Preferences Aggregation: the MAUT approach

University Paris Dauphine
LAMSADE
FRANCE

Chapter 4

Outline

Outline

- A Decision Maker (DM) is facing a decision problem, i.e., the DM has to deal with multiple alternatives and has to compare them.
- Alternatives are described on several attributes.
- A criterion is an attribute with a preference relation (monotonic attribute).
- Criteria cannot be reduced to one criterion as they are potentially in conflict.

Example (Compare two bikes on three attributes)

	Speed	Robustness	Price
Mountain bike	20 km/h	Good	500 €
Race bike	35 km/h	Middle	1000 €

Example (Compare many objects)

Couches-culottes

- +++ Très bon 20 à 17
- ++ Bon 16,5 à 13
- + Acceptable 12,5 à 10
- − Insuffisant 9,5 à 7
- −− Très insuffisant 6,5 à 0

Les pourcentages entre parenthèses expriment le poids de chaque critère dans la notation finale.

	Joone Protection Premium	Pampers Premium Protection	Pampers Baby-Dry	Naty Eco by Naty	Pampers Premium Protection Active Fit[3]	Carrefour Baby Ultra dry & stretch	Lupilu Soft & Dry	Mots d'enfants (Marque Repère) Ultra confort	Love & Green Couches hypoallergéniques[3]	Lotus Baby Touch 3 Ultra confort	Pommette (Intermarché) Ecologic
• Label bio	Non	Non	Non	FSC/AB Vincotte UE	Non	Non	FSC	PEFC	FSC	FSC	FSC/Nordic ecolabel
• Prix indicatif	64,90 €[1] 162 couches	12,60 € 50 couches	15,60 € 50 couches	18,90 € 50 couches	12,30 € 46 couches	10,80 € 56 couches	7,30 € 56 couches	8,90 € 50 couches	19,65 € 52 couches	19 € 58 couches	9 € 32 couches
• Prix pour une couche	0,40 €	0,25 €	0,31 €	0,38 €	0,27 €	0,19 €	0,13 €	0,18 €	0,38 €	0,33 €	0,28 €
Performances (60 %)	+++	++	+	+	+	++	++	+	++	++	++
• Tenue	+++	++	++	++	++	++	++	++	++	++	++
• Absorption	+++	++	+	+	+	++	++	++	++	+++	++
• Protection contre l'humidité	+++	++	+	−	+	++	+	−	++	+	+
Composition (40 %)	+++	++	+++	+++	+	+	−	−	−	−−	−−
Pesticides											
• Résidu du glyphosate	+++	+++	+++	+++	+++	+++	+++	+++	−	−−	−−
• Pesticides organochlorés	+++	+++	+++	+++	+	+	+++	+++	+++	+++	+++
Autres molécules toxiques potentielles											
• Dioxines	+++	+++	+++	+++	+++	+++	+++	+++	+++	+++	+++
• Composés organiques volatils (COV)	+++	++	+++	+++	+++	++	++	−−	+++	+++	+++
• Composés organiques halogénés adsorbables (AOX)	+++	+++	+++	+++	+++	+++	−	+++	+++	+++	−
• Allergènes	+++	+++	+++	+++	+++	+++	+++	+++	+++	+++	+++
NOTE GLOBALE (100 %)	**17**/20	**14,5**/20	**12,5**/20[2]	**12,5**/20[2]	**12,5**/20	**12,5**/20	**12**/20	**12**/20	**9,5**/20[4]	**9,5**/20[4]	**9,5**/20[4]

(1) Livraison comprise dans le prix. (2) L'appréciation globale ne peut pas être supérieure à l'appréciation sur les performances. (3) Le fabricant indique que cette référence est en fin de commercialisation. (4) L'appré ciation globale ne peut pas être supérieure à "Insuffisante" ou "Très insuffisante" en fonction de la composition du produit.

Multi-attribute formal model: Inputs

- A set of alternatives $X = X_1 \times X_2 \times \cdots \times X_n$
- There exists preferences on the values of each criterion i (utility function, qualitative preference relation $\succsim_i, \ldots$)
- A representation of the importance of each criterion or set of criteria (weights, importance relation, ...)

Multi-attribute formal model: a treatment

- Using the input information, elaborate a decision rule allowing to compare two different alternatives, i.e.,

$$\left.\begin{array}{l} x = (x_1, \ldots, x_n) \\ y = (y_1, \ldots, y_n) \end{array}\right\} \Longrightarrow x \succsim y \text{ or } y \succsim x$$

Outline

PARETO DOMINANCE

An alternative is preferred to another one if it is considered to be better on all the criteria.

$$x \succsim y \Longleftrightarrow [\forall i \in N, x_i \succsim_i y_i]$$

Example

	Speed	Robustness	Price
Bike A	10 km/h	Good	600 €
Bike B	20 km/h	Good	550 €
Bike C	19m/h	Very Good	800 €

$$B \succsim A \text{ and } not(A \succsim B) \Longrightarrow B \succ A$$

$$not(B \succsim C) \text{ and } not(C \succsim B)$$

Pareto dominance **is not so interesting**

Dominance

- An alternative $x = (x_1, \ldots, x_n)$ dominates an alternative $y = (y_1, \ldots, y_n)$ if $\forall i \in N, x_i \succsim_i y_i$.
- An alternative $x = (x_1, \ldots, x_n)$ strictly dominates an alternative $y = (y_1, \ldots, y_n)$ if $\forall i \in N, x_i \succsim_i y_i$ and $\exists i_0 \in N, x_{i_0} \succ_{i_0} y_{i_0}$.

Definition

The Pareto front is the set of all non-dominated alternatives.

Remark

- The optimal solution is necessary in the Pareto front
- In general, the Pareto front may be poor, i.e., it is not really different to the whole set of alternatives.

MULTI-OBJECTIVE OPTIMIZATION

- Principle: The “Best” alternative should be the nearest alternative to an “ideal point”.
- Usually, the “ideal point” if computed by taking the max (resp min) value on each criterion.
- Many distances are also used in the resolution of a multi-objective problem.

Weighted sum

Let be $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ two alternatives such that $x_i, y_i \in \mathbb{R}$, $\forall i \in N$. Let be w_i the weight associated to the criterion i.

$$x \succsim y \iff \sum_{i=1}^{n} w_i \ x_i \geq \sum_{i=1}^{n} w_i \ y_i$$

Example

	Speed	Robustness	Price
Bike A	8/20	18/20	12/20
Bike B	18/20	8/20	12/20
Bike C	12/20	12/20	12/20

$$w_S > w_R \Longrightarrow B \succsim A$$

$$w_R > w_S \Longrightarrow A \succsim B$$

$$\forall w_R, w_S, \text{ we have } A \succsim C \text{ or } B \succsim C$$

The majority rule

Let be $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ two alternatives. x is preferred to y if it is considered “good” on a majority of criteria.

$$x \succsim y \Longleftrightarrow \left|\{i \in N : x_i \succsim_i y_i\}\right| \geq \left|\{i \in N : y_i \succsim_i x_i\}\right|$$

Example

	Speed	Robustness	Price
Bike A	10 km/h	Good	600 €
Bike B	20 km/h	Good	550 €
Bike C	19m/h	Very Good	800 €

$$B \succsim C$$

Example (Majority rule)

	Speed	Robustness	Price
Bike A	20 km/h	Very Good	600 €
Bike B	15 km/h	Good	500 €
Bike C	25m/h	Bad	550 €

Which bike do you choose?

