Preferences Aggregation: the MAUT approach

University Paris Dauphine
LAMSADE
FRANCE

Chapter 4
Outline

1. Framework
2. The simple models
3. Introduction to MCDA
4. Multi Attribute Utility Theory
5. The additive model
6. A non-additive model: The Choquet integral
7. To conclude
Outline

1. Framework
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7. To conclude
A Decision Maker (DM) is facing a decision problem, i.e., the DM has to deal with multiple alternatives and has to compare them.

Alternatives are described on several attributes.

A criterion is an attribute with a preference relation (monotonic attribute).

Criteria cannot be reduced to one criterion as they are potentially in conflict.
Example (Compare two bikes on three attributes)

<table>
<thead>
<tr>
<th></th>
<th>Speed</th>
<th>Robustness</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mountain bike</td>
<td>20 km/h</td>
<td>Good</td>
<td>500 €</td>
</tr>
<tr>
<td>Race bike</td>
<td>35 km/h</td>
<td>Middle</td>
<td>1000 €</td>
</tr>
</tbody>
</table>
Example (Compare many objects)

<table>
<thead>
<tr>
<th>Couches-culottes</th>
<th>Joone</th>
<th>Pampers Premium Protection</th>
<th>Pampers Baby Dry</th>
<th>Naty Eco by Naty</th>
<th>Pampers Premium Protection Active FE™</th>
<th>Carrefour Baby Ultra airy 3 brezness</th>
<th>Lupilu Soft % Dry</th>
<th>Meta d'enfants (Marque Répartie) Ultra confort</th>
<th>Love &amp; Green Couches hygiénique®</th>
<th>Lotus Baby Touch 1 Ultra confort</th>
<th>Pommette (Biosphériste) Ecologic FSC/Halal coulées</th>
</tr>
</thead>
<tbody>
<tr>
<td>Label bis</td>
<td>Non</td>
<td>Non</td>
<td>Non</td>
<td>FSC/MB1 Wonna ur</td>
<td>Non</td>
<td>Non</td>
<td>Non</td>
<td>Non</td>
<td>Non</td>
<td>Non</td>
<td>Non</td>
</tr>
<tr>
<td>Prix indicatif</td>
<td>64.90 € (1)</td>
<td>102 €</td>
<td>15.60 €</td>
<td>15.90 €</td>
<td>12.60 €</td>
<td>10.50 €</td>
<td>9.60 €</td>
<td>10.90 €</td>
<td>18.65 €</td>
<td>16.90 €</td>
<td>9 €</td>
</tr>
<tr>
<td>Prix pour une couche</td>
<td>0.40 €</td>
<td>0.21 €</td>
<td>0.31 €</td>
<td>0.34 €</td>
<td>0.37 €</td>
<td>0.34 €</td>
<td>0.34 €</td>
<td>0.34 €</td>
<td>0.31 €</td>
<td>0.34 €</td>
<td>0.28 €</td>
</tr>
<tr>
<td>Performance (60%)</td>
<td>★★★</td>
<td>★★★</td>
<td>★★★</td>
<td>★★★</td>
<td>★★★</td>
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<td>★★★</td>
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<tr>
<td>Tâche</td>
<td>★★★★</td>
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<tr>
<td>Absorption</td>
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<tr>
<td>Protection contre l'humidité</td>
<td>★★★★</td>
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<td>★★★★</td>
<td>★★★★</td>
<td>★★★★</td>
<td>★★★★</td>
</tr>
<tr>
<td>Composition (40%)</td>
<td>★★★★</td>
<td>★★★★</td>
<td>★★★★</td>
<td>★★★★</td>
<td>★★★★</td>
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<td>★★★★</td>
</tr>
<tr>
<td>Pesticides</td>
<td>★★★★</td>
<td>★★★★</td>
<td>★★★★</td>
<td>★★★★</td>
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<tr>
<td>Restitutio glycolique</td>
<td>★★★★</td>
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<td>★★★★</td>
<td>★★★★</td>
<td>★★★★</td>
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<tr>
<td>Risque carcinogène</td>
<td>★★★★</td>
<td>★★★★</td>
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<td>★★★★</td>
</tr>
</tbody>
</table>

### Preferences Aggregation: the MAUT approach

(LAMSADE)
Multi-attribute formal model: Inputs

- A set of alternatives \( X = X_1 \times X_2 \times \cdots \times X_n \)

- There exists preferences on the values of each criterion \( i \) (utility function, qualitative preference relation \( \succsim_i \), . . .)

- A representation of the importance of each criterion or set of criteria (weights, importance relation, . . .)
Multi-attribute formal model: a treatment

Using the input information, elaborate a decision rule allowing to compare two different alternatives, i.e.,

\[ x = (x_1, \ldots, x_n) \]
\[ y = (y_1, \ldots, y_n) \]

\( \Rightarrow \) \( x \succsim y \) or \( y \succsim x \)
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PARETO DOMINANCE

An alternative is preferred to another one if it is considered to be better on all the criteria.

\[ x \succeq y \iff \forall i \in \mathbb{N}, x_i \succeq_i y_i \]

Example

<table>
<thead>
<tr>
<th></th>
<th>Speed</th>
<th>Robustness</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bike A</td>
<td>10 km/h</td>
<td>Good</td>
<td>600 €</td>
</tr>
<tr>
<td>Bike B</td>
<td>20 km/h</td>
<td>Good</td>
<td>550 €</td>
</tr>
<tr>
<td>Bike C</td>
<td>19 m/h</td>
<td>Very Good</td>
<td>800 €</td>
</tr>
</tbody>
</table>

\[ B \succeq A \text{ and } \neg (A \succeq B) \implies B \succ A \]

\[ \neg (B \succeq C) \text{ and } \neg (C \succeq B) \]

Pareto dominance is not so interesting
Dominance

- An alternative \( x = (x_1, \ldots, x_n) \) dominates an alternative \( y = (y_1, \ldots, y_n) \) if \( \forall i \in N, x_i \succeq_i y_i. \)

- An alternative \( x = (x_1, \ldots, x_n) \) strictly dominates an alternative \( y = (y_1, \ldots, y_n) \) if \( \forall i \in N, x_i \succeq_i y_i \) and \( \exists i_0 \in N, x_{i_0} \succ_{i_0} y_{i_0}. \)

Definition

The Pareto front is the set of all non-dominated alternatives.

Remark

- The optimal solution is necessary in the Pareto front
- In general, the Pareto front may be poor, i.e., it is not really different to the whole set of alternatives.
MULTI-OBJECTIVE OPTIMIZATION

- Principle: The “Best” alternative should be the nearest alternative to an “ideal point”.
- Usually, the “ideal point” if computed by taking the max (resp min) value on each criterion.
- Many distances are also used in the resolution of a multi-objective problem.
Weighted sum

Let be \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) two alternatives such that \( x_i, y_i \in \mathbb{R}, \forall i \in \mathbb{N} \). Let be \( w_i \) the weight associated to the criterion \( i \).

\[
x \succeq y \iff \sum_{i=1}^{n} w_i x_i \geq \sum_{i=1}^{n} w_i y_i
\]

Example

<table>
<thead>
<tr>
<th></th>
<th>Speed</th>
<th>Robustness</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bike A</td>
<td>8/20</td>
<td>18/20</td>
<td>12/20</td>
</tr>
<tr>
<td>Bike B</td>
<td>18/20</td>
<td>8/20</td>
<td>12/20</td>
</tr>
<tr>
<td>Bike C</td>
<td>12/20</td>
<td>12/20</td>
<td>12/20</td>
</tr>
</tbody>
</table>

\[
w_S > w_R \implies B \succeq A
\]

\[
w_R > w_S \implies A \succeq B
\]

\[\forall w_R, w_S, \text{ we have } A \succeq C \text{ or } B \succeq C\]
The majority rule

