Graph Algorithms

Max Flow - Min Cut

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November 13, 2025

The story so far

- (Short) Reachability problems:
  - Is there a path from s to t? (Strongly) connected components? ...
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- Algorithms:
  - Unweighted Reachability: BFS/DFS $O(m+n)$ time
  - Single-Source Shortest Paths:
    - Unweighted: BFS $O(m+n)$ time
    - Positive weights: Dijkstra $O((m+n)\log n)$ time (with min-heaps)
    - General weights: Bellman-Ford $O(mn)$ time
  - Minimum Spanning Tree:
    - Prim's and Kruskal's algorithms: $O(m \log n)$ time

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- New problem: Maximum Flow - Minimum Cut
  - Find minimum weight set of edges that **destroys** connectedness.

Problem Definition

Definition

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For an edge-weighted digraph $G = (V, A)$, a **Minimum Edge Cut** between vertices $s, t \in V$ is a set of arcs $A' \subseteq A$ such that

1. $G = (V, A \setminus A')$ has no directed path from s to t.
2. The weight of A' is minimum among sets satisfying (1).

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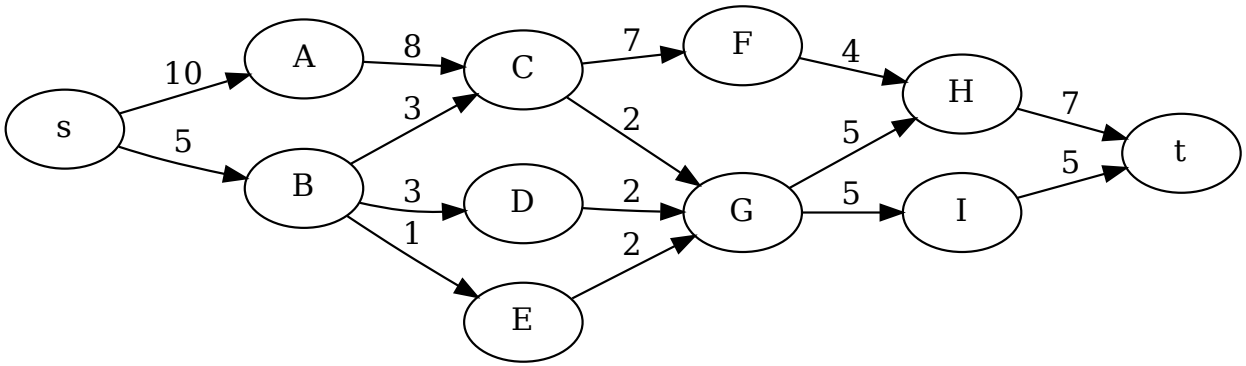
- Algorithmic problem: Given $G = (V, A)$ and $s, t \in V$, compute minimum $s - t$ cut.

Equivalent formulation:

Definition

In the Min $s - t$ Cut problem, we are given edge-weighted digraph $G = (V, A)$, $s, t \in V$ and are asked to compute a set $S \subseteq V$ with $s \in S, t \notin S$ such that $\sum_{x \in S} \sum_{y \notin S} w(xy)$ is minimum.

Example



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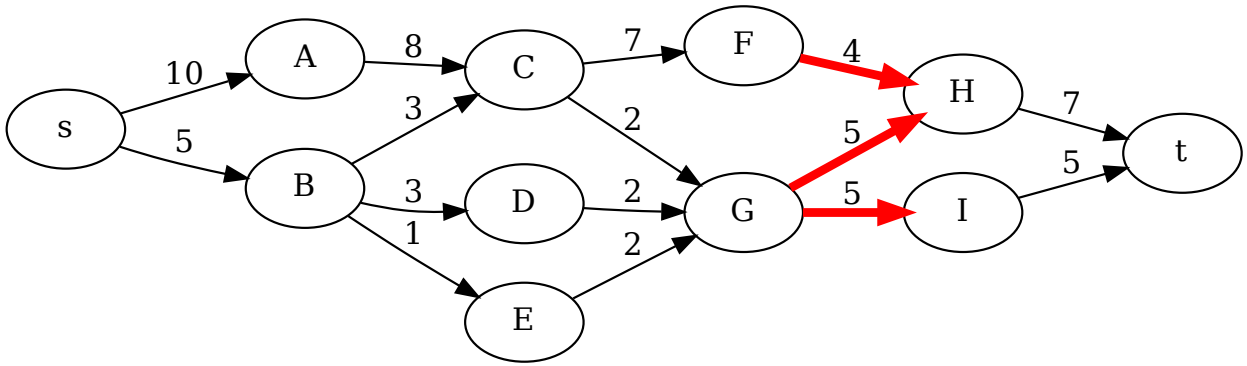
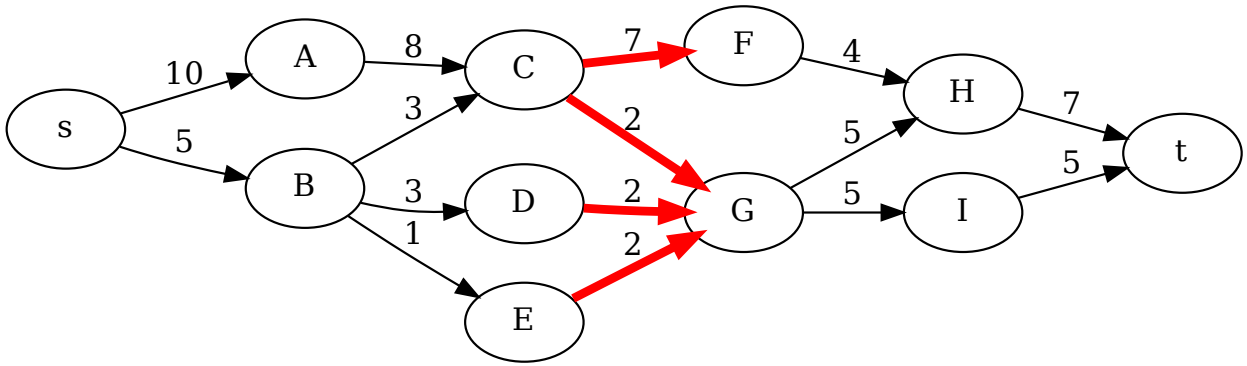


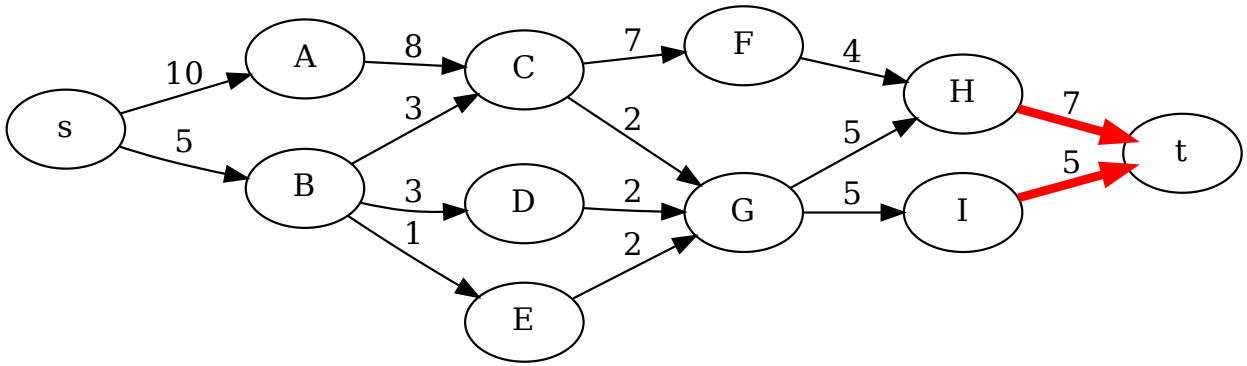
Cost: 14

Example



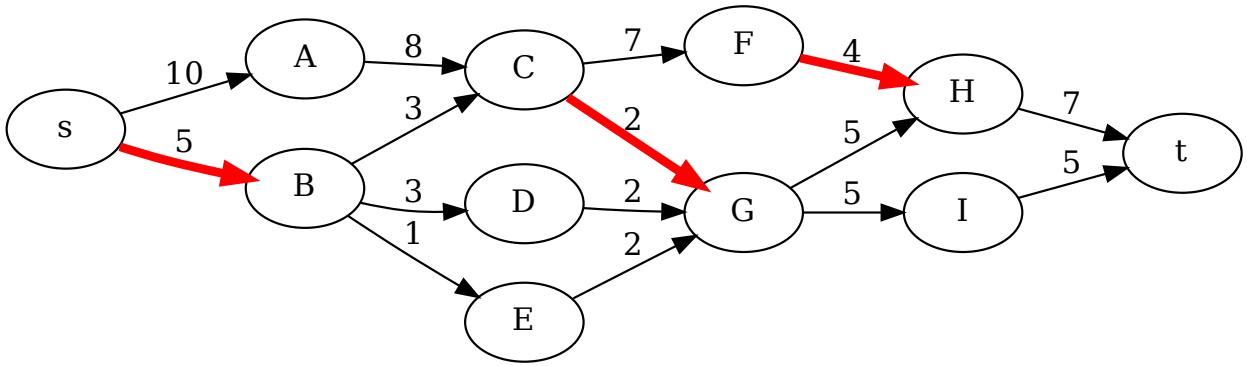
Cost: 13

Example



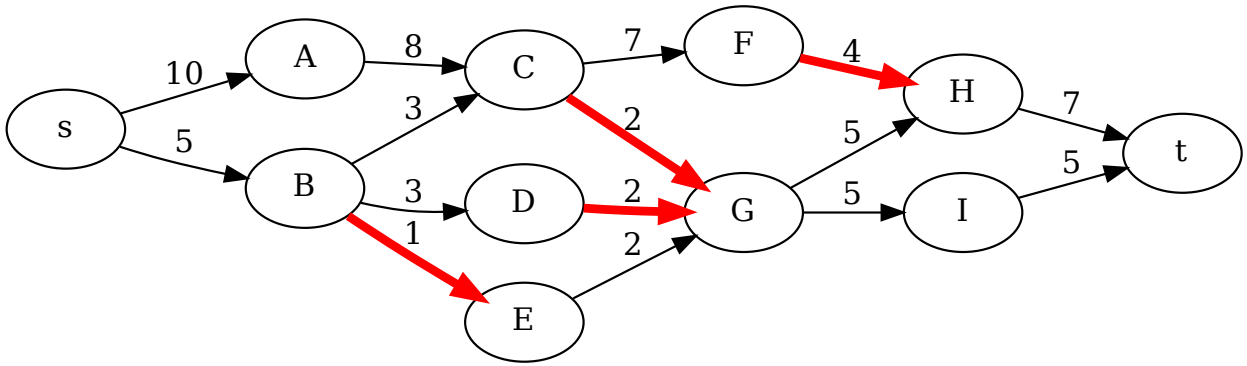
Cost: 12

Example



Cost: 11

Example



Cost: 9

Obvious (bad) algorithms

Lemma

The two formulations are equivalent.

Proof.

- Given cut-set A', define S as set of vertices reachable from s.
- Given set S, define A' as set of arcs from S to $V \setminus S$.

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- $O(2^m \cdot m)$ time: try all cut-sets
- $O(2^n \cdot m)$ time: try all sets S

Pretty terrible! Can we do polynomial time?

