Graph Algorithms

Graph Traversals and Connectivity III

Michael Lampis

September 25, 2025

Graph Traversal

Problem

Given (di)graph G, determine connectivity properties:

- Is G (strongly) connected?
- What are the (strongly) connected components of G?
- Which vertices can be reached from a given source s?
- What is the shortest path distance from (given vertex) s to (given vertex) t?

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- Today: Applications

Topological Sort

Definition of Topological Sort

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A topological sort of a digraph $G = (V, A)$ is an ordering (numbering) of the vertices with the following property: if we have an arc from a vertex numbered i to a vertex numbered j, then $i < j$.

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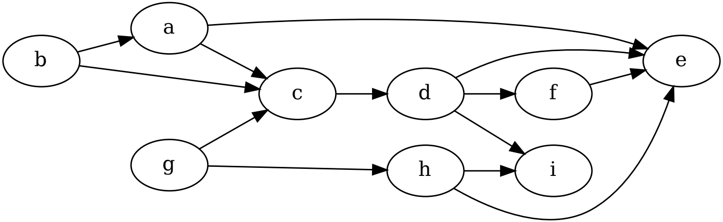
- In other words, arcs go from lower to higher numbers.
- Vertices are numbered $1, \ldots, n$.

Topological Sort – Example



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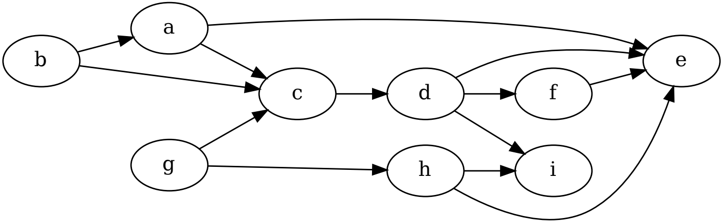

Ranking:

1	2	3	4	5	6	7	8	9
b	a	g	c	d	h	i	f	e

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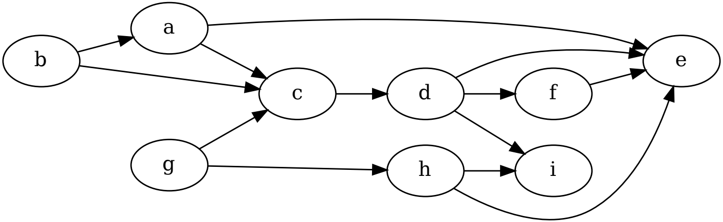

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g	b	a	c	h	d	f	e	i

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A digraph G is a Directed Acyclic Graph (DAG) if G contains no directed cycles.

NB: We also count cycles of length 2 (digons).

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Lemma

A digraph G admits a topological ordering if and only if G is a DAG.

Proof.

- $\Rightarrow$: A cycle C cannot be topo-sorted, because all vertices have positive in- and out-degree.
- $\Leftarrow$: A DAG always contains a sink v (why?), number it n, order the rest inductively.

□

Every DAG has a source/sink

Lemma

If $G = (V, A)$ is a DAG, then G contains at least one source and at least one sink.

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- Let $P = x_1, x_2, \dots, x_k$ be the **longest** directed simple path in G.
- If there exists an arc $x_k y$ with $y \in \{x_1, \dots, x_{k-1}\}$, then G is not a DAG, contradiction.

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- If there exists an arc $x_k y$ with $y \notin \{x_1, \ldots, x_{k-1}\}$, then P is not longest, contradiction.
- $\Rightarrow x_k$ is a sink (out-degree 0)
- Symmetric reasoning shows that x_1 is a source.

□

Straightforward Algorithm (for Matrices)

procedure Topo-Sort(G)
  for $i = 1$ to n **do**
    Find a source in $G \to v$
    Number of $v \leftarrow i$
    $G \leftarrow G - v$
  end for
  Output Numbers of $v \in V$
end procedure

Straightforward Algorithm (for Matrices)

procedure TOPO-SORT(G)
    for $i = 1$ to n **do**
        Find a source in $G \to v$ ▷ How?
        Number of $v \leftarrow i$ ▷ Num[v] $\leftarrow i$
        $G \leftarrow G - v$ ▷ How?
    end for
    Output Numbers of $v \in V$
end procedure

Straightforward Algorithm (for Matrices)

```
Active[v] ← 1 for all v ∈ V
procedure TOPO-SORT(G)
    for i = 1 to n do
        v ← Find-source(G,Active)
        Num[v] ← i
        Active[v] ← 0
    end for
    Output Numbers of v ∈ V
end procedure
procedure FIND-SOURCE(G,Active)
    for v ∈ V do
        if Active[v] == 1 and d⁻(v) == 0 then
            Return v
        end if
    end for
end procedure
```

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Straightforward Algorithm (for Matrices)

```
procedure CHECK-IF-SOURCE(G,Active,v)
    for u ∈ V do
        if A[u, v] == 1 and Active[u] then
            Return No
        end if
    end for
    Return Yes
end procedure
```

Straightforward Algorithm (for Matrices)

procedure CHECK-IF-SOURCE(G,Active,v) $\triangleright$ $O(n)$ time
    for $u \in V$ **do**
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        if Active[v] $==1$ and $d^-(v) == 0$ **then** $\triangleright$ $O(n)$ time
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        end if
    end for
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Straightforward Algorithm (for Matrices)

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Active[v] ← 1 for all v ∈ V
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    for i = 1 to n do                                   ▷ O(n) iterations
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    for v ∈ V do                                        ▷ O(n) iterations
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            Return v
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    end for
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Optimality for Find-Source

Lemma

Find-Source cannot be solved in $o(n^2)$ time (for adjacency matrices).

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Proof.

- Intuition: If an algorithm takes $< \binom{n}{2} - 1$ steps, there exists pair i, j for which neither $A[i, j]$ nor $A[j, i]$ was consulted, therefore impossible to know which of i, j are sources.
- Adversary argument: as long as possible, reply to an algorithm's queries by saying that a vertex has no incoming arcs, until the last step.

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□

Caution!

- This does not imply that our topological sorting algorithm is optimal!

Topological sort in linear time

```
Initialize                          ▷ DFS Initialization as before
i ← n                               ▷ Next vertex to be added to list
for v ∈ V do
    if v is White then
        DFS-Visit(G,v)
    end if
end for
procedure DFS-Visit(G,u)
    ...                             ▷ DFS as before
    u.f = t, Color u Black          ▷ When u turns black, append.
    Num[u] ← i, i − −
end procedure
```

Correctness Analysis

Lemma

G is a DAG $\Leftrightarrow$ DFS produces no backward arcs.

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G is a DAG $\Leftrightarrow$ DFS produces no backward arcs.

Theorem

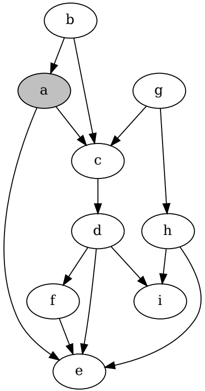
Previous algorithm is correct.

Proof.

- Consider arc uv at time when u became Gray:
  - If v was White $\Rightarrow$ v will be descendant of u $\Rightarrow$ v will become Black first $\Rightarrow$ v will be assigned higher number.
  - If v was Black $\Rightarrow$ v will be assigned higher number.
  - If v was Gray $\Rightarrow$ uv is a backward arc, contradiction!