Let be $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ two alternatives. $x$ is preferred to $y$ if it is considered “good” on a majority of criteria.

$$x \succ y \iff |\{i \in N : x_i \succ_i y_i\}| \geq |\{i \in N : y_i \succ_i x_i\}|$$

Example

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$B \succ C$
Example (Majority rule)

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<tr>
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<td>15 km/h</td>
<td>Good</td>
<td>500 €</td>
</tr>
<tr>
<td>Bike C</td>
<td>25m/h</td>
<td>Bad</td>
<td>550 €</td>
</tr>
</tbody>
</table>

Which bike do you choose?
Example (Majority rule)

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<td>Bad</td>
<td>550 €</td>
</tr>
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\[ A \succ B \]
\[ B \succ C \]
\[ C \succ A \]

\[ \implies \text{Condorcet Paradox} \]
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MultiCriteria Decision Aiding (MCDA): Difficulties

- MultiCriteria Decision Aiding is not so easy: it is not an easy task
- Every method has advantages and inconveniences: there is no “best method”
- All methods have structural bias.
Paul Valery (Artist, Writer, Poet, Philosopher (1871-1945))

- Tout ce qui est simple est faux, mais tout ce qui ne l’est pas est inutilisable

- What is simple is false. What is complex is useless.
Three types of problems in MCDA

- **Choice Problem**: choose the “best” alternative(s).

- **Ranking Problem**: rank the alternatives from the “best” to the “worst”.

- **Sorting Problem**: sort the alternatives into pre-defined categories (in general ordered categories)
Two main approaches in MCDA

- **Multi Attribute Utility Theory**: A quantitative approach “aggregate then compare” (scoring)

  \[ x \succeq y \iff U(x_1, \ldots, x_n) \geq U(y_1, \ldots, y_n) \]

- **Outranking**: qualitative approach “compare then aggregate”

  \[ x \succsim y \iff |\{ i \in N : x_i \succeq_i y_i \}| > |\{ i \in N : y_i \succeq_i x_i \}| \]
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Principle

Le be $X$ a set of alternatives evaluated on a finite set of $n$ criteria $N = \{1, \ldots, n\}$. In general, we set $X = X_1 \times X_2 \times \ldots \times X_n$.

Le be $\succeq_X$ a complete preorder on $X$ (preferences of a DM).

- $\succeq_X$ are supposed to be representable by an overall utility function:
  \[
  \forall x, y \in X, \quad x \succeq_X y \Leftrightarrow F(U(x)) \geq F(U(y))
  \]
  where
  - $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$
  - $U(x) = (u_1(x_1), \ldots, u_n(x_n))$
  - $u_i : X_i \to \mathbb{R}$ is a marginal utility function or simply called utility function or a scale on $\mathbb{R}$
  - $F : \mathbb{R}^n \to \mathbb{R}$ an aggregation function

- $F$ is generally characterized by a parameter vector $\theta$ (weight vector,\ldots).
Problems

1. How to choose the aggregation function $F$?

2. How to construct the marginal utility functions $u_i : X_i \rightarrow \mathbb{R}$?

3. The marginal utility functions $u_i : X_i \rightarrow \mathbb{R}$ should have a signification for the decision maker (see measurement theory):
   - **Ordinal scales**: Differences between values have no importance (e.g. a rank). They can represent orders and pre-orders.
   - **Cardinal scales**: Differences between values may be meaningful.
     - **Interval scales**: absolute differences between values are important.
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The additive model

- \( \succeq_X \) are supposed to be representable by an overall utility function:

\[
\forall x \in X, \quad F(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} u_i(x_i)
\]

This model is equivalent to the existence of weights \( w_i, i = 1, \ldots, n \), such that

\[
\forall x \in X, \quad F(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} w_i \cdot u_i(x_i)
\]
The additive model

- A simple method

- Additive value function involves compensation between criteria, i.e., a bad performance on a criterion $i$ could be compensated by a good performance on another criterion.

  See e.g. students evaluation based on the weighted sum.

- In the weighted sum, weights represent, in reality, the substitution rate between criteria.
The additive model

- In the weighted sum, weights represent, in reality, the substitution rate between criteria.

- E.g. for \( n = 2 \), \( w_1 = b \) \( w_2 \) means the DM is indifferent between these two alternatives \((0, b)\) and \((1, 0)\), i.e., \((0, b) \sim (1, 0)\).

- There is a total compensation between “bad” performances and “good” performances.

If we have \((a, b) \sim (a - \delta, b + \gamma)\) then gain of \(\gamma\) compensates the loss of \(\delta\).

Indeed we have

\[
aw_1 + bw_2 = (a - \delta)w_1 + (b - \gamma)w_2
\]

\[
\iff \delta w_1 = \gamma w_2
\]

\[
\frac{w_1}{w_2} = \frac{\gamma}{\delta}
\]

Implicitly, this implies that all the criteria could be express indirectly in the same unit (\(\mathbb{E}\), seconds, \ldots).
The additive model

- Requires to normalize the criteria. In general, we set $\forall i \in N, u_i : X_i \mapsto [0, 1]$.

  E.g. For a criterion to be maximized, we could choose the following normalization functions:

  - $u_i(x_i) = \frac{x_i}{\max x_i}$
  - $u_i(x_i) = \frac{x_i - \min x_i}{\max x_i - \min x_i}$
  - ...
The mutual Preferential independence

- The additive model requires to satisfy the mutual Preferential independence axiom, i.e., criteria are independent in the sense of preferences

\[ \forall i \in N, \forall z_i, t_i \in X_i, \forall x, y \in X, \]

\[ (z_i, x_{N-i}) \succeq (z_i, y_{N-i}) \iff (t_i, x_{N-i}) \succeq (t_i, y_{N-i}) \]

An attribute is preferentially independent from all other attributes when changes in the rank ordering of preferences of other attributes does not change the preference order of the attribute.
MAUT in practice

1. People suppose $\succeq_X$ representable by an overall utility function:

$$x \succeq_X y \Leftrightarrow F(U(x)) \geq F(U(y))$$

2. $F$ is generally characterized by a parameter vector $\theta$ (weight vector, . . .).

3. People ask to the DM some preferential information $\succeq_{X'}$ on a reference subset (learning set) $X' \subseteq X$.

4. The parameter vector is constructed so that $\succeq_X$ is an extension of $\succeq_{X'}$.

5. The model obtained in $X'$ will be then automatically extended to $X$. 
Example (Evaluation of diapers by the Magazine “60 millions consommateurs” in September 2018)

