, and for $k \in K$ and $s \in \mathbb{S}$, let z_s^k be a variable which takes 1 if slot s is the last slot allocated for the routing of demand k and 0 if not. The contiguous slots $s' \in \{s - w_k + 1, ..., s\}$ should be assigned to demand k whenever $z_s^k = 1$. The SA is equivalent to the following integer linear programming

$$\min \sum_{s \in \mathbb{S}} u_s, \tag{5.1}$$

subject to

$$z_s^k = 0, \quad \text{for all } k \in K \text{ and } s \in \{1, ..., w_k - 1\}, \tag{5.2}$$

$$\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1, \quad \text{for all } k \in K, \tag{5.3}$$

$$\sum_{k \in \tilde{K}_e} \sum_{s'=s}^{\min(\bar{s}, s+w_k-1)} z_s^k - u_s \leq 0, \quad \text{for all } e \in E, \text{ and } s \in \mathbb{S}, \tag{5.4}$$

$$z_s^k \geq 0, \quad \text{for all } k \in K \text{ and } s \in \mathbb{S}, \tag{5.5}$$

$$0 \leq u_s \leq 1, \quad \text{for all } s \in \mathbb{S}, \tag{5.6}$$

$$z_s^k \in \{0,1\}, \quad \text{for all } k \in K \text{ and } s \in \mathbb{S}, \tag{5.7}$$

$$u_s \in \{0,1\}, \quad \text{for all } s \in \mathbb{S}. \tag{5.8}$$

where $\tilde{K}_e$ denotes the set of demands in K passing through the edge e (i.e., $\tilde{K}_e = \{k \in K, e \in E(p_k)\}$. Equations (5.2) ensure that the demand k cannot occupy a slot s as last slot before her slot-width w_k. Inequalities (5.3) ensure than more than one interval of contiguous slots can be assigned to each demand $k \in K$. It should normally be an equation form ensuring that exactly one slot $s \in \{w_k, \ldots, \bar{s}\}$ (one interval of contiguous slots) must be assigned to demand k as last slot. Here we relax this constraint. Optimizing the spectrum-usage objective function, the equality is guaranteed at the optimum. Inequalities (5.4) express the fact that the demands passed through the same edge e, they cannot share a slot s over the edge $e \in E$ with $s \in \{1, ..., \bar{s}\}$. Inequalities (5.5)-(5.6) are the trivial inequalities, and constraints (5.7)-(5.8) are the integrality constraints.

5.3 Associated Polytope

Let $P_{sa}(G, K, \mathbb{S})$ be the polytope, convex hull of the solutions for the formulation (5.1)-(5.8). Here we study the facial structure of the polytope $P_{sa}(G, K, \mathbb{S})$.
A solution of the SA problem based on the variables (u, z) is given by two sets S_k for each demand $k \in K$ and U for the spectrum-usage of $\mathbb{S}$ where

a) S_k denotes the set of index of the last-slots selected for the demand k s.t. $|S_k| \geq 1$.

b) U denotes the set of slots allocated over the spectrum $\mathbb{S}$ such that for each demand $k \in K$ and last slot $s \in S_k \rightarrow$ each slot $s' \in \{s - w_k + 1, ..., s\}$ should be in U i.e. $s' \in U$.

5.3.1 Dimension

Let $\tilde{A}$ denote the matrix associated with the equations (5.2). We ensure that the matrix $\tilde{A}$ is of full rank given that the demands are independants, and the slots in $\mathbb{S}$ are independents for each demand $k \in K$. As a result, $rank(\tilde{A}) = \sum_{k \in K} (w_k - 1)$. Let us denote by r' the rank of the matrix $\tilde{A}$.

Proposition 5.3.1. *Consider an equation $\sigma z + \mu u = \lambda$ of $P_{sa}(G, K, \mathbb{S})$. The equation system (5.2) defines a minimal equation system for $P_{sa}(G, K, \mathbb{S})$. As a consequence, we obtain that $\mu_s = 0$ for all $s \in \mathbb{S}$ and $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$.*

Proof. To prove that $\sigma z + \mu u = \lambda$ is a linear combination of equations (5.2), it's sufficient to prove that for each demand $k \in K$, there exists for each demand $k \in K$ a $\gamma^k \in \mathbb{R}^{w_k - 1}$ s.t. $(\mu, \sigma) = \gamma \tilde{A}$. Let $u^{\mathcal{S}}$ and $z^{\mathcal{S}}$ denote the incidence vector of a solution $\mathcal{S}$ of the SA problem. Let us show that $\mu_s = 0$ for all $s \in \mathbb{S}$. Consider a slot $\tilde{s} \in \mathbb{S}$. To do so, we consider a solution $\mathcal{S}^{105} = (U^{105}, S^{105})$ in which

a) a set of last-slots S_k^{105} is assigned to each demand $k \in K$ with $|S_k^{105}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{105} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{105}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{105}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{105}$ and $s" \in S_{k'}^{105}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in S_k^{105}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\tilde{s} \notin \{s - w_k + 1, ..., s\}$ for each $k \in K$ and $s \in S_k^{105}$ (slot-assignment constraint taking into account the possibility of adding the slot $\tilde{s}$ in the set of used slots U^{105}).

$\mathcal{S}^{105}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{105}}, z^{\mathcal{S}^{105}})$ belongs to $P_{sa}(G, K, \mathbb{S})$. Based on this, we derive a solution $\mathcal{S}^{106} = (U^{106}, S^{106})$ from the solution $\mathcal{S}^{105}$ by adding the slot $\tilde{s}$ as a used slot in U^{106} without modifying the the last-slots assigned to the demands K in $\mathcal{S}^{105}$ which remain the same in the solution $\mathcal{S}^{106}$ i.e., $S_k^{105} = S_k^{106}$ for each demand $k \in K$. The solution $\mathcal{S}^{106}$ is feasible given that

a) a set of last-slots S_k^{106} is assigned to each demand $k \in K$ with $|S_k^{106}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{106} used in $\mathbb{S}$ s.t. $\tilde{s} \in U^{106}$, and for each demand k and last-slot $s \in S_k^{106}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{106}$,

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{106}$ and $s' \in S_{k'}^{106}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s \in S_k^{106}, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint).

$\mathcal{S}^{106}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{106}}, z^{\mathcal{S}^{106}})$ belongs to $P_{sa}(G, K, \mathbb{S})$. We then obtain that

$$\mu u^{\mathcal{S}^{105}} + \sigma z^{\mathcal{S}^{105}} = \mu u^{\mathcal{S}^{106}} + \sigma z^{\mathcal{S}^{106}} = \mu u^{\mathcal{S}^{105}} + \sigma z^{\mathcal{S}^{105}} + \mu_{\tilde{s}}.$$

It follows that $\mu_{\tilde{s}} = 0$ for the slot $\tilde{s} \in \mathbb{S}$. The slot $\tilde{s}$ is chosen arbitrarily in $\mathbb{S}$, we iterate the same procedure for all feasible slots in $\mathbb{S}$ s.t. we find

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Let us show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$. Consider a demand k and a slot s in $\{w_k, ..., \bar{s}\}$. To do so, we consider a solution $\mathcal{S}^{107} = (U^{107}, S^{107})$ in which

a) a set of last-slots S_k^{107} is assigned to each demand $k \in K$ with $|S_k^{107}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{107} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{107}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{107}$,

c) $\{s'-w_k+1,...,s'\}\cap\{s"-w_{k'}+1,...,s"\}=\emptyset$ for each $k,k'\in K$ and each $s'\in S_k^{107}$ and $s"\in S_{k'}^{107}$ with $E(p_k)\cap E(p_{k'})\neq\emptyset$, i.e., for each edge $e\in E$ and each slot $s"\in\mathbb{S}$ we have $\sum_{k\in\tilde{K}_e}|\{s'\in S_k^{107}, s"\in\{s'-w_k+1,...,s'\}|\leq 1$ (non-overlapping constraint),

d) and $\{s-w_k+1,...,s\}\cap\{s'-w_{k'}+1,...,s'\}=\emptyset$ for each $k'\in K$ and $s'\in S_{k'}^{107}$ with $E(p_k)\cap E(p_{k'})\neq\emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s in the set of last-slots S_k^{107} assigned to the demand k in the solution $\mathcal{S}^{107}$).

$\mathcal{S}^{107}$ is clearly feasible for the problem given that it satisfies all the SA constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{107}}, z^{\mathcal{S}^{107}})$ belongs to $P_{sa}(G,K,\mathbb{S})$. Based on this, we derive a solution $\mathcal{S}^{108}=(E^{108},S^{108})$ from the solution $\mathcal{S}^{107}$ by adding the slot s as last-slot to the demand k without modifying the last-slots assigned to the demands $K\setminus\{k\}$ in $\mathcal{S}^{107}$ remain the same in the solution $\mathcal{S}^{108}$ i.e., $S_{k'}^{107}=S_{k'}^{108}$ for each demand $k'\in K\setminus\{k\}$, and $S_k^{108}=S_k^{107}\cup\{s\}$ for the demand k. The solution $\mathcal{S}^{108}$ is feasible given that

a) a set of last-slots S_k^{108} is assigned to each demand $k\in K$ with $|S_k^{108}|\geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{108} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s\in S_k^{108}$ and $s'\in\{s-w_k+1,...,s\}$, we have $s'\in U^{108}$,

c) $\{s-w_k+1,...,s\}\cap\{s'-w_{k'}+1,...,s'\}=\emptyset$ for each $k,k'\in K$ and each $s\in S_k^{108}$ and $s'\in S_{k'}^{108}$ with $E(p_k)\cap E(p_{k'})\neq\emptyset$, i.e., for each edge $e\in E$ and each slot $s"\in\mathbb{S}$ we have $\sum_{k\in\tilde{K}_e}|\{s\in S_k^{108}, s"\in\{s-w_k+1,...,s\}|\leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(u^{\mathcal{S}^{108}}, z^{\mathcal{S}^{108}})$ belongs to $P_{sa}(G,K,\mathbb{S})$. We then obtain that

$$\mu u^{\mathcal{S}^{107}}+\sigma z^{\mathcal{S}^{107}}=\mu u^{\mathcal{S}^{108}}+\sigma z^{\mathcal{S}^{108}}=\mu u^{\mathcal{S}^{107}}+\sigma z^{\mathcal{S}^{107}}+\sigma_s^k+\sum_{\tilde{s}\in\{s,...,s-w_k+1\}\setminus U^{107}}\mu_{\tilde{s}}.$$

It follows that $\sigma_s^k=0$ for demand k and a slot $s\in\{w_k,...,\bar{s}\}$ given that $\mu_{\tilde{s}}=0$ for each $\tilde{s}\in\mathbb{S}$.
The slot s is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k,...,\bar{s}\}$ of demand k s.t. we find

$$\sigma_s^k=0, \text{ for demand } k \text{ and all slots } s\in\{w_k,...,\bar{s}\}.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K\setminus\{k\}$ such that

$$\sigma_s^{k'}=0, \text{ for all } k'\in K\setminus\{k\} \text{ and all slots } s\in\{w_{k'},...,\bar{s}\}.$$

Consequently, we conclude that

$$\sigma_s^k=0, \text{ for all } k\in K \text{ and all slots } s\in\{w_k,...,\bar{s}\},$$

On the other hand, note that the slots $s\in\{1,...,w_k-1\}$ for each demand k are independants s.t. for each demand $k\in K$, we have

$$\sum_{s=1}^{w_k-1}\sigma_s^k=\sum_{s=1}^{w_k-1}\gamma^{k,s}\rightarrow\sum_{s=1}^{w_k-1}(\sigma_s^k-\gamma^{k,s})=0$$

The only solution of this system is $\sigma_s^k = \gamma^{k,s}$ for each $s \in \{1, ..., w_k - 1\}$ for the demand k. As k is chosen arbitrarily in K, we iterate the same procedure for all $k' \in K \setminus \{k\}$. We then get that

$$\sigma_s^k = \gamma^{k,s}, \text{ for all } k \in K \text{ and all } s \in \{1, ..., w_k - 1\}. \tag{5.9}$$

As a result $(\mu, \sigma) = \gamma \tilde{A}$ which ends the proof. □

Theorem 5.3.1. *The dimension of $P_{sa}(G, K, \mathbb{S})$ is given by*

$$dim(P_{sa}(G, K, \mathbb{S})) = |K| * |\mathbb{S}| + |\mathbb{S}| - r' = |K| * |\mathbb{S}| + |\mathbb{S}| - \sum_{k \in K}(w_k - 1).$$

Proof. Given the rank of the matrix $\tilde{A}$ which equals to r' and the proposition (5.3.1). □

5.3.2 Facial Investigation

Here we study the facial structure of the basic constraints of the compact formulation (5.1)-(5.8) that are facets defining for the polyhedron $P_{sa}(G, K, \mathbb{S})$ under certain conditions.

Theorem 5.3.2. *Consider a demand $k \in K$ and a slot $s \in \{w_k, .., \bar{s}\}$. Then, the inequality $z_s^k \geq 0$ is facet defining for $P_{sa}(G, K, \mathbb{S})$.*

Proof. Let us denote $\tilde{F}_s^k$ the face induced by inequality $z_s^k \geq 0$, which is given by

$$\tilde{F}_s^k = \{(u, z) \in P_{sa}(G, K, \mathbb{S}) : z_s^k = 0\}.$$

In order to prove that inequality $z_s^k \geq 0$ is facet defining for $P_{sa}(G, K, \mathbb{S})$, we start checking that $\tilde{F}_s^k$ is a proper face, and $\tilde{F}_s^k \neq P_{sa}(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{109} = (U^{109}, S^{109})$ as below

a) a set of last-slots S_k^{109} is assigned to each demand $k \in K$ with $|S_k^{109}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{109} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s' \in S_k^{109}$ and $s" \in \{s' - w_k + 1, ..., s'\}$, we have $s" \in U^{109}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{109}$ and $s" \in S_{k'}^{109}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint),

d) and the slot s is not chosen as a last-slot for the demand k in the solution $\mathcal{S}^{109}$, i.e., $s \notin S_k^{109}$.

Obviously, $\mathcal{S}^{109}$ is feasible solution for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Moreover, the corresponding incidence vector $(u^{\mathcal{S}^{109}}, z^{\mathcal{S}^{109}})$ is belong to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}_s^k$ given that it is composed by $z_s^k = 0$. As a result, $\tilde{F}_s^k$ is not empty ($\tilde{F}_s^k \neq \emptyset$). Furthermore, given that $s \in \{w_k, ..., \bar{s}\}$ for the demand k, this means that there exists at least one feasible solution for the SA problem in which $s \in S_k$ for the demand k. As a result, $\tilde{F}_s^k \neq P_{sa}(G, K, \mathbb{S})$.
On the other hand, we know that all the solutions of $\tilde{F}_s^k$ are in $P_{sa}(G, K, \mathbb{S})$ which means that they verify the equations system (5.2) s.t. the new equations system (5.10) associated with $\tilde{F}_s^k$ is written as below

$$\begin{cases} z_s^k = 0, \text{ s.t. } k \text{ and } s \text{ are chosen arbitrarily,} \\ z_s^k = 0, \text{ for all } k \in K \text{ and all } s \in \{1, ..., w_k - 1\}. \end{cases} \tag{5.10}$$

The equation $z_s^k = 0$ is not result of equations of system (5.2) which means that the equation $z_s^k = 0$ is not redundant in the system (5.10). As a result, the matrix associated with the system (5.10) denoted as M is of full rank. As a result, the dimension of the face $\tilde{F}_s^k$ is equal to

$$dim(\tilde{F}_s^k) = |K| * |\mathbb{S}| + |\mathbb{S}| - rank(M) = |K| * |\mathbb{S}| + |\mathbb{S}| - (1 + \tilde{r}) = dim(P_{sa}(G, K, \mathbb{S})) - 1.$$

As a result, the face $\tilde{F}_s^k$ is facet defining for $P_{sa}(G, K, \mathbb{S})$.

□

Theorem 5.3.3. *Consider a slot $s \in \mathbb{S}$. Then, the inequality $u_s \geq 0$ is facet defining for $P_{sa}(G, K, \mathbb{S})$.*

Proof. Let us denote F_s the face induced by inequality $u_s \geq 0$, which is given by

$$F_s = \{(u, z) \in P_{sa}(G, K, \mathbb{S}) : u_s = 0\}.$$

In order to proof that inequality $u_s \geq 0$ is facet defining for $P_{sa}(G, K, \mathbb{S})$, we start checking that F_s is a proper face and $F_s \neq P_{sa}(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{110} = (U^{110}, S^{110})$ as below

a) a set of last-slots S_k^{110} is assigned to each demand $k \in K$ with $|S_k^{110}| \geq 1$ (contiguity and continuity constraints),
b) a set of slots U^{110} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s' \in S_k^{110}$ and $s" \in \{s' - w_k + 1, ..., s'\}$, we have $s" \in U^{110}$,
c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{110}$ and $s" \in S_{k'}^{110}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint),
d) and the slot s is not used in the solution $\mathcal{S}^{110}$, i.e., $s \notin U^{110}$ s.t. for each demand k and last-slot $s' \in S_k^{110}$ we have $s \notin \{s' - w_k + 1, ..., s'\}$.

Obviously, $\mathcal{S}^{110}$ is feasible solution for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Moreover, the corresponding incidence vector $(u^{\mathcal{S}^{110}}, z^{\mathcal{S}^{110}})$ is belong to $P_{sa}(G, K, \mathbb{S})$ and then to F_s given that it is composed by $u_s = 0$. As a result, F_s is not empty ($F_s \neq \emptyset$). Furthermore, given that $s \in \mathbb{S}$, this means that there exists at least one feasible solution for the SA problem in which s is used , i.e., $s \in U$. As a result, $F_s \neq P_{sa}(G, K, \mathbb{S})$.
On the other hand, we know that all the solutions of F_s are in $P_{sa}(G, K, \mathbb{S})$ which means that they verify the equations system (5.2) s.t. the new equations system (5.11) associated with F_s is written as below

$$\begin{cases} u_s = 0, \text{ s.t. } s \text{ is chosen arbitrarily,} \\ z_s^k = 0, \text{ for all } k \in K \text{ and all } s \in \{1, ..., w_k - 1\}. \end{cases} \tag{5.11}$$

The equation $u_s = 0$ is not result of equations of system (5.2) which means that the equation $u_s = 0$ is not redundant in the system (5.11). As a result, the matrix associated with the system (5.11) denoted as M' is of full rank. As a result, the dimension of the face F_s is equal to

$$dim(F_s) = |K| * |\mathbb{S}| + |\mathbb{S}| - rank(M') = |K| * |\mathbb{S}| + |\mathbb{S}| - (1 + \tilde{r}) = dim(P_{sa}(G, K, \mathbb{S})) - 1.$$

As a result, the face F_s is facet defining for $P_{sa}(G, K, \mathbb{S})$.