Flows

Definition

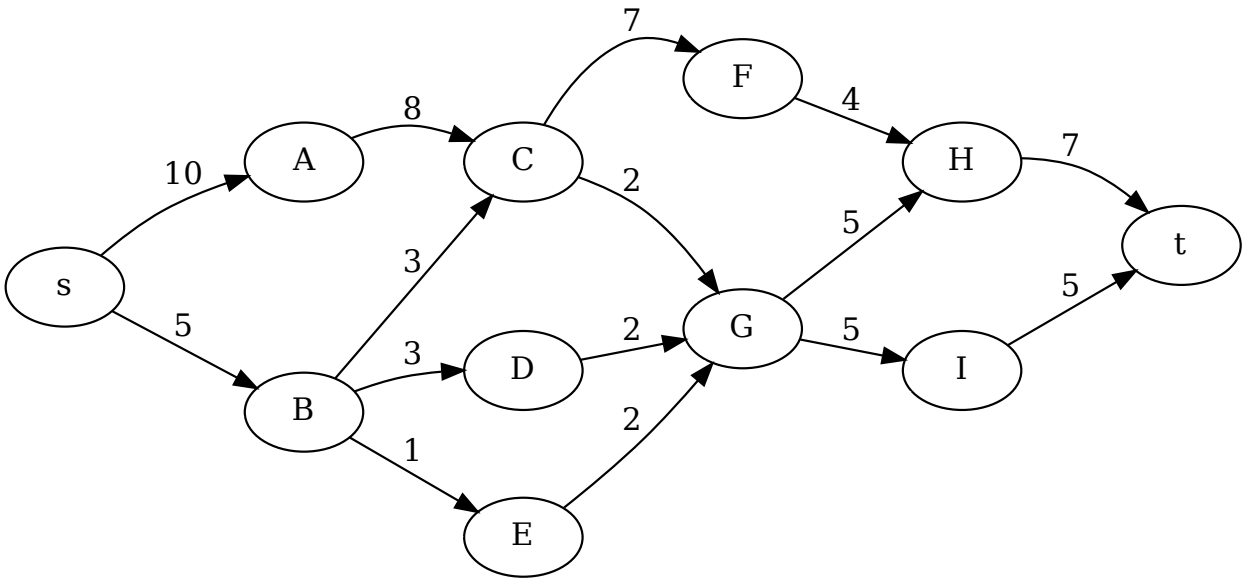
Given a digraph $G = (V, A)$, $s, t \in V$ and a capacity function $\mathbf{c} : A \to \mathbb{Q}^+$, a flow is a function $\mathbf{f} : A \to \mathbb{Q}^+$ satisfying the following:

1. Capacity-respecting: for all $a \in A$ we have $\mathbf{f}(a) \leq \mathbf{c}(a)$
2. Flow-conservation: for all $x \in V \setminus \{s, t\}$ we have $\sum_{z \in N^-(x)} \mathbf{f}(zx) = \sum_{y \in N^-(x)} \mathbf{f}(xy)$

Intuition:

- s is sending something (information, energy, etc.) to t.
- Capacities of arcs must be respected.
- Flows can be re-arranged in vertices, but must be conserved and go from s to t
- **Value** of flow $\mathbf{f}$ is total flow coming out of s
- We seek a flow of **maximum value**

Example



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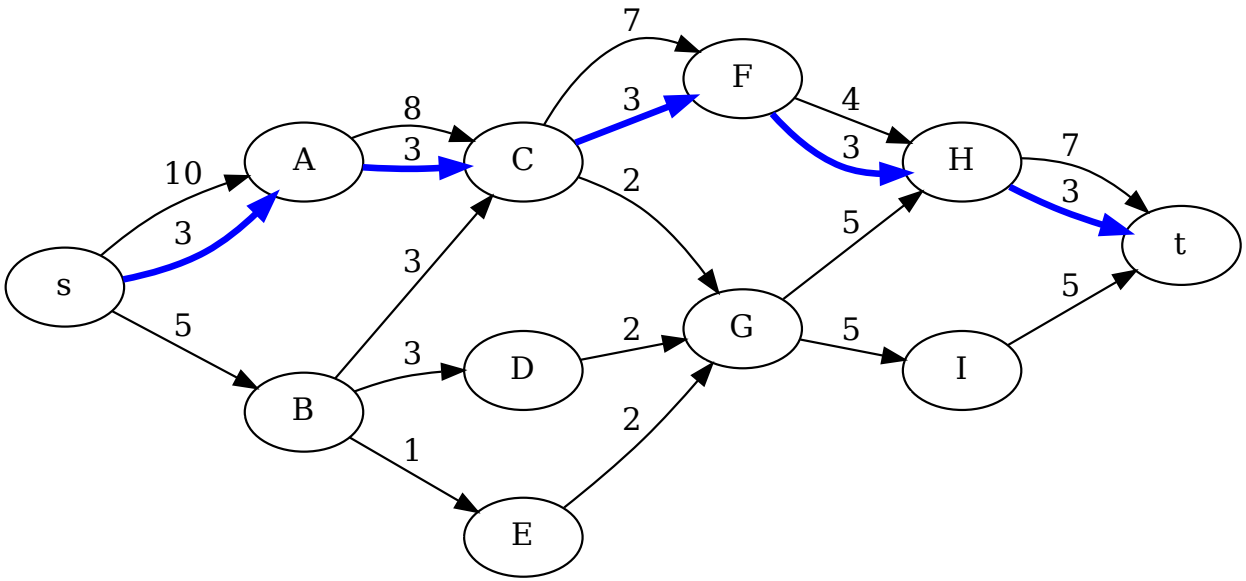

Flow: 3

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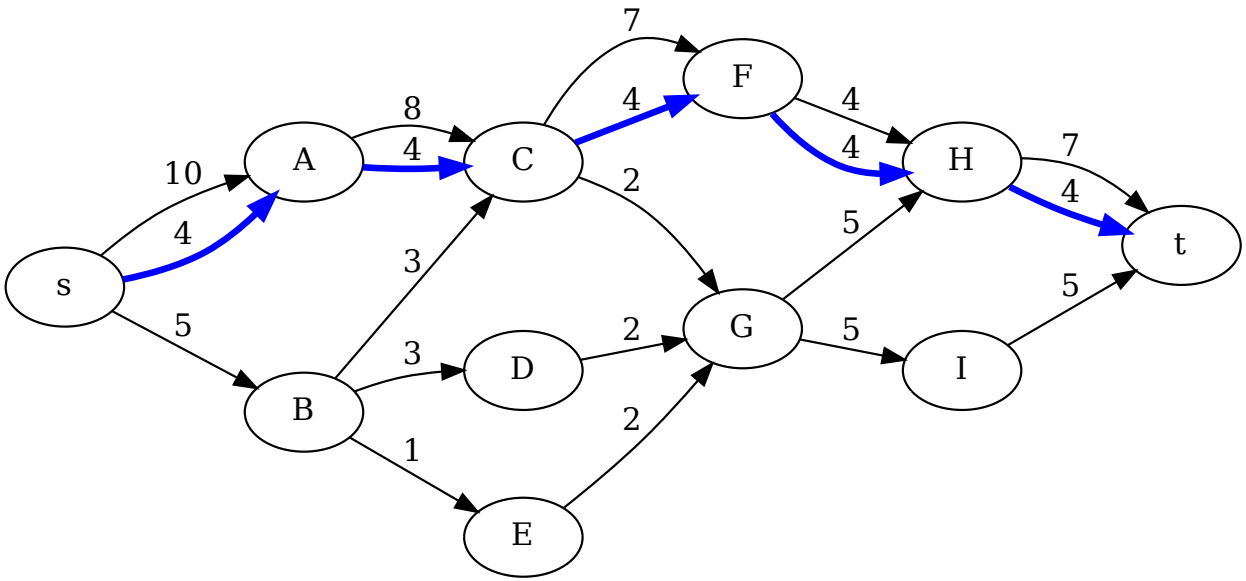
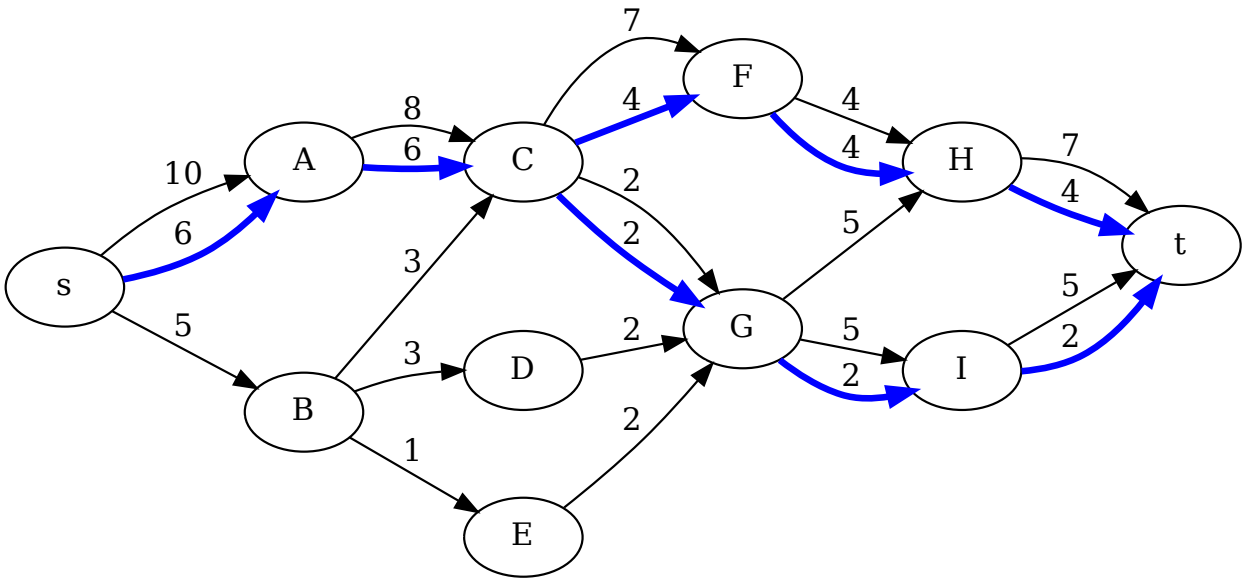