Example (Majority rule)

	Speed	Robustness	Price
Bike A	20 km/h	Very Good	600 €
Bike B	15 km/h	Good	500 €
Bike C	25m/h	Bad	550 €

$$A \succsim B$$

$$B \succsim C$$

$$C \succsim A$$

$\Longrightarrow$ Condorcet Paradox

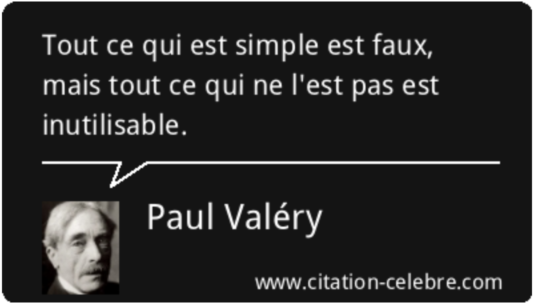
Outline

MultiCriteria Decision Aiding (MCDA): Difficulties

- MultiCriteria Decision Aiding is not so easy: it is not an easy task
- Every method has advantages and inconveniences : there is no "best method"
- All methods have structural bias.

Paul Valery (Artist, Writer, Poet, Philosopher (1871-1945))

- Tout ce qui est simple est faux, mais tout ce qui ne l'est pas est inutilisable
- What is simple is false. What is complex is useless.



Three types of problems in MCDA

- Choice Problem: choose the “best” alternative(s).
- Ranking Problem: rank the alternatives from the “best” to the “worst”.
- Sorting Problem: sort the alternatives into pre-defined categories (in general ordered categories)

Two main approaches in MCDA

- Multi Attribute Utility Theory: A quantitative approach "aggregate then compare" (scoring)

$$x \succsim y \iff U(x_1, \ldots, x_n) \geq U(y_1, \ldots, y_n)$$

- Outranking: qualitative approach "compare then aggregate"

$$x \succsim y \iff \left|\{i \in N : x_i \succsim_i y_i\}\right| \rhd \left|\{i \in N : y_i \succsim_i x_i\}\right|$$

Outline

Principle

Le be X a set of alternatives evaluated on a finite set of n criteria $N = \{1, \ldots, n\}$. In general, we set $X = X_1 \times X_2 \times \ldots \times X_n$.

Le be $\succsim_X$ a complete preorder on X (preferences of a DM).

- $\succsim_X$ are supposed to be representable by an overall utility function:

$$\forall x, y \in X, \quad x \succsim_X y \Leftrightarrow F(U(x)) \geq F(U(y))$$

  where

  - $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$
  - $U(x) = (u_1(x_1), \ldots, u_n(x_n))$
  - $u_i : X_i \to \mathbb{R}$ is a marginal utility function or simply called utility function or a scale on $\mathbb{R}$
  - $F : \mathbb{R}^n \to \mathbb{R}$ an aggregation function

- F is generally characterized by a parameter vector θ (weight vector,...).

Problems

1. How to choose the aggregation function F?
2. How to construct the marginal utility functions $u_i : X_i \to \mathbb{R}$?
3. The marginal utility functions $u_i : X_i \to \mathbb{R}$ should have a signification for the decision maker (see measurement theory):
    - Ordinal scales: Differences between values have no importance (e.g. a rank). They can represent orders and pre-orders.
    - Cardinal scales: Differences between values may be meaningful.
        - *Interval scales* : absolute differences between values are important.

Outline

The additive model

- $\succsim_X$ are supposed to be representable by an overall utility function:

$$\forall x \in X, \quad F(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} u_i(x_i)$$

- This model is equivalent to the existence of weights w_i, $i = 1, \ldots, n$, such that

$$\forall x \in X, \quad F(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} w_i \; u_i(x_i)$$

The additive model

- A simple method
- Additive value function involves compensation between criteria, i.e., a bad performance on a criterion i could be compensated by a good performance on another criterion.

  See e.g. students evaluation based on the weighted sum.
- In the weighted sum, weights represent, in reality, the substitution rate between criteria.

The additive model

- In the weighted sum, weights represent, in reality, the substitution rate between criteria.

  - E.g. for $n = 2$, $w_1 = b\ w_2$ means the DM is indifferent between these two alternatives $(0, b)$ and $(1, 0)$, i.e., $(0, b) \sim (1, 0)$.
  - There is a total compensation between "bad" performances and "good" performances.

    If we have $(a, b) \sim (a - \delta, b + \gamma)$ then gain of γ compensates the loss of δ.

    Indeed we have

    $$a\ w_1 + b\ w_2 = (a - \delta)\ w_1 + (b - \gamma)\ w_2$$
    $$\iff \delta\ w_1 = \gamma\ w_2$$
    $$\iff \frac{w_1}{w_2} = \frac{\gamma}{\delta}$$

    Implicitly, this implies that all the criteria could be express indirectly in the same unit (€, seconds, ...).

The additive model

- Requires to normalize the criteria. In general, we set $\forall i \in N, u_i : X_i \mapsto [0,1]$.

  E.g. For a criterion to be maximized, we could choose the following normalization functions:

  - $u_i(x_i) = \dfrac{x_i}{\max x_i}$
  - $u_i(x_i) = \dfrac{x_i - \min x_i}{\max x_i - \min x_i}$
  - ...

The mutual Preferential independence

- The additive model requires to satisfy the mutual Preferential independence axiom, i.e., criteria are independent in the sense of preferences

  $\forall i \in N, \forall z_i, t_i \in X_i, \forall x, y \in X,$

  $$(z_i, x_{N-i}) \succsim (z_i, y_{N-i}) \Leftrightarrow (t_i, x_{N-i}) \succsim (t_i, y_{N-i})$$

  An attribute is preferentially independent from all other attributes when changes in the rank ordering of preferences of other attributes does not change the preference order of the attribute.

MAUT in practice

1. People suppose $\succsim_X$ representable by an overall utility function:

$$x \succsim_X y \Leftrightarrow F(U(x)) \geq F(U(y))$$

2. F is generally characterized by a parameter vector θ (weight vector,...).
3. People ask to the DM some preferential information $\succsim_{X'}$ on a reference subset (learning set) $X' \subseteq X$
4. The parameter vector is constructed so that $\succsim_X$ is an extension of $\succsim_{X'}$.
5. The model obtained in X' will be then automatically extended to X.

Example (Evaluation of diapers by the Magazine "60 millions consommateurs" in September 2018)

À LA UNE HYGIÈNE BÉBÉ

Les résultats de notre essai

Couches-culottes

- +++ Très bon 20 à 17
- ++ Bon 16,5 à 13
- \+ Acceptable 12,5 à 10
- − Insuffisant 9,5 à 7
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Les pourcentages entre parenthèses expriment le poids de chaque critère dans la notation finale.

