CentraleSupélec

MASTER DSBI
DECISION MODELING

EXAM JANUARY 2020

2 hours

All the documents are allowed. The access to the Internet is prohibited (except going to my web page to download the subject of the exam to read on your computer and maybe some other documents)

Exercise 1 : Voting rules

1. Let us consider the following preferences of $n = 17$ voters, given on a set $X = \{a, b, c, d\}$ of $m = 4$ candidates :

$$\begin{array}{ll} \text{5 voters :} & c \succ b \succ a \succ d \\ \text{4 voters :} & b \succ a \succ c \succ d \\ \text{4 voters :} & d \succ b \succ c \succ a \\ \text{2 voters :} & a \succ c \succ b \succ d \\ \text{2 voters :} & d \succ b \succ a \succ c \end{array}$$

Which candidate is elected by using the following voting rules or principles :

(a) plurality voting ?

(b) plurality runoff voting (plurality with two rounds) ?

(c) Condorcet principle ?

(d) Borda principle ?

(e) Copeland principle ?

The Copeland principle associates to each candidate x a score calculated as follows :

$$S_{Cop}(a) = \sum_{\substack{y \in X \\ y \neq x}} Cop(x, y)$$

where X is the set of candidates and $Cop(x, y) = \begin{cases} +1 & \text{if a majority of voters prefers } x \text{ to } y \quad (\geq 50\%) \\ -1 & \text{if a majority of voters prefers } y \text{ to } x \quad (\geq 50\%) \\ 0 & \text{if the both two previous situations arise simultaneously} \end{cases}$

The candidate with the highest Copeland score is elected.

(f) Kramer-Simpson principle ?

The Kramer-Simpson principle associates to each candidate x a score

$$KS(x) = \min_{\substack{y \in X \\ y \neq x}} n(x, y)$$

where X is the set of candidates and $n(x, y)$ being the number of voters who prefer x to y.

The candidate with the highest Kramer-Simpson score is elected.

2. In general, does the Copeland principle elect the Condorcet winner, if this latter exists ? Justify your answer.
3. In general, does the Kramer-Simpson principle elect the Condorcet winner, if this latter exists ? Justify your answer.
4. Let us consider the following preferences of 5 voters given on a set $X = \{a, b, c, d, e\}$ of 5 candidates :

$$\begin{array}{ll} \text{1 voter :} & a \succ b \succ c \succ d \succ e \\ \text{1 voter :} & b \succ c \succ d \succ a \succ e \\ \text{1 voter :} & c \succ d \succ a \succ b \succ e \\ \text{1 voter :} & d \succ a \succ b \succ c \succ e \end{array}$$

The Borda principle is chosen to elect the winner of this election.

(a) Who is elected ?

(b) By adding 3 new voters to the previous 5 one (and we have now a total of 8 voters), is it possible to provide the preferences of these new voters such that the candidate e is elected (the winner is not necessary unique) ? Justify your answer.

(c) By adding 4 new voters to the previous one (and we have now a total of 9 voters), is it possible to provide the preferences of these new voters such that the candidate e is elected (the winner is not necessary unique) ? Justify your answer.

Exercise 2 : Ranking or sorting ?

We consider 8 students of a Master program, evaluated on 6 subjects (criteria supposed to be maximized) as follows (see Table 1) :

Alternatives	CR_1	CR_2	CR_3	CR_4	CR_5	CR_6
a_1	10	10	10	10	10	10
a_2	9	11	10	11	11	9
a_3	11	9	10	11	9	11
a_4	10	10	9	9	10	9
a_5	9	11	11	10	10	9
a_6	11	10	10	11	10	10
a_7	10	9	10	11	10	10
a_8	11	10	11	12	9	9

TABLE 1 – Evaluation of 8 students on 6 subjects.

The two parts below are independent and can be solved separately.

Part 1 : Ranking

1. We assume that, each score is given on a scale $[0, 20]$, corresponding to a value of the marginal utility function associated to a subject.

(a) Determine the ranking $\succsim_1$ of the 8 students by using a weighted sum, where the weights (ECTS) associated to the 6 subjects is the vector $W_1 = (6; 3; 2; 6; 2; 6)$.

