Preferences Aggregation: the MAUT approach

University Paris Dauphine
LAMSADE
FRANCE

Chapter 4
Outline

1. Framework
2. The simple models
3. Introduction to MCDA
4. Multi Attribute Utility Theory
5. The additive model
6. A non-additive model: The Choquet integral
7. To conclude
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A Decision Maker (DM) is facing a decision problem, i.e., the DM has to deal with multiple alternatives and has to compare them.

Alternatives are described on several attributes.

A criterion is an attribute with a preference relation (monotonic attribute).

Criteria cannot be reduced to one criterion as they are potentially in conflict.
Example (Compare two bikes on three attributes)

<table>
<thead>
<tr>
<th></th>
<th>Speed</th>
<th>Robustness</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mountain bike</td>
<td>20 km/h</td>
<td>Good</td>
<td>500 €</td>
</tr>
<tr>
<td>Race bike</td>
<td>35 km/h</td>
<td>Middle</td>
<td>1000 €</td>
</tr>
</tbody>
</table>
Example (Compare many objects)

<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Label bio</td>
<td>Non</td>
<td>Non</td>
<td>Non</td>
<td>Non</td>
<td>FSC/ABW Recette UR</td>
<td>Non</td>
<td>Non</td>
<td>Non</td>
<td>Non</td>
<td>Non</td>
<td>FSC/ABW Recette UR</td>
</tr>
<tr>
<td>Prix indicatif</td>
<td>6.60 €</td>
<td>12.60 €</td>
<td>15.60 €</td>
<td>10.30 €</td>
<td>12.30 €</td>
<td>10.60 €</td>
<td>8.90 €</td>
<td>18.65 €</td>
<td>9 €</td>
<td>9 €</td>
<td>9 €</td>
</tr>
<tr>
<td>Prix pour une couche</td>
<td>-0.40 €</td>
<td>0.21 €</td>
<td>-0.31 €</td>
<td>0.34€</td>
<td>0.17 €</td>
<td>-0.28 €</td>
<td>0.16 €</td>
<td>5.18 €</td>
<td>-0.33 €</td>
<td>5.31 €</td>
<td>0.28 €</td>
</tr>
</tbody>
</table>

Performance (50%)
- Temperature: 
- Absorption: 
- Protection contre l'humidité: 

Composition (40%)
- Pesticides: 
- Résidus de glyphosat: 

Autres molécules toxiques potentielles
- Endocrines: 
- Composés organiques volatils (COV): 
- Composés organiques halogénés chlorotrifluorométhanes (ACT): 
- Allergènes: 

(1) Coût inclus dans le prix. (2) L'appartenance globale ne peut pas être supérieure à l'appartenance sur les performances. (3) Le titre va indiquer que cette référence est fin de commerce / couche / (5) capacité (4) Coût inclus dans le prix. (5) L’appartenance globale ne peut pas être supérieure à “not classified” ou “not sufficient”.
Multi-attribute formal model: Inputs

- A set of alternatives \( X = X_1 \times X_2 \times \cdots \times X_n \)
- There exists the preferences on the values of each criterion \( i \) (utility function, qualitative preference relation \( \succsim_i \), . . .)
- A representation of the importance of each criterion or set of criteria (weights, importance relation, . . .)
Multi-attribute formal model: a treatment

Using the input information, elaborate a decision rule allowing to compare two different alternatives, i.e.,

\[
\begin{align*}
    x &= (x_1, \ldots, x_n) \\
    y &= (y_1, \ldots, y_n)
\end{align*}
\]

\[\implies x \succcurlyeq y \text{ or } y \succcurlyeq x\]
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PARETO DOMINANCE

An alternative is preferred to another one if it is considered to be better on all the criteria.

\[ x \succsim y \iff \forall i \in N, x_i \succsim_i y_i \]

Example

<table>
<thead>
<tr>
<th></th>
<th>Speed</th>
<th>Robustness</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bike A</td>
<td>10 km/h</td>
<td>Good</td>
<td>600 €</td>
</tr>
<tr>
<td>Bike B</td>
<td>20 km/h</td>
<td>Good</td>
<td>550 €</td>
</tr>
<tr>
<td>Bike C</td>
<td>19 m/h</td>
<td>Very Good</td>
<td>800 €</td>
</tr>
</tbody>
</table>

\[ B \succsim A \text{ and } \neg (A \succsim B) \implies B \succ A \]

\[ \neg(B \succsim C) \text{ and } \neg(C \succsim B) \]

Pareto dominance is not so interesting
Dominance

- An alternative $x = (x_1, \ldots, x_n)$ dominates an alternative $y = (y_1, \ldots, y_n)$ if $\forall i \in N, x_i \succeq_i y_i$.
- An alternative $x = (x_1, \ldots, x_n)$ strictly dominates an alternative $y = (y_1, \ldots, y_n)$ if $\forall i \in N, x_i \succeq_i y_i$ and $\exists i_0 \in N, x_{i_0} >_{i_0} y_{i_0}$.

Definition

The Pareto front is the set of all non-dominated alternatives.

Remark

- The optimal solution is necessary in the Pareto front
- In general, the Pareto front may be poor, i.e., it is not really different to the whole set of alternatives.
MULTI-OBJECTIVE OPTIMIZATION

- Principle: The “Best” alternative should be the nearest alternative to an “ideal point”.
- Usually, the “ideal point” if computed by taking the max (resp min) value on each criterion.
- Many distances are also used in the resolution of a multi-objective problem.
Weighted sum

Let be \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) two alternatives such that \( x_i, y_i \in \mathbb{R}, \forall i \in N \). Let be \( w_i \) the weight associated to the criterion \( i \).

\[
x \succ y \iff \sum_{i=1}^{n} w_i x_i \geq \sum_{i=1}^{n} w_i y_i
\]

Example

<table>
<thead>
<tr>
<th></th>
<th>Speed</th>
<th>Robustness</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bike A</td>
<td>8/20</td>
<td>18/20</td>
<td>12/20</td>
</tr>
<tr>
<td>Bike B</td>
<td>18/20</td>
<td>8/20</td>
<td>12/20</td>
</tr>
<tr>
<td>Bike C</td>
<td>12/20</td>
<td>12/20</td>
<td>12/20</td>
</tr>
</tbody>
</table>

\[
w_S > w_R \implies B \succ A
\]
\[
w_R > w_S \implies A \succ B
\]
\[
\forall w_R, w_S, \text{ we have } A \succ C \text{ or } B \succ C
\]
The majority rule

Let be $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ two alternatives. $x$ is preferred to $y$ if it is considered “good” on a majority of criteria.

$x \succsim y \iff \left| \{i \in N : x_i \succsim_i y_i\} \right| \geq \left| \{i \in N : y_i \succsim_i x_i\} \right|$

Example

<table>
<thead>
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$B \succsim C$
Example (Majority rule)

<table>
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<tr>
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<td>15 km/h</td>
<td>Good</td>
<td>500 €</td>
</tr>
<tr>
<td>Bike C</td>
<td>25 m/h</td>
<td>Bad</td>
<td>550 €</td>
</tr>
</tbody>
</table>

Which bike do you choose?
Example (Majority rule)

<table>
<thead>
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<th></th>
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<td>25 m/h</td>
<td>Bad</td>
<td>550 €</td>
</tr>
</tbody>
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\[ A \succ B \]
\[ B \succ C \]
\[ C \succ A \]

\[ \implies \text{Condorcet Paradox} \]
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MultiCriteria Decision Aiding (MCDA): Difficulties

- MultiCriteria Decision Ciding is not so easy: it is not an easy task
- Every method has advantages and inconveniences: there is no “best method”
- All methods have structural bias.
Paul Valery (Artist, Writer, Poet, Philosopher (1871-1945))

- Tout ce qui est simple est faux, mais tout ce qui ne l’est pas est inutilisable

- What is simple is false. What is complex is useless.
Three types of problems in MCDA

- **Choice Problem**: choose the “best” alternative(s).
- **Ranking Problem**: rank the alternatives from the “best” to the “worst”.
- **Sorting Problem**: sort the alternatives into pre-defined categories (in general ordered categories)
Two main approaches in MCDA

- **Multi Attribute Utility Theory**: A quantitative approach “aggregate then compare” (scoring)
  \[ x \succeq y \iff U(x_1, \ldots, x_n) \geq U(y_1, \ldots, y_n) \]

- **Outranking**: qualitative approach “compare then aggregate”
  \[ x \succeq y \iff |\{i \in N : x_i \succeq_i y_i\}| \triangleright |\{i \in N : y_i \succeq_i x_i\}| \]
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Principle

Le be $X$ a set of alternatives evaluated on a finite set of $n$ criteria $N = \{1, \ldots, n\}$. In general, we set $X = X_1 \times X_2 \times \ldots \times X_n$.