□

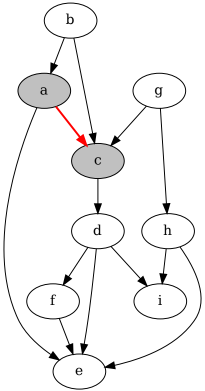
Example



Times:

	d	f
a	1	
b		
c		
d		
e		
f		
g		
h		
i		

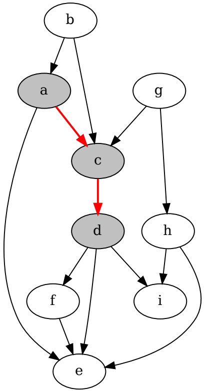
Example



Times:

	d	f
a	1	
b		
c	2	
d		
e		
f		
g		
h		
i		

Example



Times:

	d	f
a	1	
b		
c	2	
d	3	
e		
f		
g		
h		
i		

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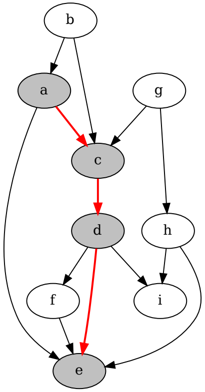
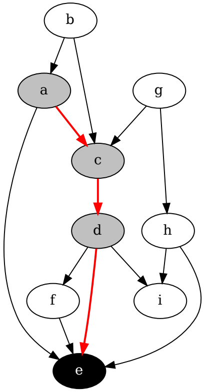


Times:

	d	f
a	1	
b		
c	2	
d	3	
e	4	
f		
g		
h		
i		

Example



Times:

	d	f
a	1	
b		
c	2	
d	3	
e	4	5
f		
g		
h		
i		

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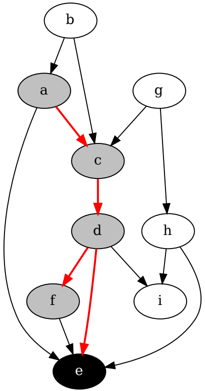

Times:

	d	f
a	1	
b		
c	2	
d	3	
e	4	5
f	6	
g		
h		
i		

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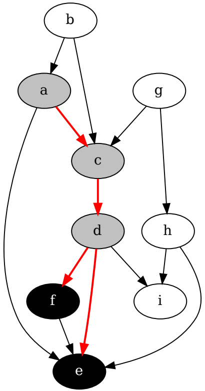

Times:

	d	f
a	1	
b		
c	2	
d	3	
e	4	5
f	6	7
g		
h		
i		

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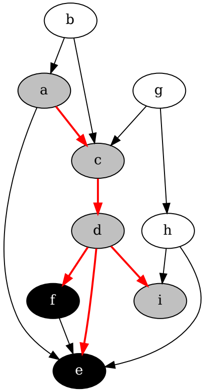
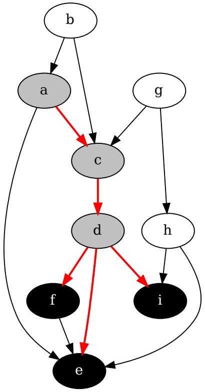


Times:

	d	f
a	1	
b		
c	2	
d	3	
e	4	5
f	6	7
g		
h		
i	8	

Example



Times:

	d	f
a	1	
b		
c	2	
d	3	
e	4	5
f	6	7
g		
h		
i	8	9

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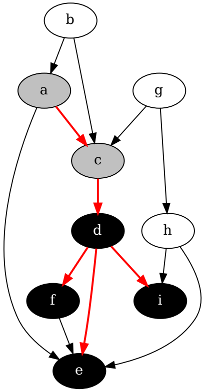
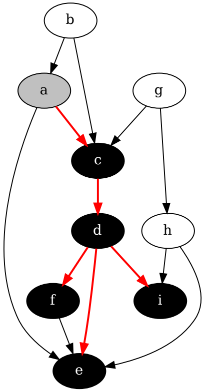


Times:

	d	f
a	1	
b		
c	2	
d	3	10
e	4	5
f	6	7
g		
h		
i	8	9

Example



Times:

	d	f
a	1	
b		
c	2	11
d	3	10
e	4	5
f	6	7
g		
h		
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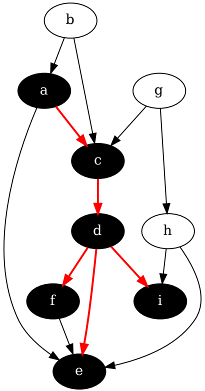
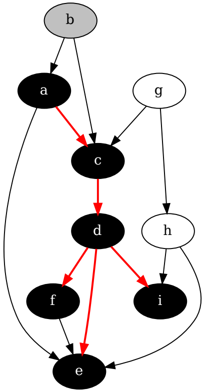


Times:

	d	f
a	1	12
b		
c	2	11
d	3	10
e	4	5
f	6	7
g		
h		
i	8	9

Example



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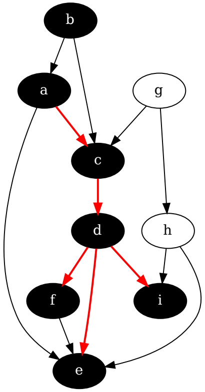

Times:

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g		
h		
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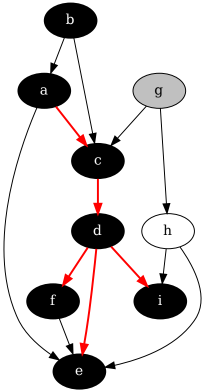
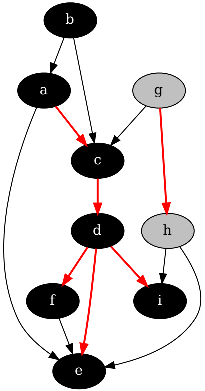
Example



Times:

	d	f
a	1	12
b	13	14
c	2	11
d	3	10
e	4	5
f	6	7
g	15	
h		
i	8	9

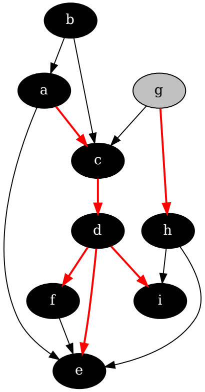
Example



Times:

	d	f
a	1	12
b	13	14
c	2	11
d	3	10
e	4	5
f	6	7
g	15	
h	16	
i	8	9

Example



Times:

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e	4	5
f	6	7
g	15	
h	16	17
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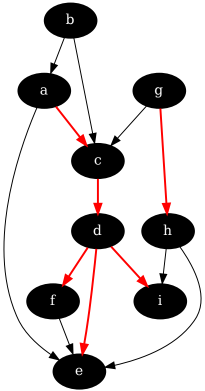
Example



Times:

	d	f
a	1	12
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d	3	10
e	4	5
f	6	7
g	15	18
h	16	17
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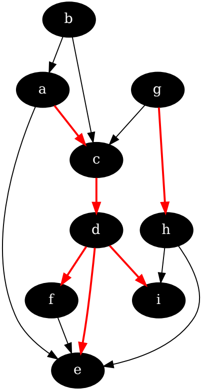
Example



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c	2	11
d	3	10
e	4	5
f	6	7
g	15	18
h	16	17
i	8	9

Ordering:

g, h, b, a, c, d, i, f, e

Strongly Connected Components

Strongly Connected Components

Definition

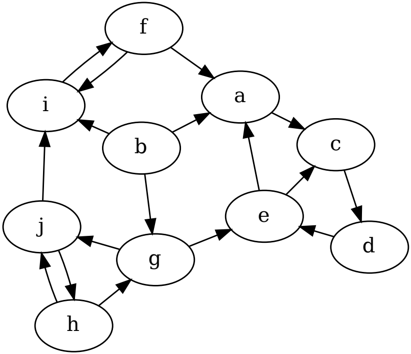
In a digraph G, a strongly connected component C is a maximal set of vertices such that for all $u, v \in C$. G contains a $u \to v$ and a $v \to u$ path.

Problem

Given a digraph $G = (V, A)$, output a partition of V into strongly connected components.

NB: For example, compute an integer $cc(v)$ for each $v \in V$ such that $cc(v) = cc(u)$ if and only if v, u are in the same SCC.