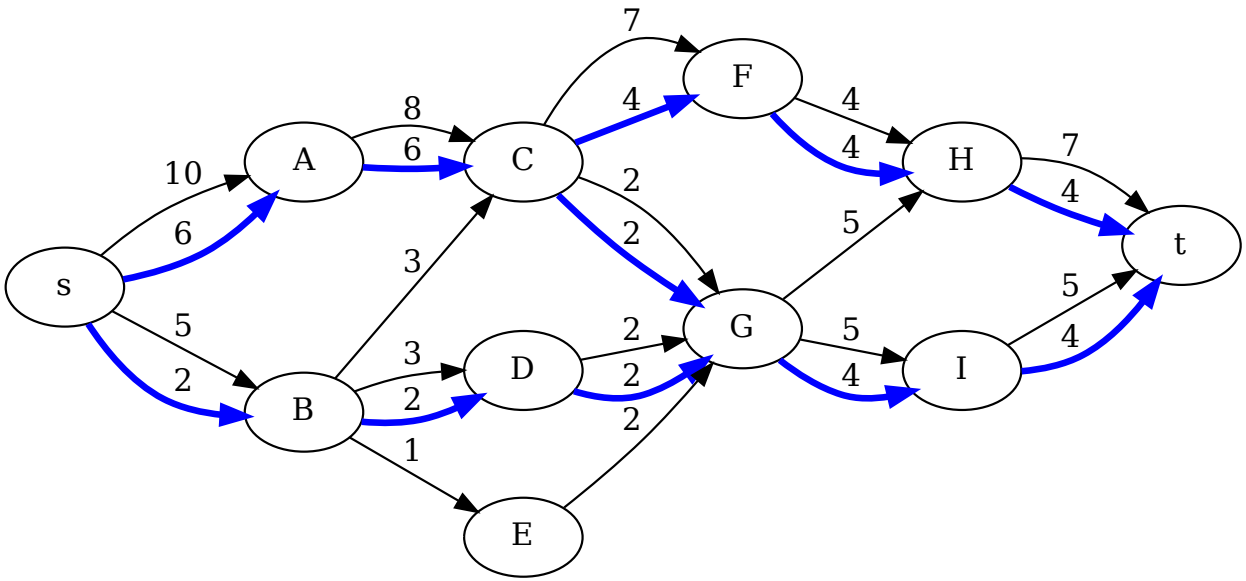
Flow: 4

Example



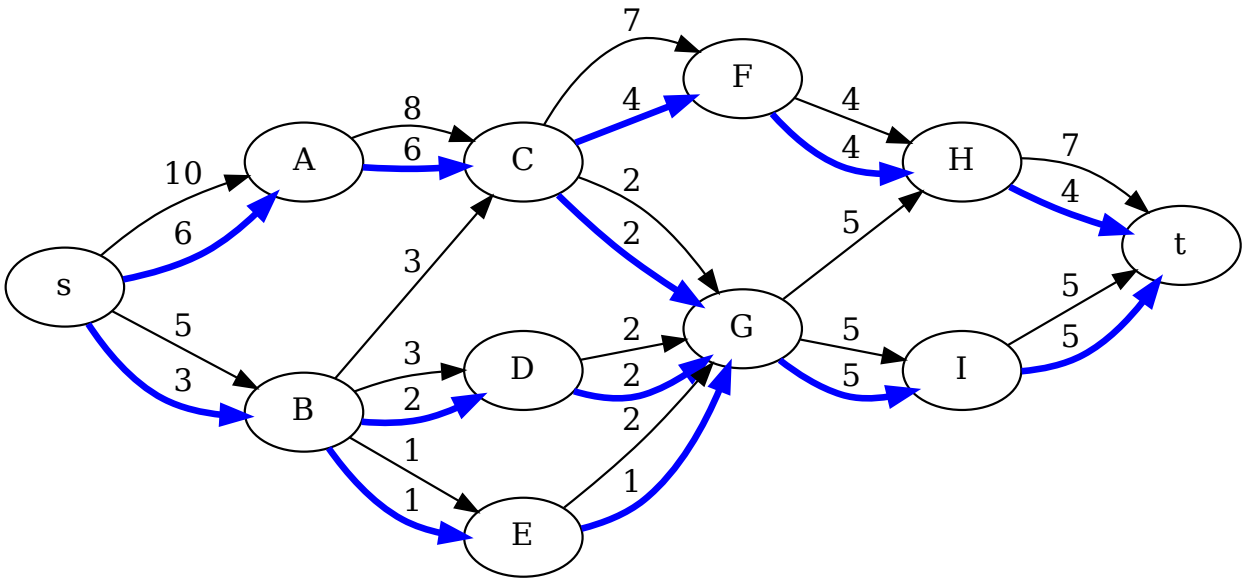
Flow: 6

Example



Flow: 8

Example



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Max-Flow Min-Cut: easy connection

Lemma

For all directed graphs $G = (V, A)$, s, t, and weight function $w : A \to \mathbb{Q}^+$ if f if a feasible flow value from s to t and c is a feasible cut from s to t, then $f \leq c$.

Corollary

If f^* is the max flow value and c^* is the minimum cut value, then $f^* \leq c^*$.

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Proves that for previous example $f^* = c^* = 9$.

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Proof.

- Let A' be a feasible cut, S the set of vertices reachable from s.
- Let $\mathbf{f}$ be a flow function of value f.
- We have $f \leq \sum_{x \in S} \sum_{y \notin S} \mathbf{f}(xy)$.
- $\sum_{x \in S} \sum_{y \notin S} \mathbf{f}(xy) \leq \sum_{x \in S} \sum_{y \notin S} \mathbf{c}(xy) = c$.

(Weak) duality

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- Common situation in combinatorial optimization:
  - We want to optimize (say, minimize) something
    - For example: Min Weight Spanning Tree (variants)
  - We find a dual objective (maximization) problem that lower bounds the solution
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- In situations where the original problem is hard to solve (NP-hard), this trick at least allows us to get an idea for how far we are from the optimal solution.
- In some **rare** situations we can find a dual problem where the optimal values of the two objectives **match**!!!
- This is the case for Max-Flow and Min-Cut

The Ford-Fulkerson Method

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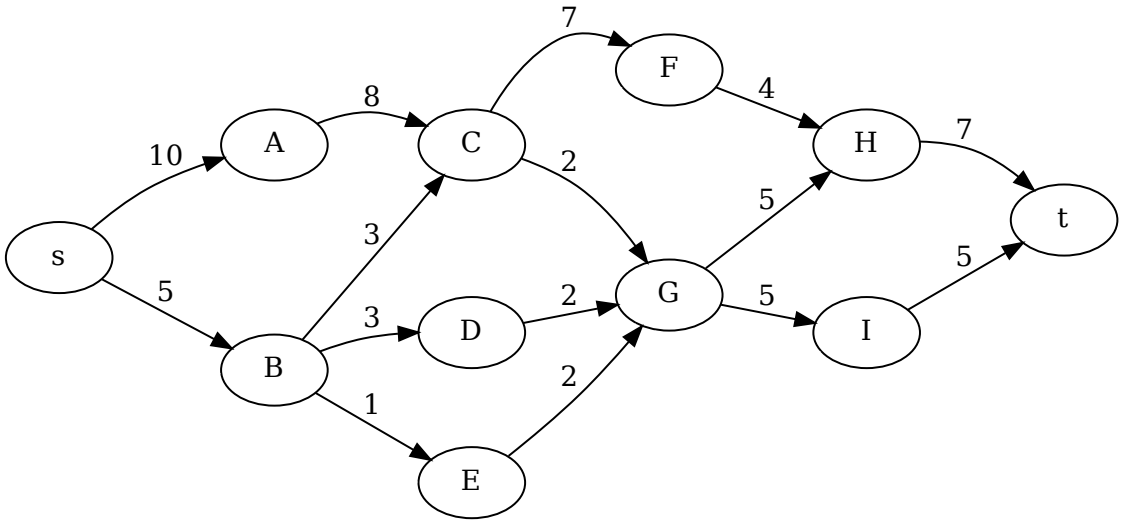
- Saturate: send as much capacity as allowed by the arc of minimum capacity in the path
  - $\Rightarrow$ this arc is then saturated (cannot take more flow)
- How to update the graph so that we respect future capacities?
  - Naïve method: subtract flow from capacities of used arcs (doesn't work!!)
  - Correct method: Construct **Residual network**
- Complexity:
  - Depends heavily on how paths are selected (more later!)

Example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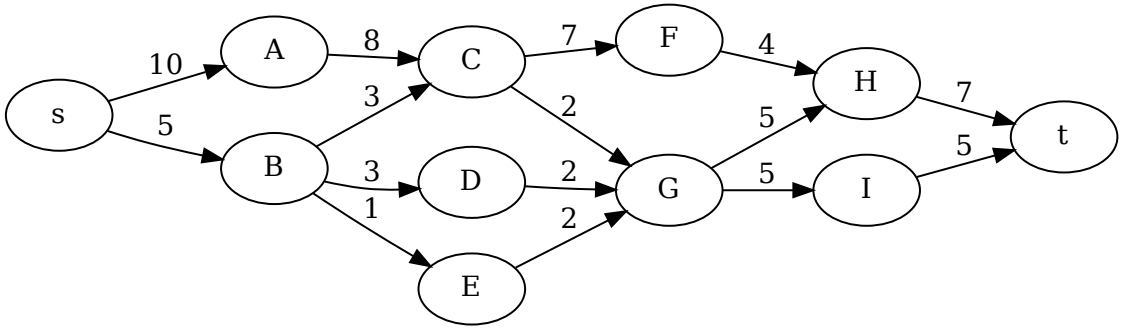
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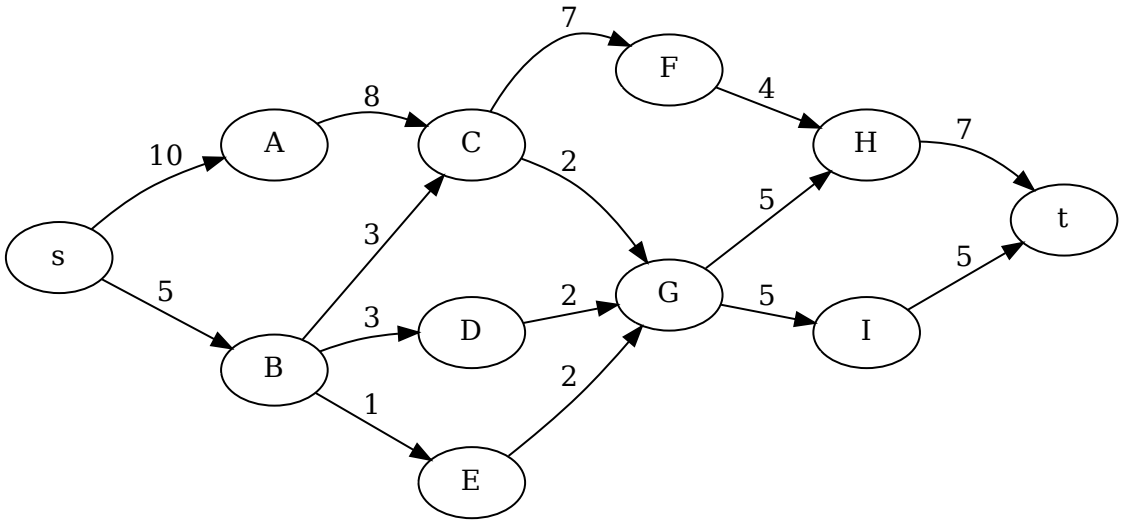
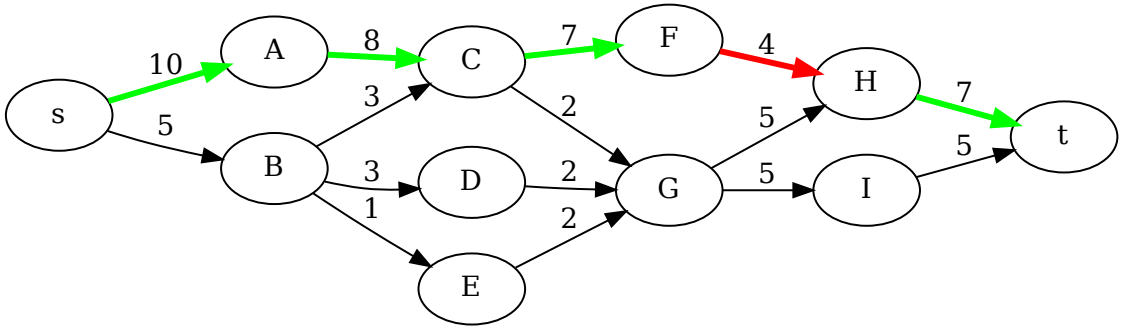