Le be $\succeq_X$ a complete preorder on $X$ (preferences of a DM).

$\succeq_X$ are supposed to be representable by an overall utility function:

$$\forall x, y \in X, \quad x \succeq_X y \iff F(U(x)) \geq F(U(y))$$

where

- $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$
- $U(x) = (u_1(x_1), \ldots, u_n(x_n))$
- $u_i : X_i \rightarrow \mathbb{R}$ is a marginal utility function or simply called utility function or a scale on $\mathbb{R}$
- $F : \mathbb{R}^n \rightarrow \mathbb{R}$ an aggregation function

$F$ is generally characterized by a parameter vector $\theta$ (weight vector,\ldots).
Problems

1. How to choose the aggregation function $F$?

2. How to construct the marginal utility functions $u_i : X_i \rightarrow \mathbb{R}$?

3. The marginal utility functions $u_i : X_i \rightarrow \mathbb{R}$ should have a signification for the decision maker (see measurement theory):
   - **Ordinal scales**: Differences between values have no importance (e.g. a rank). They can represent orders and pre-orders.
   - **Cardinal scales**: Differences between values may be meaningful.
     - *Interval scales*: absolute differences between values are important.
     - *Ratio scales*: ratio between values are important.
The additive model

\( \succeq_X \) are supposed to be representable by an overall utility function:

\[
\forall x \in X, \quad F(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} u_i(x_i)
\]

This model is equivalent to the existence of weights \( w_i, i = 1, \ldots, n \), such that

\[
\forall x \in X, \quad F(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} w_i \cdot u_i(x_i)
\]
The additive model

- A simple method

Additive value function involves compensation between criteria, i.e., a bad performance on a criterion $i$ could be compensated by a good performance on another criterion.

See e.g. students evaluation based on the weighted sum.

- In the weighted sum, weights represent, in reality, the substitution rate between criteria.
The additive model

- In the weighted sum, weights represent, in reality, the substitution rate between criteria.

  - E.g. for $n = 2$, $w_1 = b w_2$ means the DM is indifferent between these two alternatives $(0, b)$ and $(1, 0)$, i.e., $(0, b) \sim (1, 0)$.
  
  - There is a total compensation between “bad” performances and “good” performances.

  If we have $(a, b) \sim (a - \delta, b + \gamma)$ then gain of $\gamma$ compensates the loss of $\delta$.

  Indeed we have

  $$a w_1 + b w_2 = (a - \delta) w_1 + (b - \gamma) w_2$$

  $$\iff \delta w_1 = \gamma w_2$$

  $$\iff \frac{w_1}{w_2} = \frac{\gamma}{\delta}$$

  Implicitly, this implies that all the criteria could be express indirectly in the same unit ($€$, seconds, ...).
The additive model

- Requires to normalize the criteria. In general, we set $\forall i \in N, u_i : X_i \mapsto [0, 1]$.

E.g. For a criterion to be maximized, we could choose the following normalization functions:

- $u_i(x_i) = \frac{x_i}{\max x_i}$
- $u_i(x_i) = \frac{x_i - \min x_i}{\max x_i - \min x_i}$
- ...
The mutual Preferential independence

- The additive model requires to satisfy the mutual Preferential independence axiom, i.e., criteria are independent in the sense of preferences

\[ \forall i \in N, \forall z_i, t_i \in X_i, \forall x, y \in X, \]

\[ (z_i, x_{N-i}) \succ (z_i, y_{N-i}) \iff (t_i, x_{N-i}) \succ (t_i, y_{N-i}) \]

An attribute is preferentially independent from all other attributes when changes in the rank ordering of preferences of other attributes does not change the preference order of the attribute.
MAUT in practice

1. People suppose \( \succeq_X \) representable by an overall utility function:

\[
x \succeq_X y \iff F(U(x)) \geq F(U(y))
\]

2. \( F \) is generally characterized by a parameter vector \( \theta \) (weight vector, \ldots).

3. People ask to the DM some preferential information \( \succeq_{X'} \) on a reference subset (learning set) \( X' \subseteq X \).

4. The parameter vector is constructed so that \( \succeq_X \) is an extension of \( \succeq_{X'} \).

5. The model obtained in \( X' \) will be then automatically extended to \( X \).
**Example (Evaluation of diapers by the Magazine “60 millions consommateurs” in September 2018)**

### Couches-culottes

<table>
<thead>
<tr>
<th>Marque</th>
<th>Protection Premium</th>
<th>Premium Protection</th>
<th>Baby-Dry</th>
<th>Eco by Naty</th>
<th>Premium Protection Active Fit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joone</td>
<td>Très bon (20 à 17)</td>
<td>Bon (16,5 à 13)</td>
<td>Acceptable (12,5 à 10)</td>
<td>Insuffisant (9,5 à 7)</td>
<td>Très insuffisant (5,5 à 0)</td>
</tr>
<tr>
<td>Pampers</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Naty</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Label bio**
- Joone: Non
- Pampers: Non
- Pampers: Non
- Naty: FSC/AB Vincotte UE
- Pampers: Non

**Prix indicatif**
- Joone: 64,90 € (162 couches)
- Pampers: 12,60 € (50 couches)
- Pampers: 15,60 € (50 couches)
- Naty: 16,90 € (50 couches)
- Pampers: 12,30 € (46 couches)

**Prix pour une couche**
- Joone: 0,40 €
- Pampers: 0,25 €
- Pampers: 0,31 €
- Naty: 0,38 €
- Pampers: 0,27 €

**Performances (60%)**
- Joone: +++
- Pampers: ++
- Pampers: +
- Naty: +
- Pampers: +

**Tenue**
- Joone: +++
- Pampers: ++
- Pampers: +
- Naty: +
- Pampers: +

**Absorption**
- Joone: +++
- Pampers: ++
- Pampers: +
- Naty: +
- Pampers: +

**Protection contre l’humidité**
- Joone: +++
- Pampers: ++
- Pampers: +
- Naty: +
- Pampers: +

**Composition (40%)**
- Joone: +++
- Pampers: ++
- Pampers: +
- Naty: +
- Pampers: +

**Pesticides**
- Joone: +++
- Pampers: ++
- Pampers: +
- Naty: +
- Pampers: +

**Autres molécules toxiques potentielles**
- Joone: +++
- Pampers: ++
- Pampers: +
- Naty: +
- Pampers: +

**NOTE GLOBALE (100%)**
- Joone: 17/20
- Pampers: 14,5/20
- Pampers: 12,5/20
- Naty: 12,5/20
- Pampers: 12,5/20

[1] Livraison comprise dans le prix  
[2] L’appréciation globale ne peut pas être supérieure à l’appréciation sur les performances  
[3] Le fabricant indique que cette référence est en fin de commercialisation  
[4] L’appréciation
Example (Evaluation of diapers by the Magazine “60 millions consommateurs” in September 2018)