SCC Example



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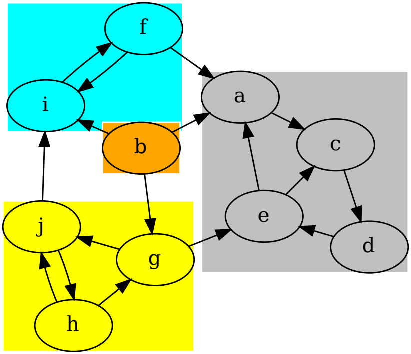

Straightforward Algorithm

```
procedure SCC(G)
    cc(v) ← −1 for all v ∈ V
    cur ← 1
    for v ∈ V do
        if cc(v) == −1 then
            cc(v) ← cur
            for u ∈ V do
                if Reach(G, u, v) ∧ Reach(G, v, u) then
                    cc(u) = cc(v)
                end if
            end for
            cur + +
        end if
    end for
    Return cc
end procedure
```

Straightforward Algorithm

```
procedure SCC(G)                                  ▷ O(n^3 + n^2 m) time
    cc(v) ← −1 for all v ∈ V
    cur ← 1
    for v ∈ V do                                  ▷ O(n) iterations
        if cc(v) == −1 then
            cc(v) ← cur
            for u ∈ V do                          ▷ O(n) iterations
                if Reach(G, u, v) ∧ Reach(G, v, u) then   ▷ O(n + m)
                    cc(u) = cc(v)
                end if
            end for
            cur++
        end if
    end for
    Return cc
end procedure
```

SCC by 2-DFS algorithm

Algorithm idea:

1. Run DFS, compute a finish time for each vertex.
2. Compute G^T.
   - Reminder: G^T is G where arcs are reversed.
3. Run DFS on G^T.
   - **Important:** Consider vertices **not** in alphabetical order, but in decreasing order of finish time from first DFS.
   - (Topological sort order, if G was a DAG).
4. Each DFS tree from the previous step is a SCC of G.

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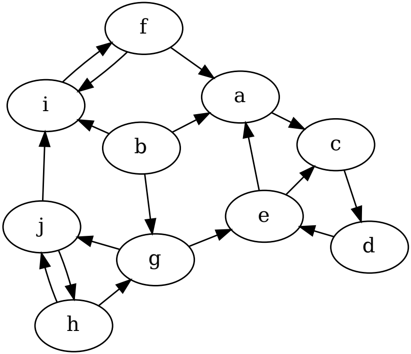
Sanity check: is this algorithm correct on DAGs?

SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	f
a		
b		
c		
d		
e		
f		
g		
h		
i		
j		

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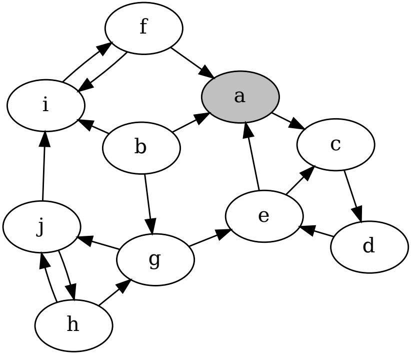
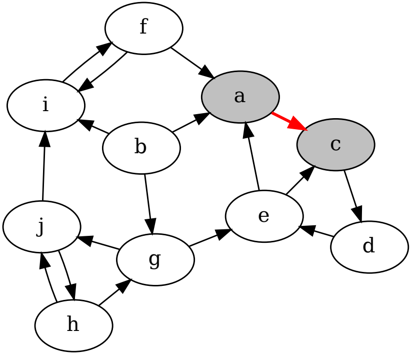


Times:

	d	*f*
a	1	
b		
c		
d		
e		
f		
g		
h		
i		
j		

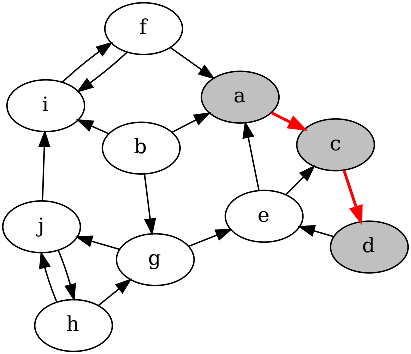
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	*f*
a	1	
b		
c	2	
d		
e		
f		
g		
h		
i		
j		

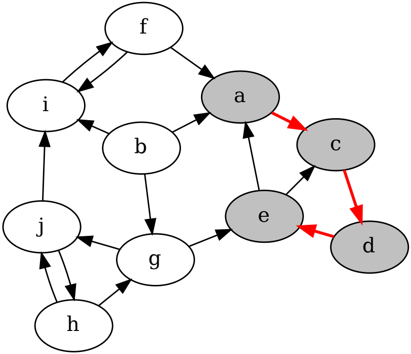
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	f
a	1	
b		
c	2	
d	3	
e		
f		
g		
h		
i		
j		

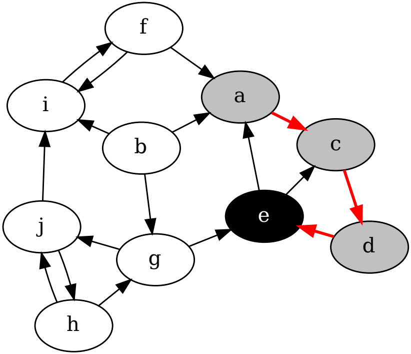
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	*f*
a	1	
b		
c	2	
d	3	
e	4	
f		
g		
h		
i		
j		

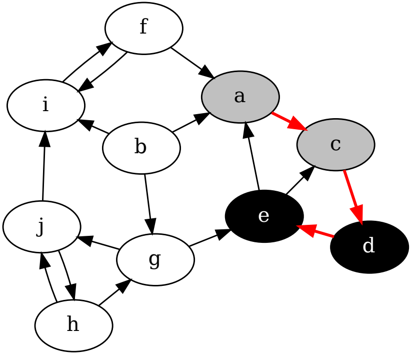
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	f
a	1	
b		
c	2	
d	3	
e	4	5
f		
g		
h		
i		
j		

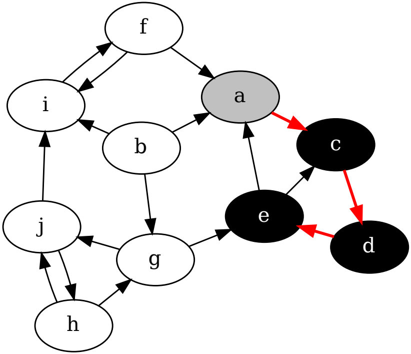
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	f
a	1	
b		
c	2	
d	3	6
e	4	5
f		
g		
h		
i		
j		

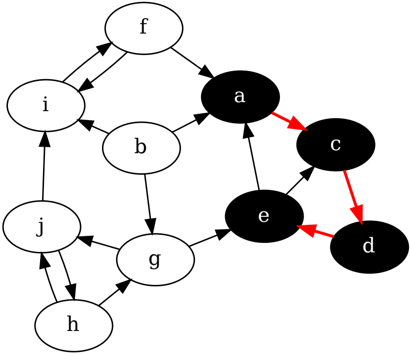
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	f
a	1	
b		
c	2	7
d	3	6
e	4	5
f		
g		
h		
i		
j		

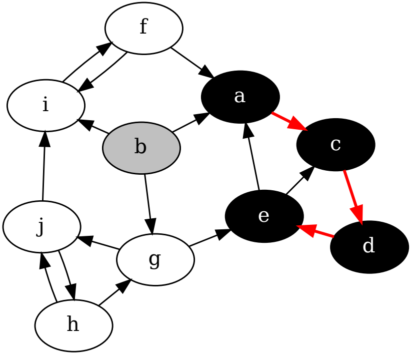
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	f
a	1	8
b		
c	2	7
d	3	6
e	4	5
f		
g		
h		
i		
j		