Find an $s \rightarrow t$ path.
Min capacity:4



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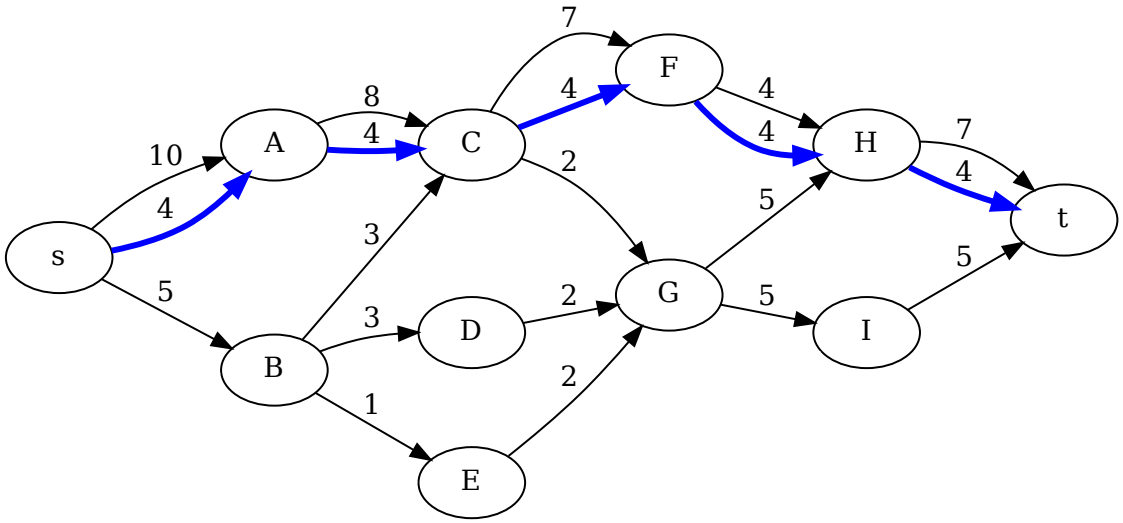
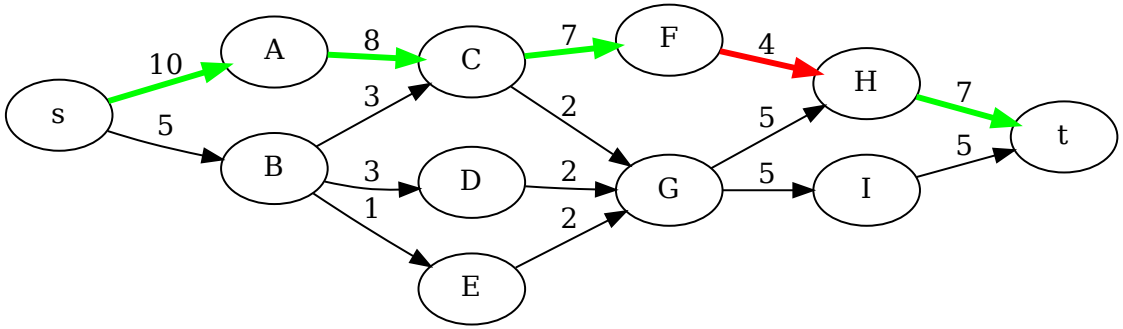


Saturate the path



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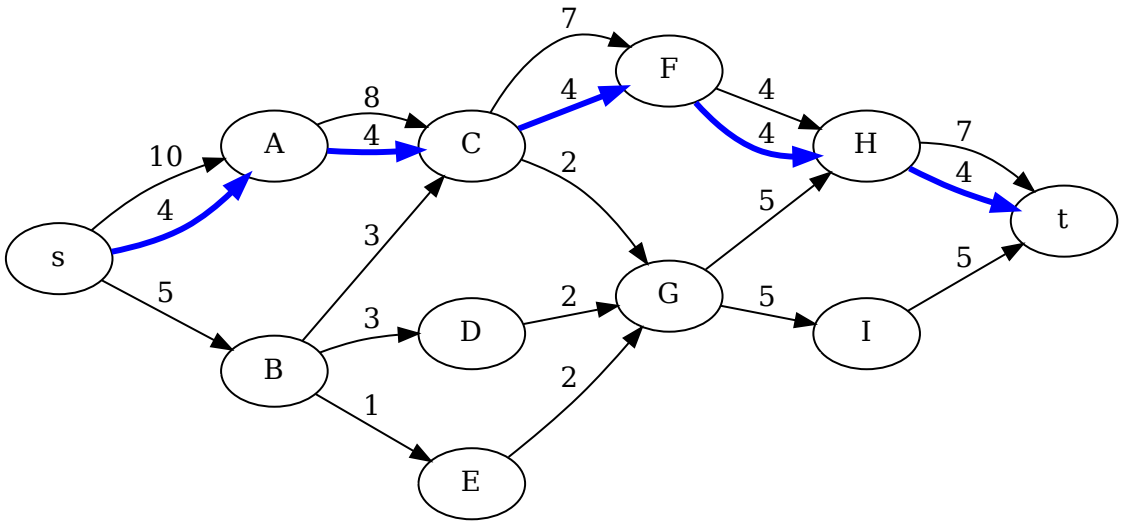
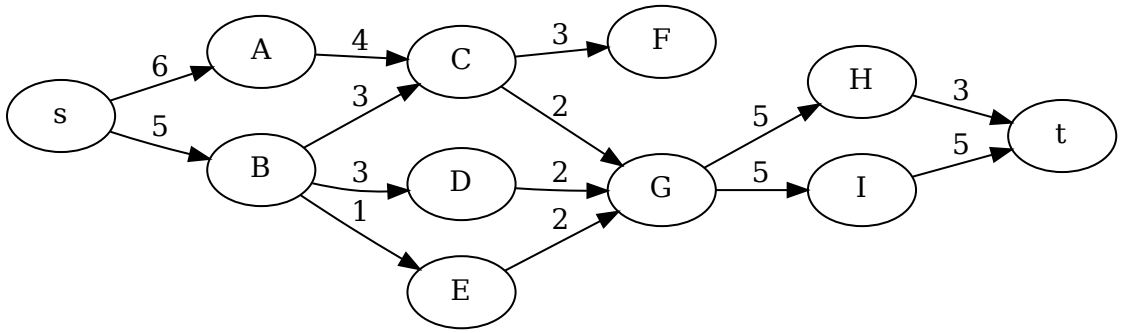


Update Capacities



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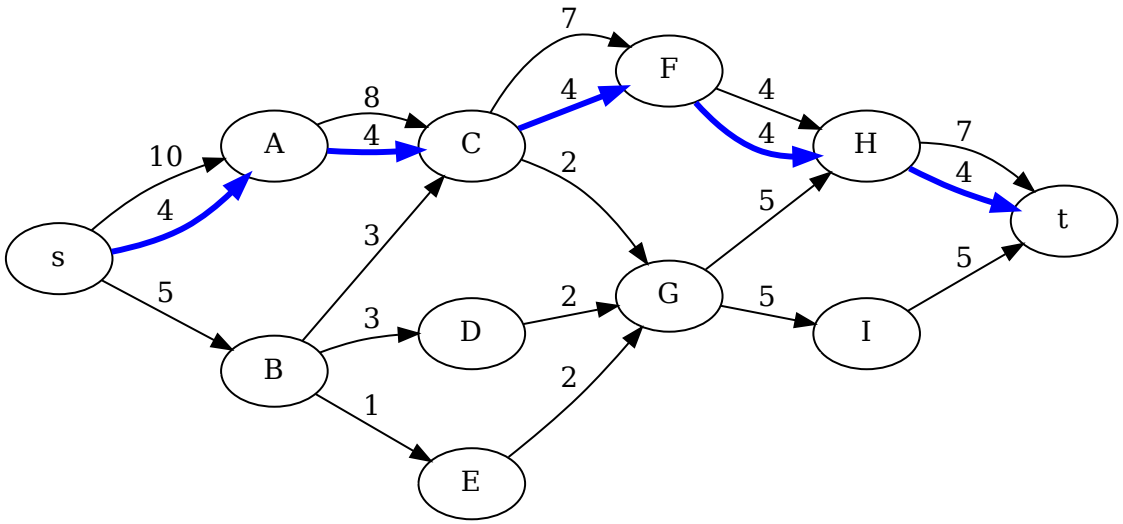
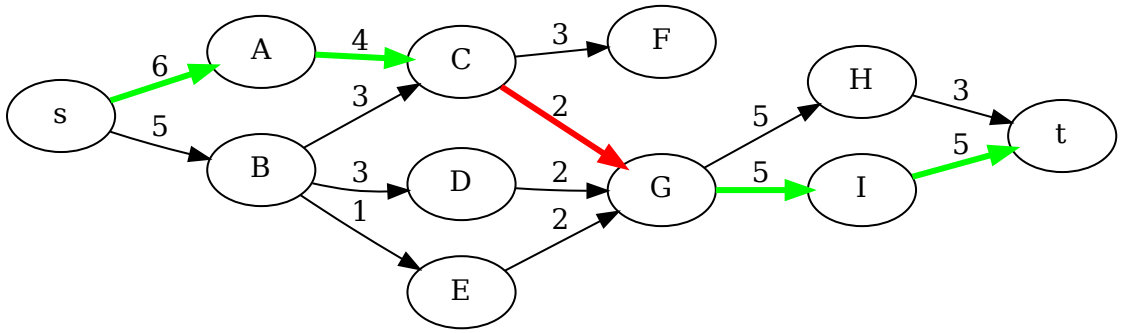


Find an $s \rightarrow t$ path.
Min capacity:2



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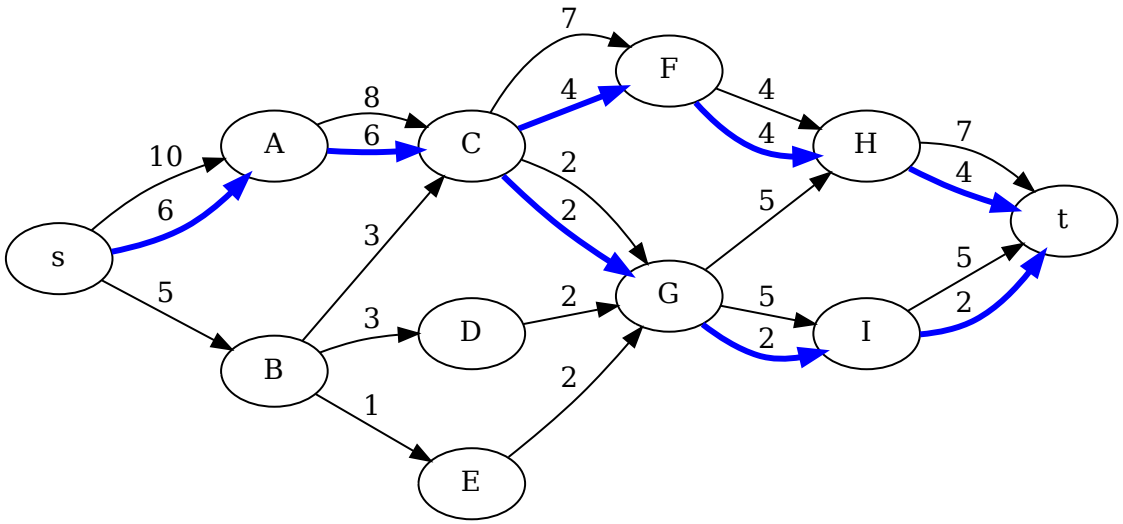