□

Theorem 5.3.4. *Consider a slot $s \in \mathbb{S}$. Then, the inequality $u_s \leq 1$ is facet defining for $P_{sa}(G, K, \mathbb{S})$.*

Proof. Let us denote F'_s the face induced by inequality $u_s \leq 1$ given by

$$F'_s = \{(u, z) \in P_{sa}(G, K, \mathbb{S}) : u_s = 1\}.$$

In order to proof that inequality $u_s \leq 1$ is facet defining for $P_{sa}(G, K, \mathbb{S})$, we start checking that F'_s is a proper face and $F'_s \neq P_{sa}(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{111} = (U^{111}, S^{111})$ as below

a) a set of last-slots S^{111}_k is assigned to each demand $k \in K$ with $|S^{111}_k| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{111} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s' \in S^{111}_k$ and $s" \in \{s' - w_k + 1, ..., s'\}$, we have $s" \in U^{111}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{111}_k$ and $s" \in S^{111}_{k'}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint),

d) and the slot s is used in the solution $\mathcal{S}^{111}$, i.e., $s \in U^{111}$ s.t. there exist at leat one demand $k \in K$ and last-slot $s' \in S^{111}_k$ with $s \in \{s' - w_k + 1, ..., s'\}$.

Obviously, $\mathcal{S}^{111}$ is feasible solution for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Moreover, the corresponding incidence vector $(u^{\mathcal{S}^{111}}, z^{\mathcal{S}^{111}})$ is belong to $P_{sa}(G, K, \mathbb{S})$ and then to F'_s given that it is composed by $u_s = 1$. As a result, F'_s is not empty ($F'_s \neq \emptyset$). Furthermore, given that $s \in \mathbb{S}$, this means that there exists at least one feasible solution for the SA problem in which s is not used , i.e., $s \notin U$. As a result, $F'_s \neq P_{sa}(G, K, \mathbb{S})$.
On the other hand, we know that all the solutions of F'_s are in $P_{sa}(G, K, \mathbb{S})$ which means that they verify the equations system (5.2) s.t. the new equations system (5.12) associated with F'_s is written as below

$$\begin{cases} u_s = 1, \text{ s.t. } s \text{ is chosen arbitrarily,} \\ z^k_s = 0, \text{ for all } k \in K \text{ and all } s \in \{1, ..., w_k - 1\}. \end{cases} \tag{5.12}$$

The equation $u_s = 1$ is not result of equations of system (5.2) which means that the equation $u_s = 1$ is not redundant in the system (5.12). As a result, the matrix associated with the system (5.12) denoted as $\tilde{M}$ is of full rank. As a result, the dimension of the face F'_s is equal to

$$dim(F'_s) = |K| * |\mathbb{S}| + |\mathbb{S}| - rank(\tilde{M}) = |K| * |\mathbb{S}| + |\mathbb{S}| - (1 + \tilde{r}) = dim(P_{sa}(G, K, \mathbb{S})) - 1.$$

As a result, the face F'_s is facet defining for $P_{sa}(G, K, \mathbb{S})$.

□

Theorem 5.3.5. *For a demand $k \in K$, inequality $\sum_{s=w_k}^{\bar{s}} z^k_s \geq 1$ is facet defining for $P_{sa}(G, K, \mathbb{S})$.*

Proof. Let us denote $\tilde{F}^k_{\mathbb{S}}$ the face induced by inequality $\sum\limits_{s=w_k}^{\bar{s}} z^k_s \geq 1$, which is given by

$$\tilde{F}^k_{\mathbb{S}} = \{(x, z) \in P_{sa}(G, K, \mathbb{S}) : \sum_{s=w_k}^{\bar{s}} z^k_s = 1\}$$

In order to proof that inequality $\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1$ is facet defining for $P_{sa}(G,K,\mathbb{S})$, we start checking that $\tilde{F}_{\mathbb{S}}^k$ is a proper face which means that it is not empty, and $\tilde{F}_{\mathbb{S}}^k \neq P_{sa}(G,K,\mathbb{S})$. We construct a solution $\mathcal{S}^{112} = (U^{112}, S^{112})$ as below

a) a set of last-slots S_k^{112} is assigned to each demand $k \in K$ with $|S_k^{112}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{112} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{112}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{112}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{112}$ and $s" \in S_{k'}^{112}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint),

d) and one slot s from the set $\{w_k, ..., \bar{s}\}$ is chosen for the demand k as a last-slot in the solution $\mathcal{S}^{112}$, i.e., $|S_k^{112}| = 1$.

Obviously, $\mathcal{S}^{112}$ is feasible solution for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Moreover, the corresponding incidence vector $(u^{\mathcal{S}^{112}}, z^{\mathcal{S}^{112}})$ is belong to $P_{sa}(G,K,\mathbb{S})$ and then to $\tilde{F}_{\mathbb{S}}^k$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$. As a result, $\tilde{F}_{\mathbb{S}}^k$ is not empty ($\tilde{F}_{\mathbb{S}}^k \neq \emptyset$). Furthermore, given that $s \in \{w_k, ..., \bar{s}\}$ for the demand k, this means that there exists at least one feasible solution for the problem in which $|S_k| \geq 2$ for the demand k. As a result, $\tilde{F}_{\mathbb{S}}^k \neq P_{sa}(G,K,\mathbb{S})$.
On the other hand, we know that all the solutions of $\tilde{F}_{\mathbb{S}}^k$ are in $P_{sa}(G,K,\mathbb{S})$ which means that they verify the equations system (5.2) s.t. the following equations system (5.13) associated with F_S^k is written as below

$$\begin{cases} \displaystyle\sum_{s=w_k}^{\bar{s}} z_s^k = 1, \text{ s.t. } k \text{ is chosen arbitrarily}, \\ z_s^k = 0, \text{ for all } k \in K \text{ and all } s \in \{1, ..., w_k - 1\}. \end{cases} \tag{5.13}$$

The system (5.13) shows that the equation $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$ is not result of equations of system (5.2) which means that the equation $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$ is not redundant in the system (5.13). As a result, the matrix associated with the system (5.13) denoted as M_3 is in full rank which implies that the dimension of the face $\tilde{F}_{\mathbb{S}}^k$ is equal to

$$dim(\tilde{F}_{\mathbb{S}}^k) = |K| * |\mathbb{S}| + |\mathbb{S}| - rank(M_3) = |K| * |\mathbb{S}| + |\mathbb{S}| - (1 + \tilde{r}) = dim(P_{sa}(G,K,\mathbb{S})) - 1.$$

As a result, the face $\tilde{F}_{\mathbb{S}}^k$ is facet defining for $P_{sa}(G,K,\mathbb{S})$.

We strengthen the proof as follows. We denote the inequality $\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1$ by $\alpha u + \beta z \leq \lambda$.
Let $\mu u + \sigma z \leq \tau$ be a valid inequality that defines a facet F of $P_{sa}(G,K,\mathbb{S})$. Suppose that $\tilde{F}_{\mathbb{S}}^k \subset F = \{(u,z) \in P_{sa}(G,K,\mathbb{S}) : \mu u + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)}$) s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma \tilde{A}$, and that

a) $\sigma_s^{k'} = 0$ for all demands $k' \in K \setminus \{k\}$ and all slots $s \in \{w_k, ..., \bar{s}\}$,

b) and $\mu_s = 0$ for all slots $s \in \mathbb{S}$,,

c) and all σ_s^k are equivalents for demand k and slots $s \in \{w_k, ..., \bar{s}\}$.

First, let show that $\mu_s = 0$ for all $s \in \mathbb{S}$. Consider a slot $\tilde{s} \in \mathbb{S}$. To do so, we consider a solution $\mathcal{S}^{113} = (U^{113}, S^{113})$ in which

a) a set of last-slots S_k^{113} is assigned to each demand $k \in K$ with $|S_k^{113}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{113} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{113}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{113}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{113}$ and $s" \in S_{k'}^{113}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in S_k^{113}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\tilde{s} \notin \{s - w_k + 1, ..., s\}$ for each $k \in K$ and $s \in S_k^{113}$ (slot-assignment constraint taking into account the possibility of adding the slot $\tilde{s}$ in the set of used slots U^{113}),

e) and one slot s from the set $\{w_k, ..., \bar{s}\}$ is chosen for the demand k as a last-slot in the solution $\mathcal{S}^{113}$, i.e., $|S_k^{113}| = 1$.

$\mathcal{S}^{113}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{113}}, z^{\mathcal{S}^{113}})$ belongs to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}_{\mathbb{S}}^k$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$. Based on this, we derive a solution $\mathcal{S}'^{113} = (U'^{113}, S'^{113})$ from the solution $\mathcal{S}^{113}$ by adding the slot $\tilde{s}$ as a used slot in U'^{113} without modifying the the last-slots assigned to the demands K in $\mathcal{S}^{113}$ which remain the same in the solution $\mathcal{S}'^{113}$ i.e., $S_k^{113} = S_k'^{113}$ for each demand $k \in K$. The solution $\mathcal{S}'^{113}$ is feasible given that

a) a set of last-slots $S_k'^{113}$ is assigned to each demand $k \in K$ with $|S_k'^{113}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U'^{113} used in $\mathbb{S}$ s.t. $\tilde{s} \in U'^{113}$, and for each demand k and last-slot $s \in S_k'^{113}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U'^{113}$,

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k'^{113}$ and $s' \in S_{k'}'^{113}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s \in S_k'^{113}, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint),

d) and one slot s from the set $\{w_k, ..., \bar{s}\}$ is chosen for the demand k as a last-slot in the solution $\mathcal{S}'^{113}$, i.e., $|S_k'^{113}| = 1$.

$\mathcal{S}'^{113}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}'^{113}}, z^{\mathcal{S}'^{113}})$ belongs to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}_{\mathbb{S}}^k$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$. We then obtain that

$$\mu u^{\mathcal{S}^{113}} + \sigma z^{\mathcal{S}^{113}} = \mu u^{\mathcal{S}'^{113}} + \sigma z^{\mathcal{S}'^{113}} = \mu u^{\mathcal{S}^{113}} + \sigma z^{\mathcal{S}^{113}} + \mu_{\tilde{s}}.$$

It follows that $\mu_{\tilde{s}} = 0$ for the slot $\tilde{s} \in \mathbb{S}$.
The slot $\tilde{s}$ is chosen arbitrarily in $\mathbb{S}$, we iterate the same procedure for all feasible slots in $\mathbb{S}$ s.t. we find

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Next, we will show that, $\sigma_{s'}^{k'} = 0$ for all $k' \in K \setminus \{k\}$ and all $s' \in \{w_{k'}, ..., \bar{s}\}$. Consider the demand k' in $K \setminus \{k\}$ and a slot s' in $\{w_{k'}, ..., \bar{s}\} \setminus \{s\}$. For that, we consider a solution $\mathcal{S}^{114} = (U^{114}, S^{114})$ in which

a) a set of last-slots S_k^{114} is assigned to each demand $k \in K$ with $|S_k^{114}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{114} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{114}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{114}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{114}$ and $s" \in S_{k'}^{114}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in S_k^{114}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_{k'} + 1, ..., s'\} \cap \{s" - w_k + 1, ..., s"\} = \emptyset$ for each $k \in K$ and $s" \in S_k^{114}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S_{k'}^{114}$ assigned to the demand k' in the solution $\mathcal{S}^{114}$),

e) and $|S_k^{114}| = 1$ for the demand k.

$\mathcal{S}^{114}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{114}}, z^{\mathcal{S}^{114}})$ is belong to F and then to $\tilde{F}_{\mathbb{S}}^k$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$. Based on this, we derive a solution $\mathcal{S}^{115} = (U^{115}, S^{115})$ from the solution $\mathcal{S}^{114}$ by adding the slot s' as last-slot to the demand k' without modifying the last-slots assigned to the demands $K \setminus \{k'\}$ in $\mathcal{S}_k^{114}$ remain the same in the solution $\mathcal{S}^{115}$ i.e., $S_k^{114} = S_k^{115}$ for each demand $k \in K \setminus \{k'\}$, and $S_{k'}^{115} = S_{k'}^{114} \cup \{s'\}$ for the demand k'. The solution $\mathcal{S}^{115}$ is feasible given that

a) a set of last-slots S_k^{115} is assigned to each demand $k \in K$ with $|S_k^{115}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{115} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{115}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{115}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{115}$ and $s" \in S_{k'}^{115}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in S_k^{115}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $|S_k^{115}| = 1$ for the demand k.

The corresponding incidence vector $(u^{\mathcal{S}^{115}}, z^{\mathcal{S}^{115}})$ is belong to F and then to $\tilde{F}_{\mathbb{S}}^k$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$. We the obtain that

$$\mu u^{\mathcal{S}^{114}} + \sigma z^{\mathcal{S}^{114}} = \mu u^{\mathcal{S}^{115}} + \sigma z^{\mathcal{S}^{115}} = \mu u^{\mathcal{S}^{114}} + \sigma z^{\mathcal{S}^{114}} + \sigma_{s'}^{k'} + \sum_{\tilde{s} \in \{s' - w_{k'} + 1, ..., s'\} \setminus U^{114}} \mu_{\tilde{s}}.$$

It follows that $\sigma_{s'}^{k'} = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ given that $\mu_{\tilde{s}} = 0$ for all $\tilde{s} \in \mathbb{S}$. The slot s' is chosen arbitrarily for the demand k', we iterate the same procedure for all feasible slots in $\{w_{k'}, ..., \bar{s}\}$ of demand k' s.t. we find

$$\sigma_{s'}^{k'} = 0, \text{ for the demand } k' \text{ and all slots } s' \in \{w_{k'}, ..., \bar{s}\}.$$

Given that the demand k' is chosen arbitrarily. We iterate the same thing for all the demands $k"$ in $K \setminus \{k, k'\}$ such that

$$\sigma_s^{k"} = 0, \text{ for all } k" \in K \setminus \{k, k'\} \text{ and all slots } s \in \{w_{k"}, ..., \bar{s}\}$$

Consequently, we conclude that

$$\sigma_{s'}^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s' \in \{w_{k'}, ..., \bar{s}\}.$$

Let's prove now that σ_s^k for demand k and slots s in $\{w_k, ..., \bar{s}\}$ are equivalent. Consider a slot $s' \in \{w_k, ..., \bar{s}\}$ s.t. $s' \notin S_k^{116}$. For that, we consider a solution $\tilde{\mathcal{S}}^{116} = (\tilde{U}^{116}, \tilde{S}^{116})$ in which

a) a set of last-slots $\tilde{S}_k^{116}$ is assigned to each demand $k \in K$ with $|\tilde{S}_k^{116}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots $\tilde{U}^{116}$ used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in \tilde{S}_k^{116}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in \tilde{U}^{116}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in \tilde{S}_k^{116}$ and $s" \in \tilde{S}_{k'}^{116}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in \tilde{S}_k^{116}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in \tilde{S}_{k'}^{116}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $\tilde{S}_k^{116}$ assigned to the demand k in the solution $\tilde{\mathcal{S}}^{116}$).,

e) and $|\tilde{S}_k^{116}| = 1$ for the demand k.

$\tilde{\mathcal{S}}^{116}$ is clearly feasible for the problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\tilde{\mathcal{S}}^{116}}, z^{\tilde{\mathcal{S}}^{116}})$ is belong to F and then to $\tilde{F}_{\mathbb{S}}^k$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$. Based on this, we derive a solution $\mathcal{S}^{117} = (U^{117}, S^{117})$ from the solution $\tilde{\mathcal{S}}^{116}$ by adding the slot s' as last-slot to the demand k and removing the last slot $s \in S_k^{116}$, i.e., $S_k^{117} = (\tilde{S}_k^{116} \setminus \{s\}) \cup \{\tilde{s}\}$ for the demand k s.t. $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S_{k'}^{117}$ with $E_k^{117} \cap E_{k'}^{117} \neq \emptyset$. The last-slots assigned to the demands $K \setminus \{k\}$ in $\tilde{\mathcal{S}}^{116}$ remain the same, i.e., $\tilde{S}_{k"}^{116} = S_{k"}^{117}$ for each demand $k" \in K \setminus \{k\}$. The solution $\mathcal{S}^{117}$ is feasible given that

a) a set of last-slots S_k^{117} is assigned to each demand $k \in K$ with $|S_k^{117}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{117} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{117}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{117}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{117}$ and $s" \in S_{k'}^{117}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in S_k^{117}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $|S_k^{117}| = 1$.

The corresponding incidence vector $(u^{\mathcal{S}^{117}}, z^{\mathcal{S}^{117}})$ is belong to F and then to $\tilde{F}_{\mathbb{S}}^k$ given that it is composed by $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$. We then obtain that

$$\mu u^{\tilde{\mathcal{S}}^{116}} + \sigma z^{\tilde{\mathcal{S}}^{116}} = \mu u^{\mathcal{S}'^{117}} + \sigma z^{\mathcal{S}'^{117}} = \mu u^{\tilde{\mathcal{S}}^{116}} + \sigma z^{\tilde{\mathcal{S}}^{116}} - \sigma_s^k + \sigma_{s'}^k - \sum_{\tilde{s} \in U^{116} \setminus U^{117}} \mu_{\tilde{s}} + \sum_{\tilde{s}' \in U^{117} \setminus U^{116}} \mu_{\tilde{s}'}.$$

It follows that $\sigma_{s'}^k = \sigma_s^k$ for demand k' and a slots $s, s' \in \{w_k, ..., \bar{s}\}$ given that $\mu_{\tilde{s}} = 0$ for all $\tilde{s} \in \mathbb{S}$.
The slot s is chosen arbitrarily for the demand k in $\{w_k, ..., \bar{s}\}$, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k s.t. we find

$$\sigma_{s'}^k = \sigma_s^k, \text{ for all slots } s, s' \in \{w_k, ..., \bar{s}\}.$$

Consequently, we obtain that $\sigma_s^k = \rho$ for demand k and slots s in $\{w_k, ..., \bar{s}\}$.
On the other hand, we use the same technique applied in the polyhedron dimension proof 5.3.1 to prove that

$$\sigma_{s'}^{k'} = \gamma^{k',s'}, \text{ for all } k' \in K \text{ and all } s' \in \{1, ..., w_{k'} - 1\}.$$

We conclude that $\mu_s = 0$ for each slot $s \in \mathbb{S}$, and for each $k' \in K$ and $s \in \mathbb{S}$

$$\sigma_s^{k'} = \begin{cases} \gamma^{k',s}, \text{if } s \in \{1, ..., w_{k'} - 1\} \\ \rho, \text{if } k' = k \text{ and } s \in \{w_{k'}, ..., \bar{s}\} \\ 0, otherwise. \end{cases}$$

As a result $(\mu, \sigma) = \sum\limits_{s=w_k}^{\bar{s}} \rho\beta_s^k + \gamma\tilde{A}$ which ends our strengthening of the proof. □

5.4 Valid Inequalities and Facets

In what follows, we present several valid inequalities for $P_{sa}(G, K, \mathbb{S})$, and prove that they are facet-defining under certain conditions.

5.4.1 Interval-Capacity-Cover Inequalities

We start this section by introducing some classes of valid inequalities related to the knapsack constraints. Let us introduce the following conflict graph.

Definition 5.4.1. *Consider the conflict graph $\tilde{G}^r$ defined as follows. For each demand $k \in K$, consider a node v_k in $\tilde{G}^r$. Two nodes v_k and $v_{k'}$ are linked by an edge in $\tilde{G}^r$ iff $E(p_k) \cap E(p_{k'}) \neq \emptyset$. This is equivalent to say that two linked nodes v_k and $v_{k'}$ means that the routing paths of the demands k, k' share an edge in G.*

Based on the definitions (2.4.3) and on the conflict graph $\tilde{G}^r$, we introduce the following inequalities.

