Graph Algorithms

Graph Traversals and Connectivity

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September 3, 2025

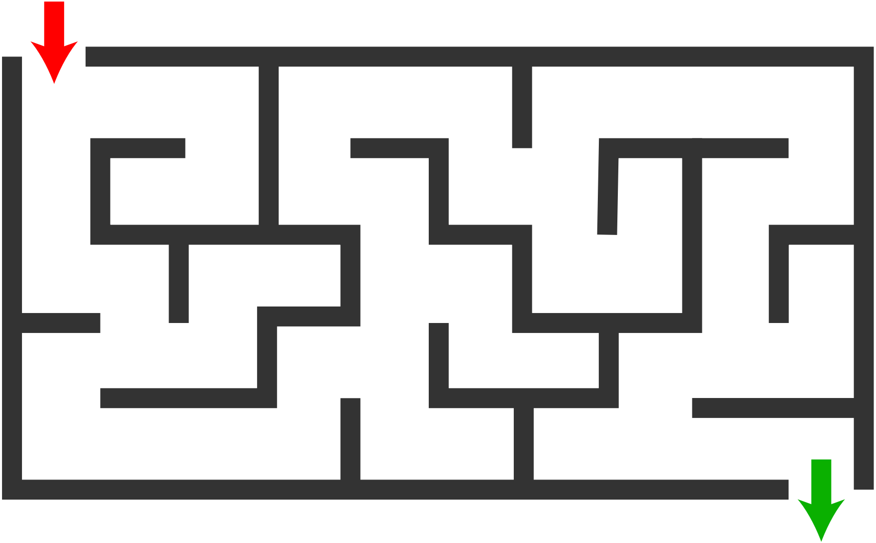
Graph Traversal

Problem

Given (di)graph G, determine connectivity properties:

- Is G (strongly) connected?
- What are the (strongly) connected components of G?
- Which vertices can be reached from a given source s?
- What is the shortest path distance from (given vertex) s to (given vertex) t?

Graph Traversal



Breadth-First Search

Breadth-first search (BFS) is a basic **graph traversal** algorithm.

- Input: G and specified source vertex s
- Output: Set of vertices reachable from s and tree of shortest paths from s to all such vertices.

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Key properties:

- Linear time and space complexity $O(n + m)$.
- Works for both graphs and digraphs.

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Key idea:

- Explore vertices in order of increasing distance from s, using a queue.

How NOT to solve graph traversal!

How NOT to solve graph traversal!

Attempt to decide reachability from s to t:

```
procedure REACH(G, s, t)
    if s = t or st ∈ E then                ▷ Base Case
        Return Yes
    end if
    for u ∈ N(s) do                        ▷ Inductive Case
        if Reach(G, u, t) then
            Return Yes
        end if
    end for
    Return No
end procedure
```

How NOT to solve graph traversal!

Counterexample:

- Suppose $G = K_3 + K_1$, t is the isolated vertex, s is in the triangle.
- Algorithm goes into infinite loop!

How NOT to solve graph traversal!

Attempt to decide reachability from s to t in k steps:

```
procedure Reach(G, s, t, k)
    if k < 0 then
        Return No
    end if
    if s = t or st ∈ E then                         ▷ Base Case
        Return Yes
    end if
    for u ∈ N(s) do                                 ▷ Inductive Case
        if Reach(G, u, t, k − 1) then
            Return Yes
        end if
    end for
    Return No
end procedure
```

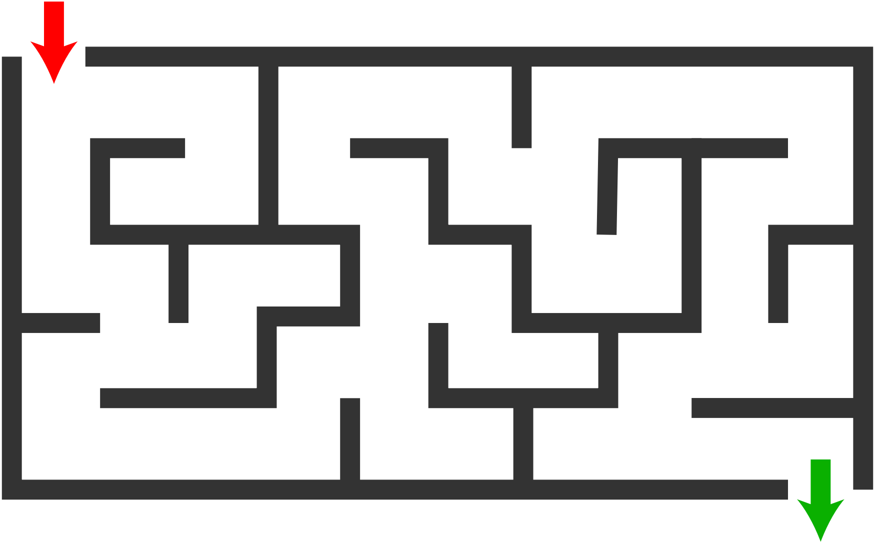
How NOT to solve graph traversal!

Counterexample:

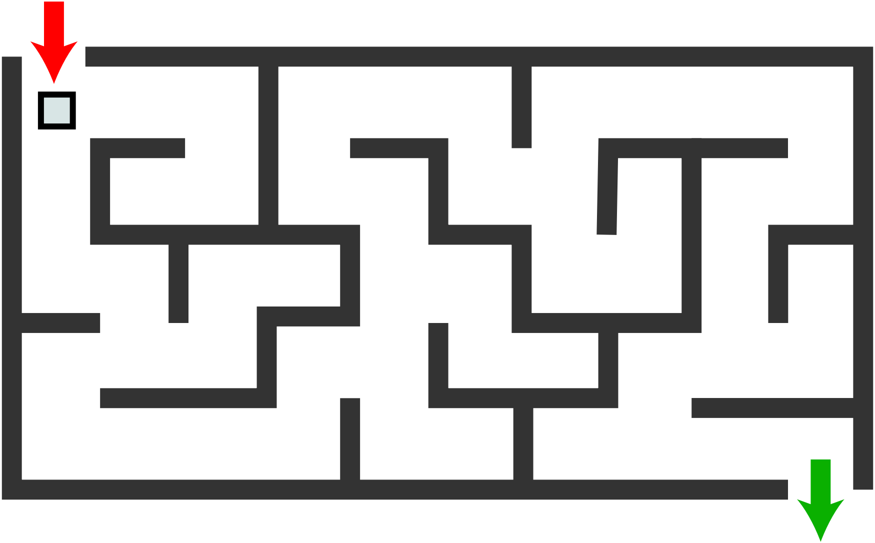
Algorithm is correct, but...