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• Label bio	Non	Non	Non	FSC/AB Vincotte UE	Non
• Prix indicatif	64,90 €[1] 162 couches	12,60 € 50 couches	15,60 € 50 couches	18,90 € 50 couches	12,30 € 46 couches
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Performances (60 %)	+++	++	+	+	+
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• Protection contre l'humidité	+++	++	+	−	+
Composition (40 %)	+++	++	+++	+++	+
Pesticides					
• Résidu du glyphosate	+++	+++	+++	+++	+++
• Pesticides organochlorés	+++	+++	+++	+++	+
Autres molécules toxiques potentielles					
• Dioxines	+++	+++	+++	+++	+++
• Composés organiques volatils (COV)	+++	++	+++	+++	+++
• Composés organiques halogénés adsorbables (AOX)	+++	+++	+++	+++	+++
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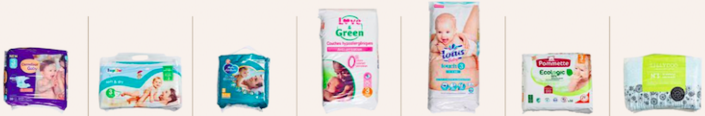
NOTE GLOBALE (100 %)	**17**/20	**14,5**/20	**12,5**/20[2]	**12,5**/20[2]	**12,5**/20

(1) Livraison comprise dans le prix. (2) L'appréciation globale ne peut pas être supérieure à l'appréciation sur les performances. (3) Le fabricant indique que cette référence est en fin de commercialisation. (4) L'appré

Example (Evaluation of diapers by the Magazine "60 millions consommateurs" in September 2018)



Carrefour Baby Ultra dry & stretch	**Lupilu** Soft & Dry	**Mots d'enfants** (Marque Repère) Ultra confort	**Love & Green** Couches hypoallergéniques[3]	**Lotus Baby** Touch 3 Ultra confort	**Pommette** (Intermarché) Ecologic	**Lillydoo** Couches bébé
Non	FSC	PEFC	FSC	FSC	FSC/Nordic ecolabel	Non
10,80 € 56 couches	7,30 € 56 couches	8,90 € 50 couches	19,65 € 52 couches	19 € 58 couches	9 € 32 couches	12 € 33 couches
0,19 €	0,13 €	0,18 €	0,38 €	0,33 €	0,28 €	0,36 €
++	++	+	++	++	++	+
++	++	++	++	++	++	+
++	++	++	++	+++	++	++
++	+	-	++	+	+	+
+	-	-	-	--	--	--
+++	+++	+++	-	--	--	--
+	+++	+++	+++	+++	+++	--
+++	+++	+++	+++	+++	+++	+++
++	++	--	+++	+++	+++	++
+++	-	+++	+++	+++	-	+++
+++	+++	+++	+++	+++	+++	+++
12,5/20	**12/20**	**12/20**	**9,5/20**[4]	**9,5/20**[4]	**9,5/20**[4]	**6,5/20**[4]

ciation globale ne peut pas être supérieure à "Insuffisante" ou "Très insuffisante" en fonction de la composition du produit.

Example (Evaluation of diapers by the Magazine "60 millions consommateurs" in September 2018)

	Performance	Composition	Global score (/20)
A- Joone	$+++$	$+++$	17
B- Pamp. Prem	$++$	$++$	14.5
C- Pamp. Baby	$+$	$+++$	12.5
D- Naty	$+$	$+++$	12.5
E- Pamp. Activ.	$+$	$+$	12.5
F- Carref. Baby	$++$	$+$	12.5
G- Lupilu	$++$	$-$	12
H- Mots d'enfants	$+$	$-$	12
I- Love & Green	$++$	$-$	9.5
K- Lotus Baby	$++$	$--$	9.5
L- Pommette	$++$	$--$	9.5
M- Lillydoo	$+$	$--$	6.5
	$w_p = 60\%$	$w_c = 40\%$	

$$+++ \equiv \text{Very good} \in [17, 20]; \; ++ \equiv \text{Good} \in [13, 16.5];$$
$$+ \equiv \text{Acceptable} \in [10, 12.5];$$
$$- \equiv \text{Insufficient} \in [7, 9.5]; \quad -- \equiv \text{Very Insufficient} \in [0, 6.5]$$

Example (Evaluation of diapers by the Magazine "60 millions consommateurs" in September 2018)

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H- Mots d'enfants	$+$	$-$	12
I- Love & Green	$++$	$-$	9.5
K- Lotus Baby	$++$	$--$	9.5
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M- Lillydoo	$+$	$--$	6.5
	$w_p = 60\%$	$w_c = 40\%$	

- Which evaluation model was used by this magazine?
- Are these preferences representable by an additive model? (by using the given utility functions and weight)

The UTA Approach

Principles

- Created by Jacquet Lagreze & Siskos in 1982 (at LAMSADE)
- The UTA (UTilités Additives) method aims at inferring one or more additive value functions from a given ranking on a reference set A_R.
- The method uses special linear programming techniques to assess these functions so that the ranking(s) obtained through these functions on A_R is (are) as consistent as possible with the given one.

The UTA Approach

UTA Principles: Input data

- A set of Criteria N
- A set of alternatives X evaluated on N
- A preorder $\succsim_{X'}$ on $X' \subseteq X$ (not necessary complete)
- For each element $x = (x_1, \ldots, x_n) \in X$, it is assumed that

$$U(x) = \sum_{i=1}^{n} u_i(x_i) \tag{1}$$

where $u_i : X_i \rightarrow \mathbb{R}$, $i = 1, \ldots, n$ are marginal utility functions

The UTA Approach

UTA Principles: Input data

- For each criterion i, $X_i = [\alpha_i, \beta_i]$ is the criterion evaluation scale such that $\alpha_i \leq \beta_i$
- The following normalization constraints, associated to the marginal utility functions, are considered:

$$\begin{cases} u_i(\alpha_i) = 0, \ \forall i = 1, \ldots, n \\ \sum_{i=1}^{n} u_i(\beta_i) = 1 \end{cases} \tag{2}$$

The UTA Approach

UTA Principles: Input data

- Each marginal value function u_i is assumed to be piecewise linear, so that the interval $[\alpha_i, \beta_i]$ is divided into $\gamma_i \geq 1$ equal sub-intervals

$$[\alpha_i = x_i^0, x_i^1], [x_i^1, x_i^2], \ldots, [x_i^{\gamma_i - 2}, x_i^{\gamma_i - 1}], [x_i^{\gamma_i - 1}, x_i^{\gamma_i} = \beta_i]$$

  where

$$x_i^j = \alpha_i + \frac{j(\beta_i - \alpha_i)}{\gamma_i}, \; j = 0, \ldots, \gamma_i$$

- Hence, using linear interpolation, the utility function associated to an element $x_i \in [x_i^j, x_i^{j+1}]$ is given by