Proposition 5.4.1. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $j \geq i + 1$. Let $K' \subset K$ be a minimal cover for interval $I = [s_i, s_j]$ s.t. K' defines a clique in $\tilde{G}^r$. Then, the inequality*

$$\sum_{k \in K'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |K'| - 1, \tag{5.14}$$

is valid for $P_{sa}(G, K, \mathbb{S})$.

Proof. The interval $I = [s_i, s_j]$ can cover at most $|K'| - 1$ demands given that K' is a minimal cover for interval $I = [s_i, s_j]$ over edge e. We start the proof by assuming that the inequality (5.14) is not valid for $P_{sa}(G, K, \mathbb{S})$. It follows that there exists a SA solution S in which $\{s_i + w_k - 1, ..., s_j\} \cap S_k = \emptyset$ for a demand $k \in K'$ s.t.

$$\sum_{k \in K'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) > |K'| - 1.$$

Since $\{s_i + w_k - 1, ..., s_j\} \cap S_k = \emptyset$ for a demand $k \in K'$ this means that $\sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) = 0$, and taking into account that K' is minimal cover for the interval $I = [s_i, s_j]$, and $\sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) \leq 1$ for each demand $k \in K'$, it follows that

$$\sum_{k' \in K' \setminus \{k\}} \sum_{s=s_i+w_{k'}-1}^{s_j} z_s^{k'}(S) \leq |K'| - 1,$$

which contradicts what we supposed before, i.e., $\sum_{k\in K'}\sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) > |K'|-1$.
Hence $\sum_{k\in K'} |S_k \cap \{s_i+w_k-1,...,s_j\}| \leq |K'|-1$.
We conclude at the end that the inequality (5.14) is valid for $P_{sa}(G,K,\mathbb{S})$. □

The inequality (5.14) can be strengthened using an extention of each minimal cover $K' \subset K$ for an interval I as follows.

Proposition 5.4.2. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1,\bar{s}]$. Let $K' \subseteq K$ be a minimal cover for interval $I = [s_i, s_j]$ s.t. K' defines a clique in $\tilde{G}^r$, and $\Xi(K')$ be a subset of demands in $K \setminus K'$ s.t. $\Xi(K') = \{k \in K \setminus K'$ s.t. $w_k \geq w_{k'}$ and $E(p_k)\cap E(p_{k'}) \neq \emptyset \quad \forall k' \in K'\}$. Then, the inequality*

$$\sum_{k\in K'}\sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{k'\in\Xi(K')}\sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq |K'|-1, \tag{5.15}$$

is valid for $P_{sa}(G,K,\mathbb{S})$.

Proof. The interval $I = [s_i, s_j]$ can cover at most $|K'|-1$ demands from the demands in $K' \cup \Xi(K')$ given that K' is a minimal cover for interval $I = [s_i, s_j]$ and the definition of the set $\Xi(K')$ s.t. for each pair (k, k') with $k \in K'$ and $k' \in \Xi(K')$, the set $(K' \setminus \{k\}) \cup \{k'\}$ stills defining minimal cover for the interval I over the edge e. Furthermore, for each quadruplet $(k, k', \tilde{k}, \tilde{k}')$ with $k, k' \in K'$ and $\tilde{k}, \tilde{k}' \in \Xi(K')$, the set $(K' \setminus \{k, k'\}) \cup \{\tilde{k}, \tilde{k}'\}$ stills defining minimal cover for the interval I given that $w_k + w_{k'} \leq w_{\tilde{k}} + w_{\tilde{k}'}$.
We strengthen the proof as follows. Let's first suppose that the inequality (5.15) is not valid for $P_{sa}(G,K,\mathbb{S})$. It follows that there exists a SA solution S in which $\{s_i+w_{k'}-1,...,s_j\}\cap S_{k'} = \emptyset$ for each demand $k' \in \Xi(K')$ s.t.

$$\sum_{k\in K'}\sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) > |K'|-1.$$

Since $\{s_i+w_{k'}-1,...,s_j\}\cap S_{k'} = \emptyset$ for each demand $k' \in \Xi(K')$ this means that $\sum_{s=s_i+w_{k'}-1}^{s_j} z_s^{k'}(S) = 0$, and taking into account the inequality (5.14), and that K' is minimal cover for the interval $I = [s_i, s_j]$, it follows that

$$\sum_{k\in K'}\sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) \leq |K'|-1,$$

which contradicts what we supposed before, i.e., $\sum_{k\in K'}\sum_{s=s_i+w_k-1}^{s_j} z_s^k(S) > |K'|-1$.
Hence $\sum_{k\in K'} |S_k \cap \{s_i+w_k-1,...,s_j\}| + \sum_{k'\in\Xi(K')} |S_{k'} \cap \{s_i+w_{k'}-1,...,s_j\}| \leq |K'|-1$.
We conclude at the end that the inequality (5.15) is valid for $P_{sa}(G,K,\mathbb{S})$.

□

Moreover, the inequality (5.14) can be more strengthened using lifting procedures proposed by Nemhauser and Wolsey in [109] without modifying its right-hand side.

Theorem 5.4.1. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1,\bar{s}]$ with $j \geq i+1$. Let $\tilde{K} \subset K$ be a minimal cover for interval $I = [s_i, s_j]$ s.t. $\tilde{K}$ defines a clique in $\tilde{G}^r$. Then, the inequality* (5.14) *is facet defining for the polytope $P_{sa}(G,K,\mathbb{S},I)$ iff there does not exist* *an interval of contiguous slots I' in $[1, \bar{s}]$ with $I \subset I'$ s.t. $\tilde{K}$ defines a minimal cover for the interval I' and a clique in $\tilde{G}^r$, where*

$$P_{sa}(G, K, \mathbb{S}, I) = \{(u, z) \in P_{sa}(G, K, \mathbb{S}) : \sum_{k' \in K \setminus \tilde{K} s.t. \ (v_k, v_{k'}) \in \tilde{G}^r, \forall k \in \tilde{K}} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = 0\}.$$

Proof. **Necessity**
If there exists an interval of contiguous slots I' in $[1, \bar{s}]$ with $I \subset I'$ s.t. $\tilde{K}$ defines a minimal cover for the interval I'. This means that $\{s_i + w_k - 1, ..., s_j\} \subset I'$. As a result, the inequality (5.14) induced by the minimal cover $\tilde{K}$ for the interval I, it is dominated by another inequality (5.14) induced by the same minimal cover $\tilde{K}$ for the interval I'. Hence, the inequality (5.14) cannot be facet defining for the polytope $P_{sa}(G, K, \mathbb{S}, I)$.
Sufficiency.
Let $\tilde{F}_{\tilde{K}}^I$ denote the face induced by the inequality (5.14), which is given by

$$\tilde{F}_{\tilde{K}}^I = \{(u, z) \in P_{sa}(G, K, \mathbb{S}) : \sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = |\tilde{K}| - 1\}.$$

In order to prove that inequality $\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |\tilde{K}| - 1$ is facet defining for $P_{sa}(G, K, \mathbb{S}, I)$, we start checking that $\tilde{F}_{\tilde{K}}^I$ is a proper face, and $\tilde{F}_{\tilde{K}}^I \neq P_{sa}(G, K, \mathbb{S}, I)$.
We construct a solution $\mathcal{S}^{119} = (U^{119}, S^{119})$ as below

a) a set of last-slots S_k^{119} is assigned to each demand $k \in K$ with $|S_k^{119}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{119} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{119}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{119}$,

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{119}$ and $s' \in S_{k'}^{119}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint),

d) and there is $|\tilde{K}| - 1$ demands from the minimal cover $\tilde{K}$ denoted by $\tilde{K}_{119}$ which are covered by the interval I (i.e., if $k \in \tilde{K}_{119}$, this means that the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{119}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{119}$ for each $k \in \tilde{K}_{119}$, and for each $s' \in S_{k'}^{119}$ for all $k' \in \tilde{K} \setminus \tilde{K}_{119}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

Obviously, $\mathcal{S}^{119}$ is a feasible solution for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Moreover, the corresponding incidence vector $(u^{\mathcal{S}^{119}}, z^{\mathcal{S}^{119}})$ is belong to $P_{sa}(G, K, \mathbb{S}, I)$ and then to $\tilde{F}_{\tilde{K}}^I$ given that it is composed by $\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = |\tilde{K}| - 1$. As a result, $\tilde{F}_{\tilde{K}}^I$ is not empty (i.e., $\tilde{F}_{\tilde{K}}^I \neq \emptyset$). Furthermore, given that $s \in \{s_i + w_k - 1, ..., s_j\}$ for each $k \in \tilde{K}$, this means that there exists at least one feasible slot assignment S_k for the demands k in $\tilde{K}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ for each $s \in S_k$ and each $k \in \tilde{K}$. This means that $\tilde{F}_{\tilde{K}}^I \neq P_{sa}(G, K, \mathbb{S}, I)$.
We denote the inequality $\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |\tilde{K}| - 1$ by $\alpha u + \beta z \leq \lambda$. Let $\mu u + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P_{sa}(G, K, \mathbb{S}, I)$. Suppose that $\tilde{F}_{\tilde{K}}^I \subset F = \{(u, z) \in P_{sa}(G, K, \mathbb{S}, I) : \mu u + \sigma z = \tau\}$. We show that there exists $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)})$ s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma \tilde{A}$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in \tilde{K}$,

b) and σ_s^k are equivalents for all $k \in \tilde{K}$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$,

c) and $\mu_s = 0$ for all slots $s \in \mathbb{S}$.

Let us show that $\mu_s = 0$ for all $s \in \mathbb{S}$. Consider a slot $\tilde{s} \in \mathbb{S}$. To do so, we consider a solution $\mathcal{S}^{120} = (U^{120}, S^{120})$ in which

a) a set of last-slots S_k^{120} is assigned to each demand $k \in K$ with $|S_k^{120}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{120} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{120}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{120}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{120}$ and $s" \in S_{k'}^{120}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in S_k^{120}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\tilde{s} \notin \{s - w_k + 1, ..., s\}$ for each $k \in K$ and $s \in S_k^{120}$ (slot-assignment constraint taking into account the possibility of adding the slot $\tilde{s}$ in the set of used slots U^{120}).

$\mathcal{S}^{120}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{120}}, z^{\mathcal{S}^{120}})$ belongs to $P_{sa}(G, K, \mathbb{S}, I)$. Based on this, we derive a solution $\mathcal{S}^{121} = (U^{121}, S^{121})$ from the solution $\mathcal{S}^{120}$ by adding the slot $\tilde{s}$ as a used slot in U^{121} without modifying the the last-slots assigned to the demands K in $\mathcal{S}^{120}$ which remain the same in the solution $\mathcal{S}^{121}$ i.e., $S_k^{120} = S_k^{121}$ for each demand $k \in K$. The solution $\mathcal{S}^{121}$ is feasible given that

a) a set of last-slots S_k^{121} is assigned to each demand $k \in K$ with $|S_k^{121}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{121} used in $\mathbb{S}$ s.t. $\tilde{s} \in U^{121}$, and for each demand k and last-slot $s \in S_k^{121}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{121}$,

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{121}$ and $s' \in S_{k'}^{121}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s \in S_k^{121}, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint),

d) and there is $|\tilde{K}| - 1$ demands from the minimal cover $\tilde{K}$ denoted by $\tilde{K}_{121}$ which are covered by the interval I (i.e., if $k \in \tilde{K}_{121}$, this means that the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{121}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{121}$ for each $k \in \tilde{K}_{121}$, and for each $s' \in S_{k'}^{121}$ for all $k' \in \tilde{K} \setminus \tilde{K}_{121}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{121}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{121}}, z^{\mathcal{S}^{121}})$ belongs to $P_{sa}(G, K, \mathbb{S}, I)$ and then to $\tilde{F}_{\tilde{K}}^{I}$ given that it is composed by $\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = |\tilde{K}| - 1$. We then obtain that

$$\mu u^{\mathcal{S}^{120}} + \sigma z^{\mathcal{S}^{120}} = \mu u^{\mathcal{S}^{121}} + \sigma z^{\mathcal{S}^{121}} = \mu u^{\mathcal{S}^{120}} + \sigma z^{\mathcal{S}^{120}} + \mu_{\tilde{s}}.$$

It follows that $\mu_{\tilde{s}} = 0$ for the slot $\tilde{s} \in \mathbb{S}$. The slot $\tilde{s}$ is chosen arbitrarily in $\mathbb{S}$, we iterate the same procedure for all feasible slots in $\mathbb{S}$ s.t. we find

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Let's us show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in \tilde{K}$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in \tilde{K}$. For that, we consider a solution $\mathcal{S}^{122} = (U^{122}, S^{122})$ in which

a) a set of last-slots S_k^{122} is assigned to each demand $k \in K$ with $|S_k^{122}| \geq 1$ (contiguity and continuity constraints),

b) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{122}_k$ and $s" \in S^{122}_{k'}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}'_e} |\{s' \in S^{122}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S^{122}_{k'}$ with $E^{122}_k \cap E^{122}_{k'} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots S^{122}_k assigned to the demand k in the solution $\mathcal{S}^{122}$),

d) and there is $|\tilde{K}| - 1$ demands from the minimal cover $\tilde{K}$ denoted by $\tilde{K}"_{122}$ which are covered by the interval I (i.e., if $k \in \tilde{K}"_{122}$, this means that the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{122}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S^{122}_k$ for each $k \in \tilde{K}"_{122}$, and for each $s' \in S^{122}_{k'}$ for all $k' \in \tilde{K} \setminus \tilde{K}"_{122}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{122}$ is clearly feasible for the problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{122}}, z^{\mathcal{S}^{122}})$ is belong to F and then to $\tilde{F}^I_{\tilde{K}}$ given that it is composed by $\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z^k_s = |\tilde{K}| - 1$. Based on this, we derive a solution $\mathcal{S}^{123} = (U^{123}, S^{123})$ from the solution $\mathcal{S}^{122}$ by adding the slot s' as last-slot to the demand k without modifying the last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{122}$, i.e., $S^{122}_{k'} = S^{123}_{k'}$ for each demand $k' \in K \setminus \{k\}$, and $S^{123}_k = S^{122}_k \cup \{s'\}$ for the demand k. The solution $\mathcal{S}^{123}$ is feasible given that

a) a set of last-slots S^{123}_k is assigned to each demand $k \in K$ with $|S^{123}_k| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{123} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S^{123}_k$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{123}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{123}_k$ and $s" \in S^{123}_{k'}$ with $E^{123}_k \cap E^{123}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E^{123}_k} |\{s' \in S^{123}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(u^{\mathcal{S}^{123}}, z^{\mathcal{S}^{123}})$ is belong to F and then to $\tilde{F}^I_{\tilde{K}}$ given that it is composed by $\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z^k_s = |\tilde{K}| - 1$. We then obtain that

$$\mu u^{\mathcal{S}^{122}} + \sigma z^{\mathcal{S}^{122}} = \mu u^{\mathcal{S}^{123}} + \sigma z^{\mathcal{S}^{123}} = \mu u^{\mathcal{S}^{122}} + \sigma z^{\mathcal{S}^{122}} + \sigma^k_{s'} + \sum_{\tilde{s} \in U^{123} \setminus U^{122}} \mu_{\tilde{s}} - \sum_{\tilde{s} \in U^{122} \setminus U^{123}} \mu_{\tilde{s}}.$$

It follows that $\sigma^k_{s'} = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in \tilde{K}$ given that $\mu_{\tilde{s}} = 0$ for all slots $\tilde{s} \in \mathbb{S}$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in \tilde{K}$ s.t. we find

$$\sigma^k_{s'} = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } s' \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } k \notin \tilde{K}.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma^{k'}_s = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_{k'} - 1, ..., s_j\} \text{ if } k' \notin \tilde{K}.$$

Consequently, we conclude that

$$\sigma^k_s = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } k \notin \tilde{K}.$$

Let prove that σ^k_s for all $k \in \tilde{K}$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$ are equivalents. Consider a demand k' and a slot $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ with $k' \in \tilde{K}$. For that, we consider a solution $\mathcal{S}^{124} = (U^{124}, S^{124})$ in which

a) a set of last-slots S_k^{124} is assigned to each demand $k \in K$ with $|S_k^{124}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{124} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{124}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{124}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{124}$ and $s" \in S_{k'}^{124}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_{e'}} |\{s' \in S_k^{124}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s \in S_k^{124}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S_{k'}^{124}$ assigned to the demand k' in the solution $\mathcal{S}^{124}$),

e) and there is $|\tilde{K}| - 1$ demands from the minimal cover $\tilde{K}$ denoted by $\tilde{K}_{124}$ which are covered by the interval I (i.e., if $k \in \tilde{K}_{124}$, this means that the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{124}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{124}$ for each $k \in \tilde{K}_{124}$, and for each $s' \in S_{k'}^{124}$ for all $k' \in \tilde{K} \setminus \tilde{K}_{124}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{124}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{124}}, z^{\mathcal{S}^{124}})$ is belong to F and then to $\tilde{F}_{\tilde{K}}^I$ given that it is composed by $\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = |\tilde{K}| - 1$. Based on this, we derive a solution $\mathcal{S}^{125} = (E^{125}, S^{125})$ from the solution $\mathcal{S}^{125}$ by adding the slot s' as last-slot to the demand k s.t. the last-slots assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{125}$ remain the same in $\mathcal{S}^{125}$, i.e., $S_{k"}^{125} = S_{k"}^{125}$ for each demand $k" \in K \setminus \{k, k'\}$, and $S_{k'}^{125} = S_{k'}^{125} \cup \{s'\}$ for the demand k', and modifying the last-slots assigned to the demand k by adding a new last-slot $\tilde{s}$ and removing the last slot $s \in S_k^{125}$ with $s \in \{s_i + w_k + 1, ..., s_j\}$ and $\tilde{s} \notin \{s_i + w_k + 1, ..., s_j\}$ for the demand k with $k \in \tilde{K}$ s.t. $S_k^{125} = (S_k^{125} \setminus \{s\}) \cup \{\tilde{s}\}$ s.t. $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}^{125}$ with $E_k^{125} \cap E_{k'}^{125} \neq \emptyset$. The solution $\mathcal{S}^{125}$ is feasible given that

a) a set of last-slots S_k^{125} is assigned to each demand $k \in K$ with $|S_k^{125}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{125} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{125}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{125}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{125}$ and $s" \in S_{k'}^{125}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_{e'}} |\{s' \in S_k^{125}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(u^{\mathcal{S}^{125}}, z^{\mathcal{S}^{125}})$ is belong to F and then to $\tilde{F}_{\tilde{K}}^I$ given that it is composed by $\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = |\tilde{K}| - 1$. We then obtain that

$$\mu u^{\mathcal{S}^{124}} + \sigma z^{\mathcal{S}^{124}} = \mu u^{\mathcal{S}^{125}} + \sigma z^{\mathcal{S}^{125}} = \mu u^{\mathcal{S}^{124}} + \sigma z^{\mathcal{S}^{124}} + \sigma_{s'}^{k'} - \sigma_s^k + \sigma_{\tilde{s}}^k + \sum_{s" \in U^{125} \setminus U^{124}} \mu_{s"} - \sum_{s" \in U^{124} \setminus U^{125}} \mu_{s"}.$$

It follows that $\sigma_{s'}^{k'} = \sigma_s^k$ for demand k' and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $k' \in \tilde{K}$ and $s' \in \{s_i + w_{k'} + 1, ..., s_j\}$ given that $\sigma_S^k = 0$ for $S \notin \{s_i + w_k - 1, ..., s_j\}$ with $k \in \tilde{K}$, and $\mu_{s"} = 0$ for all $s" \in \mathbb{S}$.