- Execution can have complexity **exponential** in n!
- Notice that we are using very little space...

BFS (partial) example



BFS (partial) example



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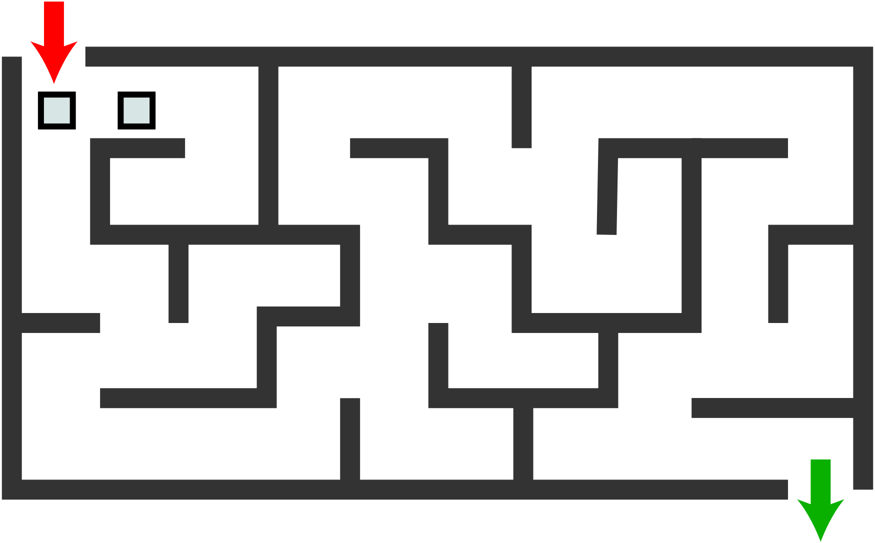

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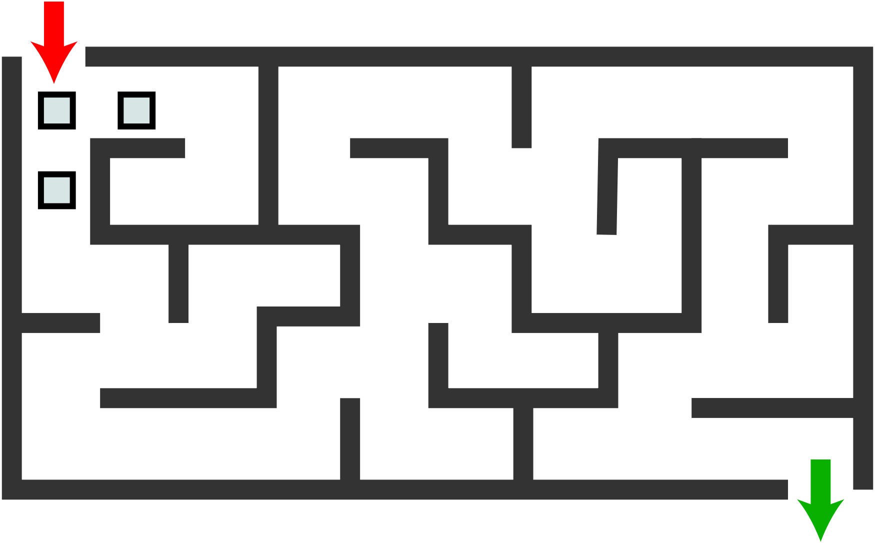

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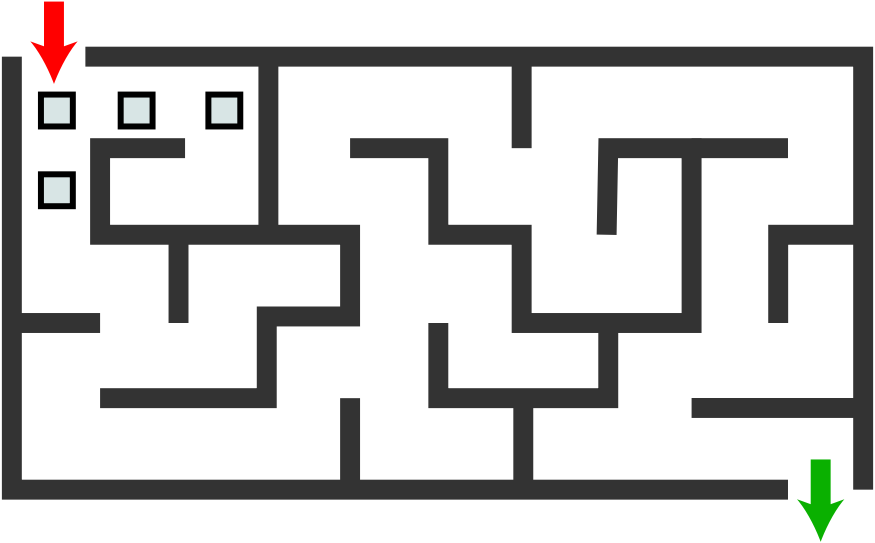

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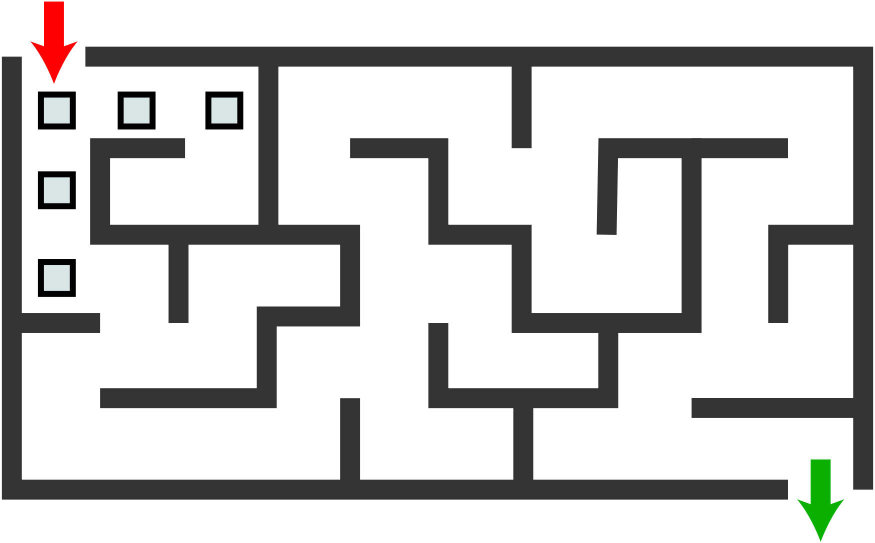