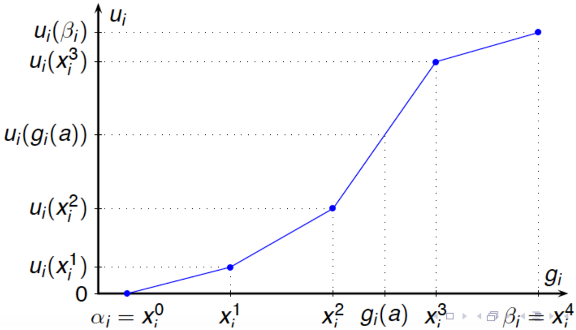
$$u_i(x_i) = u_i(x_i^j) + \frac{x_i - x_i^j}{x_i^{j+1} - x_i^j}(u_i(x_i^{j+1}) - u_i(x_i^j)). \quad (3)$$

The UTA Approach

UTA Principles: Input data

- The piecewise linear additive model is completely defined by the marginal values at the break points, i.e.

$$u_i(x_i^0) = u_i(\alpha_i),\ u_i(x_i^1), u_i(x_i^2), \ldots, u_i(x_i^{\gamma_i}) = u_i(\beta_i).$$



The UTA Approach

UTA Principles: The model

- $U(x) = \sum_{i=1}^{n} u_i(x_i)$
- For each element $x \in X'$, set

$$V(x) = U(x) + \sigma(x)$$

where $\sigma(x)$ is a nonnegative real value estimating the error of the estimation of the value $U(x)$, i.e., $\sigma(x) = V(x) - U(x)$.

The value $\sigma(x)$ will be minimized by the linear program.

UTA Principles: The linear program to solve

$$\begin{cases} \min \sum_{x \in X'} \sigma(x) \\ \\ V(x) \geq V(y) + \delta \text{ if } x \succ y \\ V(x) = V(y) \text{ if } x \sim y \\ \\ u_i(x_i^{j+1}) - u_i(x_i^j) \geq 0, \; i = 1, \ldots, n, \; j = 1, \ldots, \gamma_i - 1 \\ u_i(\alpha_i) = 0, \; \forall i = 1, \ldots, n \\ \sum_{i=1}^{n} u_i(\beta_i) = 1 \\ \sigma(x) \geq 0, \forall x \in X' \end{cases}$$

- If the optimal solution is equal to 0 then $\succsim_{X'}$ is representable by (compatible with) an additive model.
- There are many versions of the UTA method

Example (The choice of a camera)

You want to buy a camera and you have obtained the following information about six cameras evaluated on three criteria.

1. The resolution (in Millions of Pixels)
2. The price (in euros)
3. The optical zoom (a real number).

Cameras	1 : Resolution	2 : Price	3 : Zoom
a : Nikon	6	150	5
b : Sony	7	180	5
c : Panasonic	10	155	4
d : Casio	12	175	5
e : Olympus	10	160	3
f : Kodak	8	165	4

We have $X = \{a, b, c, d, e, f\}$, $N = \{1, 2, 3\}$, $X_1 = [6, 12]$, $X_2 = [150, 180]$ and $X_3 = [3, 5]$.

Your preferences on a reference subset are: $a \succ e \succ b$.

Example (The choice of a camera)

Cameras	1 : Resolution	2 : Price	3 : Zoom
a : Nikon	6	150	5
b : Sony	7	180	5
c : Panasonic	10	155	4
d : Casio	12	175	5
e : Olympus	10	160	3
f : Kodak	8	165	4

We have $X = \{a, b, c, d, e, f\}$, $N = \{1, 2, 3\}$, $X_1 = [6, 12]$, $X_2 = [150, 180]$ and $X_3 = [3, 5]$. We suppose $\gamma_1 = 2$, $\gamma_2 = 3$ and $\gamma_3 = 1$.

Hence

- For $X_1 = [6, 12]$: $x_1^0 = 6$, $x_1^1 = 9$, $x_1^2 = 12$ and $u_1(6) = 0$;
- For $X_2 = [180, 150]$: $x_2^0 = 150$, $x_2^1 = 160$, $x_2^2 = 170$, $x_2^3 = 180$ and $u_2(180) = 0$;
- For $X_3 = [3, 5]$: $x_3^0 = 3$, $x_3^1 = 5$ and $u_3(3) = 0$.

Example (The choice of a camera)

- For the criterion 1:

$$u_1(7) = u_1(6) + \tfrac{1}{3}(u_1(9) - u_1(6)) = \tfrac{1}{3}u_1(9)$$
$$u_1(8) = u_1(6) + \tfrac{2}{3}(u_1(9) - u_1(6)) = \tfrac{2}{3}u_1(9)$$
$$u_1(10) = u_1(9) + \tfrac{1}{3}(u_1(12) - u_1(9)) = \tfrac{2}{3}u_1(9) + \tfrac{1}{3}u_1(12)$$

- For the criterion 2:

$$u_2(175) = \tfrac{1}{2}u_2(170)$$
$$u_2(165) = u_2(170) + \tfrac{1}{2}(u_2(160) - u_2(170)) = \tfrac{1}{2}u_2(170) + \tfrac{1}{2}u_2(160)$$
$$u_2(155) = u_2(160) + \tfrac{1}{2}(u_2(150) - u_2(160)) = \tfrac{1}{2}u_2(160) + \tfrac{1}{2}u_2(150)$$

- For the criterion 3:

$$u_3(4) = \tfrac{1}{2}u_3(5)$$

Example (The choice of a camera)

Your preferences: $a \succ e \succ b$

$$
\begin{aligned}
\min \quad & \sigma(a) + \sigma(b) + \sigma(e) \\
\text{s.t.} \quad & \begin{cases}
u_2(150) + u_3(5) + \sigma(a) - \frac{2}{3}u_1(9) - \frac{1}{3}u_1(12) - u_2(160) - \sigma(e) \geq \delta \\
\frac{2}{3}u_1(9) + \frac{1}{3}u_1(12) + u_2(160) + \sigma(e) - \frac{1}{3}u_1(9) - u_3(5) - \sigma(b) \geq \delta \\
u_1(9) - u_1(6) \geq 0 \\
u_1(12) - u_1(9) \geq 0 \\
u_2(150) - u_2(160) \geq 0 \\
u_2(160) - u_2(170) \geq 0 \\
u_2(170) - u_2(180) \geq 0 \\
u_3(5) - u_3(3) \geq 0 \\
u_1(6) = 0 \\
u_2(180) = 0 \\
u_3(3) = 0 \\
u_1(12) + u_2(150) + u_3(5) = 1 \\
\sigma(a) \geq 0 \\
\sigma(b) \geq 0 \\
\sigma(c) \geq 0 \\
\delta = 0.001
\end{cases}
\end{aligned}
$$

Example (The choice of a camera)

Your preferences: $a \succ e \succ b$.

A solution:

$$\begin{cases} u_1(6) = 0 \\ u_1(9) = 0.1 \\ u_1(12) = 0.4 \\ u_2(180) = 0 \\ u_2(170) = 0.2 \\ u_2(160) = 0.3 \\ u_2(150) = 0.4 \\ u_3(3) = 0 \\ u_3(5) = 0.2 \end{cases}$$

Then

$$\begin{cases} u(a) = 0.6 \\ u(b) = 0.233 \\ u(c) = 0.65 \\ u(d) = 0.7 \\ u(e) = 0.5 \\ u(f) = 0.416 \end{cases}$$

The UTA^{GMS} Approach

$\mathrm{UTA}^{\mathrm{GMS}}$ Principles

- Generalizes the UTA approach
- It takes into account all additive value functions compatible with indirect preference information, while UTA is using only one such function.
- The marginal value functions are general monotone non-decreasing functions, and not piecewise linear only.