Given that the pair (k, k') are chosen arbitrary in the minimal cover $\tilde{K}$, we iterate the same procedure for all pairs (k, k') s.t. we find

$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all pairs } (k, k') \in \tilde{K}$$

with $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. We re-do the same procedure for each two slots $s, s' \in \{s_i + w_k - 1, ..., s_j\}$ for each demand $k \in K$ with $k \in \tilde{K}$ s.t.

$$\sigma_s^k = \sigma_{s'}^k, \text{for all } k \in \tilde{K} \text{ and } s, s' \in \{s_i + w_k - 1, ..., s_j\}.$$

On the other hand, we use the same technique applied in the polyhedron dimension proof of the proposition 5.3.1 to prove that for each $k \in K$ and $s \in \{1, ..., w_k - 1\}$, we have $\sigma_s^k = \gamma^{k,s}$. As a result $(\mu, \sigma) = \sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} \rho\beta_s^k + \gamma\tilde{A}$.

□

Inspiring from the inequality (5.14), we define a new valid inequality as follows.

5.4.2 Interval-Clique Inequalities

Based on the definition of the conflict graph $\tilde{G}_I^E$, we define a new conflict graph adapted to the SA problem.

Definition 5.4.2. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$. Consider the conflict graph $\tilde{G}_I'^E$ defined as follows. For each demand $k \in K$ with $w_k \leq |I|$, consider a node v_k in $\tilde{G}_I'^E$. Two nodes v_k and $v_{k'}$ are linked by an edge in $\tilde{G}_I'^E$ if $w_k + w_{k'} > |I|$ and $E(p_k) \cap E(p_{k'}) \neq \emptyset$.*

Proposition 5.4.3. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$, and C be a clique in the conflict graph $\tilde{G}_I'^E$ with $|C| \geq 3$. Then, the inequality* (2.35) *is also valid for $P_{sa}(G, K, \mathbb{S})$.*

Proof. We use the same proof of the proposition (2.4.13). □

Theorem 5.4.2. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$, and C be a clique in the conflict graph $\tilde{G}_I'^E$ with $|C| \geq 3$. Then, the inequality* (2.35) *is facet defining for $P_{sa}(G, K, \mathbb{S})$ if and only if*

a) C is a maximal clique in the conflict graph $\tilde{G}_I'^E$,

b) and there does not exist an interval of contiguous slots I' in $[1, \bar{s}]$ s.t. $I \subset I'$ with

  a) $w_k + w_{k'} \geq |I'|$ for each $k, k' \in C$,

  b) $w_k \leq |I'|$ and $2w_k \geq |I'| + 1$ for each $k \in C$.

Proof. **Neccessity.**
We distinguish two cases

a) if there exists a clique C' that contains all the demands $k \in C$. Then, the inequality (2.35) induced by the clique C is dominated by another inequality (2.35) induced by the clique C'. Hence, the inequality (2.35) cannot be facet defining for $P_{sa}(G, K, \mathbb{S})$.

b) if there exists an interval of contiguous slots I' in $[1, \bar{s}]$ s.t. $I \subset I'$ with

  a) $w_k + w_{k'} \geq |I'|$ for each $k, k' \in C$,

  b) $w_k \leq |I'|$ and $2w_k \geq |I'| + 1$ for each $k \in C$.

This means that the inequality (2.35) induced by the clique C for the interval I is dominated by the inequality (2.35) induced by the clique C for the interval I'. Hence, the inequality (2.35) cannot be facet defining for $P_{sa}(G, K, \mathbb{S})$.

Sufficiency.
Let $\tilde{F}_C^{\tilde{G}_I'^E}$ denote the face induced by the inequality (2.35), which is given by

$$\tilde{F}_C^{\tilde{G}_I'^E} = \{(u,z) \in P_{sa}(G,K,\mathbb{S}) : \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1\}.$$

In order to prove that inequality $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 1$ is facet defining for $P_{sa}(G,K,\mathbb{S})$, we start checking that $\tilde{F}_C^{\tilde{G}_I'^E}$ is a proper face, and $\tilde{F}_C^{\tilde{G}_I'^E} \neq P_{sa}(G,K,\mathbb{S})$. We construct a solution $\mathcal{S}^{126} = (U^{126}, S^{126})$ as below

a) a set of last-slots S_k^{126} is assigned to each demand $k \in K$ with $|S_k^{126}| \geq 1$ (contiguity and continuity constraints),
b) a set of slots U^{126} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{126}$ and $s' \in \{s-w_k+1,...,s\}$, we have $s' \in U^{126}$,
c) $\{s-w_k+1,...,s\} \cap \{s'-w_{k'}+1,...,s'\} = \emptyset$ for each $k,k' \in K$ and each $s \in S_k^{126}$ and $s' \in S_{k'}^{126}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint),
d) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{126}$ with $s \in \{s_i+w_k-1,...,s_j\}$, i.e., $s \in S_k^{126}$ for a node $v_k \in C$, and for each $s' \in S_{k'}^{126}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $s' \notin \{s_i+w_{k'}-1,...,s_j\}$.

Obviously, $\mathcal{S}^{126}$ is a feasible solution for the problem given that it satisfies all the constraints of our compact formulation. Moreover, the corresponding incidence vector $(u^{\mathcal{S}^{126}}, z^{\mathcal{S}^{126}})$ is belong to $P_{sa}(G,K,\mathbb{S})$ and then to $\tilde{F}_C^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. As a result, $\tilde{F}_C^{\tilde{G}_I'^E}$ is not empty (i.e., $\tilde{F}_C^{\tilde{G}_I'^E} \neq \emptyset$). Furthermore, given that $s \in \{s_i+w_k-1,...,s_j\}$ for each $v_k \in C$, this means that there exists at least one feasible slot assignment S_k for the demands k in C with $s \notin \{s_i+w_k-1,...,s_j\}$ for each $s \in S_k$ and each $v_k \in C$. This means that $\tilde{F}_C^{\tilde{G}_I'^E} \neq P_{sa}(G,K,\mathbb{S})$.
We denote the inequality $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 1$ by $\alpha u + \beta z \leq \lambda$. Let $\mu u + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P_{sa}(G,K,\mathbb{S})$. Suppose that $\tilde{F}_C^{\tilde{G}_I'^E} \subset F = \{(u,z) \in P_{sa}(G,K,\mathbb{S}) : \mu u + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k \in K}(w_k-1)})$ s.t. $(\mu,\sigma) = \rho(\alpha,\beta) + \gamma\tilde{A}$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k,...,\bar{s}\}$ with $s \notin \{s_i+w_k-1,...,s_j\}$ if $v_k \in C$,
b) and $\mu_s = 0$ for all slots $s \in \mathbb{S}$,
c) and σ_s^k are equivalents for all $v_k \in C$ and all $s \in \{s_i+w_k-1,...,s_j\}$.

Let us show that $\mu_s = 0$ for all $s \in \mathbb{S}$. Consider a slot $\tilde{s} \in \mathbb{S}$. To do so, we consider a solution $\mathcal{S}^{127} = (U^{127}, S^{127})$ in which

a) a set of last-slots S_k^{127} is assigned to each demand $k \in K$ with $|S_k^{127}| \geq 1$ (contiguity and continuity constraints),
b) a set of slots U^{127} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{127}$ and $s' \in \{s-w_k+1,...,s\}$, we have $s' \in U^{127}$,
c) $\{s'-w_k+1,...,s'\} \cap \{s"-w_{k'}+1,...,s"\} = \emptyset$ for each $k,k' \in K$ and each $s' \in S_k^{127}$ and $s" \in S_{k'}^{127}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in S_k^{127}, s" \in \{s'-w_k+1,...,s'\}| \leq 1$ (non-overlapping constraint),

d) and $\tilde{s} \notin \{s - w_k + 1, ..., s\}$ for each $k \in K$ and $s \in S_k^{127}$ (slot-assignment constraint taking into account the possibility of adding the slot $\tilde{s}$ in the set of used slots U^{127}),

e) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{127}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{127}$ for a node $v_k \in C$, and for each $s' \in S_{k'}^{127}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{127}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{127}}, z^{\mathcal{S}^{127}})$ belongs to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}_C^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. Based on this, we derive a solution $\mathcal{S}^{128} = (U^{128}, S^{128})$ from the solution $\mathcal{S}^{127}$ by adding the slot $\tilde{s}$ as a used slot in U^{128} without modifying the the last-slots assigned to the demands K in $\mathcal{S}^{127}$ which remain the same in the solution $\mathcal{S}^{128}$ i.e., $S_k^{127} = S_k^{128}$ for each demand $k \in K$. The solution $\mathcal{S}^{128}$ is feasible given that

a) a set of last-slots S_k^{128} is assigned to each demand $k \in K$ with $|S_k^{128}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{128} used in $\mathbb{S}$ s.t. $\tilde{s} \in U^{128}$, and for each demand k and last-slot $s \in S_k^{128}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{128}$,

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{128}$ and $s' \in S_{k'}^{128}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s \in S_k^{128}, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint),

d) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{128}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{128}$ for a node $v_k \in C$, and for each $s' \in S_{k'}^{128}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{128}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{128}}, z^{\mathcal{S}^{128}})$ belongs to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}_C^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. We then obtain that

$$\mu u^{\mathcal{S}^{127}} + \sigma z^{\mathcal{S}^{127}} = \mu u^{\mathcal{S}^{128}} + \sigma z^{\mathcal{S}^{128}} = \mu u^{\mathcal{S}^{127}} + \sigma z^{\mathcal{S}^{127}} + \mu_{\tilde{s}}.$$

It follows that $\mu_{\tilde{s}} = 0$ for the slot $\tilde{s} \in \mathbb{S}$. The slot $\tilde{s}$ is chosen arbitrarily in $\mathbb{S}$, we iterate the same procedure for all feasible slots in $\mathbb{S}$ s.t. we find

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Let's us show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in C$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in C$. For that, we consider a solution $\mathcal{S}^{129} = (U^{129}, S^{129})$ in which

a) a set of last-slots S_k^{129} is assigned to each demand $k \in K$ with $|S_k^{129}| \geq 1$ (contiguity and continuity constraints),

b) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{129}$ and $s" \in S_{k'}^{129}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e'} |\{s' \in S_k^{129}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S_{k'}^{129}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots S_k^{129} assigned to the demand k in the solution $\mathcal{S}^{129}$),

d) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{129}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{129}$ for a node $v_k \in C$, and for each $s' \in S_{k'}^{129}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{129}$ is clearly feasible for the problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{129}}, z^{\mathcal{S}^{129}})$ is belong to F and then to $\tilde{F}_C^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. Based on this, we derive a solution $\mathcal{S}^{130} = (U^{130}, S^{130})$ from the solution $\mathcal{S}^{129}$ by adding the slot s' as last-slot to the demand k without modifying the last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{129}$, i.e., $S_{k'}^{129} = S_{k'}^{130}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{130} = S_k^{129} \cup \{s'\}$ for the demand k. The solution $\mathcal{S}^{130}$ is feasible given that

a) a set of last-slots S_k^{130} is assigned to each demand $k \in K$ with $|S_k^{130}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{130} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{130}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{130}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{130}$ and $s" \in S_{k'}^{130}$ with $E_k^{130} \cap E_{k'}^{130} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E_k^{130}} |\{s' \in S_k^{130}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(u^{\mathcal{S}^{130}}, z^{\mathcal{S}^{130}})$ is belong to F and then to $\tilde{F}_C^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. We then obtain that

$$\mu u^{\mathcal{S}^{129}} + \sigma z^{\mathcal{S}^{129}} = \mu u^{\mathcal{S}^{130}} + \sigma z^{\mathcal{S}^{130}} = \mu u^{\mathcal{S}^{129}} + \sigma z^{\mathcal{S}^{129}} + \sigma_{s'}^k + \sum_{\tilde{s} \in U^{130} \setminus U^{129}} \mu_{\tilde{s}} - \sum_{\tilde{s} \in U^{129} \setminus U^{130}} \mu_{\tilde{s}}.$$

It follows that $\sigma_{s'}^k = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in C$ given that $\mu_{\tilde{s}} = 0$ for all slots $\tilde{s} \in \mathbb{S}$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in C$ s.t. we find

$$\sigma_{s'}^k = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } s' \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } k \notin C.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_{k'} - 1, ..., s_j\} \text{ if } k' \notin C.$$

Consequently, we conclude that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } v_k \notin C.$$

Let prove that σ_s^k for all $v_k \in C$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$ are equivalents. Consider a demand k' and a slot $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ with $v_{k'} \in C$, and a solution $\mathcal{S}^{131} = (U^{131}, S^{131})$ in which

a) a set of last-slots S_k^{131} is assigned to each demand $k \in K$ with $|S_k^{131}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{131} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{131}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{131}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{131}$ and $s" \in S_{k'}^{131}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{131}} |\{s' \in S_k^{131}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s \in S_k^{131}$ with $E_k^{131} \cap E_{k'}^{131} \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S_{k'}^{131}$ assigned to the demand k' in the solution $\mathcal{S}^{131}$),

e) and there is one demand k from the clique C (i.e., $v_k \in C$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{131}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{131}$ for a node $v_k \in C$, and for each $s' \in S_{k'}^{131}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{131}$ is clearly feasible for the problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{131}}, z^{\mathcal{S}^{131}})$ is belong to F and then to $\tilde{F}_C^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. Based on this, we derive a solution $\mathcal{S}^{132} = (U^{132}, S^{132})$ from the solution $\mathcal{S}^{131}$ by adding the slot s' as last-slot to the demand k without modifying the last-slots assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{131}$, i.e., $S_{k"}^{131} = S_{k"}^{132}$ for each demand $k" \in K \setminus \{k, k'\}$, and $S_{k'}^{132} = S_{k'}^{131} \cup \{s'\}$ for the demand k', and with modifying the last-slots assigned to the demand k by adding a new last-slot $\tilde{s}$ and removing the last slot $s \in S_k^{131}$ with $s \in \{s_i + w_k + 1, ..., s_j\}$ and $\tilde{s} \notin \{s_i + w_k + 1, ..., s_j\}$ for the demand k with $v_k \in C$ s.t. $S_k^{132} = (S_k^{131} \setminus \{s\}) \cup \{\tilde{s}\}$ s.t. $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}^{132}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$. The solution $\mathcal{S}^{132}$ is feasible given that

a) a set of last-slots S_k^{132} is assigned to each demand $k \in K$ with $|S_k^{132}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{132} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{132}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{132}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{132}$ and $s" \in S_{k'}^{132}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{132}} |\{s' \in S_k^{132}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(u^{\mathcal{S}^{132}}, z^{\mathcal{S}^{132}})$ is belong to F and then to $\tilde{F}_C^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. We then obtain that

$$\mu u^{\mathcal{S}^{131}} + \sigma z^{\mathcal{S}^{131}} = \mu u^{\mathcal{S}^{132}} + \sigma z^{\mathcal{S}^{132}} = \mu u^{\mathcal{S}^{131}} + \sigma z^{\mathcal{S}^{131}} + \sigma_{s'}^{k'} - \sigma_s^k + \sigma_{\tilde{s}}^k + \sum_{s" \in U^{132} \setminus U^{131}} \mu_{s"} - \sum_{s" \in U^{131} \setminus U^{132}} \mu_{s"}.$$

It follows that $\sigma_{s'}^{k'} = \sigma_s^k$ for demand k' and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $v_{k'} \in C$ and $s' \in \{s_i + w_{k'} + 1, ..., s_j\}$ given that $\sigma_{\tilde{s}}^k = 0$ for $\tilde{s} \notin \{s_i + w_k - 1, ..., s_j\}$ with $v_k \in C$, and $\mu_{s"} = 0$ for all $s" \in \mathbb{S}$.
Given that the pair $(v_k, v_{k'})$ are chosen arbitrary in the clique C, we iterate the same procedure for all pairs $(v_k, v_{k'})$ s.t. we find

$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all pairs } (v_k, v_{k'}) \in C,$$

with $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. We re-do the same procedure for each two slots $s, s' \in \{s_i + w_k - 1, ..., s_j\}$ for each demand $k \in K$ with $v_k \in C$ s.t.

$$\sigma_s^k = \sigma_{s'}^k, \text{for all } v_k \in C \text{ and } s, s' \in \{s_i + w_k - 1, ..., s_j\},$$
$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all } v_k, v_{k'} \in C, s \in \{s_i + w_k - 1, ..., s_j\} \text{ and } s' \in \{s_i + w_{k'} - 1, ..., s_j\}.$$

Consequently, we obtain that $\sigma_s^k = \rho$ for all $v_k \in C$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$.
On the other hand, we use the same technique applied in the polyhedron dimension proof of the proposition 5.3.1 to prove that for each $k \in K$ and $s \in \{1, ..., w_k - 1\}$, we have $\sigma_s^k = \gamma^{k,s}$.
As a result $(\mu, \sigma) = \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} \rho\beta_s^k + \gamma\tilde{A}$.

□

5.4.3 Interval-Odd-Hole Inequalities

Proposition 5.4.4. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$, and H be an odd-hole H in the conflict graph $\tilde{G}_I'^E$ with $|H| \geq 5$. Then, the inequality (2.36) is valid for $P_{sa}(G, K, \mathbb{S})$.*

Proof. We use the same proof of the proposition (5.4.3). □

Theorem 5.4.3. *Let H be an odd-hole in the conflict graph $\tilde{G}_I'^E$ with $|H| \geq 5$. Then, the inequality (2.36) is facet defining for $P_{sa}(G, K, \mathbb{S})$ if and only if*

a) for each node $v_{k'} \notin H$ in $\tilde{G}_I'^E$, there exists a node $v_k \in H$ s.t. the induced graph $\tilde{G}_I'^E((H \setminus \{v_k\}) \cup \{v_{k'}\})$ does not contain an odd-hole $H' = (H \setminus \{v_k\}) \cup \{v_{k'}\}$,

b) and there does not exist a node $v_{k'} \notin H$ in $\tilde{G}_I'^E$ s.t. $v_{k'}$ is linked with all nodes $v_k \in H$,

c) and there does not exist an interval I' of contiguous slots with $I \subset I'$ s.t. H defines also an odd-hole in the associated conflict graph $\tilde{G}_{I'}^E$.