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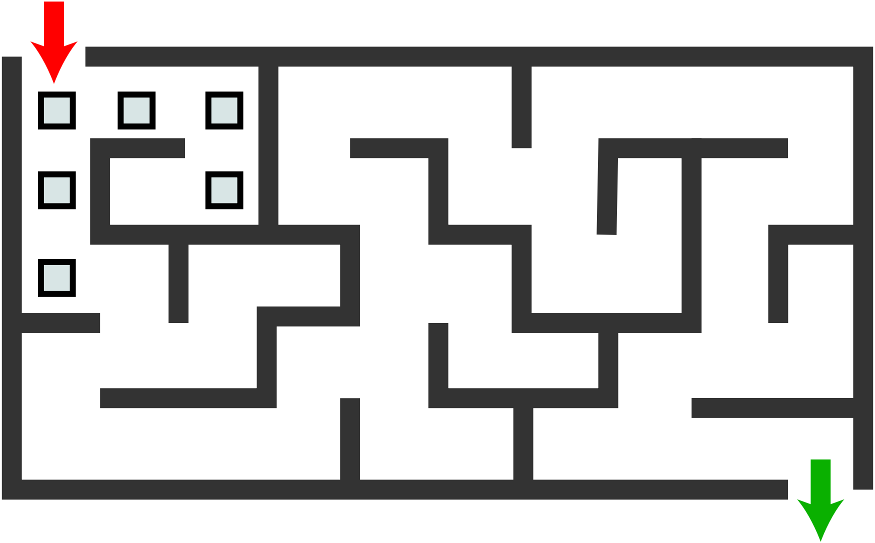

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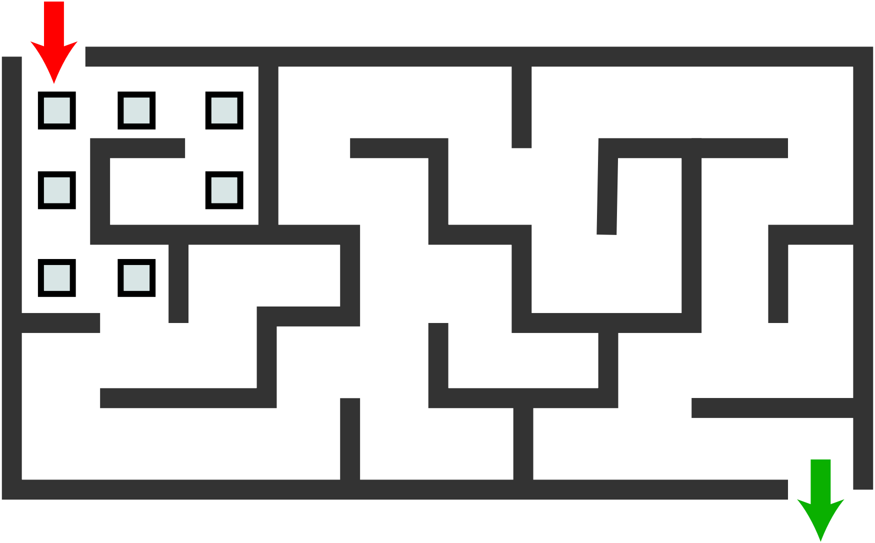

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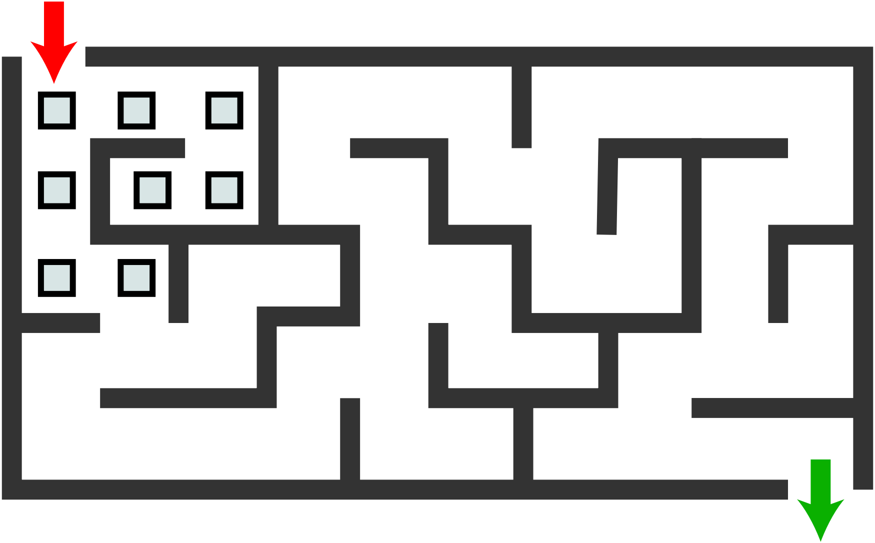

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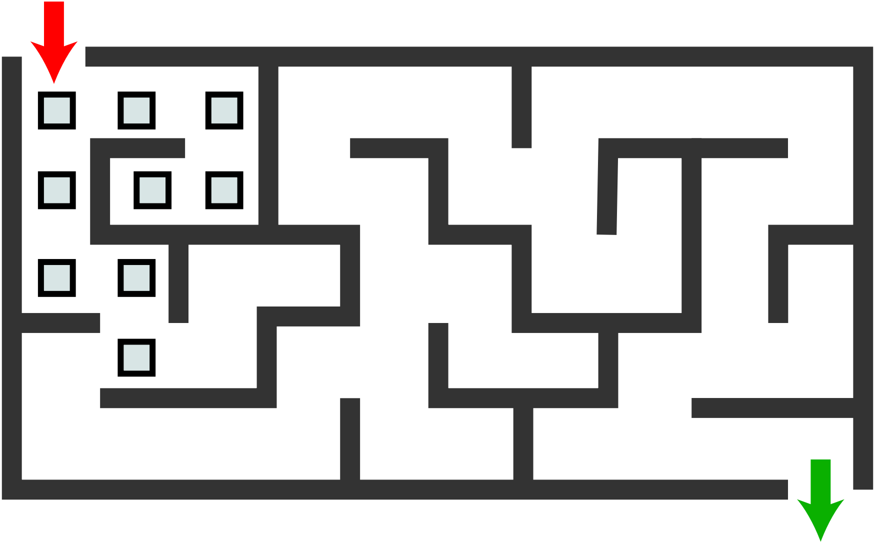