The UTA^{GMS} Approach

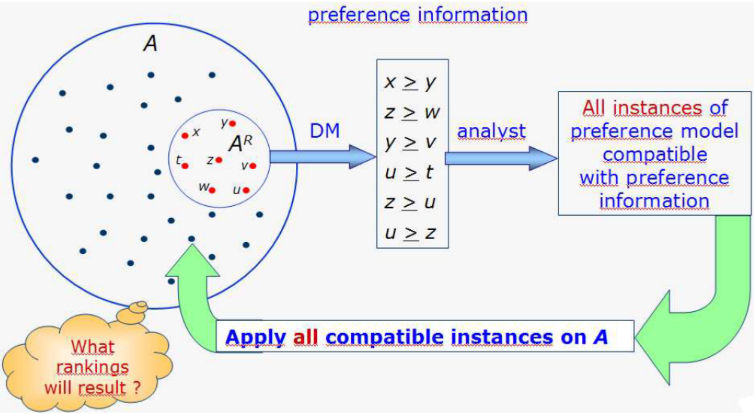
$\mathrm{UTA}^{\mathrm{GMS}}$ Principles

The method produces two rankings in the set of alternatives A, such that for any pair of alternatives $a, b \in X$

- In the necessary order, a is ranked at least as good as b if and only if, $U(a) \geq U(b)$ for all value functions compatible with the preference information.
- In the possible order, a is ranked at least as good as b if and only if, $U(a) \geq U(b)$ for at least one value function compatible with the preference information.

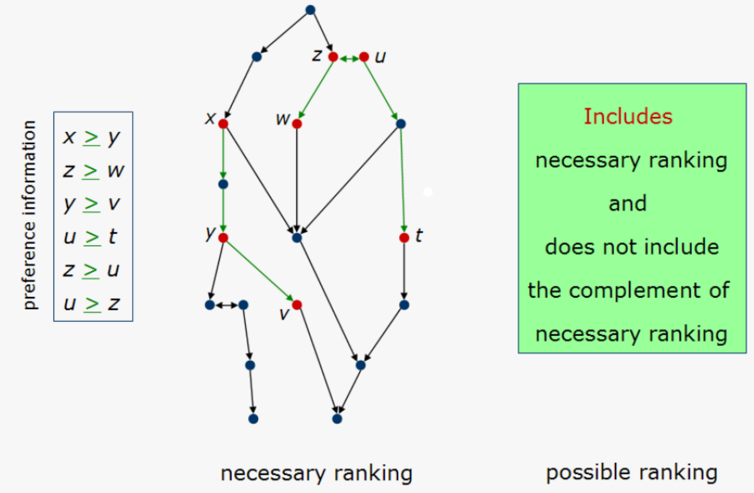
The UTA^{GMS} *Approach*

UTA Principles: Input data



The UTA^{GMS} Approach

UTA Principles: Input data



Outline

Example (A classic example of Grabisch et al. (2010))

	1 : Mathematics (M)	2 : Statistics (S)	3 : Language (L)
a	16	13	7
b	16	11	9
c	6	13	7
d	6	11	9

- For a student good in Mathematics, Language is more important than Statistics
$$\Longrightarrow a \prec b,$$
- For a student bad in Mathematics, Statistics is more important than Language
$$\Longrightarrow d \prec c.$$

Example (A classic example of Grabisch et al. (2010))

	1 : Mathematics (M)	2 : Statistics (S)	3 : Language (L)
a	16	13	7
b	16	11	9
c	6	13	7
d	6	11	9

The two preferences $a \prec b$ and $d \prec c$ lead to a contradiction with the additive model

$$\begin{cases} a \prec b \Rightarrow u_M(16)\ w_M + u_S(13)\ w_S + u_L(7)\ w_L < u_M(16)\ w_M + u_S(11)\ w_S + u_L(9)\ w_L \\ d \prec c \Rightarrow u_M(6)\ w_M + u_S(11)\ w_S + u_L(9)\ w_L < u_M(6)\ w_M + u_S(13)\ w_S + u_L(7)\ w_L. \end{cases}$$

$$i.e., \begin{cases} u_S(13)\ w_S + u_L(7)\ w_L < u_S(11)\ w_S + u_L(9)\ w_L \\ \text{and} \\ u_S(11)\ w_S + u_L(9)\ w_L < u_S(13)\ w_S + u_L(7)\ w_L \end{cases}$$

Example (A classic example of Grabisch et al. (2010))

	1 : Mathematics (M)	2 : Statistics (S)	3 : Language (L)
a	16	13	7
b	16	11	9
c	6	13	7
d	6	11	9

- The preference information $a \prec b$ and $d \prec c$ is not representable by an additive utility model.

Example (A ranking of French hospitals for weight loss surgery)

To identify the "best" hospitals in weight loss surgery, the magazine "Le Point" combines a part of the following four indicators (criteria):

- *Criterion 1 - Activity*: number of procedures performed during one year. This criterion has to be maximized.
- *Criterion 2 - Notoriety*: Its corresponds to the reputation and attractiveness of the hospital. It is a percentage of patients treated in the hospital but living in another French administrative department. More the percentage increases, more the hospital is attractive.
- *Criterion 3 - Average Length Of Stay (ALOS)*: a mean calculated by dividing the sum of inpatient days by the number of patients admissions with the same diagnosis-related group classification. If a hospital is more organized in terms of resources then its ALOS score should be low.
- *Criterion 4 - Technicality*: this particular indicator measures the ratio of procedures performed with an efficient technology compared to the same procedures performed with obsolete technology. The higher the percentage is, the more the team is trained in advanced technologies or complex surgeries.

Example (A ranking of French hospitals for weight loss surgery)

	1- Activity	2- Notoriety	3- ALOS	4- Technicality
Hospital 1 ($H1$)	200	65	3.5	85
Hospital 2 ($H2$)	450	60	4	75
Hospital 3 ($H3$)	450	50	2.5	55
Hospital 4 ($H4$)	350	50	3.5	85
Hospital 5 ($H5$)	350	55	2	75
Hospital 6 ($H6$)	150	65	2.5	80
Hospital 7 ($H7$)	200	55	2	55
Hospital 8 ($H8$)	150	60	4	80

$$H1 \succ H2; \quad H3 \succ H4; \quad H5 \succ H6; \quad H8 \succ H7.$$

Are these preferences representable by an additive function?