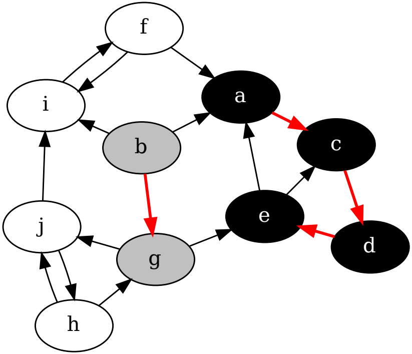
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	f
a	1	8
b	9	
c	2	7
d	3	6
e	4	5
f		
g		
h		
i		
j		

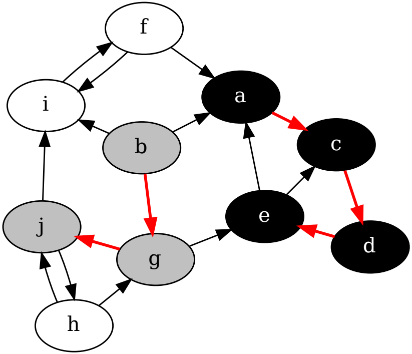
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	f
a	1	8
b	9	
c	2	7
d	3	6
e	4	5
f		
g	10	
h		
i		
j		

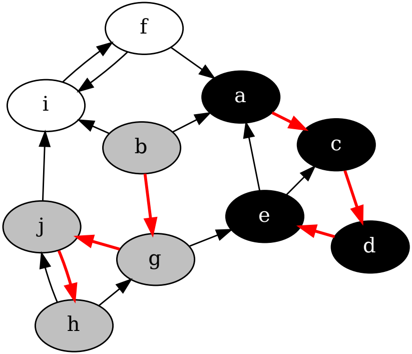
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	*f*
a	1	8
b	9	
c	2	7
d	3	6
e	4	5
f		
g	10	
h		
i		
j	11	

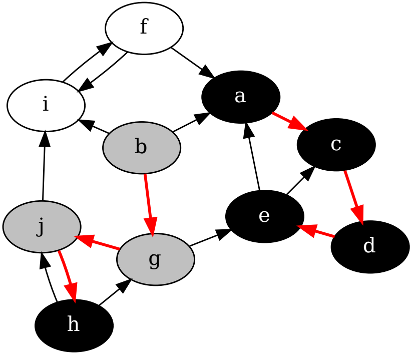
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	*f*
a	1	8
b	9	
c	2	7
d	3	6
e	4	5
f		
g	10	
h	12	
i		
j	11	

SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	*f*
a	1	8
b	9	
c	2	7
d	3	6
e	4	5
f		
g	10	
h	12	13
i		
j	11	

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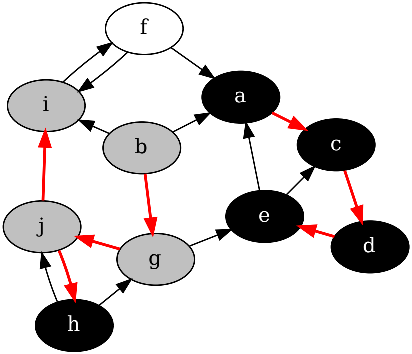
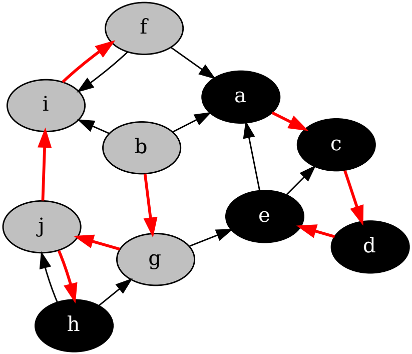


Times:

	d	*f*
a	1	8
b	9	
c	2	7
d	3	6
e	4	5
f		
g	10	
h	12	13
i	14	
j	11	

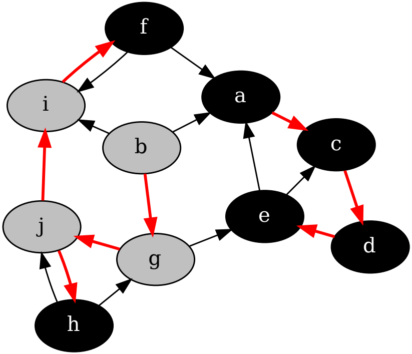
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	f
a	1	8
b	9	
c	2	7
d	3	6
e	4	5
f	15	
g	10	
h	12	13
i	14	
j	11	

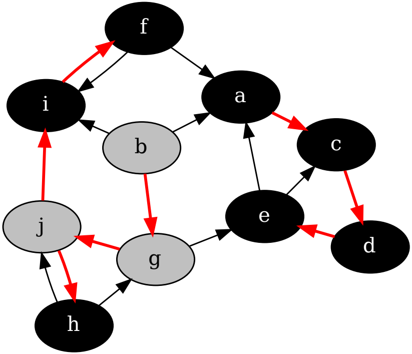
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	f
a	1	8
b	9	
c	2	7
d	3	6
e	4	5
f	15	16
g	10	
h	12	13
i	14	
j	11	

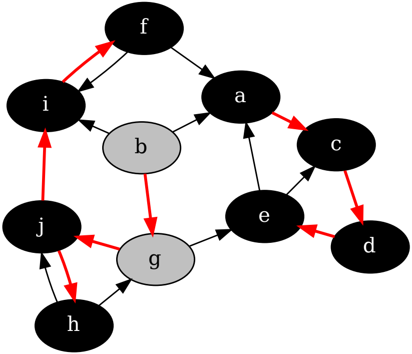
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	f
a	1	8
b	9	
c	2	7
d	3	6
e	4	5
f	15	16
g	10	
h	12	13
i	14	17
j	11	

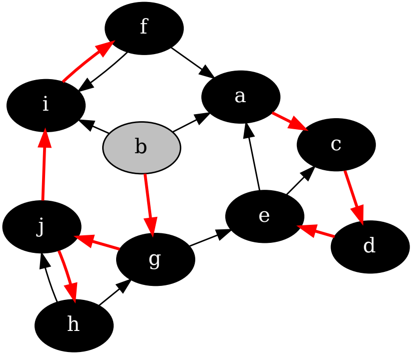
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	f
a	1	8
b	9	
c	2	7
d	3	6
e	4	5
f	15	16
g	10	
h	12	13
i	14	17
j	11	18

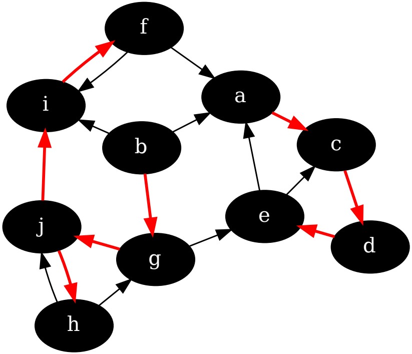
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	*f*
a	1	8
b	9	
c	2	7
d	3	6
e	4	5
f	15	16
g	10	19
h	12	13
i	14	17
j	11	18

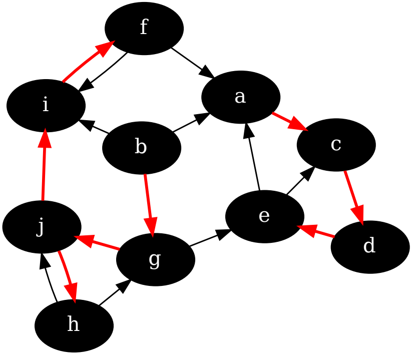
SCC by 2-DFS algorithm – Example – Phase 1



Times:

	d	f
a	1	8
b	9	20
c	2	7
d	3	6
e	4	5
f	15	16
g	10	19
h	12	13
i	14	17
j	11	18

SCC by 2-DFS algorithm – Example – Phase 1



Times:

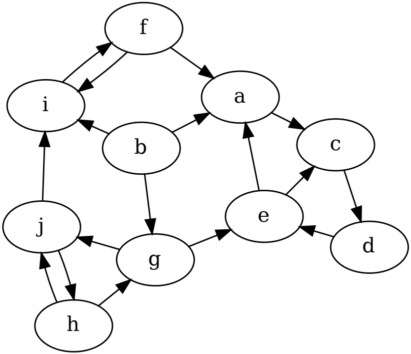
	d	f
a	1	8
b	9	20
c	2	7
d	3	6
e	4	5
f	15	16
g	10	19
h	12	13
i	14	17
j	11	18

Ordering in decreasing finish time:

b, g, j, i, f, h, a, c, d, e

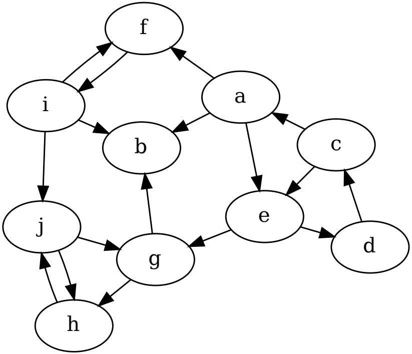
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



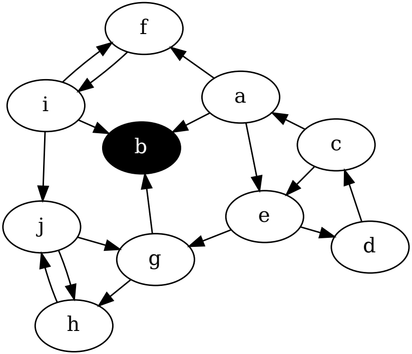
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



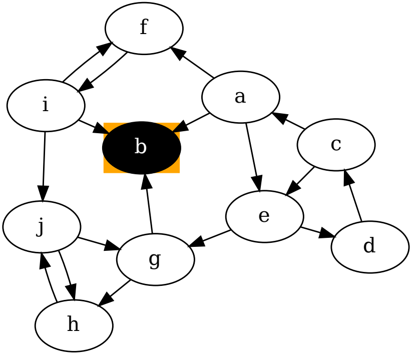
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



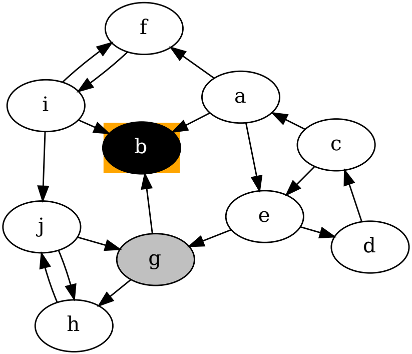
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



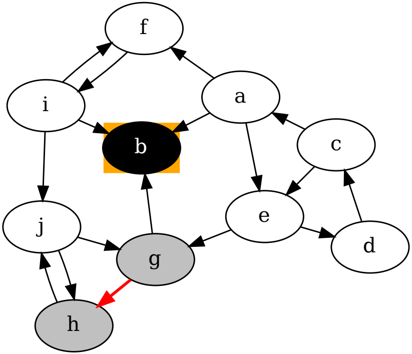
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



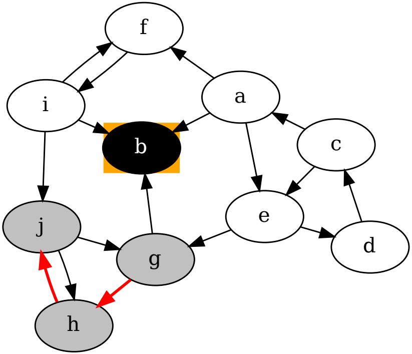
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



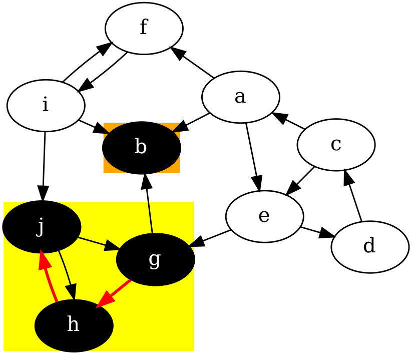
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



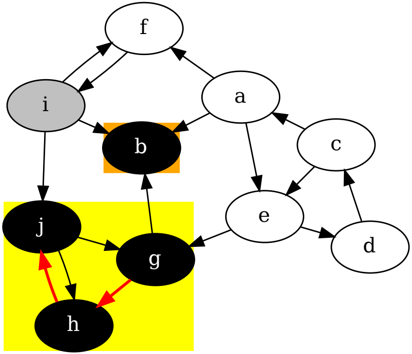
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



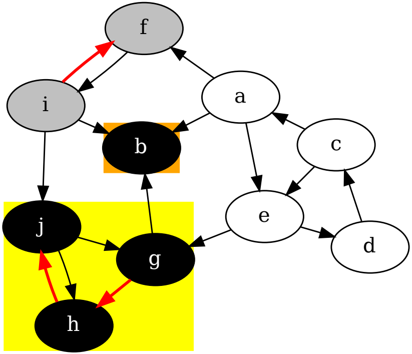
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



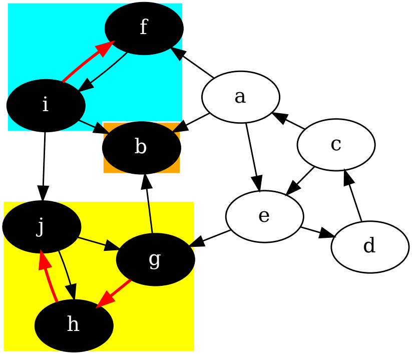
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



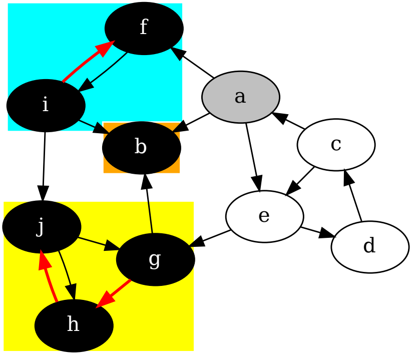
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



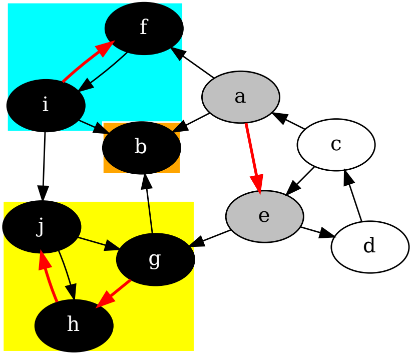
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



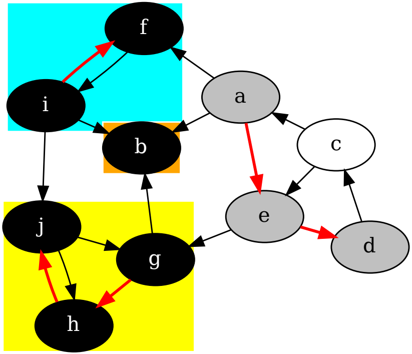
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



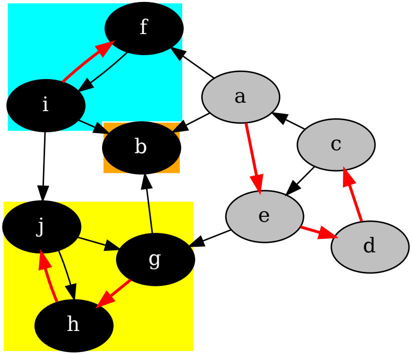
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



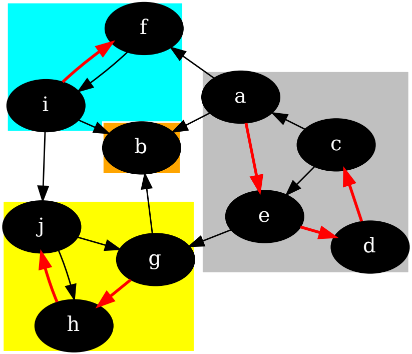
SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



SCC by 2-DFS algorithm – Example – Phase 2

Ordering: b, g, j, i, f, h, a, c, d, e



Intuition – Component digraph

Definition

If $G = (V, A)$ is a digraph, G^{SCC} is the digraph that has:

- A vertex for each SCC of G.
- An arc $C_1 C_2$ if there exist $x_1 \in C_1, x_2 \in C_2$ with $x_1 x_2 \in A$.