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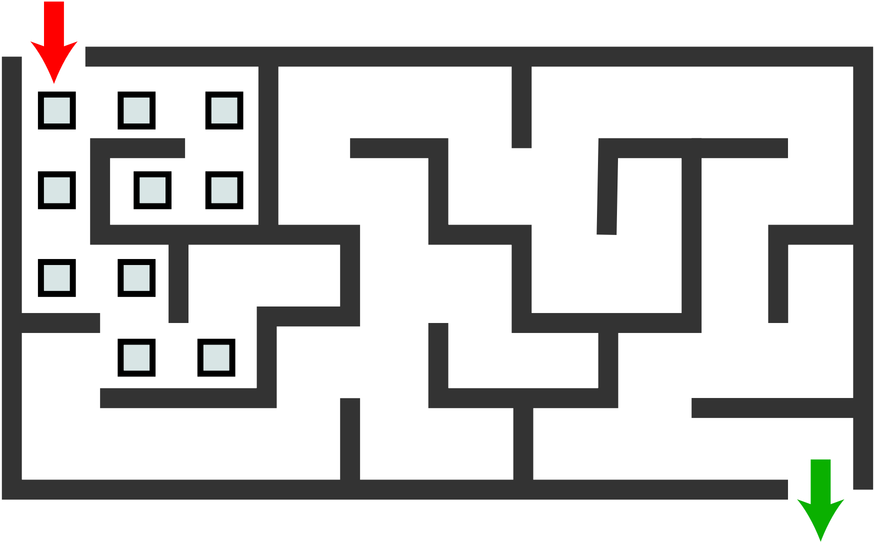

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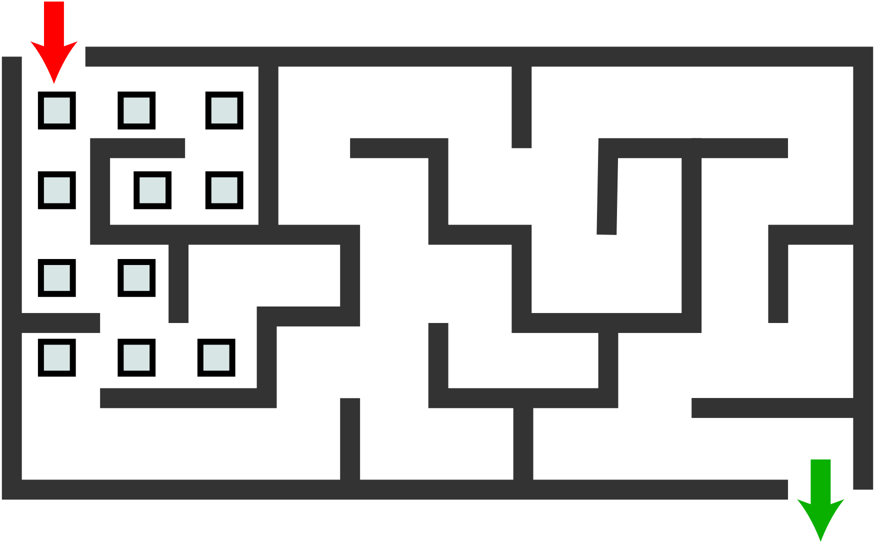


Reminder: Queues

Queue (FIFO)

Queue specifications:

- Two operations: Enqueue and Dequeue.
- Enqueue: Adds an element to the data structure.
- Dequeue: Removes and returns the **oldest** element of the data structure.
- Complexity: $O(1)$ for both operations.

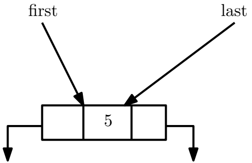
Queue example – Lists

1. Enqueue(5), Enqueue(9), Enqueue(7)
2. Dequeue $\to$ 5
3. Enqueue(4)
4. Dequeue $\to$ 9

Queue example – Lists

1. Enqueue(5), Enqueue(9), Enqueue(7)
2. Dequeue $\rightarrow$ 5
3. Enqueue(4)
4. Dequeue $\rightarrow$ 9

Queue example – Lists



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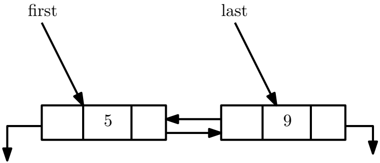

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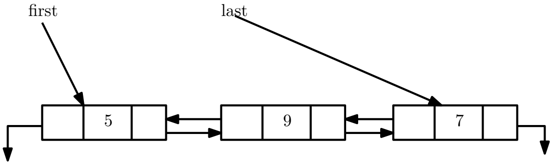

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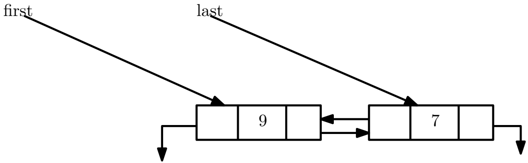

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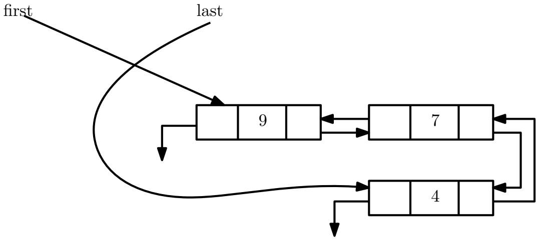

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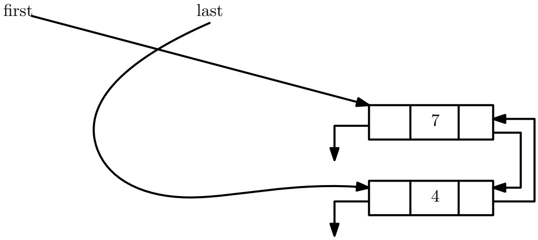

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2. Dequeue → 5
3. Enqueue(4)
4. Dequeue → 9

Queue example – Arrays

1. Enqueue(5), Enqueue(9), Enqueue(7)
2. Dequeue $\rightarrow$ 5
3. Enqueue(4)
4. Dequeue $\rightarrow$ 9

Queue example – Arrays

*	*	*	*	*	*	*	*	...

first=-1, last=-1

1. Enqueue(5), Enqueue(9), Enqueue(7)
2. Dequeue $\rightarrow$ 5
3. Enqueue(4)
4. Dequeue $\rightarrow$ 9

Queue example – Arrays

5	*	*	*	*	*	*	*	...

first=1, last=1

1. Enqueue(5), Enqueue(9), Enqueue(7)
2. Dequeue $\rightarrow$ 5
3. Enqueue(4)
4. Dequeue $\rightarrow$ 9

Queue example – Arrays

5	9	*	*	*	*	*	*	...