<table>
<thead>
<tr>
<th>Couches-culottes</th>
<th>Joone Protection Premium</th>
<th>Pampers Premium Protection</th>
<th>Pampers Baby-Dry</th>
<th>Naty Eco by Naty</th>
<th>Pampers Premium Protection Active Fit(*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Label bio</td>
<td>Non</td>
<td>Non</td>
<td>Non</td>
<td>FSC/AB Vincotte UE</td>
<td>Non</td>
</tr>
<tr>
<td>Prix indicatif</td>
<td>64,90 €(*)</td>
<td>12,60 €</td>
<td>15,60 €</td>
<td>16,90 €</td>
<td>12,30 €</td>
</tr>
<tr>
<td>162 couches</td>
<td>50 couches</td>
<td>50 couches</td>
<td>50 couches</td>
<td>46 couches</td>
<td></td>
</tr>
<tr>
<td>Prix pour une couche</td>
<td>0,40 €</td>
<td>0,25 €</td>
<td>0,31 €</td>
<td>0,38 €</td>
<td>0,27 €</td>
</tr>
<tr>
<td>Performances (60 %)</td>
<td>+++</td>
<td>++</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Tenue</td>
<td>+++</td>
<td>++</td>
<td>+</td>
<td>+</td>
<td>++</td>
</tr>
<tr>
<td>Absorption</td>
<td>+++</td>
<td>++</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Protection contre l’humidité</td>
<td>+++</td>
<td>++</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Composition (40 %)</td>
<td>+++</td>
<td>++</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Pesticides</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Résidu du glyphosate</td>
<td>+++</td>
<td>+++</td>
<td>+++</td>
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<tr>
<td>Pesticides organochlors-</td>
<td>+++</td>
<td>+++</td>
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<td>+++</td>
<td>+++</td>
</tr>
<tr>
<td>Autres molécules toxiques potentielles</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dioxines</td>
<td>+++</td>
<td>+++</td>
<td>+++</td>
<td>+++</td>
<td>+++</td>
</tr>
<tr>
<td>Composés organiques volatils (COV)</td>
<td>+++</td>
<td>+++</td>
<td>+++</td>
<td>+++</td>
<td>+++</td>
</tr>
<tr>
<td>Composés organiques halogénés adsorbables (ACK)</td>
<td>+++</td>
<td>+++</td>
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<td>+++</td>
<td>+++</td>
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<tr>
<td>Allergènes</td>
<td>+++</td>
<td>+++</td>
<td>+++</td>
<td>+++</td>
<td>+++</td>
</tr>
<tr>
<td>NOTE GLOBALE (100 %)</td>
<td><strong>17/20</strong></td>
<td><strong>14,5/20</strong></td>
<td><strong>12,5/20</strong></td>
<td><strong>12,5/20</strong></td>
<td><strong>12,5/20</strong></td>
</tr>
</tbody>
</table>

[1] Livraison comprise dans le prix
[2] L’appréciation globale ne peut pas être supérieure à l’appréciation sur les performances
[3] Le fabricant indique que cette référence est en fin de commercialisation
[4] L’appréciation...
Example (Evaluation of diapers by the Magazine “60 millions consommateurs” in September 2018)

<table>
<thead>
<tr>
<th>Product</th>
<th>Carrefour Baby Ultra dry &amp; stretch</th>
<th>Lupilu Soft &amp; Dry</th>
<th>Mots d’enfants (Marque Repère) Ultra confort</th>
<th>Love &amp; Green Couches hypoallergéniques</th>
<th>Lotus Baby Touch 3 Ultra confort</th>
<th>Pommette (Intermarché) Ecologic</th>
<th>Lillydoo Couches bébé</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non</td>
<td>Non</td>
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</tr>
<tr>
<td>10,80 € 56 couches</td>
<td>7,30 € 56 couches</td>
<td>8,90 € 50 couches</td>
<td>19,65 € 52 couches</td>
<td>19 € 58 couches</td>
<td>9 € 32 couches</td>
<td>12 € 33 couches</td>
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<tr>
<td>0,19 €</td>
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</tbody>
</table>

Note: Global assessment cannot be superior to "Insufficient" or "Very insufficient" depending on the composition of the product.
Example (Evaluation of diapers by the Magazine “60 millions consommateurs” in September 2018)

<table>
<thead>
<tr>
<th></th>
<th>Performance</th>
<th>Composition</th>
<th>Global score (/20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A- Joone</td>
<td>+++</td>
<td>+++</td>
<td>17</td>
</tr>
<tr>
<td>B- Pamp. Prem</td>
<td>++</td>
<td>++</td>
<td>14.5</td>
</tr>
<tr>
<td>C- Pamp. Baby</td>
<td>+</td>
<td>+++</td>
<td>12.5</td>
</tr>
<tr>
<td>D- Naty</td>
<td>+</td>
<td>+++</td>
<td>12.5</td>
</tr>
<tr>
<td>E- Pamp. Activ.</td>
<td>+</td>
<td>+</td>
<td>12.5</td>
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<tr>
<td>F- Carref. Baby</td>
<td>++</td>
<td>+</td>
<td>12.5</td>
</tr>
<tr>
<td>G- Lupilu</td>
<td>++</td>
<td>−</td>
<td>12</td>
</tr>
<tr>
<td>H- Mots d’enfants</td>
<td>+</td>
<td>−</td>
<td>12</td>
</tr>
<tr>
<td>I- Love &amp; Green</td>
<td>++</td>
<td>−</td>
<td>9.5</td>
</tr>
<tr>
<td>K- Lotus Baby</td>
<td>++</td>
<td>−−</td>
<td>9.5</td>
</tr>
<tr>
<td>L- Pommette</td>
<td>++</td>
<td>−−</td>
<td>9.5</td>
</tr>
<tr>
<td>M- Lillydoo</td>
<td>+</td>
<td>−−</td>
<td>6.5</td>
</tr>
</tbody>
</table>

\[ w_p = 60\% \quad w_c = 40\% \]

++ + + + \equiv \text{Very good} \in [17, 20]; ++ ++ \equiv \text{Good} \in [13, 16.5];
++ \equiv \text{Acceptable} \in [10, 12.5];
−− \equiv \text{Very Insufficient} \in [0, 6.5]
Example (Evaluation of diapers by the Magazine “60 millions consommateurs” in September 2018)

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\[ w_p = 60\% \quad w_c = 40\% \]

- Which evaluation model was used by this magazine?
- Are these preferences representable by an additive model? (by using the given utility functions and weight)
Principles

- Created by Jacquet Lagreze & Siskos in 1982 (at LAMSADE)

- The UTA (UTilités Additives) method aims at inferring one or more additive value functions from a given ranking on a reference set $A_R$.

- The method uses special linear programming techniques to assess these functions so that the ranking(s) obtained through these functions on $A_R$ is (are) as consistent as possible with the given one.
The UTA Approach

UTA Principles: Input data

- A set of Criteria \( N \)
- A set of alternatives \( X \) evaluated on \( N \)
- A preorder \( \succeq_{X'} \) on \( X' \subseteq X \) (not necessary complete)
- For each element \( x = (x_1, \ldots, x_n) \in X \), it is assumed that

\[
U(x) = \sum_{i=1}^{n} u_i(x_i) \tag{1}
\]

where \( u_i : X_i \to \mathbb{R}, i = 1, \ldots, n \) are marginal utility functions
The UTA Approach

UTA Principles: Input data

- For each criterion $i$, $X_i = [\alpha_i, \beta_i]$ is the criterion evaluation scale such that $\alpha_i \leq \beta_i$
- The following normalization constraints, associated to the marginal utility functions, are considered:

$$
\begin{cases}
  u_i(\alpha_i) = 0, \forall i = 1, \ldots, n \\
  \sum_{i=1}^{n} u_i(\beta_i) = 1
\end{cases}
$$

(2)
The UTA Approach

UTA Principles: Input data

- Each marginal value function $u_i$ is assumed to be piecewise linear, so that the interval $[\alpha_i, \beta_i]$ is divided into $\gamma_i \geq 1$ equal sub-intervals

$$[\alpha_i = x_i^0, x_i^1], [x_i^1, x_i^2], \ldots, [x_i^{\gamma_i-2}, x_i^{\gamma_i-1}], [x_i^{\gamma_i-1}, x_i^{\gamma_i} = \beta_i]$$

where

$$x_i^j = \alpha_i + \frac{j(\beta_i - \alpha_i)}{\gamma_i}, \ j = 0, \ldots, \gamma_i$$

- Hence, using linear interpolation, the utility function associated to an element $x_i \in [x_i^j, x_i^{j+1}]$ is given by

$$u_i(x_i) = u_i(x_i^j) + \frac{x_i - x_i^j}{x_i^{j+1} - x_i^j}(u_i(x_i^{j+1}) - u_i(x_i^j)). \quad (3)$$
UTA Principles: Input data

- The piecewise linear additive model is completely defined by the marginal values at the break points, i.e.

\[ u_i(x_i^0) = u_i(\alpha_i), \ u_i(x_i^1), u_i(x_i^2), \ldots, u_i(x_i^{\gamma_i}) = u_i(\beta_i). \]
UTA Principles: The model

- \( U(x) = \sum_{i=1}^{n} u_i(x_i) \)

- For each element \( x \in X' \), set
  \[
  V(x) = U(x) + \sigma(x)
  \]
  where \( \sigma(x) \) is a nonnegative real value estimating the error of the estimation of the value \( U(x) \), i.e., \( \sigma(x) = V(x) - U(x) \).