Proof. **Neccessity.**
We use the same proof presented in the proof of theorem (2.4.9).
Sufficiency.
Let $\tilde{F}_H^{\tilde{G}_I'^E}$ denote the face induced by the inequality (2.36), which is given by

$$\tilde{F}_H^{\tilde{G}_I'^E} = \{(u, z) \in P_{sa}(G, K, \mathbb{S}) : \sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H| - 1}{2}\}.$$

In order to prove that inequality $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \frac{|H|-1}{2}$ is facet defining for $P_{sa}(G, K, \mathbb{S})$, we start checking that $\tilde{F}_H^{\tilde{G}_I'^E}$ is a proper face, and $\tilde{F}_H^{\tilde{G}_I'^E} \neq P_{sa}(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{133} = (U^{133}, S^{133})$ as below

a) a set of last-slots S_k^{133} is assigned to each demand $k \in K$ with $|S_k^{133}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{133} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{133}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{133}$,

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{133}$ and $s' \in S_{k'}^{133}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint),

d) and there is $\frac{|H|-1}{2}$ demands $\tilde{H}^{133}$ from the odd-hole H (i.e., $v_k \in \tilde{H}^{133} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{133}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{133}$ for each node $v_k \in \tilde{H}^{133}$, and for each $s' \in S_{k'}^{133}$ for all $v_{k'} \in H \setminus \tilde{H}^{133}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

Obviously, $\mathcal{S}^{133}$ is a feasible solution for the problem given that it satisfies all the constraints of our compact formulation. Moreover, the corresponding incidence vector $(u^{\mathcal{S}^{133}}, z^{\mathcal{S}^{133}})$ is belong to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}_H^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}$. As a result, $\tilde{F}_H^{\tilde{G}_I'^E}$ is not empty (i.e., $\tilde{F}_H^{\tilde{G}_I'^E} \neq \emptyset$). Furthermore, given that $s \in \{s_i + w_k - 1, ..., s_j\}$ for each $v_k \in H$, this means that there exists at least one feasible slot assignment S_k for the demands k in H with $s \notin \{s_i + w_k - 1, ..., s_j\}$ for each $s \in S_k$ and each $v_k \in H$. This means that $\tilde{F}_H^{\tilde{G}_I'^E} \neq P_{sa}(G, K, \mathbb{S})$.
We denote the inequality $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \frac{|H|-1}{2}$ by $\alpha u + \beta z \leq \lambda$. Let $\mu u + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P_{sa}(G, K, \mathbb{S})$. Suppose that $\tilde{F}_H^{\tilde{G}_I'^E} \subset F = \{(u, z) \in P_{sa}(G, K, \mathbb{S}) : \mu u + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)}$ s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma \tilde{A}$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$,
b) and $\mu_s = 0$ for all slots $s \in \mathbb{S}$,
c) and σ_s^k are equivalents for all $v_k \in H$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$.

Let us show that $\mu_s = 0$ for all $s \in \mathbb{S}$. Consider a slot $\tilde{s} \in \mathbb{S}$. To do so, we consider a solution $\mathcal{S}^{134} = (U^{134}, S^{134})$ in which

a) a set of last-slots S_k^{134} is assigned to each demand $k \in K$ with $|S_k^{134}| \geq 1$ (contiguity and continuity constraints),
b) a set of slots U^{134} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{134}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{134}$,
c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{134}$ and $s" \in S_{k'}^{134}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in S_k^{134}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),
d) and $\tilde{s} \notin \{s - w_k + 1, ..., s\}$ for each $k \in K$ and $s \in S_k^{134}$ (slot-assignment constraint taking into account the possibility of adding the slot $\tilde{s}$ in the set of used slots U^{134}),
e) and there is $\frac{|H|-1}{2}$ demands $\tilde{H}^{134}$ from the odd-hole H (i.e., $v_k \in \tilde{H}^{134} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{134}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{134}$ for each node $v_k \in \tilde{H}^{134}$, and for each $s' \in S_{k'}^{134}$ for all $v_{k'} \in H \setminus \tilde{H}^{134}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{134}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{134}}, z^{\mathcal{S}^{134}})$ belongs to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}_H^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}$. Based on this, we derive a solution $\mathcal{S}^{135} = (U^{135}, S^{135})$ from the solution $\mathcal{S}^{134}$ by adding the slot $\tilde{s}$ as a used slot in U^{135} without modifying the the last-slots assigned to the demands K in $\mathcal{S}^{134}$ which remain the same in the solution $\mathcal{S}^{135}$ i.e., $S_k^{134} = S_k^{135}$ for each demand $k \in K$. The solution $\mathcal{S}^{135}$ is feasible given that

a) a set of last-slots S_k^{135} is assigned to each demand $k \in K$ with $|S_k^{135}| \geq 1$ (contiguity and continuity constraints),
b) a set of slots U^{135} used in $\mathbb{S}$ s.t. $\tilde{s} \in U^{135}$, and for each demand k and last-slot $s \in S_k^{135}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{135}$,

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{135}$ and $s' \in S_{k'}^{135}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s \in S_k^{135}, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint),

d) and there is $\frac{|H|-1}{2}$ demands $\tilde{H}^{135}$ from the odd-hole H (i.e., $v_k \in \tilde{H}^{135} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{135}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{135}$ for each node $v_k \in \tilde{H}^{135}$, and for each $s' \in S_{k'}^{135}$ for all $v_{k'} \in H \setminus \tilde{H}^{135}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{135}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{135}}, z^{\mathcal{S}^{135}})$ belongs to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}_H^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1$. We then obtain that

$$\mu u^{\mathcal{S}^{134}} + \sigma z^{\mathcal{S}^{134}} = \mu u^{\mathcal{S}^{135}} + \sigma z^{\mathcal{S}^{135}} = \mu u^{\mathcal{S}^{134}} + \sigma z^{\mathcal{S}^{134}} + \mu_{\tilde{s}}.$$

It follows that $\mu_{\tilde{s}} = 0$ for the slot $\tilde{s} \in \mathbb{S}$. The slot $\tilde{s}$ is chosen arbitrarily in $\mathbb{S}$, we iterate the same procedure for all feasible slots in $\mathbb{S}$ s.t. we find

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Let's us show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$. For that, we consider a solution $\mathcal{S}^{136} = (U^{136}, S^{136})$ in which

a) a set of last-slots S_k^{136} is assigned to each demand $k \in K$ with $|S_k^{136}| \geq 1$ (contiguity and continuity constraints),

b) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{136}$ and $s" \in S_{k'}^{136}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e'} |\{s' \in S_k^{136}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S_{k'}^{136}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots S_k^{136} assigned to the demand k in the solution $\mathcal{S}^{136}$),

d) and there is $\frac{|H|-1}{2}$ demands $\tilde{H}^{136}$ from the odd-hole H (i.e., $v_k \in \tilde{H}^{136} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{136}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{136}$ for each node $v_k \in \tilde{H}^{136}$, and for each $s' \in S_{k'}^{136}$ for all $v_{k'} \in H \setminus \tilde{H}^{136}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{136}$ is clearly feasible for the problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{136}}, z^{\mathcal{S}^{136}})$ is belong to F and then to $\tilde{F}_H^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}$. Based on this, we derive a solution $\mathcal{S}^{137} = (U^{137}, S^{137})$ from the solution $\mathcal{S}^{136}$ by adding the slot s' as last-slot to the demand k without modifying the last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{136}$, i.e., $S_{k'}^{136} = S_{k'}^{137}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{137} = S_k^{136} \cup \{s'\}$ for the demand k. The solution $\mathcal{S}^{137}$ is feasible given that

a) a set of last-slots S_k^{137} is assigned to each demand $k \in K$ with $|S_k^{137}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{137} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{137}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{137}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{137}$ and $s" \in S_{k'}^{137}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_{e'}} |\{s' \in S_k^{137}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(u^{\mathcal{S}^{137}}, z^{\mathcal{S}^{137}})$ is belong to F and then to $\tilde{F}_H^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}$. We then obtain that

$$\mu u^{\mathcal{S}^{136}} + \sigma z^{\mathcal{S}^{136}} = \mu u^{\mathcal{S}^{137}} + \sigma z^{\mathcal{S}^{137}} = \mu u^{\mathcal{S}^{136}} + \sigma z^{\mathcal{S}^{136}} + \sigma_{s'}^k + \sum_{\tilde{s} \in U^{137} \setminus U^{136}} \mu_{\tilde{s}} - \sum_{\tilde{s} \in U^{136} \setminus U^{137}} \mu_{\tilde{s}}.$$

It follows that $\sigma_{s'}^k = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$ given that $\mu_{\tilde{s}} = 0$ for all slots $\tilde{s} \in \mathbb{S}$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$ s.t. we find

$$\sigma_{s'}^k = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } s' \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } v_k \notin H.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_{k'} - 1, ..., s_j\} \text{ if } k' \notin H.$$

Let prove that $\sigma_{s'}^{k'}$ for all $v_{k'} \in H$ and all $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ are equivalents. Consider a demand k' with $v_{k'} \in H$ and a slot $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. For that, we consider a solution $\mathcal{S}^{138} = (U^{138}, S^{138})$ in which

a) a set of last-slots S_k^{138} is assigned to each demand $k \in K$ with $|S_k^{138}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{138} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{138}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{138}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{138}$ and $s" \in S_{k'}^{138}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in S_k^{138}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s' - w_{k'} + 1, ..., s'\} \cap \{s" - w_k + 1, ..., s"\} = \emptyset$ for each $k \in K$ and $s" \in S_k^{138}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S_{k'}^{138}$ assigned to the demand k' in the solution $\mathcal{S}^{138}$),

e) and there is $\frac{|H|-1}{2}$ demands $\tilde{H}^{138}$ from the odd-hole H (i.e., $v_k \in \tilde{H}^{138} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{138}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S_k^{138}$ for each node $v_k \in \tilde{H}^{138}$, and for each $s' \in S_{k'}^{138}$ for all $v_{k'} \in H \setminus \tilde{H}^{138}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{138}$ is clearly feasible for the problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{138}}, z^{\mathcal{S}^{138}})$ is belong to F and then to $\tilde{F}_H^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}$. Based on this, we derive a solution $\mathcal{S}^{139}$ from the solution $\mathcal{S}^{138}$ as belows

a) a set of slots U^{138} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{138}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{138}$,

b) remove the last-slot $\tilde{s}$ totally covered by the interval I and which has been selected by a demand $k_i \in \{v_{k_1}, ..., v_{k_r}\}$ in the solution $\mathcal{S}^{138}$ (i.e., $\tilde{s} \in S_{k_i}^{138}$ and $\tilde{s}' \in \{s_i + w_{k_i} + 1, ..., s_j\}$) s.t. each pair of nodes $(v_{k'}, v_{k_j})$ are not linked in the odd-hole H with $j \neq i$,

c) and select a new last-slot $\tilde{s}' \notin \{s_i + w_{k_i} + 1, ..., s_j\}$ for the demand k_i i.e., $S_{k_i}^{139} = (S_{k_i}^{138} \setminus \{\tilde{s}\}) \cup \{\tilde{s}'\}$ s.t. $\{\tilde{s}' - w_{k_i} - 1, ..., \tilde{s}'\} \cap \{s - w_k + 1, ..., s\} = \emptyset$ for each $k \in K$ and $s \in S_k^{138}$ with $E(p_k) \cap E(p_{k_i}) \neq \emptyset$,

d) and add the slot s' to the set of last-slots $S_{k'}^{138}$ assigned to the demand k' in the solution $\mathcal{S}^{138}$, i.e., $S_{k'}^{139} = S_{k'}^{138} \cup \{s'\}$,

e) without changing the set of last-slots assigned to the demands $K \setminus \{k', k_i\}$, i.e., $S_k^{139} = S_k^{138}$ for each demand $K \setminus \{k', k_i\}$.

The solution $\mathcal{S}^{139}$ is clearly feasible given that

a) a set of last-slots S_k^{139} is assigned to each demand $k \in K$ with $|S_k^{139}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{139} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{139}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{139}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{139}$ and $s" \in S_{k'}^{139}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in S_k^{139}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(u^{\mathcal{S}^{139}}, z^{\mathcal{S}^{139}})$ is belong to F and then to $\tilde{F}_H^{\tilde{G}_I'^E}$ given that it is composed by $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}$. We have so

$$\mu u^{\mathcal{S}^{138}} + \sigma z^{\mathcal{S}^{138}} = \mu u^{\mathcal{S}^{139}} + \sigma z^{\mathcal{S}^{139}} = \mu u^{\mathcal{S}^{138}} + \sigma z^{\mathcal{S}^{138}} + \sigma_{s'}^{k'} + \sigma_{\tilde{s}'}^{k_i} - \sigma_{\tilde{s}}^{k_i} + \sum_{s" \in U^{139} \setminus U^{138}} \mu_{s"} - \sum_{s" \in U^{138} \setminus U^{139}} \mu_{s"}.$$

This implies that $\sigma_{\tilde{s}}^{k_i} = \sigma_{s'}^{k'}$ for $v_{k_i}, v_{k'} \in H$ given that $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k + 1, ..., s_j\}$ if $v_k \in H$ given that $\mu_{s"} = 0$ for all $s" \in \mathbb{S}$. Given that the pair $(v_k, v_{k'})$ are chosen arbitrary in the odd-hole H, we iterate the same procedure for all pairs $(v_k, v_{k'})$ s.t. we find

$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all pairs } (v_k, v_{k'}) \in H, s \in \{s_i + w_k - 1, ..., s_j\} \text{ and } \{s_i + w_{k'} - 1, ..., s_j\}.$$

Consequently, we obtain that $\sigma_s^k = \rho$ for all $v_k \in H$ and all $s \in \{s_i + w_k - 1, ..., s_j\}$.
On the other hand, we use the same technique applied in the polyhedron dimension proof 5.3.1 to prove that for each $k \in K$ and $s \in \{1, ..., w_k - 1\}$, we have $\sigma_s^k = \gamma^{k,s}$. As a result $(\mu, \sigma) = \sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} \rho \beta_s^k + \gamma \tilde{A}$.

$\square$

5.4.4 Slot-Assignment-Clique Inequalities

On the other hand, we also noticed that there may exist some cases that are not covered by the inequality (2.27). For this, we provide an adapted definition of a conflict graph $\tilde{G}_S^E$ for the SA problem and its associated inequality.

Definition 5.4.3. *Let $\tilde{G}_S'^E$ be a conflict graph defined as follows. For all slot $s \in \{w_k, ..., \bar{s}\}$ and demand $k \in K$, consider a node $v_{k,s}$ in $\tilde{G}_S'^E$. Two nodes $v_{k,s}$ and $v_{k',s'}$ are linked by an edge in $\tilde{G}_S'^E$ iff $E_1^k \cap E_1^{k'} \neq \emptyset$ and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$.*

Based on the conflict graph $\tilde{G}_S'^E$, we introduced the following inequalities.

Proposition 5.4.5. *Let C be a clique in the conflict graph $\tilde{G}'^{E}_{S}$ with $|C| \geq 3$. Then, the inequality* (2.38) *is valid for $P_{sa}(G, K, \mathbb{S})$.*

Proof. We use the same proof of the proposition (2.4.16). □

Theorem 5.4.4. *Consider a clique C in the conflict graph $\tilde{G}'^{E}_{S}$. Then, the inequality* (2.38) *is facet defining for $P_{sa}(G, K, \mathbb{S})$ iff C is a maximal clique in the conflict graph $\tilde{G}'^{E}_{S}$, and there does not exist an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with*

a) $[\min_{v_{k,s} \in C}(s - w_k + 1), \max_{v_{k,s} \in C} s] \subset I$,

b) and $w_k + w_{k'} \geq |I| + 1$ *for each* $(v_k, v_{k'}) \in C$,

c) and $2w_k \geq |I| + 1$ *and* $w_k \leq |I|$ *for each* $v_k \in C$.

Proof. **Neccessity.**
If C is a not maximal clique in the conflict graph $\tilde{G}'^{E}_{S}$, this means that the inequality (2.38) can be dominated by another inequality associated with a clique C' s.t. $C \subset C'$ without changing its right-hand side. Moreover, if there exists an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with

a) $[\min_{v_{k,s} \in C}(s - w_k + 1), \max_{v_{k,s} \in C} s] \subset I$,

b) and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'}) \in C$,

c) and $2w_k \geq |I| + 1$ and $w_k \leq |I|$ for each $v_k \in C$.

Then, the inequality (2.38) is dominated by the inequality (2.35). As a result, the inequality (2.38) cannot be facet defining for $P_{sa}(G, K, \mathbb{S})$.
Sufficiency.
Let $\tilde{F}_{C}^{\tilde{G}'^{E}_{S}}$ denote the face induced by the inequality (2.35), which is given by

$$\tilde{F}_{C}^{\tilde{G}'^{E}_{S}} = \{(u, z) \in P_{sa}(G, K, \mathbb{S}) : \sum_{v_{k,s} \in C} z^k_s = 1\}.$$

In order to prove that inequality $\sum_{v_{k,s} \in C} z^k_s \leq 1$ is facet defining for $P_{sa}(G, K, \mathbb{S})$, we start checking that $\tilde{F}_{C}^{\tilde{G}'^{E}_{S}}$ is a proper face, and $\tilde{F}_{C}^{\tilde{G}'^{E}_{S}} \neq P_{sa}(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{140} = (U^{140}, S^{140})$ as below

a) a set of last-slots S^{140}_k is assigned to each demand $k \in K$ with $|S^{140}_k| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{140} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S^{140}_k$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{140}$,

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S^{140}_k$ and $s' \in S^{140}_{k'}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint),

d) and there is one node $v_{k,s}$ from the clique C s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{140}$ with $v_{k,s} \in C$, i.e., $s \in S^{140}_k$ for a node $v_{k,s} \in C$, and for each $s' \in S^{140}_{k'}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $v_{k',s'} \notin C$.

Obviously, $\mathcal{S}^{140}$ is a feasible solution for the problem given that it satisfies all the constraints of our compact formulation. Moreover, the corresponding incidence vector $(u^{\mathcal{S}^{140}}, z^{\mathcal{S}^{140}})$ is belong to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}_{C}^{\tilde{G}'^{E}_{S}}$ given that it is composed by $\sum_{v_{k,s} \in C} z^k_s = 1$. As a result, $\tilde{F}_{C}^{\tilde{G}'^{E}_{S}}$ is not empty (i.e., $\tilde{F}_{C}^{\tilde{G}'^{E}_{S}} \neq \emptyset$). Furthermore, given that $s \in \{w_k, ..., \bar{s}\}$ for each

$v_{k,s} \in C$, this means that there exists at least one feasible slot assignment S_k for the demands k in C with $v_{k,s} \notin C$ for each $s \in S_k$. This means that $\tilde{F}_C^{\tilde{G}'^E_S} \neq P_{sa}(G, K, \mathbb{S})$.
We denote the inequality $\sum_{v_{k,s} \in C} z_s^k \leq 1$ by $\alpha u + \beta z \leq \lambda$. Let $\mu u + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P_{sa}(G, K, \mathbb{S})$. Suppose that $\tilde{F}_C^{\tilde{G}'^E_S} \subset F = \{(u, z) \in P_{sa}(G, K, \mathbb{S}) : \mu u + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)})$ s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma \tilde{A}$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $v_{k,s} \notin C$,
b) and $\mu_s = 0$ for all slots $s \in \mathbb{S}$,
c) and σ_s^k are equivalents for all $v_{k,s} \in C$.