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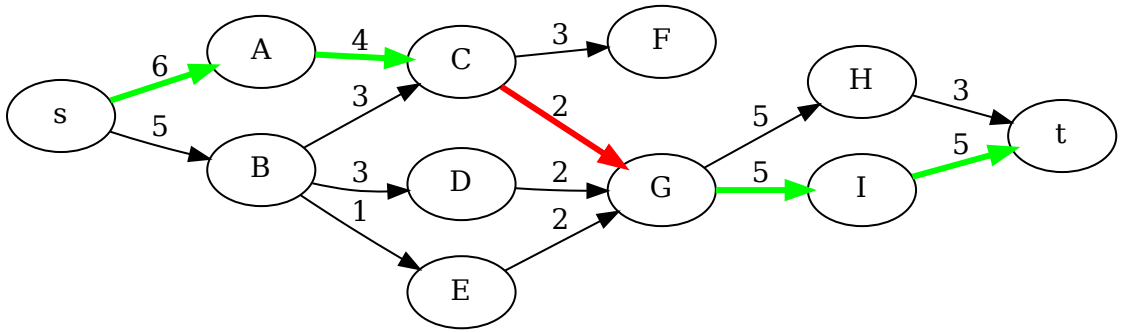

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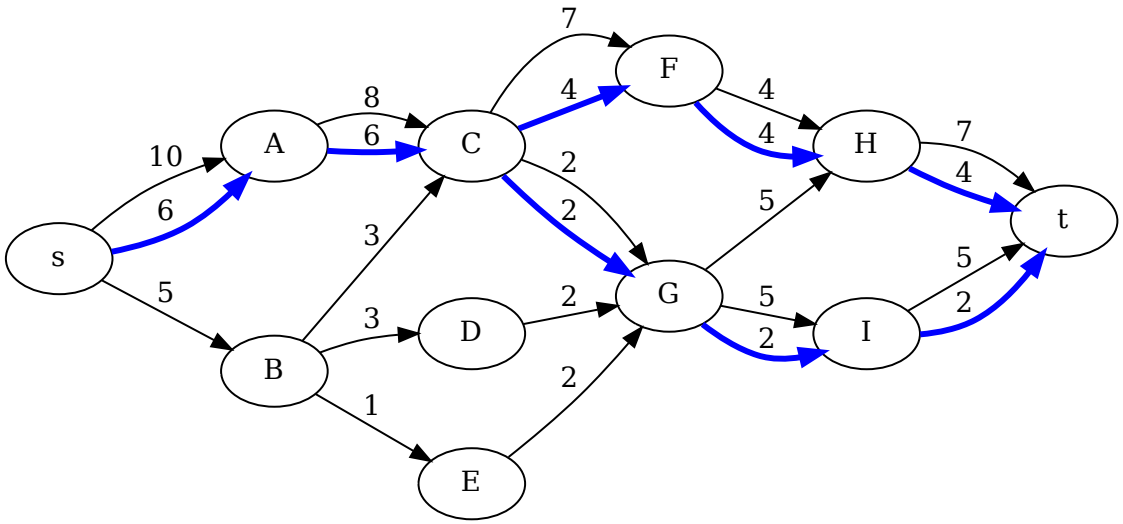

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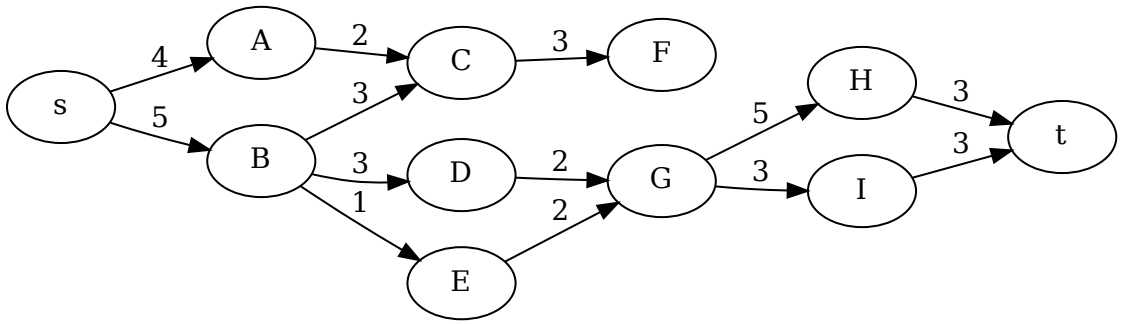