Example (A ranking of French hospitals for weight loss surgery)

	1- Activity	2- Notoriety	3- ALOS	4- Technicality
Hospital 1 ($H1$)	200	65	3.5	85
Hospital 2 ($H2$)	450	60	4	75
Hospital 3 ($H3$)	450	50	2.5	55
Hospital 4 ($H4$)	350	50	3.5	85
Hospital 5 ($H5$)	350	55	2	75
Hospital 6 ($H6$)	150	65	2.5	80
Hospital 7 ($H7$)	200	55	2	55
Hospital 8 ($H8$)	150	60	4	80

$$\begin{cases} H1 \succ H2 \Rightarrow u_1(200)w_1 + u_2(65)w_2 + u_3(3.5)w_3 + u_4(85)w_4 > u_1(450)w_1 + u_2(60)w_2 + u_3(4)w_3 + u_4(75)w_4 \\ H3 \succ H4 \Rightarrow u_1(450)w_1 + u_2(50)w_2 + u_3(2.5)w_3 + u_4(55)w_4 > u_1(350)w_1 + u_2(50)w_2 + u_3(3.5)w_3 + u_4(85)w_4 \\ H5 \succ H6 \Rightarrow u_1(350)w_1 + u_2(55)w_2 + u_3(2)w_3 + u_4(75)w_4 > u_1(150)w_1 + u_2(65)w_2 + u_3(2.5)w_3 + u_4(80)w_4 \\ H7 \prec H8 \Rightarrow u_1(200)w_1 + u_2(55)w_2 + u_3(2)w_3 + u_4(55)w_4 < u_1(150)w_1 + u_2(60)w_2 + u_3(4)w_3 + u_4(80)w_4 \end{cases}$$

The first three equations in this system lead to

$$u_1(200)w_1 + u_2(55)w_2 + u_3(2)w_3 + u_4(55)w_4 > u_1(150)w_1 + u_2(60)w_2 + u_3(4)w_3 + u_4(80)w_4$$

which contradicts the last equation.

A Limit of the additive utility model

We try another MCDA model: the Choquet integral.

The 2-additive capacity

Definition

A 2-additive capacity is a set function $\mu : 2^N \to [0,1]$ such that:

$$\sum_{\{i,j\} \subseteq N} \mu(\{i,j\}) - (n-2) \sum_{i \in N} \mu(\{i\}) = 1 \text{ (normality)}$$

$$\mu(\{i\}) \geq 0, \ \forall i \in N \text{ (nonnegativity)}$$

$\forall A \subseteq N, \ |A| \geq 2, \ \forall i \in A$

$$\sum_{j \in A \setminus \{i\}} (\mu(\{i,j\}) - \mu(\{j\})) \geq (|A|-2)\mu(\{i\}) \text{ (monotonicity).}$$

The 2-additive capacity

Notations

$$\forall i, j \in N, i \neq j,$$

$$\mu_\emptyset = \mu(\emptyset),\ \mu_i = \mu(\{i\}),\ \mu_{ij} = \mu(\{i,j\})$$

$$m^\mu(\{i,j\}) = \mu_{ij} - \mu_i - \mu_j$$

$$m^\mu(\{i\}) = \mu_i$$

m^μ is called the Möbius transform of μ.

The 2-additive capacity

Example

- $N = \{1, 2, 3\}$
- Normality constraint: $\mu_{12} + \mu_{13} + \mu_{23} - \mu_1 - \mu_2 - \mu_3 = 1$
- Nonnegativity constraints: $\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0$
- Monotonicity constraints:
  - $\mu_{12} \geq \mu_1$, $\mu_{12} \geq \mu_2$
  - $\mu_{13} \geq \mu_1$, $\mu_{13} \geq \mu_3$
  - $\mu_{23} \geq \mu_2$, $\mu_{23} \geq \mu_3$
  - $\mu_{12} + \mu_{13} \geq \mu_1 + \mu_2 + \mu_3$
  - $\mu_{12} + \mu_{23} \geq \mu_1 + \mu_2 + \mu_3$
  - $\mu_{13} + \mu_{23} \geq \mu_1 + \mu_2 + \mu_3$

The 2-additive Choquet integral

Definition

For any $x := (x_1, ..., x_n) \in X$, the expression of the 2-additive Choquet integral is:

$$C_\mu(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} \phi_i^\mu \; u_i(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} I_{ij}^\mu \; |u_i(x_i) - u_j(x_j)|$$

Where

- I_{ij}^μ = the interaction index between criteria i and j:

$$I_{ij}^\mu = \mu_{ij} - \mu_i - \mu_j.$$

- ϕ_i^μ = the importance of the criterion i ($\equiv$ Shapley index):

$$\phi_i^\mu = \mu_i + \frac{1}{2} \sum_{k \in N \setminus i} I_{ik}^\mu.$$

Interest of the 2-additive model

$$C_\mu(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} \phi_i^\mu \; u_i(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} I_{ij}^\mu \; |u_i(x_i) - u_j(x_j)|$$

- It is a generalization of the arithmetic mean ($I_{ij} = 0 \; \forall i, j \in N$)
- It is a compromise between the general Choquet integral and the arithmetic mean, i.e., offers a good compromise between flexibility of the model and complexity;
- It was used in many applications such that
  - the evaluation of discomfort in sitting position (see Grabisch et al. (2002));
  - the construction of performance measurement systems model in a supply chain context (see Berrah and Clivillé (2007), Clivillé et al. (2007));
  - complex system design (Labreuche and Pignon (2007));

Remark

The Choquet integral requires to be able to compare any element of one point of view with any element of any other point of view (commensurateness between criteria);

Interaction index $I_{ij}^{\mu} = \mu_{ij} - \mu_i - \mu_j$

Usual Interpretation

- $I_{ij}^{\mu} > 0 \Longrightarrow$ criteria i and j are *complementarity*.
- $I_{ij}^{\mu} < 0 \Longrightarrow$ criteria i and j are *redundant*.
- $I_{ij}^{\mu} = 0 \Longrightarrow$ criteria i and j are *independent* (no interaction).

The sign of the interaction index is not always stable

Example (A classic example of Grabisch et al. (2010))

	1 : Mathematics (M)	2 : Statistics (S)	3 : Language (L)
a	16	13	7
b	16	11	9
c	6	13	7
d	6	11	9

- $a \prec b$ and $d \prec c$
- $u_M(16) = 16$, $u_M(6) = 6$,
- $u_S(13) = 13$, $u_S(11) = 11$
- $u_L(7) = 7$, $u_L(9) = 9$

The sign of the interaction index is not always stable

Example (A classic example of Grabisch et al. (2010))

	Par. 1	Par. 2	Par.3	Par. 4	Par. 5	Par. 6	Par. 7	Par.8	Par. 9
$C_\mu(a)$	8.5	13.75	9.1	13.765	13.75	13.75	11.47	12.535	10.45
$C_\mu(b)$	9.5	14.25	9.7	13.995	14.25	14.25	11.93	12.785	10.75
$C_\mu(c)$	7.75	9.75	7.75	11.325	11.25	9.75	9.45	9.515	7.85
$C_\mu(d)$	7.25	9.25	7.25	10.295	9.75	9.25	8.91	9.265	7.55
μ_M	0	0.75	0	0.685	0.75	0.75	0.36	0.485	0.15
μ_S	0.25	0.5	0.25	0.73	0.75	0.5	0.465	0.455	0.25
μ_L	0	0.25	0	0.315	0	0	0.205	0.32	0
μ_{MS}	0.25	0.75	0.35	0.785	0.75	0.75	0.565	0.68	0.5
μ_{ML}	0.75	1	0.65	1	0.1	0.75	0.805	0.795	0.55
μ_{SL}	0.25	0.75	0.25	0.945	0.75	0.75	0.66	0.785	0.35
V_M^μ	0.375	0.5	0.375	0.37	0.5	0.5	0.35	0.35	0.4
V_S^μ	0.25	0.25	0.3	0.365	0.375	0.375	0.33	0.33	0.35
V_L^μ	0.375	0.25	0.325	0.265	0.125	0.125	0.32	0.32	0.25
I_{MS}^μ	0	-0.5	0.1	-0.63	-0.75	-0.5	-0.26	-0.26	0.1
I_{ML}^μ	0.75	0	0.65	0	0.25	0	0.24	-0.01	0.4
I_{SL}^μ	0	0	0	-0.1	0	0.25	-0.01	0.01	0.1

The sign of the interaction index is not always stable

How to conclude from this preference information given by the DM?