<table>
<thead>
<tr>
<th>Carrefour Baby Ultra dry &amp; stretch</th>
<th>Lupilu Soft &amp; Dry</th>
<th>Mots d’enfants (Marque Repère) Ultra confort</th>
<th>Love &amp; Green Couches hypoallergéniques</th>
<th>Lotus Baby Touch 3 Ultra confort</th>
<th>Pommette (Intermarché) Ecologic</th>
<th>Lillydoo Couches bébé</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non</td>
<td>10,80 €</td>
<td>Non</td>
<td>0,19 €</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>56 couches</td>
<td>0,13 €</td>
<td>50 couches</td>
<td>0,18 €</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FSC</td>
<td>7,30 €</td>
<td>PEFC</td>
<td>19,65 €</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>56 couches</td>
<td>0,13 €</td>
<td>52 couches</td>
<td>0,38 €</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FSC</td>
<td>8,90 €</td>
<td>FSC</td>
<td>19 €</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50 couches</td>
<td>0,18 €</td>
<td>58 couches</td>
<td>0,33 €</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FSC</td>
<td>19,65 €</td>
<td>FSC</td>
<td>9 €</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>52 couches</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>58 couches</td>
<td>0,33 €</td>
<td>33 couches</td>
<td>0,36 €</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FSC/Nordic coolabel</td>
<td>12 €</td>
<td>Non</td>
<td>0,36 €</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

![Image of diapers evaluation table]

Preferences Aggregation: the MAUT approach
Example (Evaluation of diapers by the Magazine “60 millions consommateurs” in September 2018)

<table>
<thead>
<tr>
<th></th>
<th>Performance</th>
<th>Composition</th>
<th>Global score (/20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A- Joone</td>
<td>+++</td>
<td>+++</td>
<td>17</td>
</tr>
<tr>
<td>B- Pamp. Prem</td>
<td>++</td>
<td>++</td>
<td>14.5</td>
</tr>
<tr>
<td>C- Pamp. Baby</td>
<td>+</td>
<td>+++</td>
<td>12.5</td>
</tr>
<tr>
<td>D- Naty</td>
<td>+</td>
<td>+++</td>
<td>12.5</td>
</tr>
<tr>
<td>E- Pamp. Activ.</td>
<td>+</td>
<td>+</td>
<td>12.5</td>
</tr>
<tr>
<td>F- Carref. Baby</td>
<td>++</td>
<td>+</td>
<td>12.5</td>
</tr>
<tr>
<td>G- Lupilu</td>
<td>++</td>
<td>−</td>
<td>12</td>
</tr>
<tr>
<td>H- Mots d’enfants</td>
<td>+</td>
<td>−</td>
<td>12</td>
</tr>
<tr>
<td>I- Love &amp; Green</td>
<td>++</td>
<td>−</td>
<td>9.5</td>
</tr>
<tr>
<td>K- Lotus Baby</td>
<td>++</td>
<td>−−</td>
<td>9.5</td>
</tr>
<tr>
<td>L- Pommette</td>
<td>++</td>
<td>−−</td>
<td>9.5</td>
</tr>
<tr>
<td>M- Lillydoo</td>
<td>+</td>
<td>−−</td>
<td>6.5</td>
</tr>
</tbody>
</table>

\[ w_p = 60\% \quad w_c = 40\% \]

+++ ≡ Very good ∈ [17, 20]; ++ ≡ Good ∈ [13, 16.5]; 
+ ≡ Acceptable ∈ [10, 12.5]; 
− ≡ Insufficient ∈ [7, 9.5]; 
−− ≡ Very Insufficient ∈ [0, 6.5]
Example (Evaluation of diapers by the Magazine “60 millions consommateurs” in September 2018)

<table>
<thead>
<tr>
<th></th>
<th>Performance</th>
<th>Composition</th>
<th>Global score (/20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A- Joone</td>
<td>++++</td>
<td>+++</td>
<td>17</td>
</tr>
<tr>
<td>B- Pamp. Prem</td>
<td>++</td>
<td>++</td>
<td>14.5</td>
</tr>
<tr>
<td>C- Pamp. Baby</td>
<td>+</td>
<td>+++</td>
<td>12.5</td>
</tr>
<tr>
<td>D- Naty</td>
<td>+</td>
<td>+++</td>
<td>12.5</td>
</tr>
<tr>
<td>E- Pamp. Activ.</td>
<td>+</td>
<td>+</td>
<td>12.5</td>
</tr>
<tr>
<td>F- Carref. Baby</td>
<td>++</td>
<td>+</td>
<td>12.5</td>
</tr>
<tr>
<td>G- Lupilu</td>
<td>++</td>
<td>−</td>
<td>12</td>
</tr>
<tr>
<td>H- Mots d’enfants</td>
<td>+</td>
<td>−</td>
<td>12</td>
</tr>
<tr>
<td>I- Love &amp; Green</td>
<td>++</td>
<td>−</td>
<td>9.5</td>
</tr>
<tr>
<td>K- Lotus Baby</td>
<td>++</td>
<td>−−</td>
<td>9.5</td>
</tr>
<tr>
<td>L- Pommette</td>
<td>++</td>
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<td>6.5</td>
</tr>
</tbody>
</table>

\( w_p = 60\% \quad w_c = 40\% \)

- Which evaluation model was used by this magazine?
- Are these preferences representable by an additive model? (by using the given utility functions and weight)
Objective of the UTA method

- Created by Jacquet Lagreze & Siskos in 1982 (at LAMSADEN)
- Elicit simultaneously utility functions and weights.
UTA Principles: Input data

- A set of Criteria $N$
- A set of alternatives $X$ evaluated on $N$
- A preorder $\succsim_X$ on $X' \subseteq X$ (not necessary complete)

UTA Principles: The model

- $U(x) = \sum_{i=1}^{n} w_i \ u_i(x_i)$

Where

- $\sum_{i=1}^{n} w_i = 1$
- $0 \leq u_i(x_i) \leq 1$ with $u_i$ a nondecreasing function

- Set $V(x) = U(x) + \delta(x)$ where $\delta(x)$ is a nonnegative real value.
UTA Principles: The linear program to solve

\[
\min \sum_{x \in \mathcal{X}'} \delta(x)
\]

\[
\begin{align*}
V(x) & \geq V(y) + \epsilon \iff x \succ y \\
V(x) & = V(y) \iff x \sim y \\
0 & \leq u_i(x_i) \leq 1 \\
\sum_{i=1}^{n} w_i & = 1
\end{align*}
\]

- If the optimal solution is equal to 0 then $\succeq_{\mathcal{X}'}$ is representable by (compatible with) an additive model.
- There are many versions of the UTA method.
Outline

1. Framework
2. The simple models
3. Introduction to MCDA
4. Multi Attribute Utility Theory
5. The additive model
6. A non-additive model: The Choquet integral
7. To conclude
Example (A classic example of Grabisch et al. (2010))

1: Mathematics (M)  2: Statistics (S)  3: Language (L)

<table>
<thead>
<tr>
<th></th>
<th>Math</th>
<th>Stat</th>
<th>Lang</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>16</td>
<td>13</td>
<td>7</td>
</tr>
<tr>
<td>b</td>
<td>16</td>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>c</td>
<td>6</td>
<td>13</td>
<td>7</td>
</tr>
<tr>
<td>d</td>
<td>6</td>
<td>11</td>
<td>9</td>
</tr>
</tbody>
</table>

• For a student good in Mathematics, Language is more important than Statistics
  \[ \Rightarrow a \prec b, \]