Lemma

G^{SCC} is always a DAG.

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Lemma

G^{SCC} is always a DAG.

Proof.

If there exist arcs $C_1 \to C_2$ and $C_2 \to C_1$, $C_1 \cup C_2$ is strongly connected, contradicting maximality. □

Intuition

Key ideas:

- If x finished last in first DFS of G

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- $\Rightarrow$ the SCC of x is a source in G^{SCC}

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Intuition

Key ideas:

- If x finished last in first DFS of G
- $\Rightarrow$ the SCC of x is a source in G^{SCC}
- $\Rightarrow$ the SCC of x is a sink in $(G^T)^{SCC}$
- $\Rightarrow$ a DFS in G^T from x will visit exactly the SCC of x

Sources SCC finishes last

Lemma

Let $G = (V, A)$ *be a digraph,* C_1, C_2 *two SCCs, with* $x_1 \in C_1, x_2 \in C_2$ *and* $x_1x_2 \in A$*. Then, there exists a vertex of* C_1 *which finishes* **after** *all vertices of* C_2*.*

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Let $G = (V, A)$ be a digraph, C_1, C_2 two SCCs, with $x_1 \in C_1, x_2 \in C_2$ and $x_1x_2 \in A$. Then, there exists a vertex of C_1 which finishes **after** *all vertices of C_2.*

Proof.

- First vertex of $C_1 \cup C_2$ to be discovered is $y \in C_1$:
  By White-Path theorem, all of $C_1 \cup C_2$ are y's descendants, so finish before y.

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Lemma

Let $G = (V, A)$ be a digraph, C_1, C_2 two SCCs, with $x_1 \in C_1, x_2 \in C_2$ and $x_1 x_2 \in A$. Then, there exists a vertex of C_1 which finishes **after** *all vertices of C_2.*

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- First vertex of $C_1 \cup C_2$ to be discovered is $y \in C_2$:
  By White-Path theorem, all vertices of C_2 are discovered after y and finish before y. There is no path from C_2 to C_1 (o/w G^{SCC} not a DAG), so at f_y all of C_1 still White $\Rightarrow$ finishes later than y.

□

Proof of Correctness

Proof.

Induction on number of SCCs.

- Suppose first k SCCs are correct.
- DFS for $(k+1)$-th SCC starts at x which has largest **finish** time of all White vertices.
  - All vertices of the SCC of x (C) are currently White (I.H.)
  - White-Path theorem: SCC computed contains **at least** all of C
  - Need to prove: we do not put **other stuff** in computed SCC of x

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- If $y \notin C$ is reachable from x in G^T:
  - Either y is in the first k components $\Rightarrow$ y Black
  - Or $y \in C'$ with some $C \to C'$ path in G^T

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Induction on number of SCCs.

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- DFS for $(k+1)$-th SCC starts at x which has largest **finish** time of all White vertices.
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  - Need to prove: we do not put **other stuff** in computed SCC of x
- If $y \notin C$ is reachable from x in G^T:
  - Either y is in the first k components $\Rightarrow$ y Black
  - Or $y \in C'$ with some $C \to C'$ path in G^T
  - Then, $\exists C' \to C$ path in G
  - $\Rightarrow$ some vertex of C' finishes after x, contradiction!!

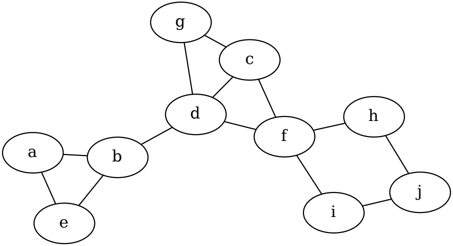
□

Articulation Points

Articulation Point

Definition

In a **undirected** graph $G = (V, E)$, $v \in V$ is a **cut vertex** or an **articulation point** if $G - v$ has more connected components than G.



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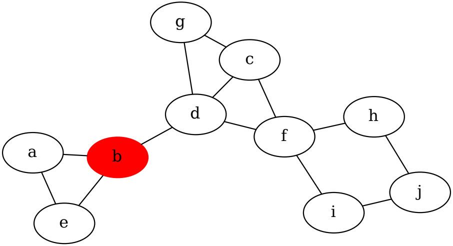

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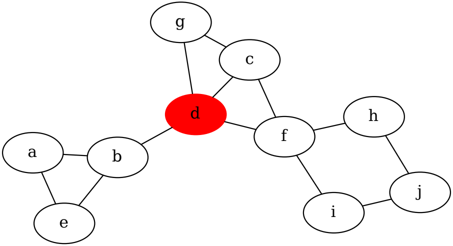

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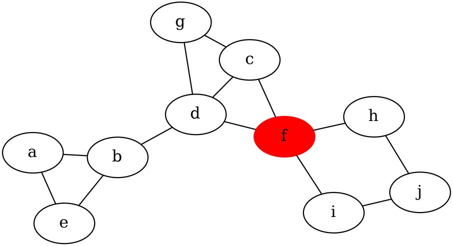
Definition

In a **undirected** graph $G = (V, E)$, $v \in V$ is a **cut vertex** or an **articulation point** if $G - v$ has more connected components than G.



Articulation Points

Problem

Given undirected graph G, output all articulation points (if any).

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Obvious algorithm:

$C \leftarrow$ number of components of G (DFS)
for $v \in V$ **do**
    if comps($G - v$)$>C$ **then**
        Add v to output
    end if
end for

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Complexity $O(mn + n^2)$ for lists, $O(n^3)$ for matrices. Better?

Articulation Points – DFS

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Lemma

The root r of a DFS tree is an articulation point if and only if r has at least two children.

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An internal vertex v of a DFS tree is an articulation point, if and only if it has a child c such that no descendant of c is adjacent to a proper ancestor of v.

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Proof.

- $\Leftarrow$: let c_1, c_2 be the two children with min discovery times. Then, c_1, c_2 in distinct components of $G - r$.
  - At time 2 only r, c_1 Gray, so if $c_1 \to c_2$ path exists, by White-Path theorem, c_2 would be descendant of c_1.

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  - At time 2 only r, c_1 Gray, so if $c_1 \to c_2$ path exists, by White-Path theorem, c_2 would be descendant of c_1.
- $\Rightarrow$: if r has one child, there is a path between any two vertices of $G - r$ (using tree edges only).

□

Articulation Points – Lemma 2

Lemma

An internal vertex v of a DFS tree is an articulation point, if and only if it has a child c such that no descendant of c is adjacent to a proper ancestor of v. (assume G connected)

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Proof.

- $\Leftarrow$: c and its descendants form a component of $G - v$ which does not contain the proper ancestors of v. (also: no cross edges)
- $\Rightarrow$: Let C_1 be the component of $G - v$ where DFS started.
  - v has earlier discovery than all vertices of $G \setminus (C_1 \cup \{v\})$
  - $\Rightarrow$ all other vertices of $G \setminus (C_1 \cup \{v\})$ are descendants of v by White-Path theorem
  - All proper ancestors of v are in C_1
  - No edges between C_1 and other components of $G - v$

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Towards an algorithm

- Execute DFS
- Decide if root is an articulation point (easy!)
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- Execute DFS
- Decide if root is an articulation point (easy!)
- Decide if each internal vertex is an articulation point
  - Problem: obvious algorithm is $O(m)$ per vertex...
  - Need to store some information so we don't repeat work...