- Mathematics & Statistics are independent? complementary? redundant?
- Mathematics & Literature are independent? complementary? redundant?
- Statistics & Literature are independent? complementary? redundant?

- **"The three subjects, taken together, are not without interaction"**

The sign of the interaction index is not always stable

Example (A ranking of French hospitals for weight loss surgery)

	1- Activity	2- Notoriety	3- ALOS	4- Technicality
Hospital 1 ($H1$)	200	65	3.5	85
Hospital 2 ($H2$)	450	60	4	75
Hospital 3 ($H3$)	450	50	2.5	55
Hospital 4 ($H4$)	350	50	3.5	85
Hospital 5 ($H5$)	350	55	2	75
Hospital 6 ($H6$)	150	65	2.5	80
Hospital 7 ($H7$)	200	55	2	55
Hospital 8 ($H8$)	150	60	4	80

$$H1 \succ H2; \quad H3 \succ H4; \quad H5 \succ H6; \quad H8 \succ H7.$$

The sign of the interaction index is not always stable

Example (A ranking of French hospitals for weight loss surgery)

	Par.1	Par.2	Par.3	Par.4	Par.5	Par.6	Par.7	Par.8
$C_\mu(H1)$	0.607	0.555	0.692	0.549	0.563	0.59	0.597	0.589
$C_\mu(H2)$	0.509	0.531	0.682	0.539	0.553	0.58	0.533	0.548
$C_\mu(H3)$	0.514	0.576	0.693	0.548	0.605	0.588	0.530	0.564
$C_\mu(H4)$	0.502	0.506	0.683	0.538	0.538	0.578	0.510	0.531
$C_\mu(H5)$	0.586	0.606	0.704	0.63	0.629	0.645	0.583	0.609
$C_\mu(H6)$	0.576	0.568	0.694	0.582	0.606	0.635	0.573	0.599
$C_\mu(H7)$	0.513	0.502	0.539	0.508	0.529	0.538	0.514	0.527
$C_\mu(H8)$	0.523	0.512	0.618	0.518	0.539	0.548	0.524	0.537
μ_1	0.02	0.19	0.453	0.093	0.263	0.192	0.065	0.143
μ_2	0.01	0.01	0.2	0.083	0.209	0.182	0.055	0.133
μ_3	0	0	0	0.073	0.2	0.172	0.045	0.123
μ_4	0.086	0.2	0.697	0.317	0.273	0.416	0.075	0.153
μ_{12}	0.02	0.2	0.653	0.176	0.472	0.375	0.143	0.153
μ_{13}	0.22	0.51	0.526	0.469	0.667	0.45	0.265	0.462
μ_{14}	0.144	0.2	0.697	0.317	0.273	0.416	0.149	0.277
μ_{23}	0.01	0.01	0.2	0.17	0.209	0.455	0.101	0.297
μ_{24}	0.75	0.68	0.697	0.636	0.711	0.663	0.75	0.765
μ_{34}	0.086	0.2	0.927	0.363	0.556	0.519	0.075	0.153
I^μ_{12}	-0.01	0	0	0	0	0	0.02	-0.12
I^μ_{13}	0.2	0.32	0.073	0.3	0.2	0.13	0.15	0.194
I^μ_{14}	0.038	-0.19	-0.453	-0.09	-0.24	-0.2	0.0078	-0.02
I^μ_{23}	-1,169	0	0	0.014	-0.2	0.1	0	0.04
I^μ_{24}	0.655	0.47	-0.2	0.236	0.22	0.06	0.61	0.477
I^μ_{34}	1.998	0	0.23	0.027	0.08	-0.07	-0.04	-0.123

Necessary and possible interaction between criteria

Hypothesis

A preference information given by the DM is representable by a 2-additive Choquet integral.

Let be $i, j \in N$, $i \neq j$ and $\mathcal{C}_{pref}$ the set of all capacities compatible with a preference information given by the DM.

1. There exists a possible positive (respectively, null, negative) interaction between i and j if there exists a capacity $\mu \in \mathcal{C}_{pref}$ such that $I^{\mu}_{ij} > 0$ (respectively, $I^{\mu}_{ij} = 0$, $I^{\mu}_{ij} < 0$).

2. There exists a necessary positive (respectively, null, negative) interaction between i and j if $I^{\mu}_{ij} > 0$ (respectively, $I^{\mu}_{ij} = 0$, $I^{\mu}_{ij} < 0$) for all capacity $\mu \in \mathcal{C}_{pref}$.

Identification of Necessary and possible interactions (Step 1):

$$\text{Minimize } Z_1 = \sum_{(x,y)\in P\cup I} (\Gamma^+_{xy} + \Gamma^-_{xy})$$

Subject to

(PL$_1$)

$$C_\mu(u(x)) - C_\mu(u(y)) + \Gamma^+_{xy} - \Gamma^-_{xy} \geq \varepsilon \quad \forall x, y \in X \text{ such that } x \; P \; y \qquad (4)$$

$$C_\mu(u(x)) - C_\mu(u(y)) + \Gamma^+_{xy} - \Gamma^-_{xy} = 0 \quad \forall x, y \in X \text{ such that } x \; I \; y \qquad (5)$$

$$\Gamma^+_{xy} \geq 0, \quad \Gamma^-_{xy} \geq 0 \qquad \forall x, y \in X \text{ such that } x \; (P \cup I) \; y \qquad (6)$$

$$\varepsilon \geq 0 \qquad (7)$$

$$\sum_{\{i,j\}\subseteq N} m(\{i,j\}) + \sum_{i\in N} m(\{i\}) = 1 \qquad (8)$$

$$m(\{i\}) \geq 0 \quad \text{for all } i \in N \qquad (9)$$

$$m(\{i\}) + \sum_{j\in A\setminus\{i\}} m(\{i,j\}) \geq 0 \quad \forall A \setminus \{i\}, \; \forall i \in N. \qquad (10)$$

- PL$_1$ is always feasible
- $Z_1^* = 0 \Longrightarrow$ we can conclude that, depending on the sign of the variable ε, the preference information $\{P, I\}$ may be representable by a 2-additive Choquet integral.
- $Z_1^* > 0 \Longrightarrow$, then there is no 2-additive Choquet integral model compatible with $\{P, I\}$.