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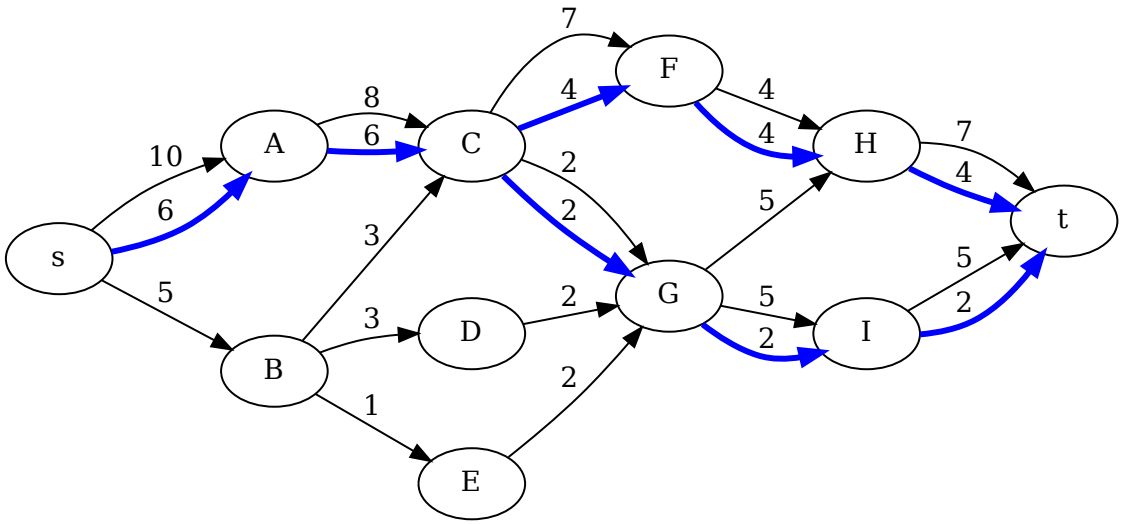
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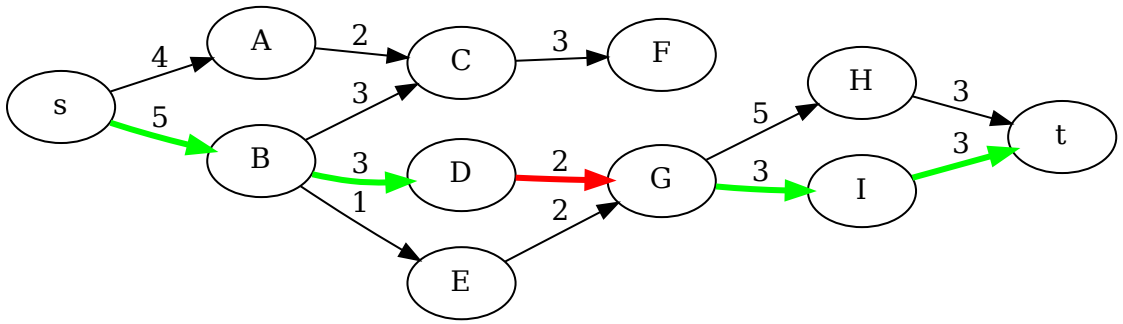
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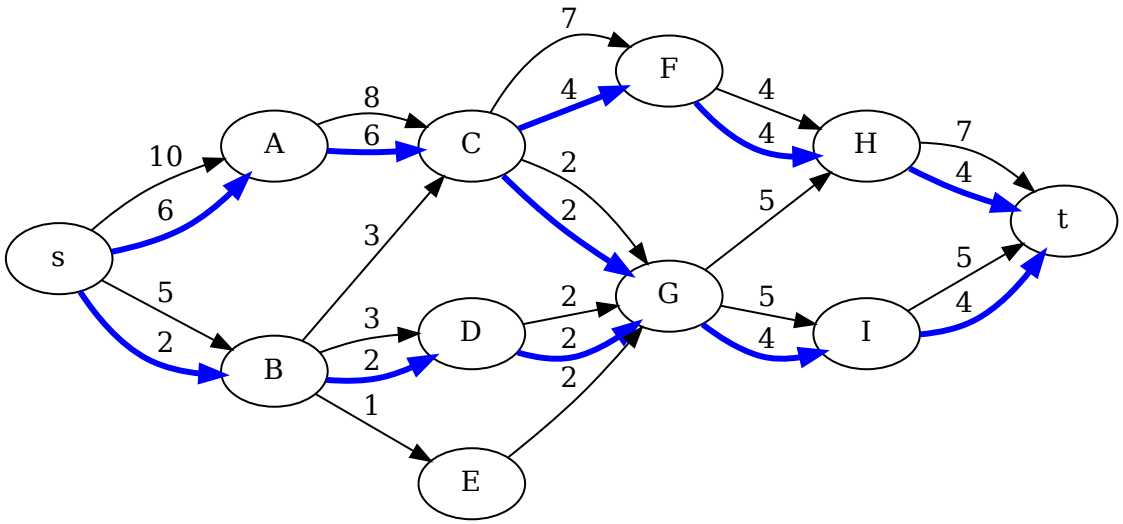
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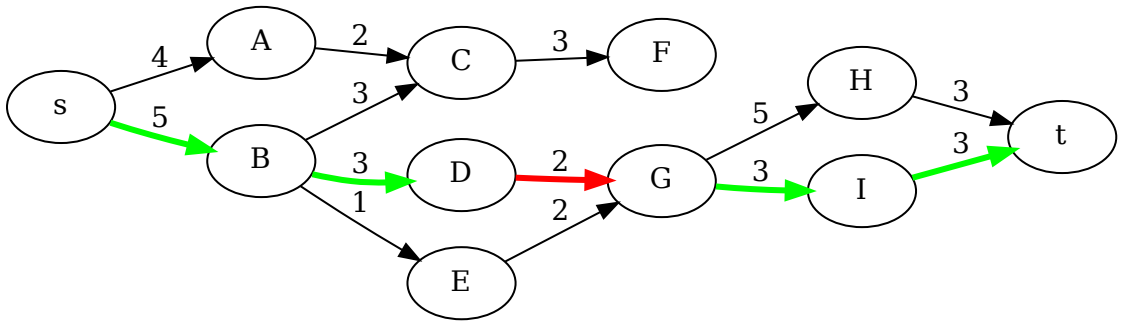
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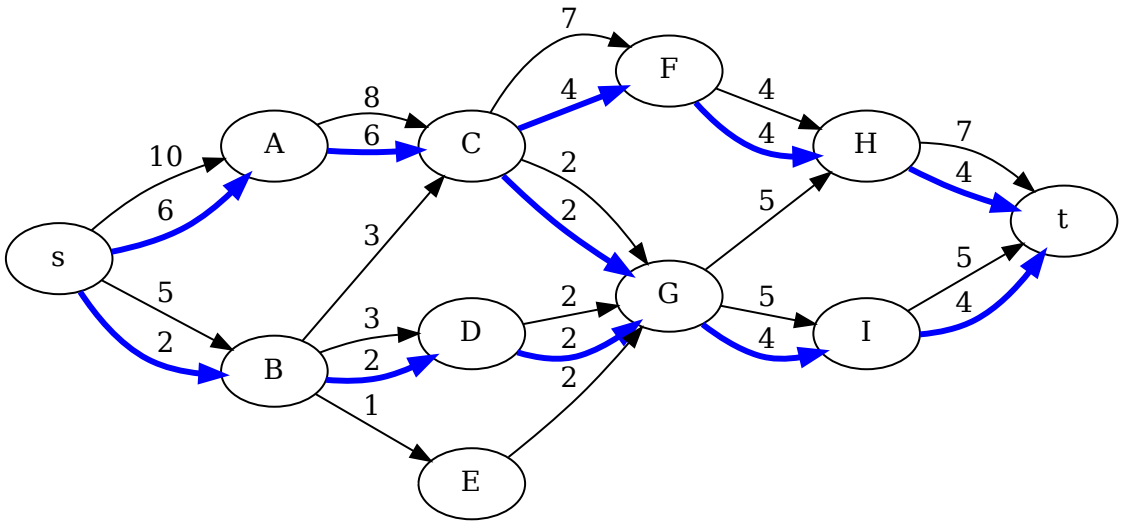
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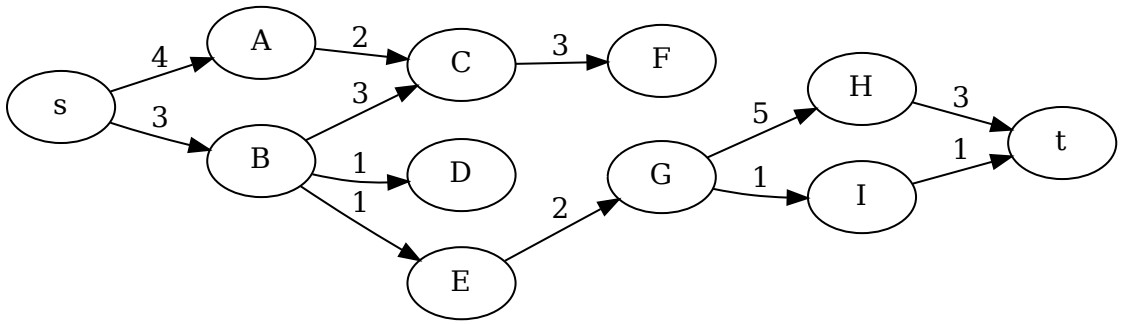
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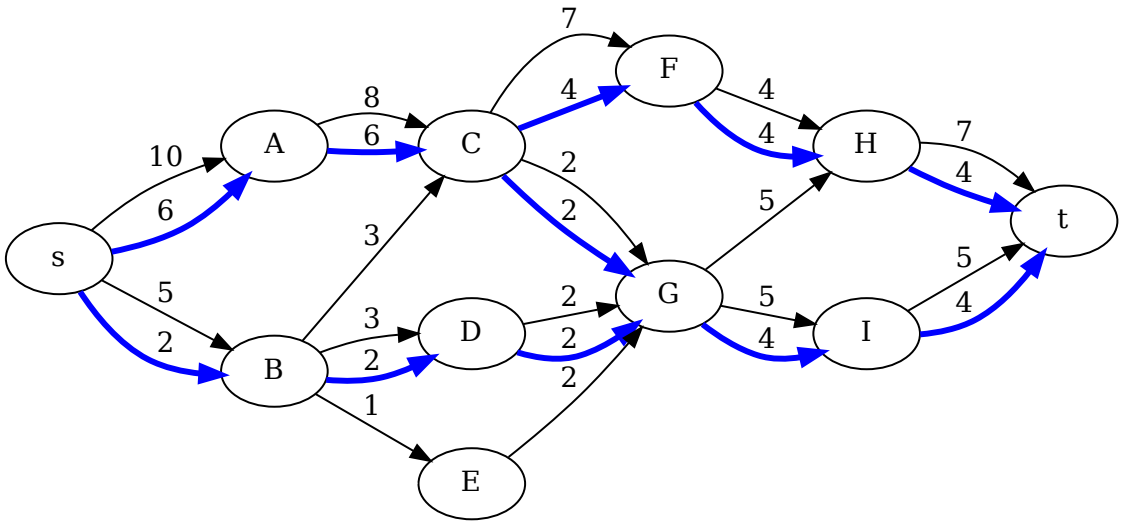
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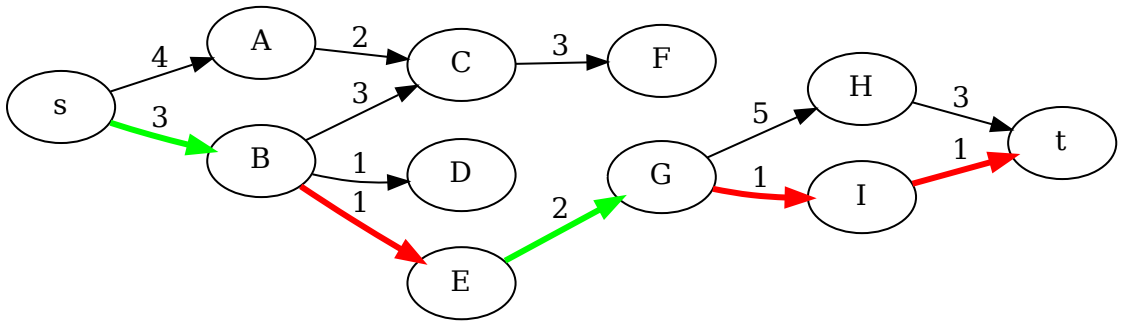
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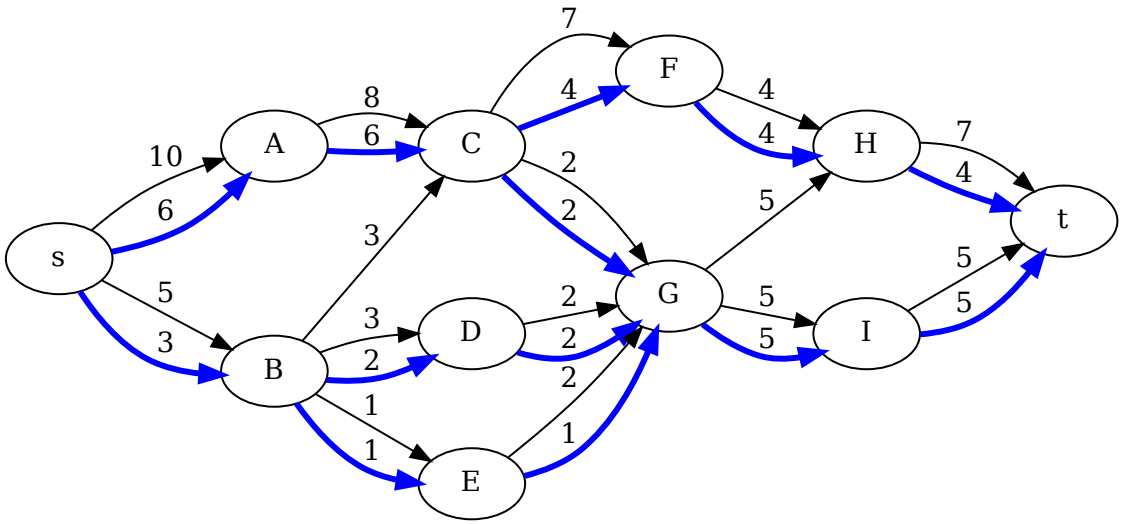
Example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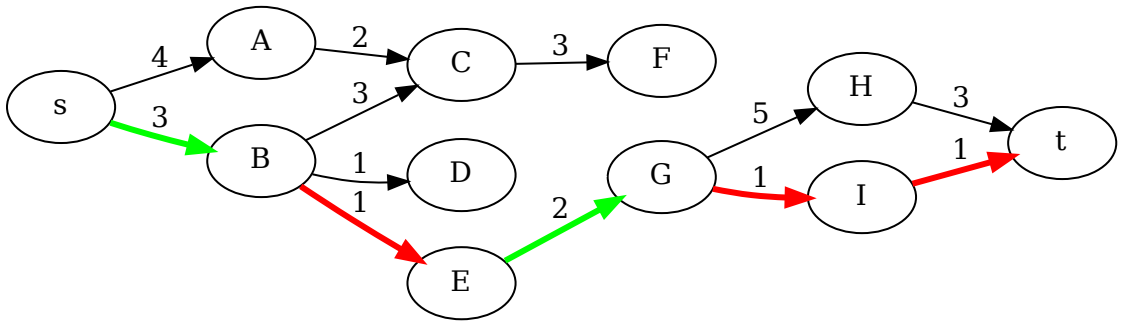
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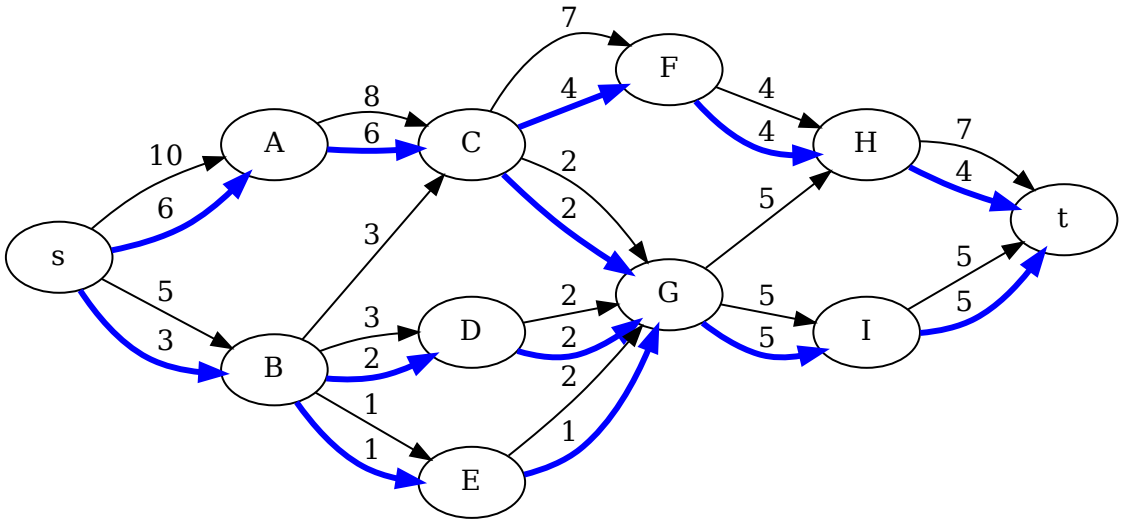
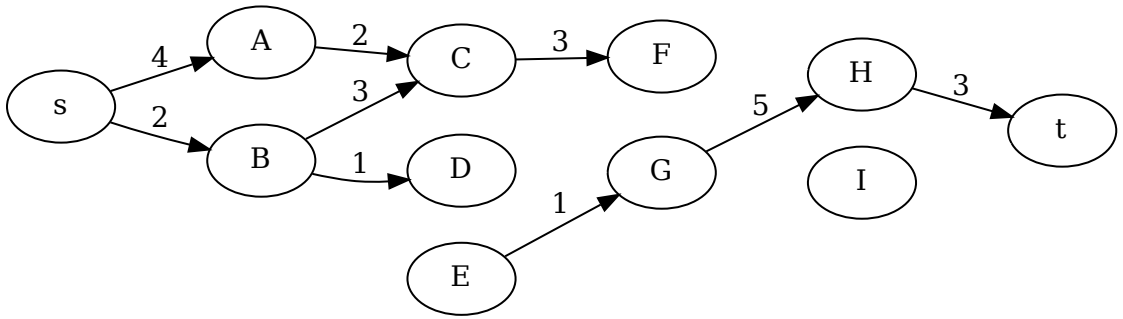
Example