Going MADD

Definition

Given G and DFS tree, we define $\mathrm{madd}(v)$ to be the **minimum ancestor-descendant discovery** time among the neighbors of v. Formally,

$$\mathrm{madd}(v) = \min_{u \in \mathrm{desc}(v), w \in N[u]} d_w$$

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- In other words: for each v, we check all the sub-trees rooted at v, and find who has the **highest** neighbor in the tree.

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- In other words: for each v, we check all the sub-trees rooted at v, and find who has the **highest** neighbor in the tree.
- Claim: For internal vertex v with child x, v is an articulation point if and only if

$$\mathrm{madd}(x) = d_v$$

.

- Claim: $\mathrm{madd}(v)$ can be computed for all vertices in linear time.

MADD in linear time

Lemma

We can compute all minimum ancestor-descendant discovery times in linear time.

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Proof.

$$\mathrm{madd}(v) = \min_{u \in \mathrm{desc}(v), w \in N[u]} d_w$$

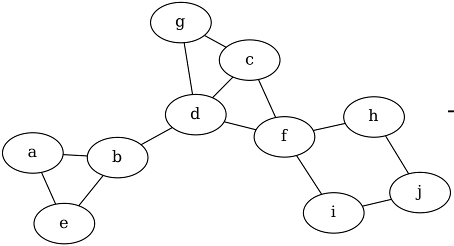
is equivalent to:

$$\mathrm{madd}(v) = \min\{(\min_{u \in \mathrm{child}(v)} \mathrm{madd}(u)), (\min_{u \in N(v)} d_u\})$$

which can be computed bottom-up by checking computed values for the children of each node, when a vertex becomes Black.

□

Articulation points – Example



Times:

	d	f	madd
a			*
b			
c			
d			
e			
f			
g			
h			
i			
j			

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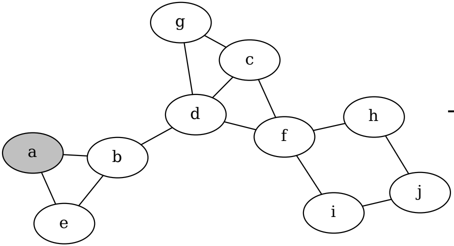

Times:

	d	f	madd
a	1		*
b			
c			
d			
e			
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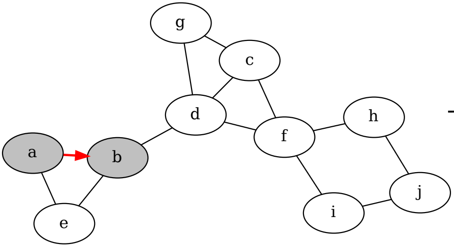

Times:

	d	f	madd
a	1		*
b	2		
c			
d			
e			
f			
g			
h			
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j			

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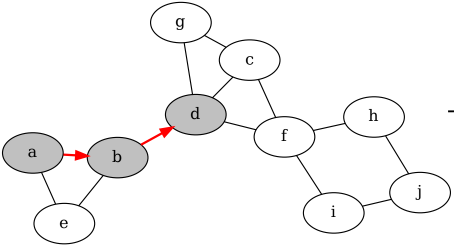

Times:

	d	f	madd
a	1		*
b	2		
c			
d	3		
e			
f			
g			
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i			
j			

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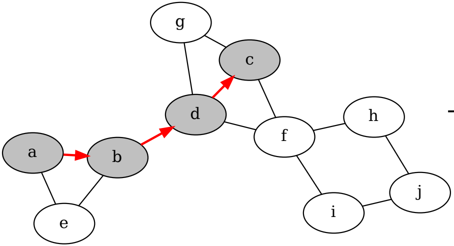

Times:

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b	2		
c	4		
d	3		
e			
f			
g			
h			
i			
j			

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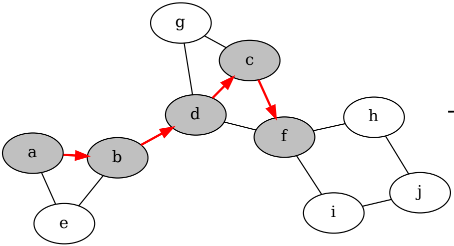

Times:

	d	f	madd
a	1		*
b	2		
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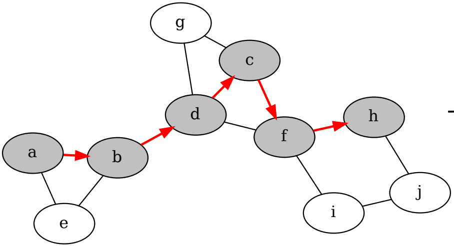

Times:

	d	f	madd
a	1		*
b	2		
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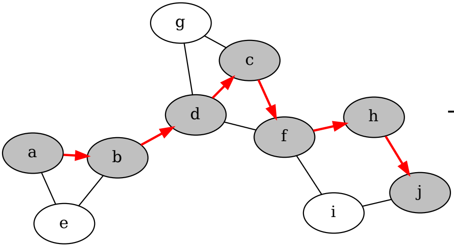
Articulation points – Example



Times:

	d	f	madd
a	1		*
b	2		
c	4		
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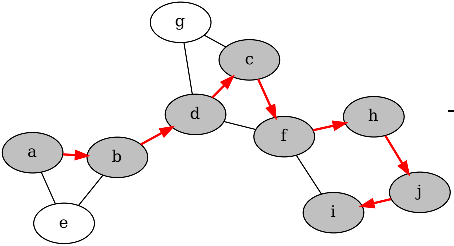

Times:

	d	f	madd
a	1		*
b	2		
c	4		
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f	5		
g			
h	6		
i	8		
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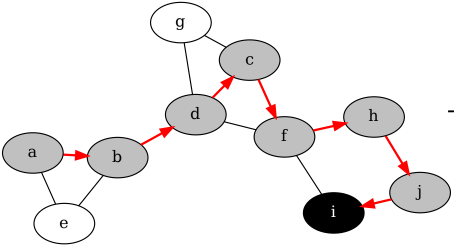

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	d	f	madd
a	1		*
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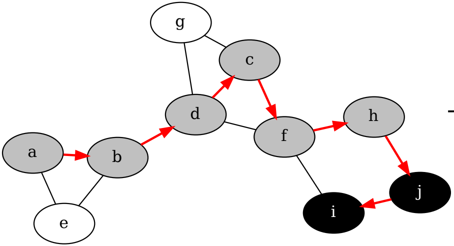

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a	1		*
b	2		
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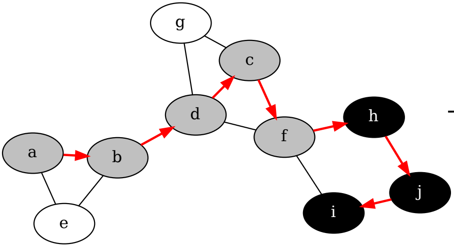

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a	1		*
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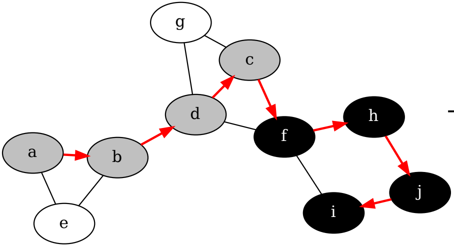