Let us show that $\mu_s = 0$ for all $s \in \mathbb{S}$. Consider a slot $\tilde{s} \in \mathbb{S}$. To do so, we consider a solution $\mathcal{S}^{141} = (U^{141}, S^{141})$ in which

a) a set of last-slots S_k^{141} is assigned to each demand $k \in K$ with $|S_k^{141}| \geq 1$ (contiguity and continuity constraints),
b) a set of slots U^{141} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{141}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{141}$,
c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{141}$ and $s" \in S_{k'}^{141}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in S_k^{141}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),
d) and $\tilde{s} \notin \{s - w_k + 1, ..., s\}$ for each $k \in K$ and $s \in S_k^{141}$ (slot-assignment constraint taking into account the possibility of adding the slot $\tilde{s}$ in the set of used slots U^{141}),
e) and there is one node $v_{k,s}$ from the clique C s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{141}$ with $v_{k,s} \in C$, i.e., $s \in S_k^{141}$ for a node $v_{k,s} \in C$, and for each $s' \in S_{k'}^{141}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $v_{k',s'} \notin C$.

$\mathcal{S}^{141}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{141}}, z^{\mathcal{S}^{141}})$ belongs to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}_C^{\tilde{G}'^E_S}$ given that it is composed by $\sum_{v_{k,s} \in C} z_s^k = 1$. Based on this, we derive a solution $\mathcal{S}^{142} = (U^{142}, S^{142})$ from the solution $\mathcal{S}^{141}$ by adding the slot $\tilde{s}$ as a used slot in U^{142} without modifying the last-slots assigned to the demands K in $\mathcal{S}^{141}$ which remain the same in the solution $\mathcal{S}^{142}$ i.e., $S_k^{141} = S_k^{142}$ for each demand $k \in K$. The solution $\mathcal{S}^{142}$ is feasible given that

a) a set of last-slots S_k^{142} is assigned to each demand $k \in K$ with $|S_k^{142}| \geq 1$ (contiguity and continuity constraints),
b) a set of slots U^{142} used in $\mathbb{S}$ s.t. $\tilde{s} \in U^{142}$, and for each demand k and last-slot $s \in S_k^{142}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{142}$,
c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{142}$ and $s' \in S_{k'}^{142}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s \in S_k^{142}, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint),
d) and there is one node $v_{k,s}$ from the clique C s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{142}$ with $v_{k,s} \in C$, i.e., $s \in S_k^{142}$ for a node $v_{k,s} \in C$, and for each $s' \in S_{k'}^{142}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $v_{k',s'} \notin C$.

$\mathcal{S}^{142}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{142}}, z^{\mathcal{S}^{142}})$ belongs to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}_C^{\tilde{G}'^E_S}$ given that it is composed by $\sum_{v_{k,s} \in C} z^k_s = 1$. We then obtain that

$$\mu u^{\mathcal{S}^{141}} + \sigma z^{\mathcal{S}^{141}} = \mu u^{\mathcal{S}^{142}} + \sigma z^{\mathcal{S}^{142}} = \mu u^{\mathcal{S}^{141}} + \sigma z^{\mathcal{S}^{141}} + \mu_{\tilde{s}}.$$

It follows that $\mu_{\tilde{s}} = 0$ for the slot $\tilde{s} \in \mathbb{S}$. The slot $\tilde{s}$ is chosen arbitrarily in $\mathbb{S}$, we iterate the same procedure for all feasible slots in $\mathbb{S}$ s.t. we find

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Let's us show that $\sigma^k_s = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$ with $v_{k,s} \notin C$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$ with $v_{k,s'} \notin C$. For that, we consider a solution $\mathcal{S}^{143} = (U^{143}, S^{143})$ in which

a) a set of last-slots S^{143}_k is assigned to each demand $k \in K$ with $|S^{143}_k| \geq 1$ (contiguity and continuity constraints),

b) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{143}_k$ and $s" \in S^{143}_{k'}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}'_e} |\{s' \in S^{143}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S^{143}_{k'}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots S^{143}_k assigned to the demand k in the solution $\mathcal{S}^{143}$),

d) and there is one node $v_{k,s}$ from the clique C s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{143}$ with $s \in \{s_i + w_k - 1, ..., s_j\}$, i.e., $s \in S^{143}_k$ for a node $v_{k,s} \in C$, and for each $s' \in S^{143}_{k'}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $v_{k',s'} \notin C$.

$\mathcal{S}^{143}$ is clearly feasible for the problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{143}}, z^{\mathcal{S}^{143}})$ is belong to F and then to $\tilde{F}_C^{\tilde{G}'^E_S}$ given that it is composed by $\sum_{v_{k,s} \in C} z^k_s = 1$. Based on this, we derive a solution $\mathcal{S}^{144} = (U^{144}, S^{144})$ from the solution $\mathcal{S}^{143}$ by adding the slot s' as last-slot to the demand k without modifying the last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{143}$, i.e., $S^{143}_{k'} = S^{144}_{k'}$ for each demand $k' \in K \setminus \{k\}$, and $S^{144}_k = S^{143}_k \cup \{s'\}$ for the demand k. The solution $\mathcal{S}^{144}$ is feasible given that

a) a set of last-slots S^{144}_k is assigned to each demand $k \in K$ with $|S^{144}_k| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{144} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S^{144}_k$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{144}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{144}_k$ and $s" \in S^{144}_{k'}$ with $E^{144}_k \cap E^{144}_{k'} \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e' \in E^{144}_k} |\{s' \in S^{144}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(u^{\mathcal{S}^{144}}, z^{\mathcal{S}^{144}})$ is belong to F and then to $\tilde{F}_C^{\tilde{G}'^E_S}$ given that it is composed by $\sum_{v_{k,s} \in C} z^k_s = 1$. We then obtain that

$$\mu u^{\mathcal{S}^{143}} + \sigma z^{\mathcal{S}^{143}} = \mu u^{\mathcal{S}^{144}} + \sigma z^{\mathcal{S}^{144}} = \mu u^{\mathcal{S}^{143}} + \sigma z^{\mathcal{S}^{143}} + \sigma^k_{s'} + \sum_{\tilde{s} \in U^{144} \setminus U^{143}} \mu_{\tilde{s}} - \sum_{\tilde{s} \in U^{143} \setminus U^{144}} \mu_{\tilde{s}}.$$

It follows that $\sigma^k_{s'} = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $v_{k,s'} \notin C$ given that $\mu_{\tilde{s}} = 0$ for all slots $\tilde{s} \in \mathbb{S}$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $v_{k,s'} \notin C$ s.t. we find

$$\sigma^k_{s'} = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s'} \notin C.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma^{k'}_s = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } v_{k',s} \notin C.$$

Consequently, we conclude that

$$\sigma^k_s = 0, \text{ for all } k \in K \text{ and all slots } s \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s} \notin C.$$

Let prove that σ^k_s for all $v_{k,s} \in C$ are equivalents. Consider a demand k' and a slot s with $v_{k',s} \in C$, and a solution $\mathcal{S}^{145} = (U^{145}, S^{145})$ in which

a) a set of last-slots S^{145}_k is assigned to each demand $k \in K$ with $|S^{145}_k| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{145} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S^{145}_k$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{145}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S^{145}_k$ and $s" \in S^{145}_{k'}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E^{145}_k} |\{s' \in S^{145}_k, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s \in S^{145}_k$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S^{145}_{k'}$ assigned to the demand k' in the solution $\mathcal{S}^{145}$),

e) and there is one node $v_{k,s}$ from the clique C s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{145}$ with $v_{k,s} \in C$, i.e., $s \in S^{145}_k$ for a node $v_{k,s} \in C$, and for each $s' \in S^{145}_{k'}$ for all $v_{k'} \in C \setminus \{v_k\}$ we have $v_{k',s'} \notin C$.

$\mathcal{S}^{145}$ is clearly feasible for the problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{145}}, z^{\mathcal{S}^{145}})$ is belong to F and then to $\tilde{F}^{\tilde{G}'^E_S}_C$ given that it is composed by $\sum_{v_{k,s} \in C} z^k_s = 1$. Based on this, we derive a solution $\mathcal{S}^{146} = (E^{146}, S^{146})$ from the solution $\mathcal{S}^{145}$ by adding the slot s' as last-slot to the demand k without modifying the last-slots assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{145}$, i.e., $S^{145}_{k"} = S^{146}_{k"}$ for each demand $k" \in K \setminus \{k, k'\}$, and $S^{146}_{k'} = S^{145}_{k'} \cup \{s'\}$ for the demand k', and with modifying the last-slots assigned to the demand k by adding a new last-slot $\tilde{s}$ and removing the last slot $s \in S^{145}_k$ with $s \in \{s_i + w_k + 1, ..., s_j\}$ and $\tilde{s} \in \{w_k, ..., \bar{s}\}$ for the demand k with $v_{k,\tilde{s}} \notin C$ s.t. $S^{146}_k = (S^{145}_k \setminus \{s\}) \cup \{\tilde{s}\}$ s.t. $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S^{146}_{k'}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$. The solution $\mathcal{S}^{146}$ is feasible given that

a) a set of last-slots S^{146}_k is assigned to each demand $k \in K$ with $|S^{146}_k| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{146} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S^{146}_k$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{146}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{146}$ and $s" \in S_{k'}^{146}$ with $E_k^{146} \cap E_{k'}^{146} \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in K, e \in E_k^{146}} |\{s' \in S_k^{146}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(u^{\mathcal{S}^{146}}, z^{\mathcal{S}^{146}})$ is belong to F and then to $\tilde{F}_C^{\tilde{G}'^E_S}$ given that it is composed by $\sum_{v_{k,s} \in C} z_s^k = 1$. We then obtain that

$$\mu u^{\mathcal{S}^{145}} + \sigma z^{\mathcal{S}^{145}} = \mu u^{\mathcal{S}^{146}} + \sigma z^{\mathcal{S}^{146}} = \mu u^{\mathcal{S}^{145}} + \sigma z^{\mathcal{S}^{145}} + \sigma_{s'}^{k'} - \sigma_s^k + \sigma_{\tilde{s}}^k + \sum_{s" \in U^{146} \setminus U^{145}} \mu_{s"} - \sum_{s" \in U^{145} \setminus U^{146}} \mu_{s"}.$$

It follows that $\sigma_{s'}^{k'} = \sigma_s^k$ for demand k' and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $v_{k',s'} \in C$ given that $\sigma_{\tilde{s}}^k = 0$ for $v_{k,\tilde{s}} \notin C$, and $\mu_{s"} = 0$ for all $s" \in \mathbb{S}$.
Given that the pair $(v_k, v_{k'})$ are chosen arbitrary in the clique C, we iterate the same procedure for all pairs $(v_k, v_{k'})$ s.t. we find

$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all pairs } (v_{k,s}, v_{k',s'}) \in C,$$

Consequently, we obtain that $\sigma_s^k = \rho$ for all $v_{k,s} \in C$.
On the other hand, we use the same technique applied in the polyhedron dimension proof of the proposition 5.3.1 to prove that for each $k \in K$ and $s \in \{1, ..., w_k - 1\}$, we have $\sigma_s^k = \gamma^{k,s}$.
As a result $(\mu, \sigma) = \sum_{v_{k,s} \in C} \rho \beta_s^k + \gamma \tilde{A}$.

□

5.4.5 Slot-Assignment-Odd-Hole Inequalities

Proposition 5.4.6. *Let H be an odd-hole in the conflict graph $\tilde{G}'^E_S$ with $|H| \geq 5$. Then, the inequality* (2.40) *is valid for $P_{sa}(G, K, \mathbb{S})$.*

Proof. We use the same proof of the proposition (2.4.12). □

Theorem 5.4.5. *Let H be an odd-hole in the conflict graph $\tilde{G}'^E_S$ with $|H| \geq 5$. Then, the inequality* (2.40) *is facet defining for $P_{sa}(G, K, \mathbb{S})$ iff*

a) for each node $v_{k',s'} \notin H$ in $\tilde{G}'^E_S$, there exists a node $v_{k,s} \in H$ s.t. the induced graph $\tilde{G}'^E_S((H \setminus \{v_{k,s}\}) \cup \{v_{k',s'}\})$ does not contain an odd-hole,

b) and there does not exist a node $v_{k',s'} \notin H$ in $\tilde{G}'^E_S$ s.t. $v_{k',s'}$ is linked with all nodes $v_{k,s} \in H$,

c) and there does not exist an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with

a) $[\min_{v_{k,s} \in H}(s - w_k + 1), \max_{v_{k,s} \in H}] \subset I$,

b) and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'})$ linked in H,

c) and $2w_k \geq |I| + 1$ and $w_k \leq |I|$ for each $v_k \in H$.

Proof. **Neccessity.**
We distinguish the following cases:

a) if for a node $v_{k',s'} \notin H$ in $\tilde{G}'^E_S$, there exists a node $v_{k,s} \in H$ s.t. the induced graph $\tilde{G}'^E_S(H \setminus \{v_{k,s}\} \cup \{v_{k',s'}\})$ contains an odd-hole $H' = (H \setminus \{v_{k,s}\}) \cup \{v_{k',s'}\}$. This implies that the inequality (2.40) can be dominated using some technics of lifting based on the following two inequalities $\sum_{v_{k,s} \in H} z_s^k \leq \frac{|H|-1}{2}$, and $\sum_{v_{k',s'} \in H'} z_{s'}^{k'} \leq \frac{|H'|-1}{2}$.

b) if there exists a node $v_{k',s'} \notin H$ in $\tilde{G}'^{E}_{S}$ s.t. $v_{k',s'}$ is linked with all nodes $v_{k,s} \in H$. This implies that the inequality (2.40) can be dominated by the following valid inequality

$$\sum_{v_{k,s} \in H} z^k_s + \frac{|H|-1}{2} z^{k'}_{s'} \leq \frac{|H|-1}{2}.$$

c) if there exists an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with

   a) $[\min_{v_{k,s} \in H}(s - w_k + 1), \max_{v_{k,s} \in H}] \subset I$,

   b) and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'})$ linked in H,

   c) and $2w_k \geq |I| + 1$ and $w_k \leq |I|$ for each $v_k \in H$.

   This implies that the inequality (2.40) is dominated by the inequality (2.36).

   If no one of these cases is verified, the inequality (2.40) can never be dominated by another inequality without changing its right-hand side. Otherwise, the inequality (2.40) cannot be facet defining for $P_{sa}(G, K, \mathbb{S})$.
   Sufficiency.

   Let $\tilde{F}^{\tilde{G}'^{E}_{S}}_{H}$ denote the face induced by the inequality (2.36), which is given by

$$\tilde{F}^{\tilde{G}'^{E}_{S}}_{H} = \{(u, z) \in P_{sa}(G, K, \mathbb{S}) : \sum_{v_{k,s} \in H} z^k_s = \frac{|H|-1}{2}\}.$$

   In order to prove that inequality $\sum_{v_{k,s} \in H} z^k_s \leq \frac{|H|-1}{2}$ is facet defining for $P_{sa}(G, K, \mathbb{S})$, we start checking that $\tilde{F}^{\tilde{G}'^{E}_{S}}_{H}$ is a proper face, and $\tilde{F}^{\tilde{G}'^{E}_{S}}_{H} \neq P_{sa}(G, K, \mathbb{S})$. We construct a solution $\mathcal{S}^{147} = (U^{147}, S^{147})$ as below

a) a set of last-slots S^{147}_k is assigned to each demand $k \in K$ with $|S^{147}_k| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{147} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S^{147}_k$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{147}$,

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S^{147}_k$ and $s' \in S^{147}_{k'}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint),

d) and there is $\frac{|H|-1}{2}$ pair of demand-last-slot $\tilde{H}^{147}$ from the odd-hole H (i.e., $v_{k,s} \in \tilde{H}^{147} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{147}$ with $v_{k,s} \in H$, i.e., $s \in S^{147}_k$ for each node $v_k \in \tilde{H}^{147}$, and for each $s' \in S^{147}_{k'}$ for all $v_{k'} \in H \setminus \tilde{H}^{147}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

   Obviously, $\mathcal{S}^{147}$ is a feasible solution for the problem given that it satisfies all the constraints of our compact formulation. Moreover, the corresponding incidence vector $(u^{\mathcal{S}^{147}}, z^{\mathcal{S}^{147}})$ is belong to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}^{\tilde{G}'^{E}_{S}}_{H}$ given that it is composed by $\sum_{v_{k,s} \in H} z^k_s = \frac{|H|-1}{2}$. As a result, $\tilde{F}^{\tilde{G}'^{E}_{S}}_{H}$ is not empty (i.e., $\tilde{F}^{\tilde{G}'^{E}_{S}}_{H} \neq \emptyset$). Furthermore, given that $s \in \{w_k, ..., \bar{s}\}$ for each $v_{k,s} \in H$, this means that there exists at least one feasible slot assignment S_k for the demands k in H with $v_{k,s} \notin H$. This means that $\tilde{F}^{\tilde{G}'^{E}_{S}}_{H} \neq P_{sa}(G, K, \mathbb{S})$.
   We denote the inequality $\sum_{v_{k,s} \in H} z^k_s \leq \frac{|H|-1}{2}$ by $\alpha u + \beta z \leq \lambda$. Let $\mu u + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $P_{sa}(G, K, \mathbb{S})$. Suppose that $\tilde{F}^{\tilde{G}'^{E}_{S}}_{H} \subset F = \{(u, z) \in$

$P_{sa}(G, K, \mathbb{S}) : \mu u + \sigma z = \tau\}$. We show that there exist $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k\in K}(w_k-1)})$ s.t. $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma\tilde{A}$, and that

a) $\sigma_s^k = 0$ for all demands $k \in K$ and all slots $s \in \{w_k, ..., \bar{s}\}$ with $v_{k,s} \in H$,

b) and $\mu_s = 0$ for all slots $s \in \mathbb{S}$,

c) and σ_s^k are equivalents for all $v_{k,s} \in H$.