first=1, last=2

1. Enqueue(5), Enqueue(9), Enqueue(7)
2. Dequeue $\rightarrow$ 5
3. Enqueue(4)
4. Dequeue $\rightarrow$ 9

Queue example – Arrays

5	9	7	*	*	*	*	*	...

first=1, last=3

1. Enqueue(5), Enqueue(9), Enqueue(7)
2. Dequeue → 5
3. Enqueue(4)
4. Dequeue → 9

Queue example – Arrays

*	9	7	*	*	*	*	*	...

first=2, last=3

1. Enqueue(5), Enqueue(9), Enqueue(7)
2. Dequeue $\rightarrow$ 5
3. Enqueue(4)
4. Dequeue $\rightarrow$ 9

Queue example – Arrays

*	9	7	4	*	*	*	*	...

first=2, last=4

1. Enqueue(5), Enqueue(9), Enqueue(7)
2. Dequeue $\rightarrow$ 5
3. Enqueue(4)
4. Dequeue $\rightarrow$ 9

Queue example – Arrays

*	*	7	4	*	*	*	*	...

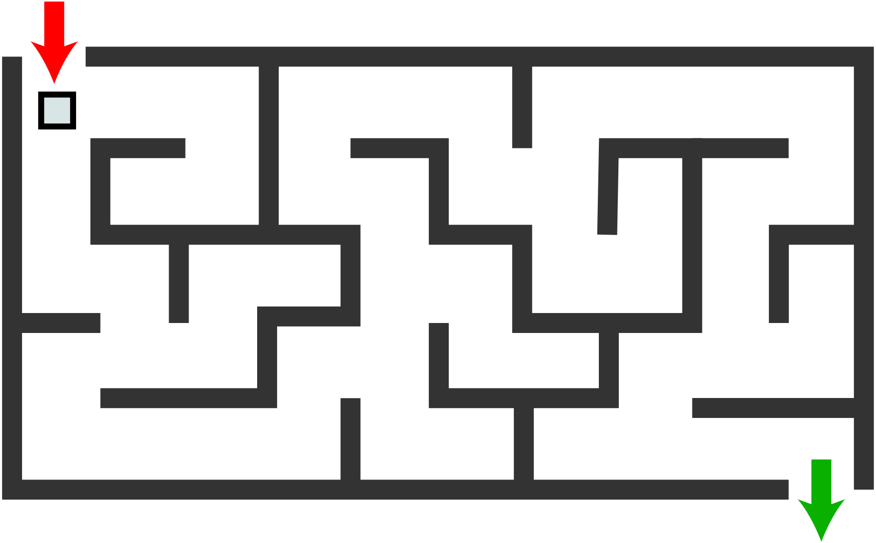
first=3, last=4

1. Enqueue(5), Enqueue(9), Enqueue(7)
2. Dequeue $\rightarrow$ 5
3. Enqueue(4)
4. Dequeue $\rightarrow$ 9

BFS Trees

- The output of BFS is a **tree** rooted at s.
- For each vertex v of the tree we calculate
  - The shortest-path distance from s to v.
  - The predecessor (parent) of v is a shortest path from s to v.

BFS Trees



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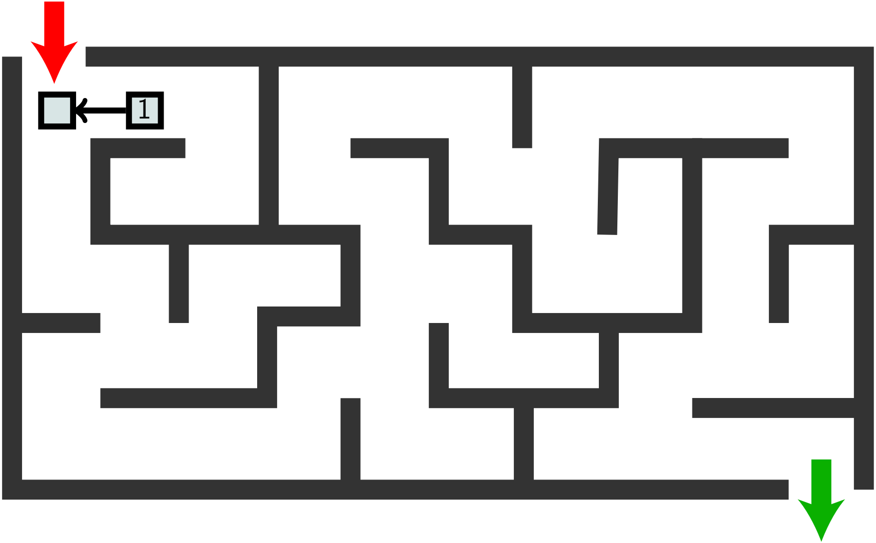

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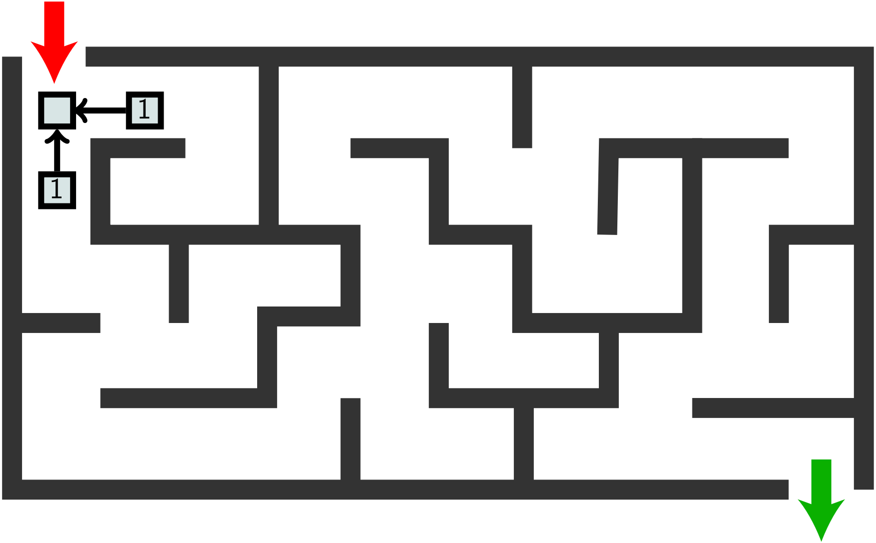

