CentraleSupélec

MASTER DSBI
DECISION MODELING
PRACTICAL WORKS

VOTING RULES : EXERCISES

Exercise 1

We consider the following profile (9 voters and 4 candidates) where the preferences of the last voter are unknown :

4 voters :	$c \succ d \succ a \succ b$
2 voters :	$a \succ b \succ d \succ c$
2 voters :	$b \succ a \succ c \succ d$
1 voter :	$? \succ ? \succ ? \succ ?$

1. Do we necessarily know the preferences of the last voter, in order to determine the result of the elections in a UK system (plurality) and French system (plurality with runoff) ? If yes, gives these preferences and the results of these elections.
2. Does the Condorcet winner exist in this election ?
3. Which preferences the last voter should have in order to elect a as the Condorcet winner ? Same question with b, c or d.

Exercise 2

We suppose here that each voter can express his preferences through a complete pre-order on a set of candidates. Given two candidates a and b, we set $x_{ab} = |\{i : a \succ_i b\}|$, i.e., the number of voters which prefer strictly a to b.

A candidate a is globally (socially) preferred to a candidate b, denote by $a \succ b$, if $x_{ab} - x_{ba} > 0$. a and b are considered indifferent, denoted by $a \sim b$, if $not(a \succ b)$ and $not(b \succ a)$.

We consider the following profile (5 voters and 4 candidates) :

1 voter :	$a \sim_1 c \succ_1 b \sim_1 d$
1 voter :	$a \sim_2 d \succ_2 b \succ_2 c$
1 voter :	$d \succ_3 c \succ_3 a \sim_3 b$
1 voter :	$a \succ_4 b \succ_4 d \succ_4 c$
1 voter :	$b \succ_5 c \succ_5 d \succ_5 a$

1. Is there a Condorcet winner ? If yes, is it unique ?
2. Is there a candidate which is never beaten ?

Exercise 3

Let X be a set of candidates (alternatives) and $N = \{1; \dots; n\}$ be a set of voters. A preference profile $R = (R_1; \dots; R_n)$ is an n-tuple containing a preference relation R_i for every voter $i \in N$. For a given profile $R = (R_1; \dots; R_n)$ and two distinct elements $a, b \in X$, let us define the following numbers $n_R(a, b)$ and $m_R(a, b)$:

$$n_R(a,b) = \left|\{i \in N : a\ R_i\ b\}\right| \quad \text{and} \quad m_R(a,b) = n_R(a,b) - n_R(b,a)$$

The majority relation R_M on X is defined as follows :

$$\forall a, b \in X,\ a\ R_M\ b \text{ if and only if } m_R(a,b) \geq 0$$

We denote by P_M the strict part of R_M, i.e.,

$$\forall a, b \in X,\ a\ P_M\ b \text{ if and only if } m_R(a,b) > 0$$

Let R_M^* denote the transitive closure of the majority relation, i.e., a binary relation such that for all $a, b \in X$, $a\ R_M^*\ b$ if and only if there exists $k \in \mathbb{N}^*$ and $a_1; a_2; \ldots; a_k \in X$ with $a_1 = a$ and $a_k = b$ such that $a_i\ R_M\ a_{i+1}$ for all $i < k$.

A Social Choice Function (SCF) is a function f that maps a preference profile R to a nonempty subset of alternatives, i.e., $f(R) \neq \emptyset$ and $f(R) \subseteq X$. We consider the following SFCs :

- *Copeland (CO)* : The Copeland score of an alternative a, $S_{co}(a)$, is defined as the number of alternatives that a beats in pairwise majority comparisons, i.e., $S_{co}(a) = \{\Big|\{b \in X : a\ P_M\ b\}\Big|\}$. The function CO returns the alternatives with the highest Copeland score.
- *Condorcet rule (COND)* : The function COND returns the Condorcet winner if it exists, and the set X of all alternatives otherwise.
- *Borda rule (BO)* : The Borda score of an alternative a is defined as $S_{BO}(a) = \sum_{b \in X \setminus \{a\}} n_R(a,b)$. The function BO returns the alternatives with the highest Borda score.
- *Top Cycle (TC)* : Given a profile R, the top cycle rule TC returns the maximal elements of R_M^*, i.e., $TC(R) = \{a \in X : a\ R_M^*\ b,\ \forall b \in X\}$

Let us consider the following profile $\mathcal{B} = (\mathcal{B}_1; \mathcal{B}_2; \mathcal{B}_3; \mathcal{B}_4; \mathcal{B}_5)$, of five voters, on $X = \{a, b, c, d, e\}$:

$$\begin{array}{l} \mathcal{B}_1 : e \succ d \succ a \succ b \succ c \\ \mathcal{B}_2 : b \succ c \succ e \succ a \succ d \\ \mathcal{B}_3 : b \succ c \succ e \succ a \succ d \\ \mathcal{B}_4 : e \succ a \succ b \succ d \succ c \\ \mathcal{B}_5 : d \succ c \succ a \succ b \succ e \end{array}$$

1. Compute the results of the functions CO, COND, BO and TC applied to this profile $\mathcal{B}$.
2. Prove that : we always have
$$TC(R) \subseteq COND(R), \quad \text{for all profiles } R \text{ on } X$$
3. Could we have $CO(R) \subseteq BO(R)$ for all profiles R defined on X ? Justify your answers.