Times:

	d	f	madd
a	1		*
b	2		
c	4		
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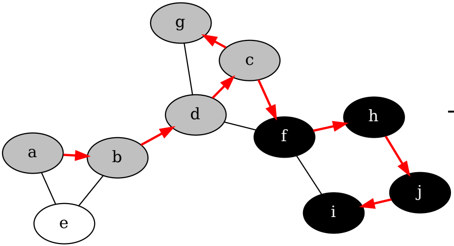

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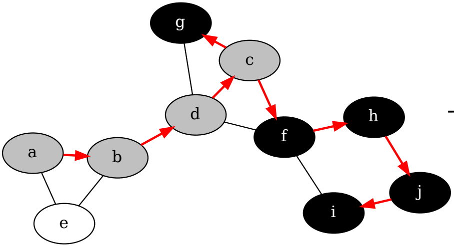

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b	2		
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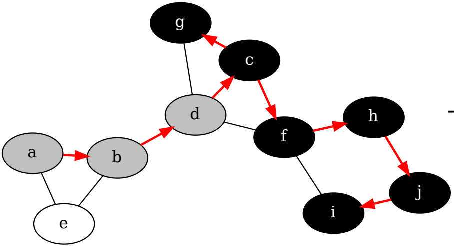

Times:

	d	f	madd
a	1		*
b	2		
c	4	15	
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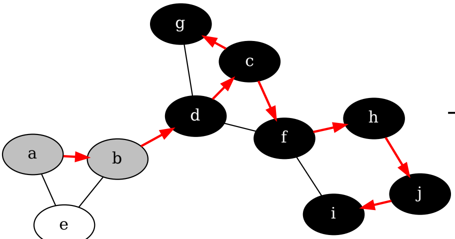

Times:

	d	f	madd
a	1		*
b	2		
c	4	15	
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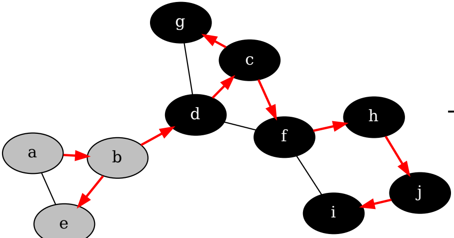

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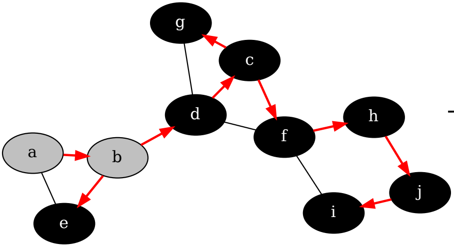

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	d	f	madd
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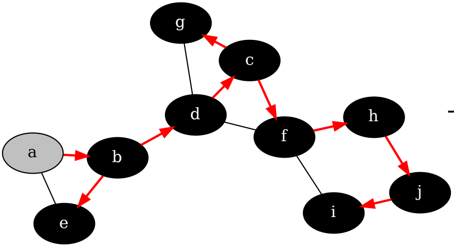

Times:

	d	f	madd
a	1		*
b	2	19	
c	4	15	
d	3	16	
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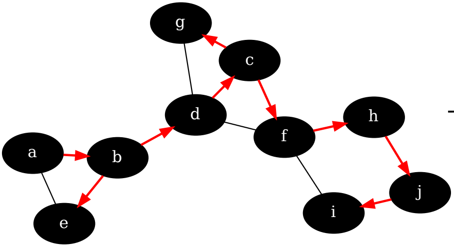

Times:

	d	f	madd
a	1	20	*
b	2	19	
c	4	15	
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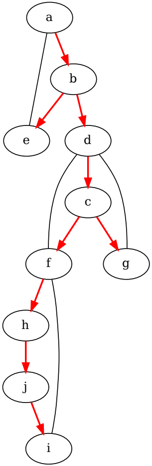

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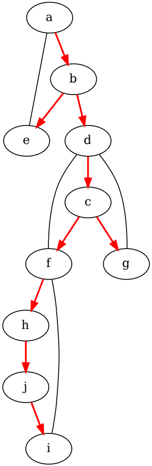

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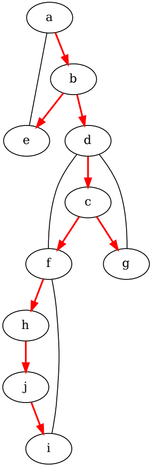

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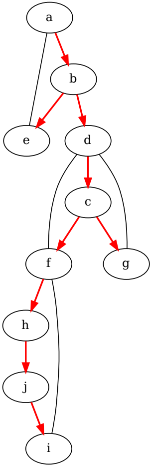

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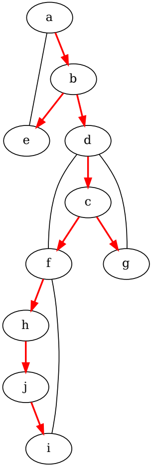

Times:

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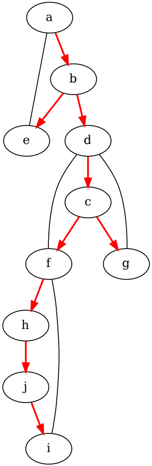

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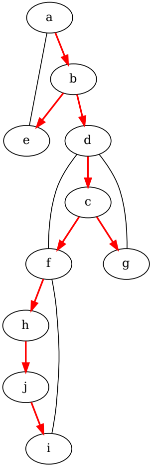

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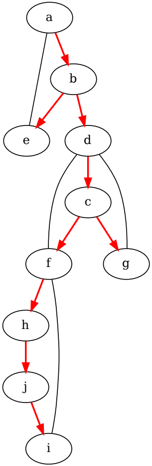

Times:

	d	f	madd
a	1	20	*
b	2	19	
c	4	15	3
d	3	16	2
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g	13	14	3
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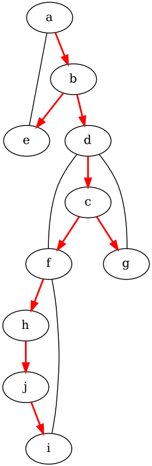

Times:

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c	4	15	3
d	3	16	2
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g	13	14	3
h	6	11	5
i	8	9	5
j	7	10	5

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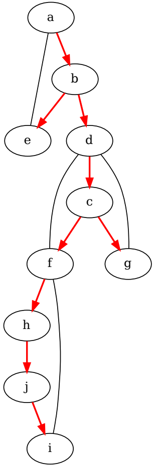

Times:

	d	f	madd
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b	2	19	1
c	4	15	3
d	3	16	2
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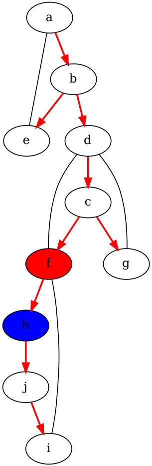

Times:

	d	f	madd
a	1	20	*
b	2	19	1
c	4	15	3
d	3	16	2
e	17	18	1
f	5	12	3
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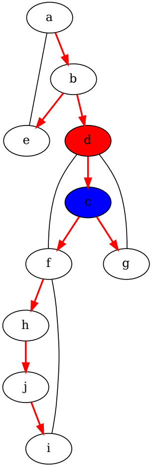
i	8	9	5
j	7	10	5

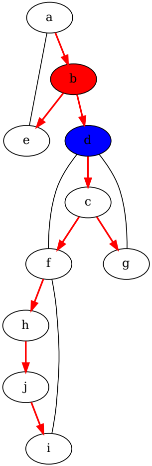
Articulation points – Example



Times:

	d	f	madd
a	1	20	*
b	2	19	1
c	4	15	3
d	3	16	2
e	17	18	1
f	5	12	3
g	13	14	3
h	6	11	5
i	8	9	5
j	7	10	5

Articulation points – Example



Times:

	d	f	madd
a	1	20	*
b	2	19	1
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h	6	11	5
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j	7	10	5