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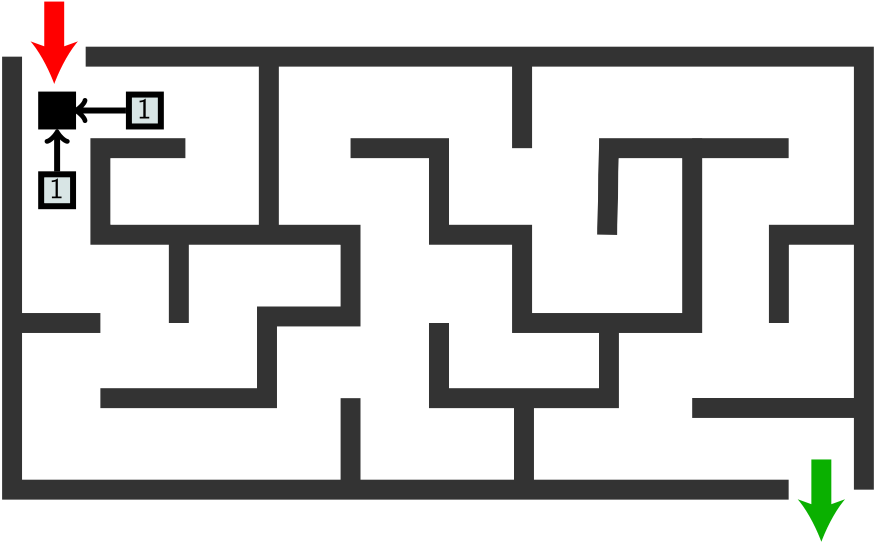

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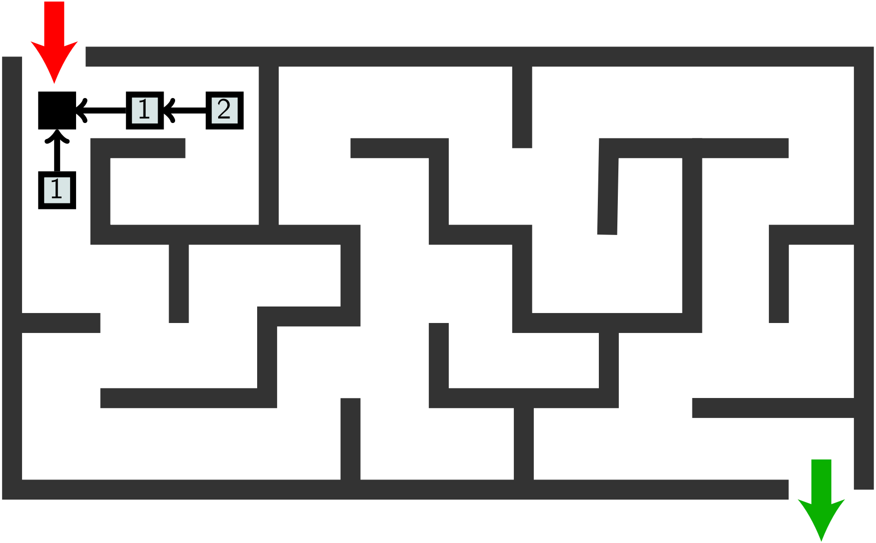
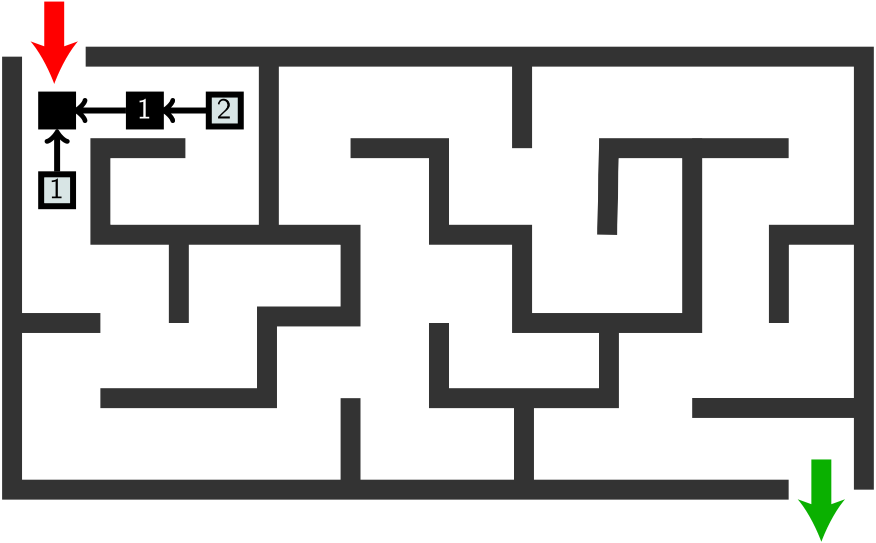


BFS Trees



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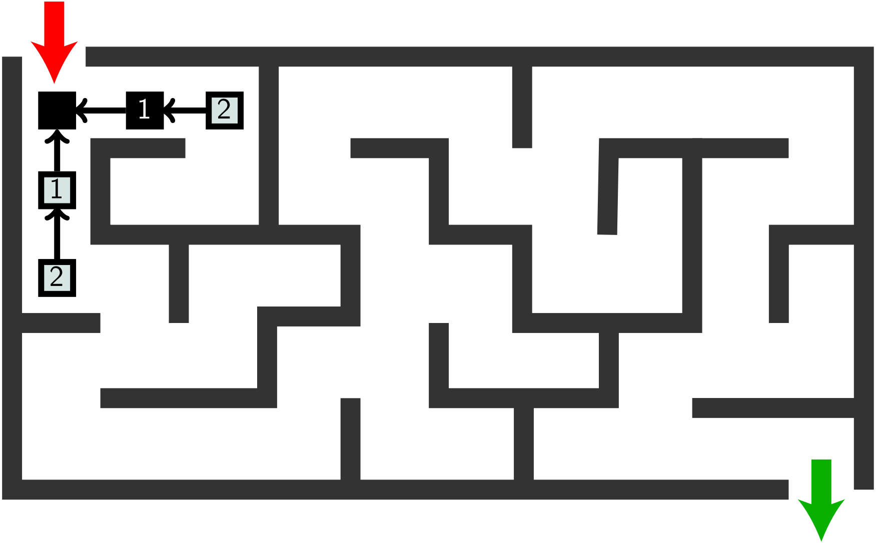
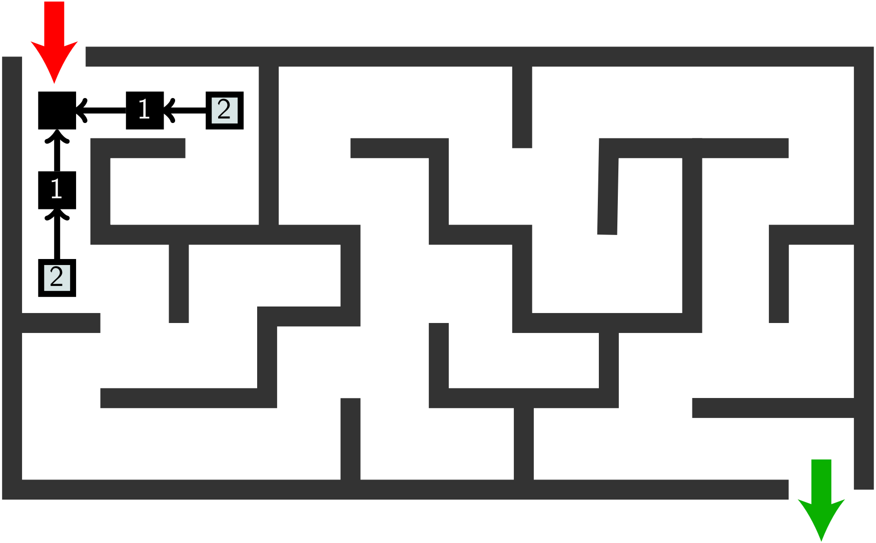