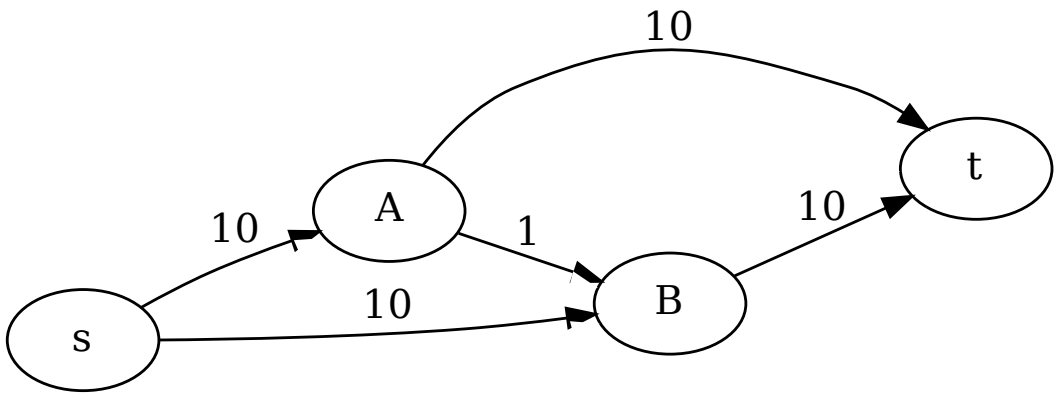


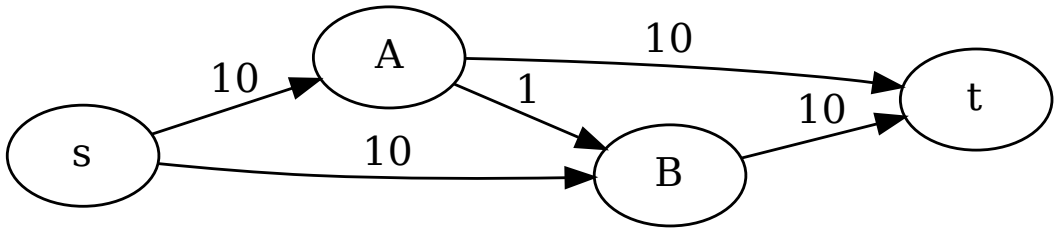
No path exists
$\Rightarrow$ current flow
is max

How to update capacities

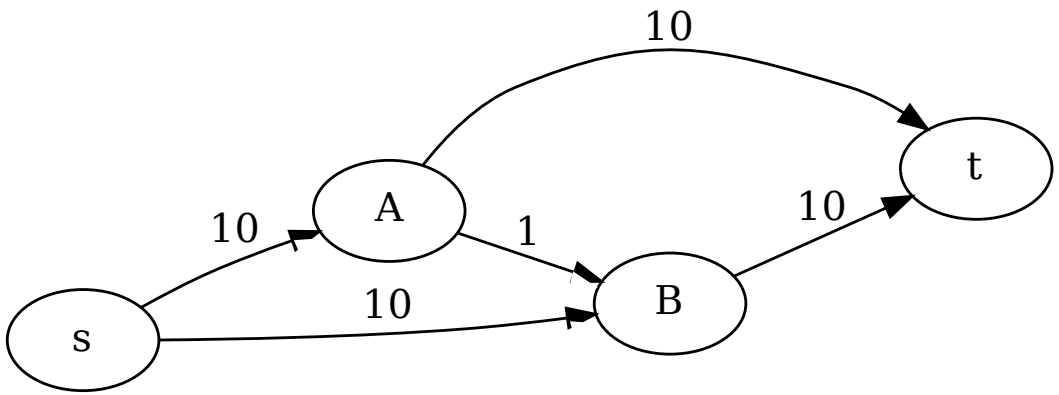
- Each time we find a path we subtract its flow from the capacities of used arcs (good!).
- However, the algorithm as currently formulated is **not** correct!
  - Intuitive reason: once we commit to a path, we never decrease the flow along its arcs.
  - However, we don't know beforehand if the optimal solution should saturate the path.
- To fix this, we need to update capacities in a way that allows us to later on decrease the flow along a used arc, as needed.

Counter-example

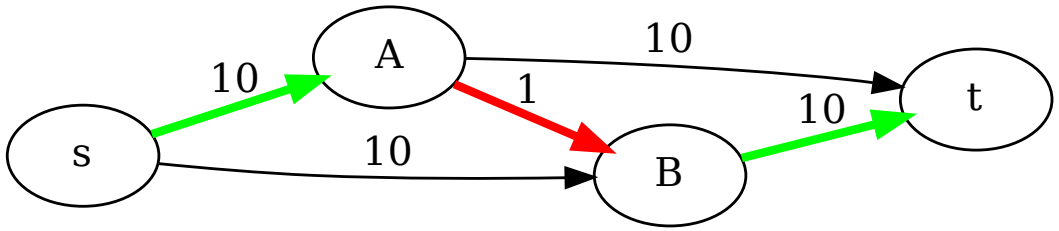




Counter-example



Find an $s \to t$ path.
Min capacity:1



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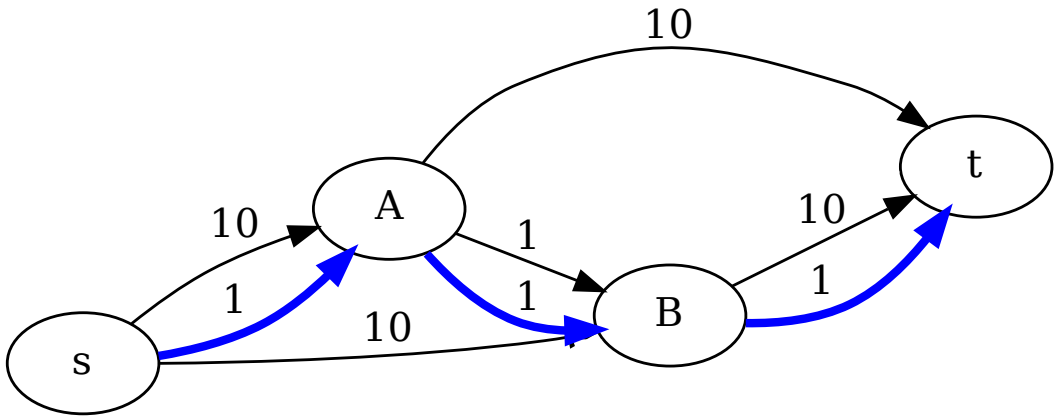
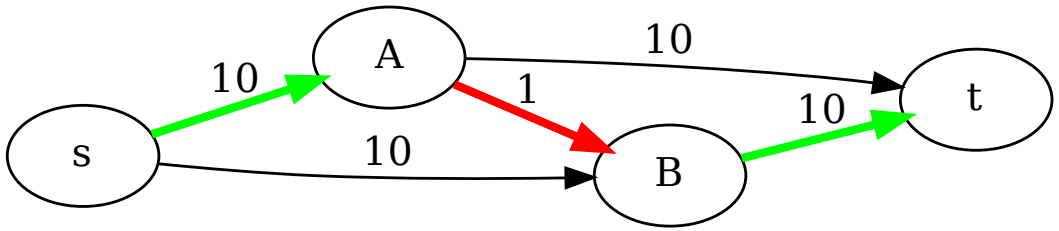


Saturate the path



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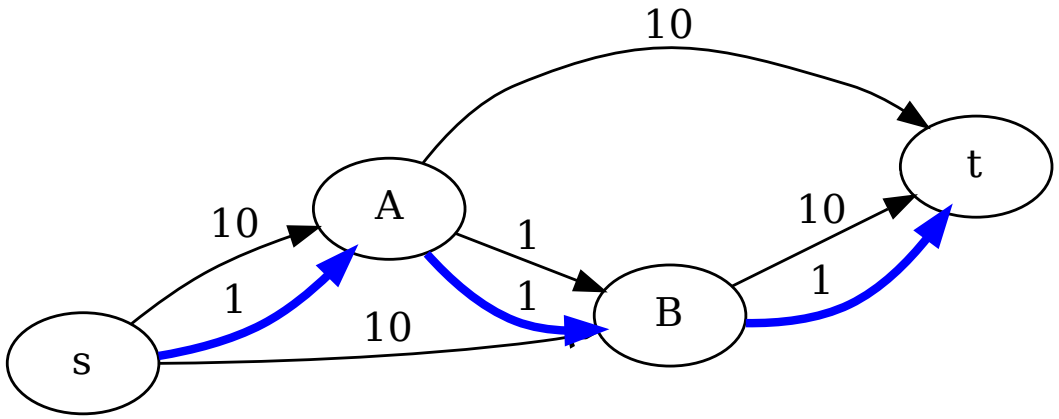
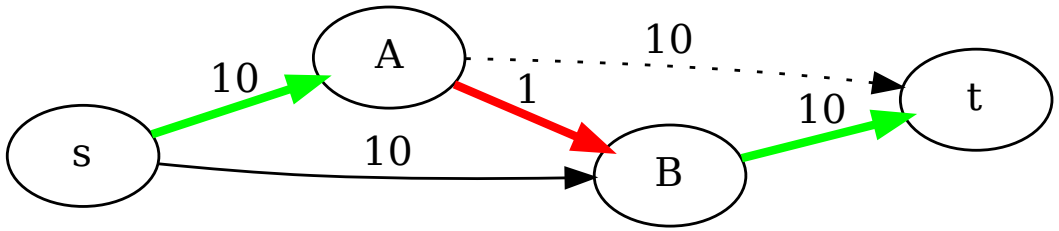
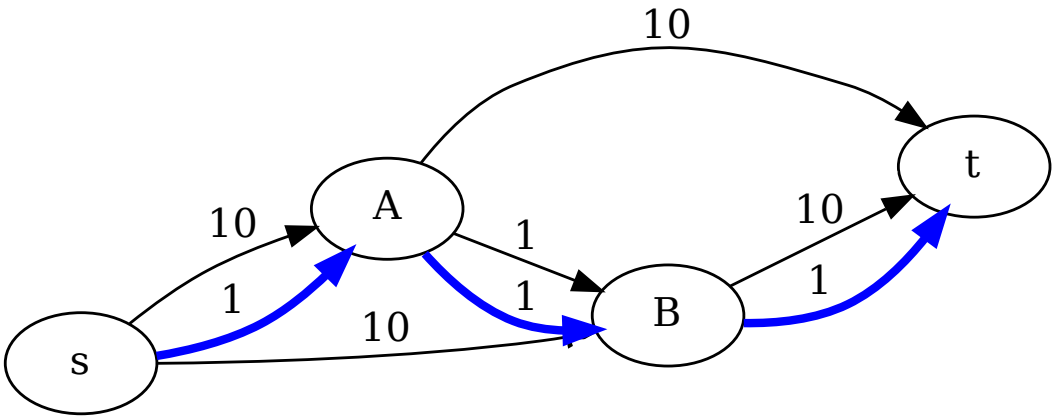