• For a student bad in Mathematics, Statistics is more important than Language
  \[ \Rightarrow d \prec c. \]
Example (A classic example of Grabisch et al. (2010))

\[1: \text{Mathematics (M)} \quad 2: \text{Statistics (S)} \quad 3: \text{Language (L)}\]

\[
\begin{array}{ccc}
16 & 13 & 7 \\
16 & 11 & 9 \\
6 & 13 & 7 \\
6 & 11 & 9 \\
\end{array}
\]

The two preferences \(a \prec b\) and \(d \prec c\) lead to a contradiction with the additive model

\[
\begin{cases}
\quad a \prec b \Rightarrow u_M(16) w_M + u_S(13) w_S + u_L(7) w_L < u_M(16) w_M + u_S(11) w_S + u_L(9) w_L \\
\quad d \prec c \Rightarrow u_M(6) w_M + u_S(11) w_S + u_L(9) w_L < u_M(6) w_M + u_S(13) w_S + u_L(7) w_L.
\end{cases}
\]

\(i.e.,\)

\[
\begin{cases}
\quad u_S(13) w_S + u_L(7) w_L < u_S(11) w_S + u_L(9) w_L \\
\quad \text{and} \quad u_S(11) w_S + u_L(9) w_L < u_S(13) w_S + u_L(7) w_L
\end{cases}
\]
Example (A classic example of Grabisch et al. (2010))

1 : Mathematics (M)  2 : Statistics (S)  3 : Language (L)

<table>
<thead>
<tr>
<th></th>
<th>Mathematics</th>
<th>Statistics</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>16</td>
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<td>11</td>
<td>9</td>
</tr>
<tr>
<td>c</td>
<td>6</td>
<td>13</td>
<td>7</td>
</tr>
<tr>
<td>d</td>
<td>6</td>
<td>11</td>
<td>9</td>
</tr>
</tbody>
</table>

- The preference information \( a \prec b \) and \( d \prec c \) is not representable by an additive utility model.
Example (A ranking of French hospitals for weight loss surgery)

To identify the “best” hospitals in weight loss surgery, the magazine “Le Point” combines a part of the following four indicators (criteria):

- **Criterion 1 - Activity**: number of procedures performed during one year. This criterion has to be maximized.
- **Criterion 2 - Notoriety**: its corresponds to the reputation and attractiveness of the hospital. It is a percentage of patients treated in the hospital but living in another French administrative department. More the percentage increases, more the hospital is attractive.
- **Criterion 3 - Average Length Of Stay (ALOS)**: a mean calculated by dividing the sum of inpatient days by the number of patients admissions with the same diagnosis-related group classification. If a hospital is more organized in terms of resources then its ALOS score should be low.
- **Criterion 4 - Technicality**: this particular indicator measures the ratio of procedures performed with an efficient technology compared to the same procedures performed with obsolete technology. The higher the percentage is, the more the team is trained in advanced technologies or complex surgeries.
Example (A ranking of French hospitals for weight loss surgery)

<table>
<thead>
<tr>
<th>Hospital</th>
<th>Activity</th>
<th>Notoriety</th>
<th>ALOS</th>
<th>Technicality</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1</td>
<td>200</td>
<td>65</td>
<td>3.5</td>
<td>85</td>
</tr>
<tr>
<td>H2</td>
<td>450</td>
<td>60</td>
<td>4</td>
<td>75</td>
</tr>
<tr>
<td>H3</td>
<td>450</td>
<td>50</td>
<td>2.5</td>
<td>55</td>
</tr>
<tr>
<td>H4</td>
<td>350</td>
<td>50</td>
<td>3.5</td>
<td>85</td>
</tr>
<tr>
<td>H5</td>
<td>350</td>
<td>55</td>
<td>2</td>
<td>75</td>
</tr>
<tr>
<td>H6</td>
<td>150</td>
<td>65</td>
<td>2.5</td>
<td>80</td>
</tr>
<tr>
<td>H7</td>
<td>200</td>
<td>55</td>
<td>2</td>
<td>55</td>
</tr>
<tr>
<td>H8</td>
<td>150</td>
<td>60</td>
<td>4</td>
<td>80</td>
</tr>
</tbody>
</table>

$H1 \succ H2; \quad H3 \succ H4; \quad H5 \succ H6; \quad H8 \succ H7.$

Are these preferences representable by an additive function?
Example (A ranking of French hospitals for weight loss surgery)

<table>
<thead>
<tr>
<th></th>
<th>1- Activity</th>
<th>2- Notoriety</th>
<th>3- ALOS</th>
<th>4- Technicality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hospital 1</td>
<td>200</td>
<td>65</td>
<td>3.5</td>
<td>85</td>
</tr>
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<td>450</td>
<td>60</td>
<td>4</td>
<td>75</td>
</tr>
<tr>
<td>Hospital 3</td>
<td>450</td>
<td>50</td>
<td>2.5</td>
<td>55</td>
</tr>
<tr>
<td>Hospital 4</td>
<td>350</td>
<td>50</td>
<td>3.5</td>
<td>85</td>
</tr>
<tr>
<td>Hospital 5</td>
<td>350</td>
<td>55</td>
<td>2</td>
<td>75</td>
</tr>
<tr>
<td>Hospital 6</td>
<td>150</td>
<td>65</td>
<td>2.5</td>
<td>80</td>
</tr>
<tr>
<td>Hospital 7</td>
<td>200</td>
<td>55</td>
<td>2</td>
<td>55</td>
</tr>
<tr>
<td>Hospital 8</td>
<td>150</td>
<td>60</td>
<td>4</td>
<td>80</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
H1 \succ H2 & \Rightarrow u_1(200)w_1 + u_2(65)w_2 + u_3(3.5)w_3 + u_4(85)w_4 > u_1(450)w_1 + u_2(60)w_2 + u_3(4)w_3 + u_4(75)w_4 \\
H3 \succ H4 & \Rightarrow u_1(450)w_1 + u_2(50)w_2 + u_3(2.5)w_3 + u_4(55)w_4 > u_1(350)w_1 + u_2(50)w_2 + u_3(3.5)w_3 + u_4(85)w_4 \\
H5 \succ H6 & \Rightarrow u_1(350)w_1 + u_2(55)w_2 + u_3(2)w_3 + u_4(75)w_4 > u_1(150)w_1 + u_2(65)w_2 + u_3(2.5)w_3 + u_4(80)w_4 \\
H7 \prec H8 & \Rightarrow u_1(200)w_1 + u_2(55)w_2 + u_3(2)w_3 + u_4(55)w_4 < u_1(150)w_1 + u_2(65)w_2 + u_3(4)w_3 + u_4(80)w_4
\end{align*}
\]

The first three equations in this system lead to

\[
 u_1(200)w_1 + u_2(55)w_2 + u_3(2)w_3 + u_4(55)w_4 > u_1(150)w_1 + u_2(60)w_2 + u_3(4)w_3 + u_4(80)w_4
\]

which contradicts the last equation.
We try another MCDA model: the Choquet integral.
Definition

A capacity (or fuzzy measure) on $N$ is a set function $\mu : 2^N \to [0, 1]$ satisfying the three properties:

1. $\mu(\emptyset) = 0$
2. $\mu(N) = 1$
3. $\forall A, B \in 2^N, [A \subseteq B \Rightarrow \mu(A) \leq \mu(B)]$ (monotonicity).

Definition

Let be $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$. The Choquet integral of $x$ w.r.t. $\mu$ is defined by:

$$C_\mu(x) := \sum_{i=1}^{n} (x_{\tau(i)} - x_{\tau(i-1)}) \mu(\{\tau(i), \ldots, \tau(n)\})$$

where $\tau$ is a permutation on $N$ such that $x_{\tau(1)} \leq x_{\tau(2)} \leq \cdots \leq x_{\tau(n-1)} \leq x_{\tau(n)}$, and $x_{\tau(0)} := 0$. 