Let us show that $\mu_s = 0$ for all $s \in \mathbb{S}$. Consider a slot $\tilde{s} \in \mathbb{S}$. To do so, we consider a solution $\mathcal{S}^{148} = (U^{148}, S^{148})$ in which

a) a set of last-slots S_k^{148} is assigned to each demand $k \in K$ with $|S_k^{148}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{148} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{148}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{148}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{148}$ and $s" \in S_{k'}^{148}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in\tilde{K}_e} |\{s' \in S_k^{148}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\tilde{s} \notin \{s - w_k + 1, ..., s\}$ for each $k \in K$ and $s \in S_k^{148}$ (slot-assignment constraint taking into account the possibility of adding the slot $\tilde{s}$ in the set of used slots U^{148}),

e) and there is $\frac{|H|-1}{2}$ pair of demand-last-slots $\tilde{H}^{148}$ from the odd-hole H (i.e., $v_{k,s} \in \tilde{H}^{148} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{148}$ with $v_{k,s} \in H$, i.e., $s \in S_k^{148}$ for each node $v_k \in \tilde{H}^{148}$, and for each $s' \in S_{k'}^{148}$ for all $v_{k'} \in H \setminus \tilde{H}^{148}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{148}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{148}}, z^{\mathcal{S}^{148}})$ belongs to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}_H^{\tilde{G}'^E_S}$ given that it is composed by $\sum_{v_{k,s}\in H} z_s^k = \frac{|H|-1}{2}$. Based on this, we derive a solution $\mathcal{S}^{149} = (U^{149}, S^{149})$ from the solution $\mathcal{S}^{148}$ by adding the slot $\tilde{s}$ as a used slot in U^{149} without modifying the the last-slots assigned to the demands K in $\mathcal{S}^{148}$ which remain the same in the solution $\mathcal{S}^{149}$ i.e., $S_k^{148} = S_k^{149}$ for each demand $k \in K$. The solution $\mathcal{S}^{149}$ is feasible given that

a) a set of last-slots S_k^{149} is assigned to each demand $k \in K$ with $|S_k^{149}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{149} used in $\mathbb{S}$ s.t. $\tilde{s} \in U^{149}$, and for each demand k and last-slot $s \in S_k^{149}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{149}$,

c) $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k, k' \in K$ and each $s \in S_k^{149}$ and $s' \in S_{k'}^{149}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k\in\tilde{K}_e} |\{s \in S_k^{149}, s" \in \{s - w_k + 1, ..., s\}| \leq 1$ (non-overlapping constraint),

d) and there is $\frac{|H|-1}{2}$ demands $\tilde{H}^{149}$ from the odd-hole H (i.e., $v_k \in \tilde{H}^{149} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{149}$ with $v_{k,s} \in H$, i.e., $s \in S_k^{149}$ for each node $v_k \in \tilde{H}^{149}$, and for each $s' \in S_{k'}^{149}$ for all $v_{k'} \in H \setminus \tilde{H}^{149}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{149}$ is clearly feasible for the SA problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{149}}, z^{\mathcal{S}^{149}})$ belongs to $P_{sa}(G, K, \mathbb{S})$ and then to $\tilde{F}_H^{\tilde{G}'^E_S}$ given that it is composed by $\sum_{v_{k,s}\in H} z_s^k = 1$. We then obtain that

$$\mu u^{\mathcal{S}^{148}} + \sigma z^{\mathcal{S}^{148}} = \mu u^{\mathcal{S}^{149}} + \sigma z^{\mathcal{S}^{149}} = \mu u^{\mathcal{S}^{148}} + \sigma z^{\mathcal{S}^{148}} + \mu_{\tilde{s}}.$$

It follows that $\mu_{\tilde{s}} = 0$ for the slot $\tilde{s} \in \mathbb{S}$. The slot $\tilde{s}$ is chosen arbitrarily in $\mathbb{S}$, we iterate the same procedure for all feasible slots in $\mathbb{S}$ s.t. we find

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Let's us show that $\sigma_s^k = 0$ for all $k \in K$ and all $s \in \{w_k, ..., \bar{s}\}$ with $v_{k,s} \notin H$. Consider the demand k and a slot s' in $\{w_k, ..., \bar{s}\}$ with $v_{k,s'} \notin H$. For that, we consider a solution $\mathcal{S}^{150} = (U^{150}, S^{150})$ in which

a) a set of last-slots S_k^{150} is assigned to each demand $k \in K$ with $|S_k^{150}| \geq 1$ (contiguity and continuity constraints),

b) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{150}$ and $s" \in S_{k'}^{150}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e'} |\{s' \in S_k^{150}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

c) and $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S_{k'}^{150}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots S_k^{150} assigned to the demand k in the solution $\mathcal{S}^{150}$),

d) and there is $\frac{|H|-1}{2}$ pair of demand-last-slot $\tilde{H}^{150}$ from the odd-hole H (i.e., $v_{k,s} \in \tilde{H}^{150} \subset H$ s.t. the demand k selects a slot s as last-slot in the solution $\mathcal{S}^{150}$ with $v_{k,s} \in H$, i.e., $s \in S_k^{150}$ for each node $v_k \in \tilde{H}^{150}$, and for each $s' \in S_{k'}^{150}$ for all $v_{k'} \in H \setminus \tilde{H}^{150}$ we have $s' \notin \{s_i + w_{k'} - 1, ..., s_j\}$.

$\mathcal{S}^{150}$ is clearly feasible for the problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{150}}, z^{\mathcal{S}^{150}})$ is belong to F and then to $\tilde{F}_H^{\tilde{G}_S'^E}$ given that it is composed by $\sum_{v_{k,s} \in H} z_s^k = \frac{|H|-1}{2}$. Based on this, we derive a solution $\mathcal{S}^{151} = (U^{151}, S^{151})$ from the solution $\mathcal{S}^{150}$ by adding the slot s' as last-slot to the demand k without modifying the last-slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{150}$, i.e., $S_{k'}^{150} = S_{k'}^{151}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{151} = S_k^{150} \cup \{s'\}$ for the demand k. The solution $\mathcal{S}^{151}$ is feasible given that

a) a set of last-slots S_k^{151} is assigned to each demand $k \in K$ with $|S_k^{151}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{151} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{151}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{151}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{151}$ and $s" \in S_{k'}^{151}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e' \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_{e'}} |\{s' \in S_k^{151}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(u^{\mathcal{S}^{151}}, z^{\mathcal{S}^{151}})$ is belong to F and then to $\tilde{F}_H^{\tilde{G}_S'^E}$ given that it is composed by $\sum_{v_{k,s} \in H} z_s^k = \frac{|H|-1}{2}$. We then obtain that

$$\mu u^{\mathcal{S}^{150}} + \sigma z^{\mathcal{S}^{150}} = \mu u^{\mathcal{S}^{151}} + \sigma z^{\mathcal{S}^{151}} = \mu u^{\mathcal{S}^{150}} + \sigma z^{\mathcal{S}^{150}} + \sigma_{s'}^k + \sum_{\tilde{s} \in U^{151} \setminus U^{150}} \mu_{\tilde{s}} - \sum_{\tilde{s} \in U^{150} \setminus U^{151}} \mu_{\tilde{s}}.$$

It follows that $\sigma_{s'}^k = 0$ for demand k and a slot $s' \in \{w_k, ..., \bar{s}\}$ with $v_{k,s'} \notin H$ given that $\mu_{\tilde{s}} = 0$ for all slots $\tilde{s} \in \mathbb{S}$.
The slot s' is chosen arbitrarily for the demand k, we iterate the same procedure for all feasible slots in $\{w_k, ..., \bar{s}\}$ of demand k with $v_{k,s'} \notin H$ s.t. we find

$$\sigma_{s'}^k = 0, \text{ for demand } k \text{ and all slots } s' \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s'} \notin H.$$

Given that the demand k is chosen arbitrarily. We iterate the same thing for all the demands k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and all slots } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } v_{k,s'} \notin H.$$

Let's prove that σ_s^k for all $v_{k,s} \in H$ are equivalents. Consider a node $v_{k',s'}$ in H. For that, we consider a solution $\mathcal{S}^{152} = (U^{152}, S^{152})$ in which

a) a set of last-slots S_k^{152} is assigned to each demand $k \in K$ with $|S_k^{152}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{152} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{152}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{152}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{152}$ and $s" \in S_{k'}^{152}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in S_k^{152}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint),

d) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s \in S_k^{152}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (non-overlapping constraint taking into account the possibility of adding the slot s' in the set of last-slots $S_{k'}^{152}$ assigned to the demand k'),

e) and there is $\frac{|H|-1}{2}$ pairs of demand k and slot s from the odd-hole H (i.e., $v_{k,s} \in H$ s.t. the demand k selects the slot s as last-slot in the solution $\mathcal{S}^{152}$ denoted by $\tilde{H}^{152}$, i.e., $s \in \mathcal{S}_k^{152}$ for each $v_{k,s} \in \tilde{H}^{152}$, and $s' \notin \mathcal{S}_{k'}^{152}$ for all $v_{k',s'} \in H \setminus \tilde{H}^{152}$.

$\mathcal{S}^{152}$ is clearly feasible for the problem given that it satisfies all the constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{152}}, z^{\mathcal{S}^{152}})$ is belong to F and then to $\tilde{F}_H^{\tilde{G}_S'^E}$ given that it is composed by $\sum_{v_{k,s} \in H} z_s^k = \frac{|H|-1}{2}$. Based on this, we derive a solution $\mathcal{S}^{153} = (U^{153}, S^{153})$ from the solution $\mathcal{S}^{152}$ s.t.

a) the last-slots assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{152}$ remain the same in $\mathcal{S}^{153}$, i.e., $S_{k"}^{152} = S_{k"}^{153}$ for each demand $k" \in K \setminus \{k, k'\}$, where k is a demand with $v_{k,s} \in \tilde{H}^{152}$ and $s \in S_k^{152}$ s.t. $v_{k',s'}$ is not linked with any node $v_{k",s"} \in \tilde{H}^{152} \setminus \{v_{k,s}\}$,

b) and adding the slot s' as last-slot to the demand k', i.e., $S_{k'}^{153} = S_{k'}^{152} \cup \{s'\}$ for the demand k',

c) and modifying the last-slots assigned to the demand k by adding a new last-slot $\tilde{s}$ and removing the last slot $s \in S_k^{152}$ with $v_{k,s} \in H$ and $v_{k,\tilde{s}} \notin H$ s.t. $S_k^{153} = (S_k^{152} \setminus \{s\}) \cup \{\tilde{s}\}$ for the demand k s.t. $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}^{153}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$.

The solution $\mathcal{S}^{153}$ is feasible given that

a) a set of last-slots S_k^{153} is assigned to each demand $k \in K$ with $|S_k^{153}| \geq 1$ (contiguity and continuity constraints),

b) a set of slots U^{153} used in $\mathbb{S}$ s.t. for each demand k and last-slot $s \in S_k^{153}$ and $s' \in \{s - w_k + 1, ..., s\}$, we have $s' \in U^{153}$,

c) $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k, k' \in K$ and each $s' \in S_k^{153}$ and $s" \in S_{k'}^{153}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$, i.e., for each edge $e \in E$ and each slot $s" \in \mathbb{S}$ we have $\sum_{k \in \tilde{K}_e} |\{s' \in S_k^{153}, s" \in \{s' - w_k + 1, ..., s'\}| \leq 1$ (non-overlapping constraint).

The corresponding incidence vector $(u^{\mathcal{S}^{153}}, z^{\mathcal{S}^{153}})$ is belong to F and then to $\tilde{F}_H^{\tilde{G}'^E_S}$ given that it is composed by $\sum_{v_{k,s} \in H} z_s^k = \frac{|H|-1}{2}$. We then obtain that

$$\mu u^{\mathcal{S}^{152}} + \sigma z^{\mathcal{S}^{152}} = \mu u^{\mathcal{S}^{153}} + \sigma z^{\mathcal{S}^{153}} = \mu u^{\mathcal{S}^{152}} + \sigma z^{\mathcal{S}^{152}} + \sigma_{s'}^{k'} - \sigma_s^k + \sigma_{\tilde{s}}^k + \sum_{s" \in U^{153} \setminus U^{152}} \mu_{s"} - \sum_{s" \in U^{152} \setminus U^{153}} \mu_{s"}.$$

It follows that $\sigma_{s'}^{k'} = \sigma_s^k$ for demand k' and a slot $s' \in \{w_{k'}, ..., \bar{s}\}$ with $v_{k',s'} \in H$ given that $\sigma_{\tilde{s}}^k = 0$ for $v_{k,\tilde{s}} \notin H$, and $\mu_{s"} = 0$ for all $s" \in \mathbb{S}$.
Consequently, we obtain that $\sigma_s^k = \rho$ for all $v_{k,s} \in H$.
On the other hand, we use the same technique applied in the polyhedron dimension proof 5.3.1 to prove that for each $k \in K$ and $s \in \{1, ..., w_k - 1\}$, we have $\sigma_s^k = \gamma^{k,s}$. As a result $(\mu, \sigma) = \sum_{v_{k,s} \in H} \rho \beta_s^k + \gamma \tilde{A}$. □

In the next section, we will derive some symmetry breaking inequalities for the SA sub-problem in which some symmetrical solutions may appeared.

5.5 Symmetry-Breaking Inequalities

Let us introduce some families of symmetry-breaking inequalities to remove symmetrical solutions obtained when solving the SA sub-problem.

Proposition 5.5.1. *We ensure that for all slot $s \in \{1, ..., \bar{s} - 1\}$*

$$u_s - u_{s+1} \geq 0, \tag{5.16}$$

which means that a slot $s+1$ can be used if and only if the slot s is used.

Similar idea was proposed by Mendez-Diaz et al. in [105] and [104] to break the symmetry for the vertex coloring problem.

Proposition 5.5.2. *To strengthen the inequality* (5.16)*, we ensure that for all slot $s \in \{1, ..., \bar{s} - 1\}$*

$$\sum_{k \in K} \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} 2^{|K|-k} z_{s'}^k \geq \sum_{k \in K} \sum_{s'=s+1}^{\min(s+w_k, \bar{s})} 2^{|K|-k} z_{s'}^k. \tag{5.17}$$

Similar idea was proposed by Friedman in [57]. However, the coefficient $2^{|K|-k}$ can provoques some numerical intractabilities for the computer machine [12]. For that, we introduce the following inequality.

Proposition 5.5.3. *We ensure that for all slot $s \in \{1, ..., \bar{s} - 1\}$*

$$\sum_{k \in K} \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z_{s'}^k \geq \sum_{k \in K} \sum_{s'=s+1}^{\min(s+w_k, \bar{s})} z_{s'}^k, \tag{5.18}$$

which means that the number of intervals of contiguous slots allocated which cover the slot $s+1$ (cardinality of slot-usage) cannot be greater than the number of channels allocated which cover the slot s.

Similar idea was proposed by Mendez-Diaz et al. in [105] and [104] to break the symmetry for the vertex coloring problem. Our inequalities and those of Mendez-Diaz et al. in [105] and [104] differ in their right and left hand sides.

Proposition 5.5.4. *Due to the inequality* (5.16)*, we ensure that for all $k \in K$, and all $s^0 \in \{1, ..., \bar{s}-1\}$ and all $s \in \{s^0, ..., \bar{s}\}$*

$$\sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq u_{s^0}, \tag{5.19}$$

which means that for a slot $S^0 \in \{1, ..., \bar{s}-1\}$, a demand k can allocate a slot in the sub-spectrum $\{S^0, ..., \bar{s}\}$ if the slot S^0 is used.

Similar idea was proposed by Mendez-Diaz et al. in [105] for the vertex coloring problem. Inequalities (5.19)} and those of Mendez-Diaz et al. in [105] differ in their left hand sides.

5.6 Lower Bounds

Here we propose some lower bounds issus from the conflict graph $\tilde{G}^r$. They can be seen as a valid inequalities for the polytope $P_{sa}(G, K, \mathbb{S})$.

Proposition 5.6.1. *Consider an edge $e \in E$. Then, the inequality*

$$\sum_{s \in \mathbb{S}} u_s \geq \sum_{k \in \tilde{K}_e} w_k, \text{ for all } e \in E, \tag{5.20}$$

is valid for $P_{sa}(G, K, \mathbb{S})$.

Proof. Inequality (5.20) ensures that the number of slots used in the spectrum $\mathbb{S}$ is greater than the flow over all the edges (the flow for an edge e is equal to the number of slots that should be used over the edge e). □

The inequality (5.20) can be generalized as follows using the conflict graph $\tilde{G}^r$.

Proposition 5.6.2. *Let C be a clique in $\tilde{G}^r$. Then, the inequality*

$$\sum_{s \in \mathbb{S}} u_s \geq \sum_{v_k \in C} w_k, \tag{5.21}$$

is valid for $P_{sa}(G, K, \mathbb{S})$.

Proof. It's trivial given the definition of the clique C in the conflict graph $\tilde{G}^r$ s.t. we know in advance that the demands in C share an edge in E which means that they cannot share a slot in $\mathbb{S}$ (non-overlapping constraint). Hence, the number of allocated slots $\sum_{s \in \mathbb{S}} u_s$ is at least equal to the number of requested slots of the demands in C.

□

5.7 Upper Bounds

Let us introduce the following weighted conflict graph in which a positive integer called weight is assigned to each node.

Definition 5.7.1. *Consider the conflict graph $\tilde{G}^r_w$ defined as follows. For each demand $k \in K$, consider a node v_k in $\tilde{G}^r_w$. Two nodes v_k and $v_{k'}$ are linked by an edge in $\tilde{G}^r_w$ iff $E(p_k) \cap E(p_{k'}) \neq \emptyset$. Each node v_k is associated with a positive weight which equals to the requested number of slots w_k of demand k.*

Definition 5.7.2. *Let C be a clique in $\tilde{G}^r_w$. It's known to be the maximum weight clique in $\tilde{G}^r_w$ if the total weight of the nodes in C ($\sum_{v_k \in C} w_k$) defines the maximum total weight over all cliques in $\tilde{G}^r_w$, i.e., $\sum_{v_k \in C} w_k \geq \sum_{v_{k'} \in C'} w_{k'}$ for all clique C' in $\tilde{G}^r_w$.*

Based on these definitions, we introduce the following inequality and showing that computing the upper bound for the SA is equivalent to solving the Maximum Weighted Clique Problem (MWC) which is well known to be an NP-hard problem [6].

Proposition 5.7.1. *Let C be the maximum weighted clique in $\tilde{G}^r_w$. Then, the upper bound is defined as follows*

$$\sum_{s \in \mathbb{S}} u_s \leq \sum_{v_k \in C} w_k, \tag{5.22}$$

Proof. It's trivial given the definition of the maximum weighted clique C in the conflict graph $\tilde{G}^r_w$ s.t. the maximum number of allocated slots $\sum_{s \in \mathbb{S}} u_s$ is at most equal to the number of requested slots of the demands in C. □

The inequality (5.22) is not valid for $P_{sa}(G, K, \mathbb{S})$ given that there exist some feasible solutions in $P_{sa}(G, K, \mathbb{S})$ which violate the inequality (5.22) when for example a slot $s \in \mathbb{S}$ is used (i.e., $u_s = 1$) but there is no demand $k \in K$ which use the slot s (i.e., $\sum_{k \in K} \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z^k_{s'}$). On the other hand, we ensure that all the optimization algorithms developed to solve the MWC problem can be used to compute the upper bound based on the conflict graph $\tilde{G}^r_w$.
Based on the inequalities (5.21) and (5.22), we conclude that the minimum number of slots to be used by the set of demands K while satisfying the SA constraints, it's equal to the total weight of the maximum weighted clique in the conflict graph $\tilde{G}^r_w$.
Based on the theoretical results presented in this chapter, we devise a Branch-and-Bound (B&B) and Branch-and-Cut algorithms to solve the SA problem. Moreover, we study the effectiveness of these algorithms and assess the impact of the valid inequalities on the effectiveness of the Branch-and-Cut algorithm.

5.8 Branch-and-Cut Algorithm

5.8.1 Description

Here we describe the Branch-and-Cut algorithm. We consider the following linear problem which can be seen as a strenghtned formulation for the compact formulation (5.1)-(5.8)

$$\min \sum_{s\in\mathbb{S}} u_s, \tag{5.23}$$

$$z_s^k = 0, \quad \text{for all } k\in K \text{ and } s\in\{1,...,w_k-1\}, \tag{5.24}$$

$$\sum_{s=w_k}^{\bar{s}} z_s^k = 1, \quad \text{for all } k\in K, \tag{5.25}$$

$$\sum_{k\in\tilde{K}_e} \sum_{s'=s}^{\min(\bar{s},s+w_k-1)} z_s^k - u_s \le 0, \quad \text{for all } e\in E, \text{ and } s\in\mathbb{S}, \tag{5.26}$$

$$u_s - \sum_{k\in K} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \le 0, \quad \text{for all } s\in\mathbb{S}, \tag{5.27}$$

$$z_s^k \ge 0, \quad \text{for all } k\in K \text{ and } s\in\mathbb{S}, \tag{5.28}$$

$$0 \le u_s \le 1, \quad \text{for all } s\in\mathbb{S}, \tag{5.29}$$

$$z_s^k \in \{0,1\}, \quad \text{for all } k\in K \text{ and } s\in\mathbb{S}, \tag{5.30}$$

$$u_s \in \{0,1\}, \quad \text{for all } s\in\mathbb{S}. \tag{5.31}$$

The inequality (5.27) ensures that if slot s is not used by at least one demand, its associated variable u_s is forced to be equal to zero.