The value \( \sigma(x) \) will be minimized by the linear program.
UTA Principles: The linear program to solve

\[
\begin{aligned}
& \min \sum_{x \in X'} \sigma(x) \\
& V(x) \geq V(y) + \delta \text{ if } x \succ y \\
& V(x) = V(y) \text{ if } x \sim y \\
& u_i(x_i^{j+1}) - u_i(x_i^j) \geq 0, \ i = 1, \ldots, n, \ j = 1, \ldots, \gamma_i - 1 \\
& u_i(\alpha_i) = 0, \ \forall i = 1, \ldots, n \\
& \sum_{i=1}^n u_i(\beta_i) = 1 \\
& \sigma(x) \geq 0, \ \forall x \in X'
\end{aligned}
\]

- If the optimal solution is equal to 0 then \(\succsim_{X'}\) is representable by (compatible with) an additive model.

- There are many versions of the UTA method
Example (The choice of a camera)

You want to buy a camera and you have obtained the following information about six cameras evaluated on three criteria.

1. The resolution (in Millions of Pixels)
2. The price (in euros)
3. The optical zoom (a real number).

<table>
<thead>
<tr>
<th>Cameras</th>
<th>1 : Resolution</th>
<th>2 : Price</th>
<th>3 : Zoom</th>
</tr>
</thead>
<tbody>
<tr>
<td>a : Nikon</td>
<td>6</td>
<td>150</td>
<td>5</td>
</tr>
<tr>
<td>b : Sony</td>
<td>7</td>
<td>180</td>
<td>5</td>
</tr>
<tr>
<td>c : Panasonic</td>
<td>10</td>
<td>155</td>
<td>4</td>
</tr>
<tr>
<td>d : Casio</td>
<td>12</td>
<td>175</td>
<td>5</td>
</tr>
<tr>
<td>e : Olympus</td>
<td>10</td>
<td>160</td>
<td>3</td>
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<tr>
<td>f : Kodak</td>
<td>8</td>
<td>165</td>
<td>4</td>
</tr>
</tbody>
</table>

We have $X = \{a, b, c, d, e, f\}$, $N = \{1, 2, 3\}$, $X_1 = [6, 12]$, $X_2 = [150, 180]$ and $X_3 = [3, 5]$.

Your preferences on a reference subset are: $a \succ e \succ b$. 
Example (The choice of a camera)

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</tbody>
</table>

We have $X = \{a, b, c, d, e, f\}$, $N = \{1, 2, 3\}$, $X_1 = [6, 12]$, $X_2 = [150, 180]$ and $X_3 = [3, 5]$. We suppose $\gamma_1 = 2$, $\gamma_2 = 3$ and $\gamma_3 = 1$.

Hence

- For $X_1 = [6, 12]$: $x_1^0 = 6$, $x_1^1 = 9$, $x_1^2 = 12$ and $u_1(6) = 0$;
- For $X_2 = [180, 150]$: $x_2^0 = 150$, $x_2^1 = 160$, $x_2^2 = 170$, $x_2^3 = 180$ and $u_2(180) = 0$;
- For $X_3 = [3, 5]$: $x_3^0 = 3$, $x_3^1 = 5$ and $u_3(3) = 0$. 

(LAMSADE)
Example (The choice of a camera)

- For the criterion 1:
  
  \[ u_1(7) = u_1(6) + \frac{1}{3}(u_1(9) - u_1(6)) = \frac{1}{3}u_1(9) \]
  
  \[ u_1(8) = u_1(6) + \frac{2}{3}(u_1(9) - u_1(6)) = \frac{2}{3}u_1(9) \]
  
  \[ u_1(10) = u_1(9) + \frac{1}{3}(u_1(12) - u_1(9)) = \frac{2}{3}u_1(9) + \frac{1}{3}u_1(12) \]

- For the criterion 2:
  
  \[ u_2(175) = \frac{1}{2}u_2(170) \]
  
  \[ u_2(165) = u_2(170) + \frac{1}{2}(u_2(160) - u_2(170)) = \frac{1}{2}u_2(170) + \frac{1}{2}u_2(160) \]
  
  \[ u_2(155) = u_2(160) + \frac{1}{2}(u_2(150) - u_2(160)) = \frac{1}{2}u_2(160) + \frac{1}{2}u_2(150) \]

- For the criterion 3:
  
  \[ u_3(4) = \frac{1}{2}u_3(5) \]
Example (The choice of a camera)

Your preferences: \( a \succ e \succ b \)

\[
\begin{align*}
\min \quad & \sigma(a) + \sigma(b) + \sigma(e) \\
\text{s.t.} \quad & u_2(150) + u_3(5) + \sigma(a) - \frac{2}{3} u_1(9) - \frac{1}{3} u_1(12) - u_2(160) - \sigma(e) \geq \delta \\
& \frac{2}{3} u_1(9) + \frac{1}{3} u_1(12) + u_2(160) + \sigma(e) - \frac{1}{3} u_1(9) - u_3(5) - \sigma(b) \geq \delta \\
& u_1(9) - u_1(6) \geq 0 \\
& u_1(12) - u_1(9) \geq 0 \\
& u_2(150) - u_2(160) \geq 0 \\
& u_2(160) - u_2(170) \geq 0 \\
& u_2(170) - u_2(180) \geq 0 \\
& u_3(5) - u_3(3) \geq 0 \\
& u_1(6) = 0 \\
& u_2(180) = 0 \\
& u_3(3) = 0 \\
& u_1(12) + u_2(150) + u_3(5) = 1 \\
& \sigma(a) \geq 0 \\
& \sigma(b) \geq 0 \\
& \sigma(c) \geq 0 \\
& \delta = 0.001
\end{align*}
\]
Example (The choice of a camera)

Your preferences: $a \succ e \succ b$.

A solution:

\[
\begin{align*}
&u_1(6) = 0 \\
&u_1(9) = 0.1 \\
&u_1(12) = 0.4 \\
&u_2(180) = 0 \\
&u_2(170) = 0.2 \\
&u_2(160) = 0.3 \\
&u_2(150) = 0.4 \\
&u_3(3) = 0 \\
&u_3(5) = 0.2
\end{align*}
\]

Then

\[
\begin{align*}
&u(a) = 0.6 \\
&u(b) = 0.233 \\
&u(c) = 0.65 \\
&u(d) = 0.7 \\
&u(e) = 0.5 \\
&u(f) = 0.416
\end{align*}
\]
The UTA\textsuperscript{GMS} Approach

\textbf{UTA}\textsuperscript{GMS} Principles

- Generalizes the UTA approach
- It takes into account all additive value functions compatible with indirect preference information, while UTA is using only one such function.
- The marginal value functions are general monotone non-decreasing functions, and not piecewise linear only.
The UTA\textsuperscript{GMS} Approach

UTA\textsuperscript{GMS} Principles

The method produces two rankings in the set of alternatives \( A \), such that for any pair of alternatives \( a, b \in X \)

- In the necessary order, \( a \) is ranked at least as good as \( b \) if and only if, \( U(a) \geq U(b) \) for all value functions compatible with the preference information.