(LAMSADE) Preferences Aggregation: the MAUT approach
Definition

Let be \( x := (x_1, ..., x_n) \in \mathbb{R}^n \). The Choquet integral of \( x \) w.r.t. \( \mu \) is defined by:

\[
C_\mu(x) := \sum_{i=1}^{n} (x_{\tau(i)} - x_{\tau(i-1)}) \mu(\{\tau(i), \ldots, \tau(n)\})
\]

where \( \tau \) is a permutation on \( N \) such that \( x_{\tau(1)} \leq x_{\tau(2)} \leq \cdots \leq x_{\tau(n-1)} \leq x_{\tau(n)} \), and \( x_{\tau(0)} := 0 \).

Example

Compute the Choquet integral of the following alternatives

<table>
<thead>
<tr>
<th></th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>7</td>
<td>17</td>
<td>14</td>
</tr>
<tr>
<td>b</td>
<td>9</td>
<td>8</td>
<td>12</td>
</tr>
</tbody>
</table>

with the capacity \( \mu \):

\[
\mu(N) = 1; \ \mu(\emptyset) = 0; \ \mu(\{1\}) = 0; \ \mu(\{2\}) = \mu(\{3\}) = \mu(\{2, 3\}) = \mu(\{1, 3\}) = \frac{1}{2}; \ \mu(\{1, 2\}) = 1
\]
Example

Compute the Choquet integral of the following alternatives

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
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<tbody>
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</tr>
<tr>
<td>$b$</td>
<td>9</td>
<td>8</td>
<td>12</td>
</tr>
</tbody>
</table>

with the capacity $\mu$:

$\mu(N) = 1; \mu(\emptyset) = 0; \mu(\{1\}) = 0; \mu(\{2\}) = \mu(\{3\}) = \mu(\{2, 3\}) = \mu(\{1, 3\}) = \frac{1}{2}; \mu(\{1, 2\}) = 1$

- $C_\mu(a) = 7 + 7\mu(\{2, 3\}) + 3\mu(\{2\}) = 12$
- $C_\mu(b) = 8 + 1\mu(\{1, 3\}) + 3\mu(\{3\}) = 10$
Definition (Möbius transform)

Let $\mu$ be a capacity on $\mathcal{N}$. The Möbius transform of $\mu$ is a function $m : 2^\mathcal{N} \rightarrow \mathbb{R}$ defined by

$$m(T) := \sum_{K \subseteq T} (-1)^{|T \setminus K|} \mu(K), \forall T \in 2^\mathcal{N}.$$ 

Remark

The link between a capacity $\mu$ and its Möbius transform $m^\mu$ is given by

$$\mu(T) : \sum_{K \subseteq T} m^\mu(K), \forall T \in 2^\mathcal{N}.$$
Definition

Let be $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$. The Choquet integral of $x$ w.r.t. $m^\mu$ (the Möbius transform of a capacity $\mu$) is defined by:

$$C_{m^\mu}(x) := \sum_{T \subseteq N} m^\mu(T) \min_{i \in T} x_i$$
Definition (Möbius transform)

Let $\mu$ be a capacity on $N$. The Möbius transform of $\mu$ is a function $m : 2^N \rightarrow \mathbb{R}$ defined by

$$m(T) := \sum_{K \subseteq T} (-1)^{|T \setminus K|} \mu(K), \forall T \in 2^N.$$ 

Example

$$C_{m^\mu}(x) := \sum_{T \subseteq N} m^\mu(T) \min_{i \in T} x_i$$

Compute the Choquet integral of the following alternatives

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>7</td>
<td>17</td>
<td>14</td>
</tr>
<tr>
<td>b</td>
<td>9</td>
<td>8</td>
<td>12</td>
</tr>
</tbody>
</table>

by using the Möbius transform $m^\mu$ associated to the following capacity $\mu$:

$\mu(N) = 1; \mu(\emptyset) = 0; \mu(\{1\}) = 0; \mu(\{2\}) = \mu(\{3\}) = \mu(\{2, 3\}) = \mu(\{1, 3\}) = \frac{1}{2};\mu(\{1, 2\}) = 1$
Definition (A 2-additive capacity)

It is a capacity $\mu$ such that its Möbius transform $m : 2^N \to \mathbb{R}$ satisfies the following two conditions:

- For all subset $T$ of $N$ such that $|T| > 2$, $m(T) = 0$;
- There exists a subset $B$ of $N$ such that $|B| = 2$ and $m(B) \neq 0$. 
The 2-additive monotonicity conditions in terms of Möbius transform $m$

\[
\sum_{\{i,j\} \subseteq N} m(\{i,j\}) + \sum_{i \in N} m(\{i\}) = 1 \quad \text{(normality)}
\]

\[
m(\{i\}) \geq 0 \quad \text{for all } i \in N \quad \text{(nonnegativity)}
\]

\[
m(\{i\}) + \sum_{j \in A \setminus \{i\}} m(\{i,j\}) \geq 0 \quad \forall A \setminus \{i\}, \forall i \in N \quad \text{(monotonicity)}.
\]
The 2-additive monotonicity conditions in terms of capacity $\mu$

$$\sum_{\{i,j\}\subseteq N} \mu(\{i,j\}) - (n - 2)\sum_{i\in N} \mu(\{i\}) = 1 \text{ (normality)}$$

$$\mu(\{i\}) \geq 0, \forall i \in N \text{ (nonnegativity)}$$

$$\forall A \subseteq N, |A| \geq 2, \forall i \in A$$

$$\sum_{j \in A \setminus \{i\}} (\mu(\{i,j\}) - \mu(\{j\})) \geq (|A| - 2)\mu(\{i\}) \text{ (monotonicity).}$$
The 2-additive capacity

Definition

A 2-additive capacity is a set function \( \mu : 2^N \rightarrow [0, 1] \) such that:

\[
\sum_{\{i,j\} \subseteq N} \mu({\{i,j\}}) - (n - 2) \sum_{i \in N} \mu({\{i\}}) = 1 \quad \text{(normality)}
\]

\[
\mu({\{i\}}) \geq 0, \quad \forall i \in N \quad \text{(nonnegativity)}
\]

\[
\forall A \subseteq N, \quad |A| \geq 2, \quad \forall i \in A
\]

\[
\sum_{j \in A \setminus \{i\}} (\mu({\{i,j\}}) - \mu({\{j\}})) \geq (|A| - 2) \mu({\{i\}}) \quad \text{(monotonicity)}.
\]
The 2-additive capacity

Notations

\[ \forall i, j \in \mathbb{N}, i \neq j, \mu_{\emptyset} = \mu(\emptyset), \mu_i = \mu(\{i\}) \text{ and } \mu_{ij} = \mu(\{i,j\}) \]
The 2-additive capacity

Example

- $N = \{1, 2, 3\}$
- Normality constraint: $\mu_{12} + \mu_{13} + \mu_{23} - \mu_1 - \mu_2 - \mu_3 = 1$
- Nonnegativity constraints: $\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0$
- Monotonicity constraints:
  - $\mu_{12} \geq \mu_1, \mu_{12} \geq \mu_2$
  - $\mu_{13} \geq \mu_1, \mu_{13} \geq \mu_3$
  - $\mu_{23} \geq \mu_2, \mu_{23} \geq \mu_3$
  - $\mu_{12} + \mu_{13} \geq \mu_1 + \mu_2 + \mu_3$
  - $\mu_{12} + \mu_{23} \geq \mu_1 + \mu_2 + \mu_3$
  - $\mu_{13} + \mu_{23} \geq \mu_1 + \mu_2 + \mu_3$
The 2-additive Choquet integral

Definition

For any $x := (x_1, \ldots, x_n) \in X$, the expression of the 2-additive Choquet integral is:

$$C_\mu(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} \phi^\mu_i u_i(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} l_{ij}^\mu |u_i(x_i) - u_j(x_j)|$$