On the other hand, and to boost the performance of the B&B algorithm, we already introduced several classes of valid inequalities to obtain tighter LP bounds. Based on this, and at each iteration in a certain level of the B&B algorithm, one can identify one or more than one violated inequality by the current fractional solution for a given class of valid inequalities. Algorithm 6 summarizes the different steps of the Branch-and-Cut algorithm taking into account additional valid inequalities for a given class of valid inequalities.

Note that the separation procedures of the valid inequalities presented in this chapter are still the same as those presented in the chapter 2 for the C-RSA. However, we need to present the separation procedure for the interval-capacity-cover inequalities (5.14) as follows. Given a fractional solution $(\bar{u}, \bar{z})$. We first consider an interval of contiguous slots $I = [s_i, s_j]$ which is identified by generating two slots s_i and s_j randomly in $\mathbb{S}$ with $s_j \ge s_i + 2\max_{k\in K} w_k$. The separation problem associated with the inequality (5.14) is NP-hard [125] given that it consists in identifying a cover $\tilde{K}^*$ for the interval $I = [s_i, s_j]$, s.t. $\sum_{k\in\tilde{K}^*} \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > |\tilde{K}^*| - 1$. For that, we use a greedy algorithm introduced by Nemhauser and Sigismondi in [110] as follows. We first select a demand $k \in K$ having the largest number of requested slot w_k with $\sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > 0$, and then set $\tilde{K}^*$ to $\tilde{K}^* = \{k\}$. After that, we iteratively add each demand $k' \in K \setminus \tilde{K}^*$ to $\tilde{K}^*$ with $\sum_{s'=s_i+w_{k'}-1}^{s_j} \bar{z}_{s'}^{k'} > 0$ and demand k' share an edge with all the demands already added $\tilde{K}^*$, until a cover $\tilde{K}^*$ is obtained for the interval I over the edge e with $\sum_{k\in\tilde{K}^*} w_k > |I|$. We further derive a minimal cover from the cover $\tilde{K}^*$ by deleting each demand $k \in \tilde{K}^*$ if $\sum_{k'\in\tilde{K}^*\setminus\{k\}} w_{k'} \le |I|$. We then add the inequality (5.14) induced by the minimal cover $\tilde{K}^*$ for the interval I if it is violated, i.e., we add the following valid inequality to the current LP

$$\sum_{k\in\tilde{K}^*} \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k \le |\tilde{K}^*| - 1.$$

Algorithm 6: Branch-and-Cut Algorithm for the SA

Data: An undirected, loopless, and connected graph $G = (V, E)$, a spectrum $\mathbb{S}$, a multi-set K of demands, and a given class of valid inequality

Result: Optimal solution for the SA problem

Stop= FALSE;
while *STOP==FALSE* **do**
    Solve the linear program LP of the SA;
    Let (u^*, z^*) be the optimal solution of LP;
    if *there exist inequalities from the given class that are violated by the current solution* (u^*, z^*) **then**
        Add them to LP ;
    end
    else
        STOP = TRUE;
    end
end
Consider the optimal solution (u^*, z^*) of LP ;
if (u^*, z^*) *is integer for the SA* **then**
    (u^*, z^*) is an optimal solution for the SA;
    End of the Branch-and-Cut algorithm ;
end
else
    Create two Sub-problems by branching one some variables or constraints ;
end
for *each Sub-problem not yet solved* **do**
    go to 2 ;
end
return the best optimal solution (u^*, z^*) for the SA;

5.8.2 Primal Heuristic

Let us present now a primal heuristic useful to boost the performance of the Branch-and-Cut algorithm. It is based on a hybrid method between a local search algorithm and a greedy-algorithm. Given an optimal fractional solution $(\bar{u}, \bar{z})$ in a certain node of the B&C tree, it consists in constructing an integral solution and "feasible" if possible from this fractional solution. To do so, we first use a local search algorithm to generate at each iteration a sequence of demands L numeroted with $L = 1', 2', ..., |K|' - 1, |K|'$. Based on this sequence of demands, our greedy algorithm selects a slot s for each demand $k' \in L$ with $\bar{z}_s^{k'} \neq 0$, while respecting the non-overlapping constraint with the set of demands that precede the demand k' in the list L (i.e., the demands $1', 2, ..., k' - 1$). However, if there does not exist such slot s for the demand k', we then select a slot s for the demand $k' \in L$ with $\bar{z}_s^{k'} = 0$ with $s \in \{w_{k'}, ..., \bar{s}\}$ while respecting the non-overlapping constraint with the set of demands that precede the demand k' in the list L. After that, we compute the total number of slots in $\mathbb{S}$ used by the set of demands K in the final solution $\mathcal{S}$ given by the greedy-algorithm (i.e., $\sum_{s \in \mathbb{S}} u_s$). Our local search algorithm generates a new sequence by doing some permutation of demands in the last sequence of demands, if the value of the solution given by greedy-algorithm is smaller than the value of the best solution found until the current iteration. Otherwise, we stop the algorithm, and we give in output the best solution found during our primal heuristic induced by the best sequence of demands having the smallest value of the total number of slots in $\mathbb{S}$ used compared with the others generated sequences.

5.9 Computational Study

5.9.1 Implementation's Feature

We use C++ to implement the B&B and B&C algorithms under Linux using the "Solving Constraint Integer Programs" (SCIP 6.0.2) framework s.t. Cplex 12.9 is used for the resolution of the linear relaxation at each node in the B&B and B&C trees. These have been also tested on LIMOS high-performance servers with a memory size limited to 64 Gb while benefiting from parallelism by activating 8 threads, and with a CPU time limited to 5 hours (18000 s). We use the same topologies presented in the Figure 3.2.2, and the same instances used in the section 3.2.2.

5.9.2 Computational Results

Preliminary results show that introducing some family of valid inequalities enables reducing the average number of nodes in the B&C tree, and also the average CPU time for several instances. On the other hand, the results show that the odd-hole inequalities (2.36) and (2.40) are efficient compared with those of clique-based inequalities (2.38), (2.35), and cover-based inequalities (5.14). As a result, we combine these families of valid inequalities s.t. their separation is performed along with the B&C algorithm in the following order

a) interval-odd-hole inequalities (2.36),

b) slot-assignment-odd-hole inequalities (2.40),

c) interval-clique inequalities (2.38),

d) slot-assignment-clique inequalities (2.38),

e) interval-capacity-cover inequalities (5.14).

We also consider the valid inequalities (5.21) introduced previously that are shown to be as a precomputed lower bounds for the SA. We generate some of them during the preprocessing stage as follows. For each demand $k \in K$, we use a greedy algorithm introduced by Nemhauser and Sigismondi in [110] to generate a maximum clique in $\tilde{G}^r$ containning demand k. We first set $\tilde{C}^k$ to $\tilde{C}^k = \{k\}$. After that, we iteratively add each demand $k' \in K \setminus \tilde{C}^k$ to $\tilde{C}^k$ s.t. the demand k' must share an edge with all the demands already generated in $\tilde{C}^k$. We further add the inequality (5.21) induced by the clique $\tilde{C}^k$ for the demand k to the compact formulation (5.1)-(5.8)

$$\sum_{s \in \mathbb{S}} u_s \geq \sum_{k' \in \tilde{C}^k} w_{k'}.$$

Based on this, we provide a comparative study between B&B (without additional valid inequalities) and B&C (with additional valid inequalities) algorithms. To do so, we evaluate the impact of valid inequalities used within the B&C algorithm. We present some computational results using several instances with a number of demand ranges in $\{10, 20, 30, 40, 50, 100, 150, 200, 250, 300\}$ and $\bar{s}$ up to 320 slots. We use two types of topologies: real, and realistic ones from SND-LIB already described in Table 3.2.2. Our series of computational results presented in Table 5.1, it concerns a comparison between the results obtained for the B&B (without additional valid inequalities) and B&C (without or with additional valid inequalities) algorithms using SCIP. We consider 4 criteria in the different tables, average number of nodes in the enumeration tree (Nb_Nd), average gap (Gap) which represents the relative error between the lower bound gotten at the end of the resolution and best upper bound, average number of violated inequalities added during the resolution (Nbr_Cuts), average Cpu time computation (T_Cpu). For each instance, we use SCIP without our additional valid inequalities and with using its proper cut generation (denoted by B&B_SCIP in the different tables), and SCIP using our valid inequalities and disabling its proper cut generation (denoted by B&C_SCIP).
The results show that introducing valid inequalities allows solving several instances to optimality that are not solved to optimality using the B&B algorithm even if SCIP use its proper cuts. Furthermore, we noticed that they enabled reducing the average number of nodes in the B&C tree for several instances s.t. there exist some cases that we are able to solve some instances in the root of the B&C tree which requires several branching when using the B&B. On the other hand, and looking at the instances that are not solved to optimality (i.e., gap $\geq 0,00$), adding valid inequalities decreases the average gap for several instances and much more for the large instances with a number of demands $|K| \geq 150$. However, there exist a few instances very rare for example the triplet (German, 300, 320) in which adding valid inequalities does not improve the results of the B&B algorithm. Based on these results, we ensure that using the valid inequalities allows obtaining tighter LP bounds and improve the effectiveness of the B&B algorithm s.t. the B&C algorithm is able to beat the B&B algorithm even if SCIP use its proper cuts that are shown to be very efficient in the literature.

5.10 Concluding Remarks

In this chapter, we studied the Spectrum Assignment sub-problem. We introduced an integer linear programming compact formulation for the problem. We investigated the facial structure of the associated polyhedron and derived valid inequalities that are facet-defining under sufficient conditions. We further discuss their separation problems. Using the polyhedral results and the separation procedures, we devise Branch-and-Cut (BC) algorithm to solve the problem. We also present some experimental results and show the effectiveness of the valid inequalities.

Instances			B&B_SCIP			B&C_SCIP_With_Add_Valid_Ineq			
Topology	$\|K\|$	$\|\mathbb{S}\|$	Nbr_Nd	Gap	T_Cpu	Nbr_Nd	Gap	Nbr_Cuts	T_Cpu
German	10	30	1	0,00	0,03	1	0,00	0	0,02
German	20	45	1	0,00	0,53	1	0,00	11	0,66
German	30	70	7	0,00	1,47	1	0,00	5	1,81
German	40	90	2,5	0,00	1,78	5	0,00	5	15,89
German	50	110	1	0,00	0,87	1	0,00	2,5	9,34
German	100	140	1	0,00	12,92	1	0,00	6,25	90,94
German	150	210	1,75	0,00	43,22	1	0,00	0,75	118,59
German	200	260	1	0,00	176,01	9,5	0,00	2,5	992,34
German	250	320	21	0,00	380,74	9	0,00	12	2148,45
German	300	320	6	0,00	2584,4	13,25	0,08	8,5	8801,17
Nsfnet	10	15	1	0,00	0,02	1	0,00	0	0,01
Nsfnet	20	40	2,5	0,00	1,83	1,5	0,00	0	0,53
Nsfnet	30	30	4	0,00	2,92	2,25	0,00	3,75	4,14
Nsfnet	40	70	4,5	0,00	2,13	28,5	0,00	32	16,01
Nsfnet	50	50	9	0,00	4,47	4,75	0,00	19,25	12,61
Nsfnet	100	120	14469	0,94	4552,24	5090,25	0,00	20,25	1565,48
Nsfnet	150	160	10,75	0,00	215,01	66	0,00	6,5	841,26
Nsfnet	200	210	37	0,00	986,26	23	0,00	2,75	2035,74
Nsfnet	250	285	138	1,00	6535,05	397,5	0,00	3,75	7999,81
Nsfnet	300	320	20,5	1,81	9932,57	27	1,02	25,5	12712,35
Spain	10	15	1	0,00	0,02	1	0,00	0,25	0,03
Spain	20	20	1	0,00	0,14	1	0,00	4	0,08
Spain	30	25	1	0,00	0,62	1	0,00	1	0,33
Spain	40	60	1	0,00	1,27	5,75	0,00	17,5	8,44
Spain	50	35	476,25	0,00	34,36	3,75	0,00	10	7,17
Spain	100	120	169,5	0,97	4782,16	2359,5	1,09	17,75	4810,79
Spain	150	160	106,75	0,84	10722,14	59,5	0,28	37,5	8804,63
Spain	200	200	26	1,60	5866,57	652,75	1,20	50,5	5829,44
Spain	250	240	1	0,00	3444,44	25	0,00	21,75	6528,09
Spain	300	280	1,25	12,46	14696,38	15,25	5,43	127	17456,46
Pioro40	10	40	1	0,00	0,02	1	0,00	0	0,03
Pioro40	20	40	1	0,00	0,53	1	0,00	0	0,10
Pioro40	30	40	1	0,00	3,74	1	0,00	5,5	0,57
Pioro40	40	40	4	0,00	1,32	3	0,00	12,5	5,84
Pioro40	50	80	5	0,00	2,66	1,25	0,00	15,75	13,52
Pioro40	100	80	3	0,00	44,31	18,5	0,00	77,5	2769,13
Pioro40	150	160	56	1,95	9335,82	57	0,00	48,75	9169,93
Pioro40	200	280	1	0,14	4934,59	1,25	0,00	28,5	3023,14
Pioro40	250	280	1	0,00	3782,08	1	0,00	73,5	2580
Pioro40	300	320	4,25	0,18	10548,18	3,25	0,36	96	13502,49
India35	10	80	1	0,00	0,04	1	0,00	0	0,05
India35	20	40	1	0,00	0,08	1	0,00	0	0,12
India35	30	40	2	0,00	3,52	1,25	0,00	12	7,13
India35	40	80	4,5	0,00	4,43	1,25	0,00	0	5,82
India35	50	160	1	0,00	7,64	6	0,00	6,25	136,81
India35	100	240	13,5	1,55	13278,76	10,75	4,67	36,5	9333,74
India35	150	400	8	4,71	18000	12,75	1,61	56	11878
India35	200	280	1	10,58	13577,39	7,75	4,81	61,5	16122,26
India35	250	280	1	1,45	18000	2	0,72	62	18000
India35	300	320	1	1,80	16858,20	1,25	4,19	53	18000
Brain161	10	40	1	0,00	0,08	1,00	0,00	0,00	0,09
Brain161	20	40	1	0,00	0,04	1,00	0,00	0,00	0,06
Brain161	30	40	1	0,00	0,36	1,00	0,00	0,00	0,18
Brain161	40	80	6,75	0,00	11,91	5,50	0,00	9,25	19,92
Brain161	50	120	9	0,00	25,23	7,50	0,00	26,50	55,13
Brain161	100	160	65	0,00	3297,48	18,75	0,00	58,00	1887,77
Brain161	150	320	58,5	0,26	10284,04	11,75	0,14	52,00	8796,04
Brain161	200	400	8	0,40	12172,23	1,00	0,18	30,25	10407,16
Brain161	250	480	1	0,86	13492,92	1,00	0,35	0,00	18000
Brain161	300	320	1	1,30	18000	1,00	0,32	127,50	18000

Table 5.1: Table of comparison between: B&B Algorithm (Without Additional Valid Inequalities and With SCIP Cuts) Vs B&C Algorithm (With Additional Valid Inequalities and Without SCIP Cuts) using SCIP.

Conclusion

In this thesis, we have studied the Constrained-Routing and Spectrum Assignment (C-RSA) problem related to the dimensioning and designing of Spectrally Flexible Optical Networks (SFONs). It's well known to be an NP-hard problem. The main aim of this thesis was to provide a deep polyhedral investigation and design a cutting plane method for the problem and handle large-sized instances.
First, we have proposed an integer linear programming formulation namely cut formulation. We have investigated the related polytope defined by the convex hull of all its solutions. Moreover, we have identified several classes of valid inequalities for the polytope and studied their facial structure. We further have discussed their separation problems. We have also proposed a primal heuristic to obtain tighter primal bounds and enhance the resolution of the problem. These results are used to devise a Branch-and-Cut (B&C) algorithm for the C-RSA problem, along with some computational results are presented using two types of instances: random and realistic ones with $|K|$ up to 300 and $|\mathbb{S}|$ up to 320. They are composed of two types of graphs (topologies): real graphs and realistic ones from SND-LIB with $|V|$ up to 161 and $|E|$ up to $|166|$. The results have shown the significant improvement allowed by introducing the valid inequalities on obtaining tighter LP bounds and improving the effectiveness of the B&C algorithm.
In the second part of the thesis, we have discussed an extended formulation based on the so-called path formulation. It can be seen as a reformulation of the cut formulation using the so-called path variables. We have developed a column generation algorithm to solve its linear relaxation. We have shown that the pricing problem is equivalent to the resource-constrained shortest path problem, which is well known to be NP-hard. For this, we have developed a pseudo-polynomial algorithm based on dynamic programming enabled solving the pricing problem in polynomial time. Using this, we have devised Branch-and-Price and Branch-and-Cut-and-Price algorithms. The results show that the Branch-and-Cut-and-Price performs very well compared with the Branch-and-Price. Hence, the significant impact and the power of the introduced valid inequalities allowed improving the effectiveness of the B&C&P algorithm. On the other, we have presented a comparative study between the B&C, B&P, and B&C&P algorithms. The results have shown that the B&C&P algorithm is able to provide optimal solutions for several instances, which is not the case for the B&C algorithm within the CPU time limit (5 hours). Moreover, both B&C and B&P algorithms perform well. However, some instances are still difficult to solve with both B&C, B&P and B&C&P algorithms. For this, some enhancements are further investigated and integrated into our algorithms. They are based on a warm-start algorithm using some metaheuristics, and a primal heuristic using a hybrid method between a greedy algorithm and local search algorithm that is shown to be very useful to obtain good primal bounds. Moreover, we introduce some symmetry-breaking inequalities that allow avoiding the equivalents sub-problems in the different enumeration trees of B&C, B&P, and B&C&P algorithms.
Afterward, we have studied the Spectrum Assignment (SA) sub-problem when the routing

is trivial or a routing path is pre-selected for each demand. First, we have presented a compact formulation for the SA. We have carried out an investigation of the associated polytope. Moreover, we have identified several valid inequalities for the polytope, some of them come from those that are already proposed for the C-RSA. We have proved that they are facet defining under certain necessary and sufficient conditions. They were further incorporated within a Branch-and-Cut algorithm. The results have shown the efficiency of the valid inequalities allowed enhancing the resolution of the SA problem. Hence, the Branch-and-Cut is shown to be very performant compared with the Branch-and-Bound algorithm.
Finally, it would be interesting to further investigate a combination of the different algorithms with machine learning and reinforcement learning algorithms to well manage the B&C, B&P, and B&C&P trees and particularly for

a) the node selection [41][51],

b) variable selection and branching rule [7][51],

c) column selection [56][171],

d) cut selection [75][166],

e) and provide a deeper comparative study between the algorithms [1].

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