- In the possible order, \( a \) is ranked at least as good as \( b \) if and only if, \( U(a) \geq U(b) \) for at least one value function compatible with the preference information.
The UTA\textsuperscript{GMS} Approach

UTA Principles: Input data

- Preference information
- All instances of preference model compatible with preference information
- Apply all compatible instances on A

What rankings will result?
The UTA$^GMS$ Approach

UTA Principles: Input data

- $x \geq y$
- $z \geq w$
- $y \geq v$
- $u \geq t$
- $z \geq u$
- $u \geq z$

Includes necessary ranking and does not include the complement of necessary ranking.
Outline

1. Framework
2. The simple models
3. Introduction to MCDA
4. Multi Attribute Utility Theory
5. The additive model
6. A non-additive model: The Choquet integral
7. To conclude
Example (A classic example of Grabisch et al. (2010))

1 : Mathematics (M)  2 : Statistics (S)  3 : Language (L)

\[
\begin{align*}
    a & \quad 16 & & 13 & & 7 \\
    b & \quad 16 & & 11 & & 9 \\
    c & \quad 6 & & 13 & & 7 \\
    d & \quad 6 & & 11 & & 9 \\
\end{align*}
\]

• For a student good in Mathematics, Language is more important than Statistics

\[ \implies a \prec b, \]

• For a student bad in Mathematics, Statistics is more important than Language

\[ \implies d \prec c. \]
Example (A classic example of Grabisch et al. (2010))

1 : Mathematics (M)  2 : Statistics (S)  3 : Language (L)

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<tr>
<td>d</td>
<td>6</td>
<td>11</td>
<td>9</td>
</tr>
</tbody>
</table>

The two preferences $a \prec b$ and $d \prec c$ lead to a contradiction with the additive model

\[
\begin{align*}
  a \prec b & \Rightarrow u_M(16) w_M + u_S(13) w_S + u_L(7) w_L < u_M(16) w_M + u_S(11) w_S + u_L(9) w_L \\
  d \prec c & \Rightarrow u_M(6) w_M + u_S(11) w_S + u_L(9) w_L < u_M(6) w_M + u_S(13) w_S + u_L(7) w_L.
\end{align*}
\]

i.e.,

\[
\begin{align*}
  u_S(13) w_S + u_L(7) w_L & < u_S(11) w_S + u_L(9) w_L \\
  \text{and} \\
  u_S(11) w_S + u_L(9) w_L & < u_S(13) w_S + u_L(7) w_L
\end{align*}
\]
Example (A classic example of Grabisch et al. (2010))

1: Mathematics (M)  2: Statistics (S)  3: Language (L)

<table>
<thead>
<tr>
<th></th>
<th>M</th>
<th>S</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>16</td>
<td>13</td>
<td>7</td>
</tr>
<tr>
<td>b</td>
<td>16</td>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>c</td>
<td>6</td>
<td>13</td>
<td>7</td>
</tr>
<tr>
<td>d</td>
<td>6</td>
<td>11</td>
<td>9</td>
</tr>
</tbody>
</table>

- The preference information $a \prec b$ and $d \prec c$ is not representable by an additive utility model.
Example (A ranking of French hospitals for weight loss surgery)

To identify the “best” hospitals in weight loss surgery, the magazine “Le Point” combines a part of the following four indicators (criteria):

- **Criterion 1 - Activity**: number of procedures performed during one year. This criterion has to be maximized.

- **Criterion 2 - Notoriety**: its corresponds to the reputation and attractiveness of the hospital. It is a percentage of patients treated in the hospital but living in another French administrative department. More the percentage increases, more the hospital is attractive.

- **Criterion 3 - Average Length Of Stay (ALOS)**: a mean calculated by dividing the sum of inpatient days by the number of patients admissions with the same diagnosis-related group classification. If a hospital is more organized in terms of resources then its ALOS score should be low.

- **Criterion 4 - Technicality**: this particular indicator measures the ratio of procedures performed with an efficient technology compared to the same procedures performed with obsolete technology. The higher the percentage is, the more the team is trained in advanced technologies or complex surgeries.
Example (A ranking of French hospitals for weight loss surgery)

<table>
<thead>
<tr>
<th>Hospital</th>
<th>Activity</th>
<th>Notoriety</th>
<th>ALOS</th>
<th>Technicality</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1</td>
<td>200</td>
<td>65</td>
<td>3.5</td>
<td>85</td>
</tr>
<tr>
<td>H2</td>
<td>450</td>
<td>60</td>
<td>4</td>
<td>75</td>
</tr>
<tr>
<td>H3</td>
<td>450</td>
<td>50</td>
<td>2.5</td>
<td>55</td>
</tr>
<tr>
<td>H4</td>
<td>350</td>
<td>50</td>
<td>3.5</td>
<td>85</td>
</tr>
<tr>
<td>H5</td>
<td>350</td>
<td>55</td>
<td>2</td>
<td>75</td>
</tr>
<tr>
<td>H6</td>
<td>150</td>
<td>65</td>
<td>2.5</td>
<td>80</td>
</tr>
<tr>
<td>H7</td>
<td>200</td>
<td>55</td>
<td>2</td>
<td>55</td>
</tr>
<tr>
<td>H8</td>
<td>150</td>
<td>60</td>
<td>4</td>
<td>80</td>
</tr>
</tbody>
</table>

\[ H1 \succ H2; \quad H3 \succ H4; \quad H5 \succ H6; \quad H8 \succ H7. \]

Are these preferences representable by an additive function?
Example (A ranking of French hospitals for weight loss surgery)

<table>
<thead>
<tr>
<th>Hospital 1 (H1)</th>
<th>1- Activity</th>
<th>2- Notoriety</th>
<th>3- ALOS</th>
<th>4- Technicality</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>200</td>
<td>65</td>
<td>3.5</td>
<td>85</td>
</tr>
<tr>
<td>Hospital 2 (H2)</td>
<td>450</td>
<td>60</td>
<td>4</td>
<td>75</td>
</tr>
<tr>
<td>Hospital 3 (H3)</td>
<td>450</td>
<td>50</td>
<td>2.5</td>
<td>55</td>
</tr>
<tr>
<td>Hospital 4 (H4)</td>
<td>350</td>
<td>50</td>
<td>3.5</td>
<td>85</td>
</tr>
<tr>
<td>Hospital 5 (H5)</td>
<td>350</td>
<td>55</td>
<td>2</td>
<td>75</td>
</tr>
<tr>
<td>Hospital 6 (H6)</td>
<td>150</td>
<td>65</td>
<td>2.5</td>
<td>80</td>
</tr>
<tr>
<td>Hospital 7 (H7)</td>
<td>200</td>
<td>55</td>
<td>2</td>
<td>55</td>
</tr>
<tr>
<td>Hospital 8 (H8)</td>
<td>150</td>
<td>60</td>
<td>4</td>
<td>80</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
H1 & \succ H2 \Rightarrow u_1(200)w_1 + u_2(65)w_2 + u_3(3.5)w_3 + u_4(85)w_4 > u_1(450)w_1 + u_2(60)w_2 + u_3(4)w_3 + u_4(75)w_4 \\
H3 & \succ H4 \Rightarrow u_1(450)w_1 + u_2(50)w_2 + u_3(2.5)w_3 + u_4(55)w_4 > u_1(350)w_1 + u_2(50)w_2 + u_3(3.5)w_3 + u_4(85)w_4 \\
H5 & \succ H6 \Rightarrow u_1(350)w_1 + u_2(55)w_2 + u_3(2)w_3 + u_4(75)w_4 > u_1(150)w_1 + u_2(65)w_2 + u_3(2.5)w_3 + u_4(80)w_4 \\
H7 & \prec H8 \Rightarrow u_1(200)w_1 + u_2(55)w_2 + u_3(2)w_3 + u_4(55)w_4 < u_1(150)w_1 + u_2(60)w_2 + u_3(4)w_3 + u_4(80)w_4
\end{align*}
\]