Where

- $l_{ij}^\mu =$ the interaction index between criteria $i$ and $j$:
  $$l_{ij}^\mu = m_{ij}^\mu = \mu_{ij} - \mu_i - \mu_j.$$ 

- $\phi^\mu_i =$ the importance of the criterion $i$ ($\equiv$ Shapley index):
  $$\phi^\mu_i = \mu_i + \frac{1}{2} \sum_{k \in N \setminus i} l_{ik}^\mu.$$ 

(LAMSADE)
Interest of the 2-additive model

\[ C_\mu(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} \phi_i^\mu u_i(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} I_{ij}^\mu |u_i(x_i) - u_j(x_j)| \]

- It is a generalization of the arithmetic mean \((l_{ij} = 0 \ \forall i, j \in N)\)

- It is a compromise between the general Choquet integral and the arithmetic mean, i.e., offers a good compromise between flexibility of the model and complexity;

- It was used in many applications such that
  - the evaluation of discomfort in sitting position (see Grabisch et al. (2002));
  - the construction of performance measurement systems model in a supply chain context (see Berrah and Clivillé (2007), Clivillé et al. (2007));
  - complex system design (Labreuche and Pignon (2007));
Remark

The Choquet integral requires to be able to compare any element of one point of view with any element of any other point of view (commensurateness between criteria);
A non-additive model: The Choquet integral

\[ I_{ij}^\mu = \mu_{ij} - \mu_i - \mu_j \]

Usual Interpretation

- \( I_{ij}^\mu > 0 \iff \) criteria \( i \) and \( j \) are \textit{complementarity}.
- \( I_{ij}^\mu < 0 \iff \) criteria \( i \) and \( j \) are \textit{redundant}.
- \( I_{ij}^\mu = 0 \iff \) criteria \( i \) and \( j \) are \textit{independent} (no interaction).
The sign of the interaction index is not always stable

Example (A classic example of Grabisch et al. (2010))

<table>
<thead>
<tr>
<th></th>
<th>Mathematics (M)</th>
<th></th>
<th>Statistics (S)</th>
<th></th>
<th>Language (L)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>16</td>
<td>b</td>
<td>13</td>
<td>c</td>
<td>7</td>
</tr>
<tr>
<td>b</td>
<td>16</td>
<td>d</td>
<td>11</td>
<td>c</td>
<td>9</td>
</tr>
<tr>
<td>c</td>
<td>6</td>
<td></td>
<td>13</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>d</td>
<td>6</td>
<td></td>
<td>11</td>
<td></td>
<td>9</td>
</tr>
</tbody>
</table>

- $a \prec b$ and $d \prec c$
- $u_M(16) = 16, \ u_M(6) = 6,$
- $u_S(13) = 13, \ u_S(11) = 11$
- $u_L(7) = 7, \ u_L(9) = 9$
The sign of the interaction index is not always stable

Example (A classic example of Grabisch et al. (2010))

<table>
<thead>
<tr>
<th>Par. 1</th>
<th>Par. 2</th>
<th>Par. 3</th>
<th>Par. 4</th>
<th>Par. 5</th>
<th>Par. 6</th>
<th>Par. 7</th>
<th>Par. 8</th>
<th>Par. 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{\mu}(a)$</td>
<td>8.5</td>
<td>13.75</td>
<td>9.1</td>
<td>13.765</td>
<td>13.75</td>
<td>13.75</td>
<td>11.47</td>
<td>12.535</td>
</tr>
<tr>
<td>$C_{\mu}(c)$</td>
<td>7.75</td>
<td>9.75</td>
<td>7.75</td>
<td>11.325</td>
<td>11.25</td>
<td>9.75</td>
<td>9.45</td>
<td>9.515</td>
</tr>
</tbody>
</table>

| $\mu_M$ | 0 | 0.75 | 0 | 0.685 | 0.75 | 0.75 | 0.36 | 0.485 | 0.15 |
| $\mu_S$ | 0.25 | 0.5 | 0.25 | 0.73 | 0.75 | 0.5 | 0.465 | 0.455 | 0.25 |
| $\mu_L$ | 0 | 0.25 | 0 | 0.315 | 0 | 0 | 0.205 | 0.32 | 0 |
| $\mu_{MS}$ | 0.25 | 0.75 | 0.35 | 0.785 | 0.75 | 0.75 | 0.565 | 0.68 | 0.5 |
| $\mu_{ML}$ | 0.75 | 1 | 0.65 | 1 | 0.1 | 0.75 | 0.805 | 0.795 | 0.55 |
| $\mu_{SL}$ | 0.25 | 0.75 | 0.25 | 0.945 | 0.75 | 0.75 | 0.66 | 0.785 | 0.35 |

| $V_{\mu}^{\mu}$ | 0.375 | 0.5 | 0.375 | 0.37 | 0.5 | 0.5 | 0.35 | 0.35 | 0.4 |
| $V_{\mu}^{M}$ | 0.25 | 0.25 | 0.3 | 0.365 | 0.375 | 0.375 | 0.33 | 0.33 | 0.35 |
| $V_{\mu}^{S}$ | 0.375 | 0.25 | 0.325 | 0.265 | 0.125 | 0.125 | 0.32 | 0.32 | 0.25 |

| $I_{\mu}^{MS}$ | 0 | -0.5 | 0.1 | -0.63 | -0.75 | -0.5 | -0.26 | -0.26 | 0.1 |
| $I_{\mu}^{ML}$ | 0.75 | 0 | 0.65 | 0 | 0.25 | 0 | 0.24 | -0.01 | 0.4 |
| $I_{\mu}^{SL}$ | 0 | 0 | 0 | -0.1 | 0 | 0.25 | -0.01 | 0.01 | 0.1 |
The sign of the interaction index is not always stable

How to conclude from this preference information given by the DM?

- Mathematics & Statistics are independent? complementary? redundant?
- Mathematics & Literature are independent? complementary? redundant?
- Statistics & Literature are independent? complementary? redundant?

- “The three subjects, taken together, are not without interaction”
The sign of the interaction index is not always stable

Example (A ranking of French hospitals for weight loss surgery)

<table>
<thead>
<tr>
<th>Hospital</th>
<th>Activity</th>
<th>Notoriety</th>
<th>ALOS</th>
<th>Technicality</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1</td>
<td>200</td>
<td>65</td>
<td>3.5</td>
<td>85</td>
</tr>
<tr>
<td>H2</td>
<td>450</td>
<td>60</td>
<td>4</td>
<td>75</td>
</tr>
<tr>
<td>H3</td>
<td>450</td>
<td>50</td>
<td>2.5</td>
<td>55</td>
</tr>
<tr>
<td>H4</td>
<td>350</td>
<td>50</td>
<td>3.5</td>
<td>85</td>
</tr>
<tr>
<td>H5</td>
<td>350</td>
<td>55</td>
<td>2</td>
<td>75</td>
</tr>
<tr>
<td>H6</td>
<td>150</td>
<td>65</td>
<td>2.5</td>
<td>80</td>
</tr>
<tr>
<td>H7</td>
<td>200</td>
<td>55</td>
<td>2</td>
<td>55</td>
</tr>
<tr>
<td>H8</td>
<td>150</td>
<td>60</td>
<td>4</td>
<td>80</td>
</tr>
</tbody>
</table>

H1 ≻ H2; H3 ≻ H4; H5 ≻ H6; H8 ≻ H7.
The sign of the interaction index is not always stable

Example (A ranking of French hospitals for weight loss surgery)