BFS Trees



The algorithm

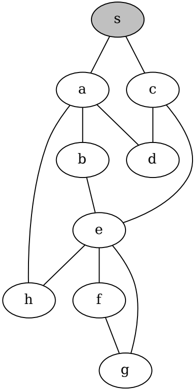
BFS

```
for $v \in V \setminus \{s\}$ do                                        ▷ initialize
    Color $v$ White, Parent of $v \leftarrow$ NULL, $\mathrm{dist}(s, v) \leftarrow \infty$
end for
$\mathrm{dist}(s, s) \leftarrow 0$, Parent of $s \leftarrow$ NULL, Color $s$ Gray
Enqueue($s$)                                                            ▷ end of initialization
while Queue not empty do                                                ▷ main loop
    $v \leftarrow$ Dequeue
    for each $u \in N(v)$ do
        if $u$ is White then
            $\mathrm{dist}(s, u) \leftarrow \mathrm{dist}(s, v) + 1$, $v$ becomes parent of $u$
            Color $u$ Gray
            Enqueue($u$)
        end if
    end for
    Color $v$ Black
end while
```

BFS High-level ideas

- Vertices are colored White, Gray, or Black
  - White: undiscovered
  - Gray: discovered, not processed yet
  - Black: finished
- As vertices are discovered, they are added to the queue
- FIFO$\rightarrow$ vertices further away are processed later (proof?)

Example



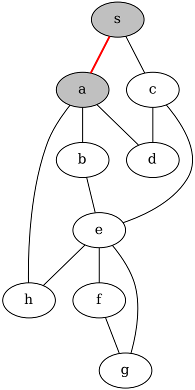
Distances:

a	b	c	d	e	f	g	h
∞	∞	∞	∞	∞	∞	∞	∞

Queue:

s

Example



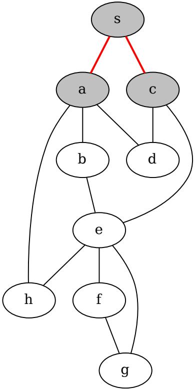
Distances:

a	b	c	d	e	f	g	h
1	∞	∞	∞	∞	∞	∞	∞

Queue:

a

Example



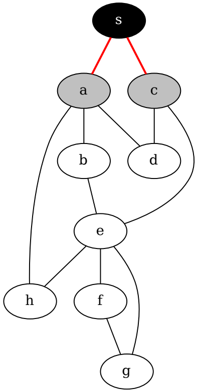
Distances:

a	b	c	d	e	f	g	h
1	∞	1	∞	∞	∞	∞	∞

Queue:

a, c

Example



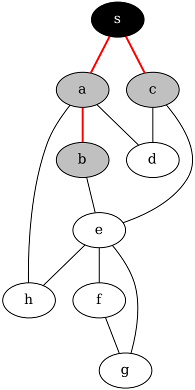
Distances:

a	b	c	d	e	f	g	h
1	∞	1	∞	∞	∞	∞	∞

Queue:

a, c

Example



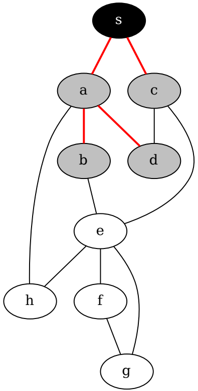
Distances:

a	b	c	d	e	f	g	h
1	2	1	∞	∞	∞	∞	∞

Queue:

c, b

Example



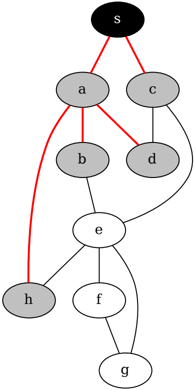
Distances:

a	b	c	d	e	f	g	h
1	2	1	2	∞	∞	∞	∞

Queue:

c, b, d

Example



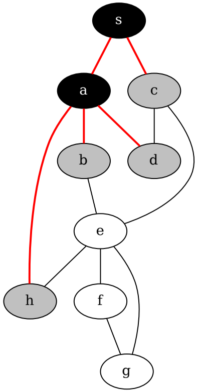
Distances:

a	b	c	d	e	f	g	h
1	2	1	2	∞	∞	∞	2

Queue:

c, b, d, h

Example



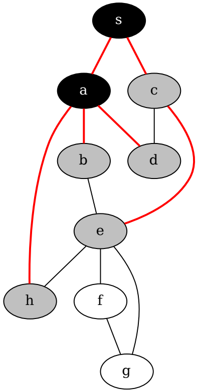
Distances:

a	b	c	d	e	f	g	h
1	2	1	2	∞	∞	∞	2

Queue:

c, b, d, h

Example



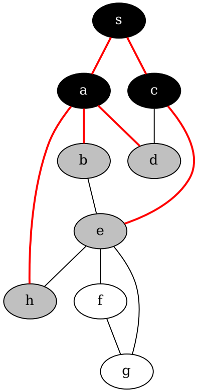
Distances:

a	b	c	d	e	f	g	h
1	2	1	2	2	∞	∞	2

Queue:

b, d, h, e

Example



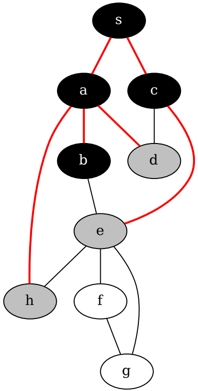
Distances:

a	b	c	d	e	f	g	h
1	2	1	2	2	∞	∞	2

Queue:

b, d, h, e

Example



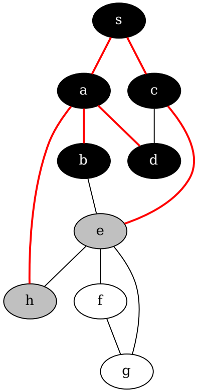
Distances:

a	b	c	d	e	f	g	h
1	2	1	2	2	∞	∞	2

Queue:

d, h, e

Example



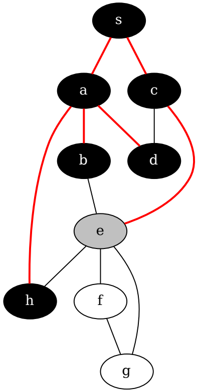
Distances:

a	b	c	d	e	f	g	h
1	2	1	2	2	∞	∞	2

Queue:

h, e

Example



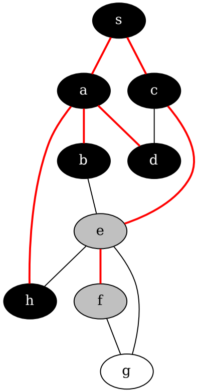
Distances:

a	b	c	d	e	f	g	h
1	2	1	2	2	∞	∞	2

Queue:

e

Example



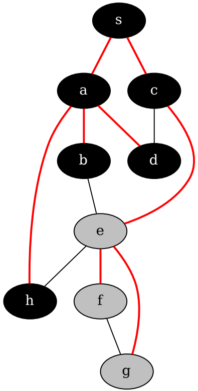
Distances:

a	b	c	d	e	f	g	h
1	2	1	2	2	3	∞	2

Queue:

f

Example



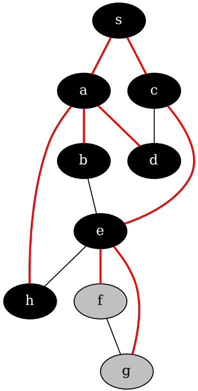
Distances:

a	b	c	d	e	f	g	h
1	2	1	2	2	3	3	2

Queue:

f, g

Example



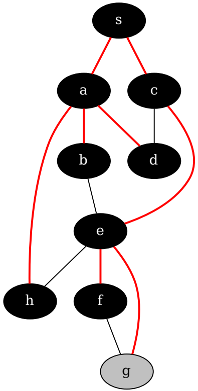
Distances:

a	b	c	d	e	f	g	h
1	2	1	2	2	3	3	2

Queue:

f, g

Example



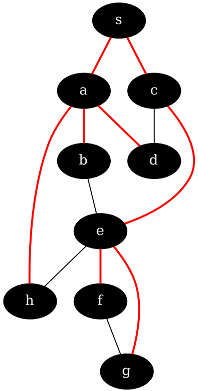
Distances:

a	b	c	d	e	f	g	h
1	2	1	2	2	3	3	2

Queue:

g

Example



Distances:

a	b	c	d	e	f	g	h
1	2	1	2	2	3	3	2

Queue:

$\emptyset$

Analysis of BFS

Running Time

- Initialization takes $O(n)$ time (and space).

Running Time

- Initialization takes $O(n)$ time (and space).
- Vertex colors go from White$\rightarrow$Gray$\rightarrow$Black
  - $O(n)$ color changes.

Running Time

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- $\Rightarrow$ Total space: $O(n)$ and total time $O(n+m)$.

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Would like to establish that:

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Define:

- $\mathrm{dist}(s, v)$: the (correct) shortest-path distance from s to v.
- $d_{BFS}(v)$: the distance from s to v computed by BFS.

We want:

$$\forall v \in V: \ \mathrm{dist}(s, v) = d_{BFS}(v)$$

The easy part

Lemma

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Proof.

Induction on $d_{BFS}(v)$:

- Base case: $d_{BFS}(v) = 0$ only applies to s and is clearly correct.
- Suppose lemma correct if $d_{BFS}(v) \leq k$ and we have a vertex v for which $d_{BFS}(v) = k + 1$.
  - For some u (dequeued before v) we have $d_{BFS}(v) = d_{BFS}(u) + 1 \Rightarrow d_{BFS}(u) = k$.
  - Inductive hypothesis: $\mathrm{dist}(s, u) \leq d_{BFS}(u) = k$.
  - Therefore, $\mathrm{dist}(s, v) \leq \mathrm{dist}(s, u) + 1 \leq k + 1$.

□

Analyzing the queue

Lemma

Vertices are placed into the queue in non-decreasing order of d_{BFS}. *Furthermore, for any* u, v *which are simultaneously in the queue we have* $|d_{BFS}(u) - d_{BFS}(v)| \leq 1$.

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Proof.

By induction on number of queue operations:

- Base case: empty queue, vacuously true
- General case: statement is true, then d_{BFS} values in the queue are of the form $k, k, \ldots, k, k+1, \ldots, k+1$ (or all vertices have same value).
- Dequeue operation $\rightarrow$ property still true.
- Enqueue operation $\rightarrow$ we add a neighbor of a just removed vertex and increase its distance by 1 $\rightarrow$ property still true.

□

The other part of the inequality

Lemma

For all $v \in V: \ \mathrm{dist}(s, v) \geq d_{BFS}(v)$

The other part of the inequality

Lemma

For all $v \in V: \ \mathrm{dist}(s, v) \geq d_{BFS}(v)$

Proof.

Suppose for $v \in V$ we have $\mathrm{dist}(s, v) < d_{BFS}(v)$. Among all such v, pick one with **minimum** $\mathrm{dist}(s, v)$.

- Let u be the last vertex before v in a shortest $s \to v$ path.
  - $\mathrm{dist}(s, u) = \mathrm{dist}(s, v) - 1 \Rightarrow \mathrm{dist}(s, u) = d_{BFS}(u)$.
- When u was dequeued, v was:
  - White: then $d_{BFS}(v) = d_{BFS}(u) + 1$, which is correct!
  - Gray: then $|d_{BFS}(v) - d_{BFS}(u)| \leq 1$, contradiction!
  - Black: then $d_{BFS}(v) \leq d_{BFS}(u)$, contradiction!

□

Implementation details

- BFS trees are not unique and depend on order in adjacency lists
  - Can the edge *cd* be added to the BFS tree in our example? What about *be*?
  - In most examples, we will assume that adjacency lists are sorted alphabetically.
- BFS can be executed also when graph is given in adjacency matrix form
  - Complexity $O(n^2)$ (why?)
- BFS also works for directed graphs
  - In line 8 we look at list of outneighbors.