The first three equations in this system lead to

\[
 u_1(200)w_1 + u_2(55)w_2 + u_3(2)w_3 + u_4(55)w_4 > u_1(150)w_1 + u_2(60)w_2 + u_3(4)w_3 + u_4(80)w_4
\]

which contradicts the last equation.
A Limit of the additive utility model

We try another MCDA model: the Choquet integral.
The 2-additive capacity

Definition

A 2-additive capacity is a set function $\mu : 2^N \rightarrow [0, 1]$ such that:

\[
\sum_{\{i,j\} \subseteq N} \mu(\{i,j\}) - (n - 2) \sum_{i \in N} \mu(\{i\}) = 1 \quad \text{(normality)}
\]

\[
\mu(\{i\}) \geq 0, \ \forall i \in N \quad \text{(nonnegativity)}
\]

\[
\forall A \subseteq N, \ |A| \geq 2, \ \forall i \in A
\]

\[
\sum_{j \in A \setminus \{i\}} (\mu(\{i,j\}) - \mu(\{j\})) \geq (|A| - 2)\mu(\{i\}) \quad \text{(monotonicity)}.
\]
The 2-additive capacity

Notations

\[ \forall i, j \in N, i \neq j, \]
\[ \mu_\emptyset = \mu(\emptyset), \mu_i = \mu(\{i\}), \mu_{ij} = \mu(\{i,j\}) \]
\[ m^\mu(\{i,j\}) = \mu_{ij} - \mu_i - \mu_j \]
\[ m^\mu(\{i\}) = \mu_i \]

\( m^\mu \) is called the Möbius transform of \( \mu \).
The 2-additive capacity

Example

- $N = \{1, 2, 3\}$
- **Normality constraint:** $\mu_{12} + \mu_{13} + \mu_{23} - \mu_1 - \mu_2 - \mu_3 = 1$
- **Nonnegativity constraints:** $\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0$
- **Monotonicity constraints:**
  - $\mu_{12} \geq \mu_1, \mu_{12} \geq \mu_2$
  - $\mu_{13} \geq \mu_1, \mu_{13} \geq \mu_3$
  - $\mu_{23} \geq \mu_2, \mu_{23} \geq \mu_3$
  - $\mu_{12} + \mu_{13} \geq \mu_1 + \mu_2 + \mu_3$
  - $\mu_{12} + \mu_{23} \geq \mu_1 + \mu_2 + \mu_3$
  - $\mu_{13} + \mu_{23} \geq \mu_1 + \mu_2 + \mu_3$
The 2-additive Choquet integral

Definition

For any \( x := (x_1, \ldots, x_n) \in X \), the expression of the 2-additive Choquet integral is:

\[
C_{\mu}(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} \phi_\mu^i u_i(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} I_{\mu}^{ij} |u_i(x_i) - u_j(x_j)|
\]

Where

- \( I_{\mu}^{ij} \) = the interaction index between criteria \( i \) and \( j \):
  \[
  I_{\mu}^{ij} = \mu_{ij} - \mu_i - \mu_j.
  \]
- \( \phi_\mu^i \) = the importance of the criterion \( i \) (≡ Shapley index):
  \[
  \phi_\mu^i = \mu_i + \frac{1}{2} \sum_{k \in N \setminus i} I_{\mu}^{ik}.
  \]
Interest of the 2-additive model

$$C_\mu(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} \phi_i^{\mu} u_i(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} l_{ij}^{\mu} |u_i(x_i) - u_j(x_j)|$$

- It is a generalization of the arithmetic mean ($l_{ij} = 0 \ \forall i, j \in N$)
- It is a compromise between the general Choquet integral and the arithmetic mean, i.e., offers a good compromise between flexibility of the model and complexity;
- It was used in many applications such that
  - the evaluation of discomfort in sitting position (see Grabisch et al. (2002));
  - the construction of performance measurement systems model in a supply chain context (see Berrah and Clivillé (2007), Clivillé et al. (2007));
  - complex system design (Labreuche and Pignon (2007));
Remark

The Choquet integral requires to be able to compare any element of one point of view with any element of any other point of view (commensurateness between criteria);
Interaction index $I_{ij}^\mu = \mu_{ij} - \mu_i - \mu_j$

Usual Interpretation

- $I_{ij}^\mu > 0$ $\implies$ criteria $i$ and $j$ are complementarity.
- $I_{ij}^\mu < 0$ $\implies$ criteria $i$ and $j$ are redundant.
- $I_{ij}^\mu = 0$ $\implies$ criteria $i$ and $j$ are independent (no interaction).
The sign of the interaction index is not always stable

Example (A classic example of Grabisch et al. (2010))

\[
\begin{array}{ccc}
1: \text{Mathematics (M)} & 2: \text{Statistics (S)} & 3: \text{Language (L)} \\
a & 16 & 13 & 7 \\
b & 16 & 11 & 9 \\
c & 6 & 13 & 7 \\
d & 6 & 11 & 9 \\
\end{array}
\]

- \(a \prec b\) and \(d \prec c\)
- \(u_M(16) = 16, \ u_M(6) = 6,\)
- \(u_S(13) = 13, \ u_S(11) = 11\)
- \(u_L(7) = 7, \ u_L(9) = 9\)
The sign of the interaction index is not always stable

Example (A classic example of Grabisch et al. (2010))

<table>
<thead>
<tr>
<th></th>
<th>Par. 1</th>
<th>Par. 2</th>
<th>Par. 3</th>
<th>Par. 4</th>
<th>Par. 5</th>
<th>Par. 6</th>
<th>Par. 7</th>
<th>Par. 8</th>
<th>Par. 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_\mu(a)$</td>
<td>8.5</td>
<td>13.75</td>
<td>9.1</td>
<td>13.765</td>
<td>13.75</td>
<td>13.75</td>
<td>11.47</td>
<td>12.535</td>
<td>10.45</td>
</tr>
<tr>
<td>$C_\mu(c)$</td>
<td>7.75</td>
<td>9.75</td>
<td>7.75</td>
<td>11.325</td>
<td>11.25</td>
<td>9.75</td>
<td>9.45</td>
<td>9.515</td>
<td>7.85</td>
</tr>
</tbody>
</table>

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_M$</td>
<td>0</td>
<td>0.75</td>
<td>0</td>
<td>0.685</td>
<td>0.75</td>
<td>0.75</td>
<td>0.36</td>
<td>0.485</td>
<td>0.15</td>
</tr>
<tr>
<td>$\mu_S$</td>
<td>0.25</td>
<td>0.5</td>
<td>0.25</td>
<td>0.73</td>
<td>0.75</td>
<td>0.5</td>
<td>0.465</td>
<td>0.455</td>
<td>0.25</td>
</tr>
<tr>
<td>$\mu_L$</td>
<td>0</td>
<td>0.25</td>
<td>0</td>
<td>0.315</td>
<td>0</td>
<td>0</td>
<td>0.205</td>
<td>0.32</td>
<td>0</td>
</tr>
<tr>
<td>$\mu_{MS}$</td>
<td>0.25</td>
<td>0.75</td>
<td>0.35</td>
<td>0.785</td>
<td>0.75</td>
<td>0.75</td>
<td>0.565</td>
<td>0.68</td>
<td>0.5</td>
</tr>
<tr>
<td>$\mu_{ML}$</td>
<td>0.75</td>
<td>1</td>
<td>0.65</td>
<td>1</td>
<td>0.1</td>
<td>0.75</td>
<td>0.805</td>
<td>0.795</td>
<td>0.55</td>
</tr>
<tr>
<td>$\mu_{SL}$</td>
<td>0.25</td>
<td>0.75</td>
<td>0.25</td>
<td>0.945</td>
<td>0.75</td>
<td>0.75</td>
<td>0.66</td>
<td>0.785</td>
<td>0.35</td>
</tr>
</tbody>
</table>