<table>
<thead>
<tr>
<th>C_μ(H1)</th>
<th>C_μ(H2)</th>
<th>C_μ(H3)</th>
<th>C_μ(H4)</th>
<th>C_μ(H5)</th>
<th>C_μ(H6)</th>
<th>C_μ(H7)</th>
<th>C_μ(H8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.607</td>
<td>0.509</td>
<td>0.514</td>
<td>0.502</td>
<td>0.586</td>
<td>0.576</td>
<td>0.513</td>
<td>0.523</td>
</tr>
<tr>
<td>0.555</td>
<td>0.531</td>
<td>0.576</td>
<td>0.506</td>
<td>0.606</td>
<td>0.568</td>
<td>0.502</td>
<td>0.512</td>
</tr>
<tr>
<td>0.692</td>
<td>0.682</td>
<td>0.693</td>
<td>0.683</td>
<td>0.704</td>
<td>0.694</td>
<td>0.539</td>
<td>0.618</td>
</tr>
<tr>
<td>0.549</td>
<td>0.539</td>
<td>0.548</td>
<td>0.538</td>
<td>0.63</td>
<td>0.582</td>
<td>0.508</td>
<td>0.518</td>
</tr>
<tr>
<td>0.563</td>
<td>0.553</td>
<td>0.605</td>
<td>0.538</td>
<td>0.629</td>
<td>0.606</td>
<td>0.529</td>
<td>0.518</td>
</tr>
<tr>
<td>0.59</td>
<td>0.58</td>
<td>0.588</td>
<td>0.578</td>
<td>0.645</td>
<td>0.635</td>
<td>0.538</td>
<td>0.548</td>
</tr>
<tr>
<td>0.597</td>
<td>0.533</td>
<td>0.530</td>
<td>0.510</td>
<td>0.583</td>
<td>0.573</td>
<td>0.514</td>
<td>0.524</td>
</tr>
<tr>
<td>0.589</td>
<td>0.548</td>
<td>0.564</td>
<td>0.531</td>
<td>0.609</td>
<td>0.599</td>
<td>0.527</td>
<td>0.537</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>µ_1</th>
<th>µ_2</th>
<th>µ_3</th>
<th>µ_4</th>
<th>µ_12</th>
<th>µ_13</th>
<th>µ_14</th>
<th>µ_23</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.01</td>
<td>0</td>
<td>0.086</td>
<td>0.02</td>
<td>0.22</td>
<td>0.144</td>
<td>0.01</td>
</tr>
<tr>
<td>0.19</td>
<td>0.01</td>
<td>0.2</td>
<td>0.073</td>
<td>0.2</td>
<td>0.51</td>
<td>0.2</td>
<td>0.01</td>
</tr>
<tr>
<td>0.453</td>
<td>0.02</td>
<td>0.697</td>
<td>0.317</td>
<td>0.653</td>
<td>0.526</td>
<td>0.697</td>
<td>0.927</td>
</tr>
<tr>
<td>0.093</td>
<td>0.083</td>
<td>0.317</td>
<td>0.273</td>
<td>0.176</td>
<td>0.469</td>
<td>0.317</td>
<td>0.363</td>
</tr>
<tr>
<td>0.263</td>
<td>0.209</td>
<td>0.472</td>
<td>0.375</td>
<td>0.667</td>
<td>0.45</td>
<td>0.472</td>
<td>0.363</td>
</tr>
<tr>
<td>0.192</td>
<td>0.182</td>
<td>0.416</td>
<td>0.075</td>
<td>0.469</td>
<td>0.265</td>
<td>0.416</td>
<td>0.075</td>
</tr>
<tr>
<td>0.065</td>
<td>0.055</td>
<td>0.133</td>
<td>0.153</td>
<td>0.143</td>
<td>0.153</td>
<td>0.149</td>
<td>0.153</td>
</tr>
<tr>
<td>0.143</td>
<td>0.133</td>
<td>0.153</td>
<td>0.153</td>
<td>0.153</td>
<td>0.153</td>
<td>0.277</td>
<td>0.153</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>I_µ_12</th>
<th>I_µ_13</th>
<th>I_µ_14</th>
<th>I_µ_23</th>
<th>I_µ_24</th>
<th>I_µ_34</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.01</td>
<td>-0.01</td>
<td>0.038</td>
<td>-0.01</td>
<td>0.75</td>
<td>0.086</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-0.19</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-0.453</td>
<td>-0.2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0.014</td>
<td>-0.2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0.47</td>
<td>0.236</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0.23</td>
<td>0.027</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(LAMSADE) Preferences Aggregation: the MAUT approach Chapter 4 67 / 79
Hypothesis

A preference information given by the DM is representable by a 2-additive Choquet integral.

Let be $i, j \in N$, $i \neq j$ and $C_{pref}$ the set of all capacities compatible with a preference information given by the DM.

1. There exists a possible positive (respectively, null, negative) interaction between $i$ and $j$ if there exists a capacity $\mu \in C_{pref}$ such that $I_{ij}^\mu > 0$ (respectively, $I_{ij}^\mu = 0$, $I_{ij}^\mu < 0$).

2. There exists a necessary positive (respectively, null, negative) interaction between $i$ and $j$ if $I_{ij}^\mu > 0$ (respectively, $I_{ij}^\mu = 0$, $I_{ij}^\mu < 0$) for all capacity $\mu \in C_{pref}$. 
Identification of Necessary and possible interactions (Step 1):

Minimize $Z_1 = \sum_{(x,y) \in P \cup I} (\Gamma_{xy}^+ + \Gamma_{xy}^-)$

Subject to

$C_\mu(u(x)) - C_\mu(u(y)) + \Gamma_{xy}^+ - \Gamma_{xy}^- \geq \varepsilon \quad \forall x, y \in X$ such that $x P y$ \hspace{1cm} (1)

$C_\mu(u(x)) - C_\mu(u(y)) + \Gamma_{xy}^+ - \Gamma_{xy}^- = 0 \quad \forall x, y \in X$ such that $x I y$ \hspace{1cm} (2)

$\Gamma_{xy}^+ \geq 0, \Gamma_{xy}^- \geq 0 \quad \forall x, y \in X$ such that $x (P \cup I) y$ \hspace{1cm} (3)

$\varepsilon \geq 0$ \hspace{1cm} (4)

$\sum_{\{i,j\} \subseteq N} m(\{i,j\}) + \sum_{i \in N} m(\{i\}) = 1$ \hspace{1cm} (5)

$m(\{i\}) \geq 0 \quad \text{for all } i \in N$ \hspace{1cm} (6)

$m(\{i\}) + \sum_{j \in A \setminus \{i\}} m(\{i,j\}) \geq 0 \quad \forall A \setminus \{i\}, \forall i \in N.$ \hspace{1cm} (7)

- $PL_1$ is always feasible
- $Z_1^* = 0 \implies$ we can conclude that, depending on the sign of the variable $\varepsilon$, the preference information $\{P, I\}$ may be representable by a 2-additive Choquet integral.
- $Z_1^* > 0 \implies$ then there is no 2-additive Choquet integral model compatible with $\{P, I\}$. 
Identification of Necessary and possible interactions (Step 2):

Maximize \( Z_2 = \varepsilon \)

Subject to

\[
C_{\mu}(u(x)) - C_{\mu}(u(y)) \geq \varepsilon \quad \forall x, y \in X \text{ such that } x P y \quad (8)
\]

\[
C_{\mu}(u(x)) - C_{\mu}(u(y)) = 0 \quad \forall x, y \in X \text{ such that } x I y \quad (9)
\]

\[
\varepsilon \geq 0 \quad (10)
\]

\[
\sum_{\{i,j\} \subseteq N} m(\{i,j\}) + \sum_{i \in N} m(\{i\}) = 1 \quad (11)
\]

\[
m(\{i\}) \geq 0 \quad \text{for all } i \in N \quad (12)
\]

\[
m(\{i\}) + \sum_{j \in A \setminus \{i\}} m(\{i,j\}) \geq 0 \quad \forall A \setminus \{i\}, \forall i \in N. \quad (13)
\]