(b) Determine the ranking $\succsim_2$ of the 8 students by using a weighted sum where the weights (ECTS) associated to the 6 subjects is the vector $W_2 = (3; 4; 2; 6; 2; 4)$.

Are the rankings $\succsim_1$ and $\succsim_2$ different ?

(c) In fact, the director of the Master has some preferences given as follows :
- the students a_3 and a_2 are judged equivalent ;
- the student a_2 is strictly preferred to the student a_1.

Does a weight vector W_3 exist such that these preferences are representable by a weighted sum ? Justify your answer and if yes, give the ranking $\succsim_3$ obtained by applying W_3 to the student's dataset.

(d) Prove that the following preferences, of the director of the Master, are not representable by a weighted sum :
- student a_1 is strictly preferred to the student a_3 ;
- student a_7 is strictly preferred to the studenta_1 ;
- criterion 6 is more important than criterion 5.

2. In this question, we assume that each score is normalized to [0,1] (e.g. by dividing it by 20) and the last two criteria are excluded to the global evaluation (CR_5 and CR_6) and only the four first criteria are considered.

(a) Determine, by yourself, a 2-additive capacity μ on the 4 criteria such that at least two interaction indices are not null.

(b) Give the ranking $\succsim_4$ obtained by applying the 2-additive capacity μ to the updated data of the 8 students.

Part 2 : Sorting

We aim at developing a multi-criteria method assigning the 8 students to some ordered categories. The envisaged method is based on the elaboration of an outranking relation, as it is done, for instance, in MR-Sort method. However, unlike MR-Sort where each alternative is compared to reference profiles representing the boundaries of the categories, the outranking relation here is defined on the given set of the alternatives.

In the sequel, we denote by $A = \{a, b, c, \ldots\}$ the set of alternatives to assign and N the set of n criteria.

The outranking relation $\succsim$ means "at least as good as", with $\succ$ its asymmetric part and $\sim$ is symmetric part.

Two alternatives a and b are said "incomparable" if [$\text{not}(a \succsim b)$ and $\text{not}(b \succsim a)$].

The p categories we consider are denoted $C_1, C_2, \ldots, C_p$ (C_1 and C_p being respectively the worst and the best category).

- $C(a)$ represents the category where the alternative a is assigned.
- $C(a) \geq C(b)$ means that a is assigned to a category greater than the category where b is assigned.
- $C(a) > C(b)$ means that a is assigned to a category strictly greater than the category where b is assigned.

1. Let us consider the following assignment principle :

$$\forall a, b \in A, \quad C(a) > C(b) \Rightarrow a \succsim b \tag{1}$$

Prove that, if this principle is used then two incomparable alternatives are necessary assigned to the same category.

2. Let us consider the following assignment principle :

$$\forall a, b \in A, \quad a \succsim b \Rightarrow C(a) \geq C(b) \tag{2}$$

Prove that, this principle implies $a \succ b \Rightarrow C(a) > C(b)$ and $a \sim b \Rightarrow C(a) = C(b)$.

3. In this question, we consider the previous assignment principle given by Equation (2).

   We suppose that the outranking relation is defined by

$$a \succsim b \Leftrightarrow \sum_{i|g_i(a)\geq g_i(b)} w_i \geq \lambda$$

   where $w_i \geq 0$ represents the weight associated to the criterion i ($\sum_{i\in N} w_i = 1$), $g_i(a)$ represents the value of the alternative a on the criterion i and $\lambda \in [0.5; 1]$ is the majority threshold.

   To assign the 8 students of the Master, we consider the following parameters and preferences :

   - We have 4 categories $C_1; C_2; C_3$ and C_4 ;
   - The majority threshold is $\lambda = \frac{2}{3}$;
   - The weight vector is $W_5 = (0.1; 0.2; 0.05; 0.4; 0.05; 0.2)$
   - The students a_3 and a_2 belong to the category C_3 ;
   - The student a_1 belongs to the category C_2.

   Determine the assignments of the other students by using these preferences and the adopted assignment principle.