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$V^\mu_M$</td>
<td>0.375</td>
<td>0.5</td>
<td>0.375</td>
<td>0.37</td>
<td>0.5</td>
<td>0.5</td>
<td>0.35</td>
<td>0.35</td>
<td>0.4</td>
</tr>
<tr>
<td>$V^\mu_S$</td>
<td>0.25</td>
<td>0.25</td>
<td>0.3</td>
<td>0.365</td>
<td>0.375</td>
<td>0.375</td>
<td>0.33</td>
<td>0.33</td>
<td>0.35</td>
</tr>
<tr>
<td>$V^\mu_L$</td>
<td>0.375</td>
<td>0.25</td>
<td>0.325</td>
<td>0.265</td>
<td>0.125</td>
<td>0.125</td>
<td>0.32</td>
<td>0.32</td>
<td>0.25</td>
</tr>
</tbody>
</table>

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$I^\mu_{MS}$</td>
<td>0</td>
<td>-0.5</td>
<td>0.1</td>
<td>-0.63</td>
<td>-0.75</td>
<td>-0.5</td>
<td>-0.26</td>
<td>-0.26</td>
<td>0.1</td>
</tr>
<tr>
<td>$I^\mu_{ML}$</td>
<td>0.75</td>
<td>0</td>
<td>0.65</td>
<td>0</td>
<td>0.25</td>
<td>0</td>
<td>0.24</td>
<td>-0.01</td>
<td>0.4</td>
</tr>
<tr>
<td>$I^\mu_{SL}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-0.1</td>
<td>0</td>
<td>0.25</td>
<td>-0.01</td>
<td>0.01</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Preferences Aggregation: the MAUT approach
The sign of the interaction index is not always stable

How to conclude from this preference information given by the DM?

- Mathematics & Statistics are independent? complementary? redundant?
- Mathematics & Literature are independent? complementary? redundant?
- Statistics & Literature are independent? complementary? redundant?

- “The three subjects, taken together, are not without interaction”
The sign of the interaction index is not always stable

Example (A ranking of French hospitals for weight loss surgery)

<table>
<thead>
<tr>
<th></th>
<th>1- Activity</th>
<th>2- Notoriety</th>
<th>3- ALOS</th>
<th>4- Technicality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hospital 1 (H1)</td>
<td>200</td>
<td>65</td>
<td>3.5</td>
<td>85</td>
</tr>
<tr>
<td>Hospital 2 (H2)</td>
<td>450</td>
<td>60</td>
<td>4</td>
<td>75</td>
</tr>
<tr>
<td>Hospital 3 (H3)</td>
<td>450</td>
<td>50</td>
<td>2.5</td>
<td>55</td>
</tr>
<tr>
<td>Hospital 4 (H4)</td>
<td>350</td>
<td>50</td>
<td>3.5</td>
<td>85</td>
</tr>
<tr>
<td>Hospital 5 (H5)</td>
<td>350</td>
<td>55</td>
<td>2</td>
<td>75</td>
</tr>
<tr>
<td>Hospital 6 (H6)</td>
<td>150</td>
<td>65</td>
<td>2.5</td>
<td>80</td>
</tr>
<tr>
<td>Hospital 7 (H7)</td>
<td>200</td>
<td>55</td>
<td>2</td>
<td>55</td>
</tr>
<tr>
<td>Hospital 8 (H8)</td>
<td>150</td>
<td>60</td>
<td>4</td>
<td>80</td>
</tr>
</tbody>
</table>

\[ H1 \succ H2; \quad H3 \succ H4; \quad H5 \succ H6; \quad H8 \succ H7. \]
**The sign of the interaction index is not always stable**

Example (A ranking of French hospitals for weight loss surgery)

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<th>Par.6</th>
<th>Par.7</th>
<th>Par.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{\mu}(H1)$</td>
<td>0.607</td>
<td>0.555</td>
<td>0.692</td>
<td>0.549</td>
<td>0.563</td>
<td>0.59</td>
<td>0.597</td>
<td>0.589</td>
</tr>
<tr>
<td>$C_{\mu}(H2)$</td>
<td>0.509</td>
<td>0.531</td>
<td>0.682</td>
<td>0.539</td>
<td>0.553</td>
<td>0.58</td>
<td>0.533</td>
<td>0.548</td>
</tr>
<tr>
<td>$C_{\mu}(H3)$</td>
<td>0.514</td>
<td>0.576</td>
<td>0.693</td>
<td>0.548</td>
<td>0.605</td>
<td>0.588</td>
<td>0.530</td>
<td>0.564</td>
</tr>
<tr>
<td>$C_{\mu}(H4)$</td>
<td>0.502</td>
<td>0.506</td>
<td>0.683</td>
<td>0.538</td>
<td>0.538</td>
<td>0.578</td>
<td>0.510</td>
<td>0.531</td>
</tr>
<tr>
<td>$C_{\mu}(H5)$</td>
<td>0.586</td>
<td>0.606</td>
<td>0.704</td>
<td>0.63</td>
<td>0.629</td>
<td>0.645</td>
<td>0.583</td>
<td>0.609</td>
</tr>
<tr>
<td>$C_{\mu}(H6)$</td>
<td>0.576</td>
<td>0.568</td>
<td>0.694</td>
<td>0.582</td>
<td>0.606</td>
<td>0.635</td>
<td>0.573</td>
<td>0.599</td>
</tr>
<tr>
<td>$C_{\mu}(H7)$</td>
<td>0.513</td>
<td>0.502</td>
<td>0.539</td>
<td>0.508</td>
<td>0.529</td>
<td>0.538</td>
<td>0.514</td>
<td>0.527</td>
</tr>
<tr>
<td>$C_{\mu}(H8)$</td>
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<td>0.539</td>
<td>0.548</td>
<td>0.524</td>
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<tr>
<td>$\mu_1$</td>
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<td>0.19</td>
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<td>0.01</td>
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<td>0.209</td>
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<td>0.273</td>
<td>0.416</td>
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<td>0.472</td>
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<td>0.765</td>
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<td>0.519</td>
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<td>0.47</td>
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<td>0.08</td>
<td>-0.07</td>
<td>-0.04</td>
<td>-0.123</td>
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</table>
**Necessary and possible interaction between criteria**

**Hypothesis**

A preference information given by the DM is representable by a 2-additive Choquet integral.

Let be $i, j \in \mathbb{N}, i \neq j$ and $C_{\text{pref}}$ the set of all capacities compatible with a preference information given by the DM.

1. **There exists a possible positive (respectively, null, negative) interaction between $i$ and $j$ if there exists a capacity $\mu \in C_{\text{pref}}$ such that $I_{ij}^{\mu} > 0$ (respectively, $I_{ij}^{\mu} = 0$, $I_{ij}^{\mu} < 0$).**

2. **There exists a necessary positive (respectively, null, negative) interaction between $i$ and $j$ if $I_{ij}^{\mu} > 0$ (respectively, $I_{ij}^{\mu} = 0$, $I_{ij}^{\mu} < 0$) for all capacity $\mu \in C_{\text{pref}}$.**
Identification of Necessary and possible interactions (Step 1):

Minimize $Z_1 = \sum_{(x,y) \in P \cup I} (\Gamma_{xy}^+ + \Gamma_{xy}^-)$

Subject to

$$C_\mu(u(x)) - C_\mu(u(y)) + \Gamma_{xy}^+ - \Gamma_{xy}^- \geq \varepsilon \quad \forall x, y \in X \text{ such that } xP y$$ (4)

$$C_\mu(u(x)) - C_\mu(u(y)) + \Gamma_{xy}^+ - \Gamma_{xy}^- = 0 \quad \forall x, y \in X \text{ such that } xI y$$ (5)

$$\Gamma_{xy}^+ \geq 0, \quad \Gamma_{xy}^- \geq 0 \quad \forall x, y \in X \text{ such that } x(P \cup I) y$$ (6)

$$\varepsilon \geq 0$$ (7)

$$\sum_{\{i,j\} \subseteq N} m(\{i,j\}) + \sum_{i \in N} m(\{i\}) = 1$$ (8)

$$m(\{i\}) \geq 0 \quad \text{for all } i \in N$$ (9)

$$m(\{i\}) + \sum_{j \in A \setminus \{i\}} m(\{i,j\}) \geq 0 \quad \forall A \setminus \{i\}, \forall i \in N.$$ (10)

- $PL_1$ is always feasible
- $Z_1^* = 0$ $\implies$ we can conclude that, depending on the sign of the variable $\varepsilon$, the preference information $\{P, I\}$ may be representable by a 2-additive Choquet integral.
- $Z_1^* > 0$ $\implies$, then there is no 2-additive Choquet integral model compatible with $\{P, I\}$. 