- \( PL_2 \) is always feasible since the Step 1 was solved before ((PL2) is launched when \( Z_1^* = 0 \))
- \( Z_2^* = 0 \implies \), there is no 2-additive Choquet integral model compatible with \( \{P, I\} \).
- \( Z_2^* > 0 \implies \), \( \{P, I\} \) is representable by a 2-additive Choquet integral.
Identification of Necessary and possible interactions (Step 3):

Maximize $Z_3 = \varepsilon$
Subject to

$$m({i,j}) \geq 0 \quad \text{(respectively } m({i,j}) \leq 0) \quad (14)$$
$$C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \quad \forall x, y \in X \text{ such that } x P y \quad (15)$$
$$C_\mu(u(x)) - C_\mu(u(y)) = 0 \quad \forall x, y \in X \text{ such that } x I y \quad (16)$$
$$\varepsilon \geq 0 \quad (17)$$
$$\sum_{\{i,j\} \subseteq N} m({i,j}) + \sum_{i \in N} m({i}) = 1 \quad (18)$$
$$m({i}) \geq 0 \quad \text{for all } i \in N \quad (19)$$
$$m({i}) + \sum_{j \in A \setminus \{i\}} m({i,j}) \geq 0 \quad \forall A \setminus \{i\}, \forall i \in N. \quad (20)$$

- If $PL_{N-N}^{ij}$ (respectively $PL_{N-P}^{ij}$) is not feasible, then there is a necessary negative (respectively positive) interaction between $i$ and $j$.

- If $PL_{N-N}^{ij}$ (respectively $PL_{N-P}^{ij}$) is feasible and the optimal solution $Z_3^* = 0$, then the constraint (15) is satisfied with $\varepsilon = 0$. Therefore, we can conclude that there is a necessary negative (respectively positive) interaction between $i$ and $j$.

- If $PL_{N-N}^{ij}$ (respectively $PL_{N-P}^{ij}$) is feasible and the optimal solution $Z_3^* > 0$, then there is no necessary negative (respectively positive) interaction between $i$ and $j$. 
Example

The linear program $PL_{N-N}^{MS}$ (respectively $PL_{N-P}^{MS}$) corresponding to the test of the existence of a necessary negative (respectively positive) interaction between the Mathematics ($M$) and Statistics ($S$) is the following:

Maximize $Z_3 = \varepsilon$

Subject to

\[
\begin{align*}
    m(\{M, S\}) &\geq 0 \quad (\text{respectively } m(\{M, S\}) \leq 0) \\
    \varepsilon &\geq 0 \\
    C_\mu(u(b)) - C_\mu(u(a)) &\geq \varepsilon \\
    C_\mu(u(c)) - C_\mu(u(d)) &\geq \varepsilon \\
    m(\{M, S\}) + m(\{M, L\}) + m(\{S, L\}) + m(\{M\}) + m(\{S\}) + m(\{L\}) &\leq 1 \\
    m(\{M\}) &\geq 0 \\
    m(\{S\}) &\geq 0 \\
    m(\{L\}) &\geq 0 \\
    m(\{M\}) + m(\{M, S\}) &\geq 0 \\
    m(\{M\}) + m(\{M, L\}) &\geq 0 \\
    m(\{M\}) + m(\{M, S\}) + m(\{M, L\}) &\geq 0 \\
    m(\{S\}) + m(\{M, S\}) &\geq 0 \\
    m(\{S\}) + m(\{S, L\}) &\geq 0 \\
    m(\{S\}) + m(\{M, S\}) + m(\{S, L\}) &\geq 0 \\
    m(\{L\}) + m(\{M, L\}) &\geq 0 \\
    m(\{L\}) + m(\{S, L\}) &\geq 0 \\
    m(\{L\}) + m(\{S, L\}) + m(\{M, L\}) &\geq 0
\end{align*}
\]
Example

\[ Z_3 = \varepsilon \]

<table>
<thead>
<tr>
<th>Optimal solution $Z_3^*$</th>
<th>$M$</th>
<th>$S$</th>
<th>$L$</th>
<th>${M, S}$</th>
<th>${M, L}$</th>
<th>${S, L}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.667</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mobius transform $m$</th>
<th>0</th>
<th>0.33</th>
<th>0.33</th>
<th>0</th>
<th>0.67</th>
<th>-0.33</th>
</tr>
</thead>
<tbody>
<tr>
<td>Importance index $V_i$</td>
<td>0.33</td>
<td>0.17</td>
<td>0.5</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Interaction index $I_{ij}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0.67</td>
<td>-0.33</td>
</tr>
</tbody>
</table>

Table: Results of $PL_{N-N}^{MS}$ testing necessary negative interaction between Mathematics and Statistics
Example

$$Z_3 = \varepsilon, M, S, L, \{M, S\}, \{M, L\}, \{S, L\}$$

<table>
<thead>
<tr>
<th></th>
<th>$Z_3^*$</th>
<th>$M$</th>
<th>$S$</th>
<th>$L$</th>
<th>${M, S}$</th>
<th>${M, L}$</th>
<th>${S, L}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal solution</td>
<td>$1$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>Mobius transform $m$</td>
<td>$0.5$</td>
<td>$0.5$</td>
<td>$0$</td>
<td>$-0.5$</td>
<td>$0.5$</td>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>Importance index $V_i$</td>
<td>$0.5$</td>
<td>$0.25$</td>
<td>$0.25$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td></td>
</tr>
<tr>
<td>Interaction index $I_{ij}$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-0.5$</td>
<td>$0.5$</td>
<td>$0$</td>
<td></td>
</tr>
</tbody>
</table>

Table: Results of $PL_{N-P}^{MS}$ testing necessary positive interaction between Mathematics and Statistics
Proposition (What’s about necessary null interactions)

Suppose there exists a 2-additive Choquet integral representing a preference information \( \{P, I\} \) on \( X \).

If the relation \( I = \emptyset \), then there is no necessary null interaction.

Example (Application to the transparency of ParcourSup)

Examen de chaque dossier

Si celle-ci se déroule normalement, les candidats recevront trois propositions : « oui », « oui, si » ou « non ». Dans les formations sélectives (IUT, BTS, CPGE, etc.) pas de surprise : les candidatures seront examinées une par une et classées en fonction des critères préétablis et indiqués sur Parcoursup.

À l'université en revanche, on trouvera deux cas de figure. Dans les filières en tension (dont le nombre de places est inférieur au nombre de candidatures) toutes les candidatures devront être classées, sans ex-aquo possible. C'est ici l'une des principales nouveautés par rapport à APB, pensée avec l'objectif de supprimer le tirage au sort pratiqué en dernier recours ces dernières années.

Un logiciel d’aide à la décision

Pour chaque formation, une commission de voeu constituée d’enseignants et de responsables de formations devra ainsi paramétrer un logiciel fourni par le ministère de l’Enseignement supérieur, de la Recherche et de l’Innovation. Appelé « outil d’aide à la décision », ce programme regroupe
Outline

1. Framework
2. The simple models
3. Introduction to MCDA
4. Multi Attribute Utility Theory
5. The additive model
6. A non-additive model: The Choquet integral
7. To conclude
Analysis of three MCDA phenomena

- **PRESCRIPTIVE APPROACH**: To help a decision maker by the proposal of a solution obtained by a model.

- **DESCRIPTIVE APPROACH**: To describe a decision maker’s preferences by the chosen model.

- **ELICITATION**: The elicitation of the decision maker’s preferences consists in obtaining parameters of a decisional model which explain the past decisions in order to help in the future decisions.
Parameter's elicitation

- **Option 1: Explicit elicitation**
  - Explain the model to the DM
  - Let the DM choose the parameters

- **Option 2: Implicit elicitation**
  - Present some (possibly fictitious) alternatives to the DM and ask him to compare them
  - Deduct the parameters of the model by solving an optimization program
Some references