Identification of Necessary and possible interactions (Step 2):

$$
(\text{PL}_2) \quad
\begin{aligned}
&\text{Maximize } Z_2 = \varepsilon \\
&\text{Subject to} \\
&\quad C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \quad \forall x, y \in X \text{ such that } x \; P \; y && (11) \\
&\quad C_\mu(u(x)) - C_\mu(u(y)) = 0 \quad \forall x, y \in X \text{ such that } x \; I \; y && (12) \\
&\quad \varepsilon \geq 0 && (13) \\
&\quad \sum_{\{i,j\} \subseteq N} m(\{i,j\}) + \sum_{i \in N} m(\{i\}) = 1 && (14) \\
&\quad m(\{i\}) \geq 0 \quad \text{for all } i \in N && (15) \\
&\quad m(\{i\}) + \sum_{j \in A \setminus \{i\}} m(\{i,j\}) \geq 0 \quad \forall A \setminus \{i\}, \; \forall i \in N. && (16)
\end{aligned}
$$

- PL_2 is always feasible since the Step 1 was solved before ((PL_2) is launched when $Z_1^* = 0$)
- $Z_2^* = 0 \Longrightarrow$, there is no 2-additive Choquet integral model compatible with $\{P, I\}$.
- $Z_2^* > 0 \Longrightarrow$, $\{P, I\}$ is representable by a 2-additive Choquet integral.

Identification of Necessary and possible interactions (Step 3):

$$
(PL^{ij}_{\text{N-N}}) \quad
\begin{aligned}
&\text{Maximize } Z_3 = \varepsilon \\
&\text{Subject to} \\
&\quad m(\{i,j\}) \geq 0 \quad (\text{respectively } m(\{i,j\}) \leq 0) && (17)\\
&\quad C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \quad \forall x, y \in X \text{ such that } x \; P \; y && (18)\\
&\quad C_\mu(u(x)) - C_\mu(u(y)) = 0 \quad \forall x, y \in X \text{ such that } x \; I \; y && (19)\\
&\quad \varepsilon \geq 0 && (20)\\
&\quad \sum_{\{i,j\} \subseteq N} m(\{i,j\}) + \sum_{i \in N} m(\{i\}) = 1 && (21)\\
&\quad m(\{i\}) \geq 0 \quad \text{for all } i \in N && (22)\\
&\quad m(\{i\}) + \sum_{j \in A \setminus \{i\}} m(\{i,j\}) \geq 0 \quad \forall A \setminus \{i\}, \; \forall i \in N. && (23)
\end{aligned}
$$

- If $PL^{ij}_{\text{N-N}}$ (respectively $PL^{ij}_{\text{N-P}}$) is not feasible, then there is a necessary negative (respectively positive) interaction between i and j.
- If $PL^{ij}_{\text{N-N}}$ (respectively $PL^{ij}_{\text{N-P}}$) is feasible and the optimal solution $Z_3^* = 0$, then the constraint (18) is satisfied with $\varepsilon = 0$. Therefore, we can conclude that there is a necessary negative (respectively positive) interaction between i and j.
- If $PL^{ij}_{\text{N-N}}$ (respectively $PL^{ij}_{\text{N-P}}$) is feasible and the optimal solution $Z_3^* > 0$, then there is no necessary negative (respectively positive) interaction between i and j.

Example

The linear program $PL^{MS}_{\text{N-N}}$ (respectively $PL^{MS}_{\text{N-P}}$) corresponding to the test of the existence of a necessary negative (respectively positive) interaction between the Mathematics (M) and Statistics (S) is the following:

$$
\begin{array}{l}
\text{Maximize} \quad Z_3 = \varepsilon \\
\text{Subject to} \\
\left\{
\begin{array}{l}
m(\{M,S\}) \geq 0 \quad (\text{respectively } m(\{M,S\}) \leq 0) \\
\varepsilon \geq 0 \\
C_\mu(u(b)) - C_\mu(u(a)) \geq \varepsilon \\
C_\mu(u(c)) - C_\mu(u(d)) \geq \varepsilon \\
m(\{M,S\}) + m(\{M,L\}) + m(\{S,L\}) + m(\{M\}) + m(\{S\}) + m(\{L\}) = 1 \\
m(\{M\}) \geq 0 \quad m(\{S\}) \geq 0 \quad m(\{L\}) \geq 0 \\
m(\{M\}) + m(\{M,S\}) \geq 0 \\
m(\{M\}) + m(\{M,L\}) \geq 0 \\
m(\{M\}) + m(\{M,S\}) + m(\{M,L\}) \geq 0 \\
m(\{S\}) + m(\{M,S\}) \geq 0 \\
m(\{S\}) + m(\{S,L\}) \geq 0 \\
m(\{S\}) + m(\{M,S\}) + m(\{S,L\}) \geq 0 \\
m(\{L\}) + m(\{M,L\}) \geq 0 \\
m(\{L\}) + m(\{S,L\}) \geq 0 \\
m(\{L\}) + m(\{S,L\}) + m(\{M,L\}) \geq 0
\end{array}
\right.
\end{array}
$$

Example

	$\mathbf{Z_3} = \varepsilon$	M	S	L	$\{\mathbf{M}, \mathbf{S}\}$	$\{M, L\}$	$\{S, L\}$
Optimal solution Z_3^*	**0.667**	-	-	-	-	-	-
Mobius transform m		0	0.33	0.33	0	0.67	-0.33
Importance index V_i		0.33	0.17	0.5	-	-	-
Interaction index I_{ij}		-	-	-	**0**	0.67	-0.33

Table: Results of $PL^{MS}_{\text{N-N}}$ testing necessary negative interaction between Mathematics and Statistics

Example

	$\mathbf{Z_3} = \varepsilon$	M	S	L	$\{\mathbf{M}, \mathbf{S}\}$	$\{M, L\}$	$\{S, L\}$
Optimal solution Z_3^*	**1**	-	-	-	-	-	-
Mobius transform m		0.5	0.5	0	-0.5	0.5	0
Importance index V_i		0.5	0.25	0.25	-	-	-
Interaction index I_{ij}		-	-	-	$-\mathbf{0.5}$	0.5	0

Table: Results of $PL_{\text{N-P}}^{MS}$ testing necessary positive interaction between Mathematics and Statistics

Outline

Analysis of three MCDA phenomena

- PRESCRIPTIVE APPROACH: To help a decision maker by the proposal of a solution obtained by a model
- DESCRIPTIVE APPROACH: To describe a decision maker's preferences by the chosen model.
- ELICITATION: The elicitation of the decision maker's preferences consists in obtaining parameters of a decisional model which explain the past decisions in order to help in the future decisions.

Parameter's elicitation

- Option 1: Explicit elicitation
  - Explain the model to the DM
  - Let the DM choose the parameters
- Option 2: Implicit elicitation
  - Present some (possibly fictitious) alternatives to the DM and ask him to compare them
  - Deduct the parameters of the model by solving an optimization program

Some references

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- Keeney-Raiffa, "Decisions with multiple objectives preferences and trade-off", 1976, Wiley
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