(LAMSAD)
Identification of Necessary and possible interactions (Step 2):

Maximize $Z_2 = \varepsilon$

Subject to

\[ C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \quad \forall x, y \in X \text{ such that } x \ P \ y \quad (11) \]
\[ C_\mu(u(x)) - C_\mu(u(y)) = 0 \quad \forall x, y \in X \text{ such that } x \ I \ y \quad (12) \]
\[ \varepsilon \geq 0 \quad (13) \]
\[ \sum_{\{i,j\} \subseteq N} m(\{i,j\}) + \sum_{i \in N} m(\{i\}) = 1 \quad (14) \]
\[ m(\{i\}) \geq 0 \quad \text{for all } i \in N \quad (15) \]
\[ m(\{i\}) + \sum_{j \in A \setminus \{i\}} m(\{i,j\}) \geq 0 \quad \forall A \setminus \{i\}, \forall i \in N. \quad (16) \]

- $PL_2$ is always feasible since the Step 1 was solved before ($PL_2$) is launched when $Z_1^* = 0$.
- $Z_2^* = 0 \implies$, there is no 2-additive Choquet integral model compatible with $\{P, I\}$.
- $Z_2^* > 0 \implies$, $\{P, I\}$ is representable by a 2-additive Choquet integral.
Identification of Necessary and possible interactions (Step 3):

Maximize $Z_3 = \varepsilon$

Subject to

- $m(\{i,j\}) \geq 0$ (respectively $m(\{i,j\}) \leq 0$)  \hspace{1cm} (17)
- $C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \hspace{1cm} \forall x, y \in X$ such that $x \not P y$ \hspace{1cm} (18)
- $C_\mu(u(x)) - C_\mu(u(y)) = 0 \hspace{1cm} \forall x, y \in X$ such that $x \not I y$ \hspace{1cm} (19)
- $\varepsilon \geq 0$ \hspace{1cm} (20)
- \[
\sum_{\{i,j\} \subseteq N} m(\{i,j\}) + \sum_{i \in N} m(\{i\}) = 1
\] \hspace{1cm} (21)
- $m(\{i\}) \geq 0$ for all $i \in N$ \hspace{1cm} (22)
- $m(\{i\}) + \sum_{j \in A \setminus \{i\}} m(\{i,j\}) \geq 0 \hspace{1cm} \forall A \setminus \{i\}, \forall i \in N.$ \hspace{1cm} (23)

- If $PL_{N-N}^{ij}$ (respectively $PL_{N-P}^{ij}$) is not feasible, then there is a necessary negative (respectively positive) interaction between $i$ and $j$.

- If $PL_{N-N}^{ij}$ (respectively $PL_{N-P}^{ij}$) is feasible and the optimal solution $Z_3^* = 0$, then the constraint (18) is satisfied with $\varepsilon = 0$. Therefore, we can conclude that there is a necessary negative (respectively positive) interaction between $i$ and $j$.

- If $PL_{N-N}^{ij}$ (respectively $PL_{N-P}^{ij}$) is feasible and the optimal solution $Z_3^* > 0$, then there is no necessary negative (respectively positive) interaction between $i$ and $j$. 

(LAMSADE)
Example

The linear program $PL_{N-N}^{MS}$ (respectively $PL_{N-P}^{MS}$) corresponding to the test of the existence of a necessary negative (respectively positive) interaction between the Mathematics (M) and Statistics (S) is the following:

Maximize $Z_3 = \varepsilon$

Subject to

\[
\begin{aligned}
& m(\{M, S\}) \geq 0 \quad (\text{respectively } m(\{M, S\}) \leq 0) \\
& \varepsilon \geq 0 \\
& C_\mu(u(b)) - C_\mu(u(a)) \geq \varepsilon \\
& C_\mu(u(c)) - C_\mu(u(d)) \geq \varepsilon \\
& m(\{M, S\}) + m(\{M, L\}) + m(\{S, L\}) + m(\{M\}) + m(\{S\}) + m(\{L\}) = 1 \\
& m(\{M\}) \geq 0 \quad m(\{S\}) \geq 0 \quad m(\{L\}) \geq 0 \\
& m(\{M\}) + m(\{M, S\}) \geq 0 \\
& m(\{M\}) + m(\{M, L\}) \geq 0 \\
& m(\{M\}) + m(\{M, S\}) + m(\{M, L\}) \geq 0 \\
& m(\{S\}) + m(\{M, S\}) \geq 0 \\
& m(\{S\}) + m(\{S, L\}) \geq 0 \\
& m(\{S\}) + m(\{M, S\}) + m(\{S, L\}) \geq 0 \\
& m(\{L\}) + m(\{M, L\}) \geq 0 \\
& m(\{L\}) + m(\{S, L\}) \geq 0 \\
& m(\{L\}) + m(\{S, L\}) + m(\{M, L\}) \geq 0
\end{aligned}
\]
Example

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<tr>
<th></th>
<th>$Z_3 = \varepsilon$</th>
<th>$M$</th>
<th>$S$</th>
<th>$L$</th>
<th>${M, S}$</th>
<th>${M, L}$</th>
<th>${S, L}$</th>
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<tr>
<td>Optimal solution $Z_3^*$</td>
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<td>-</td>
<td>-</td>
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<td>-</td>
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<td>0.33</td>
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<tr>
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<td>-</td>
<td>0</td>
<td>0.67</td>
<td>-0.33</td>
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Table: Results of $PL_{N-N}^{MS}$ testing necessary negative interaction between Mathematics and Statistics
Example

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<th>( M )</th>
<th>( S )</th>
<th>( L )</th>
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<th>( {M, L} )</th>
<th>( {S, L} )</th>
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<tr>
<td>Mobius transform ( m )</td>
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<td>0.25</td>
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<tr>
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Table: Results of \( PL_{N-P}^{MS} \) testing necessary positive interaction between Mathematics and Statistics
Outline

1. Framework
2. The simple models
3. Introduction to MCDA
4. Multi Attribute Utility Theory
5. The additive model
6. A non-additive model: The Choquet integral
7. To conclude
To conclude

Analysis of three MCDA phenomena

- **PRESCRIPTIVE APPROACH**: To help a decision maker by the proposal of a solution obtained by a model.

- **DESCRIPTIVE APPROACH**: To describe a decision maker’s preferences by the chosen model.

- **ELICITATION**: The elicitation of the decision maker’s preferences consists in obtaining parameters of a decisional model which explain the past decisions in order to help in the future decisions.
Parameter’s elicitation

- **Option 1: Explicit elicitation**
  - Explain the model to the DM
  - Let the DM choose the parameters

- **Option 2: Implicit elicitation**
  - Present some (possibly fictitious) alternatives to the DM and ask him to compare them
  - Deduct the parameters of the model by solving an optimization program
Some references