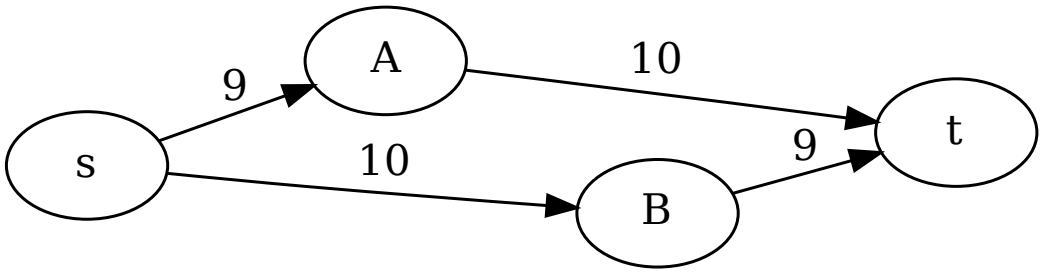


Note: this path looks stupid, but may in fact be the shortest path (e.g. if we subdivide At many times)

Counter-example



Update Capacities



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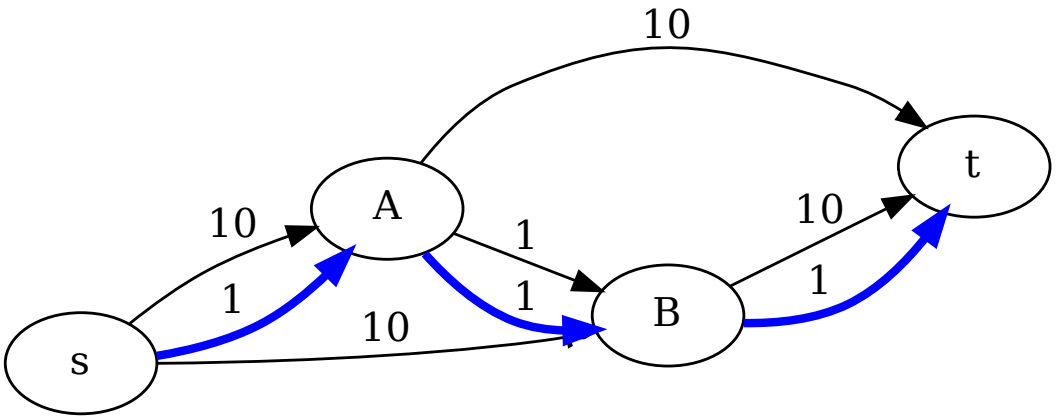

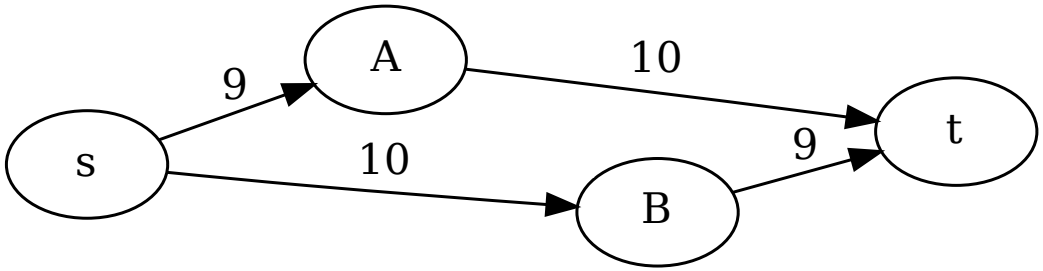

Now we have a problem!
New graph has max-flow=18, so in total we get a flow of 19.
Max=20 (!!)

Residual Networks

Definition

Let $G = (V, A), s, t \in V, \mathbf{c} : A \to \mathbb{Q}^+$ be a Max-Flow instance, $\mathbf{f} : A \to \mathbb{Q}^+$ a valid flow on G. The residual network $G = (V, A), \mathbf{c}' : A \to \mathbb{Q}^+$ is defined as follows:

- For each $uv \in A$ we set
  1. $\mathbf{c}'(uv) = \mathbf{c}(uv) - \mathbf{f}(uv)$
  2. $\mathbf{c}'(vu) = \mathbf{c}(uv) + \mathbf{f}(uv)$

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(To ease notation, assume that when $ij \in A$, then $ji \in A$, possibly with capacity 0)

Residual Network: intuition

- We currently have a flow $\mathbf{f}$.
- If $\mathbf{f}(uv) > 0$ we decrease $\mathbf{c}'(uv)$ by $\mathbf{f}(uv)$
  - Makes sense, as we have used some of the capacity of uv.

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  - We do this, even if vu previously did not exist (had capacity 0).
  - Intuition, if we later send flow through vu, we can simulate this by decreasing the flow through uv.
  - Increasing $\mathbf{f}(vu)$ by 1 is the same as decreasing $\mathbf{uv}$ by 1 for the flow-conservation constraints.

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  - Increasing $\mathbf{f}(vu)$ by 1 is the same as decreasing $\mathbf{uv}$ by 1 for the flow-conservation constraints.
- Thanks to this idea, we don't **commit** when we add a path to the solution (the flow can be reversed later).

Ford-Fulkerson method

1: Initialize flow $\mathbf{f} := 0$, residual graph $G' := G$.
2: **while** G' has an $s \to t$ path **do**
3: Find P, an $s \to t$ path in G' ▷ BFS/DFS
4: Find c_p, min capacity of P in G'
5: For $uv \in P$ set $\mathbf{f}(uv) := \mathbf{f}(uv) + c_p$ ▷ Eliminate cycles!
6: Update residual graph G'
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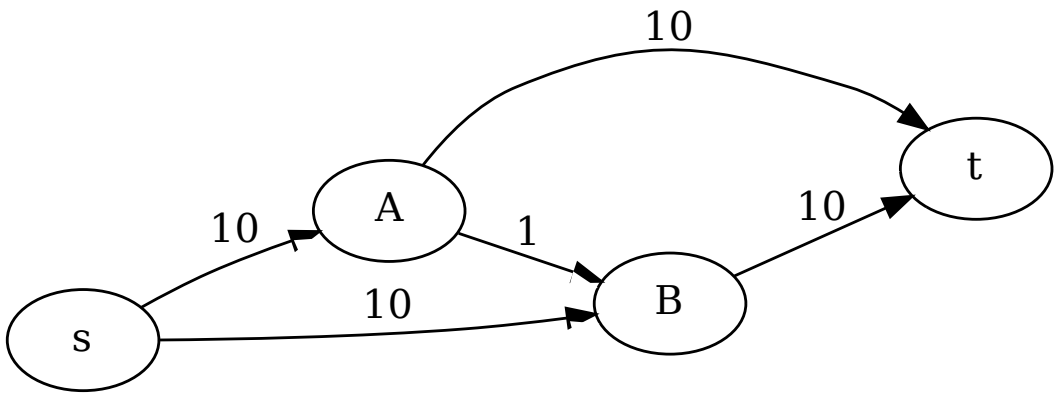
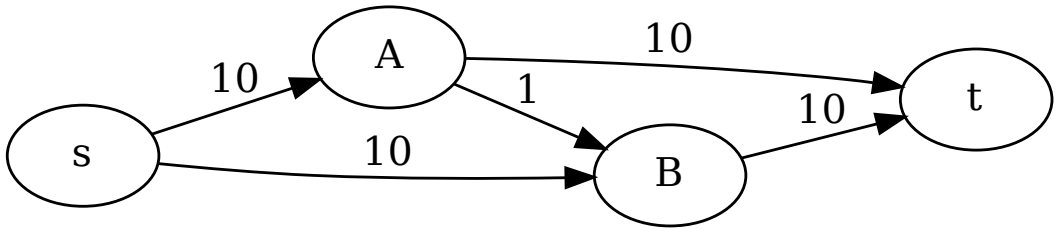
Eliminate cycles:

1: **if** $\mathbf{f}(uv) > 0$ and $\mathbf{f}(vu) > 0$ **then**
2: $\mathbf{f}(uv) -= \min\{\mathbf{f}(uv), \mathbf{f}(vu)\}$
3: $\mathbf{f}(vu) -= \min\{\mathbf{f}(uv), \mathbf{f}(vu)\}$
4: **end if**

Counter-example fixed





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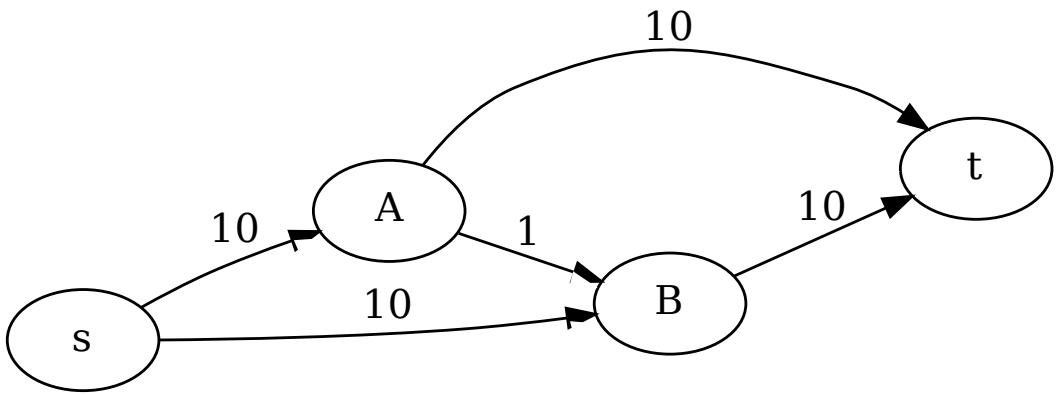

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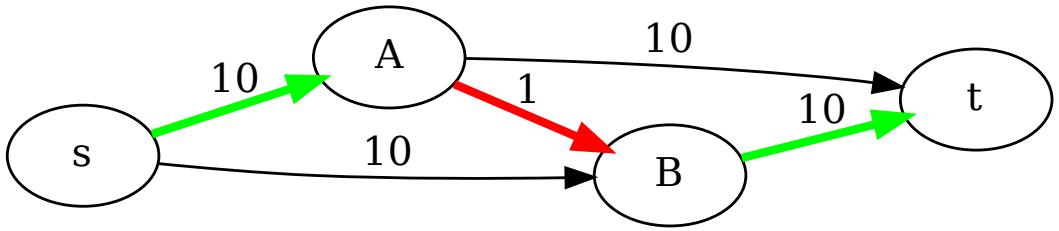

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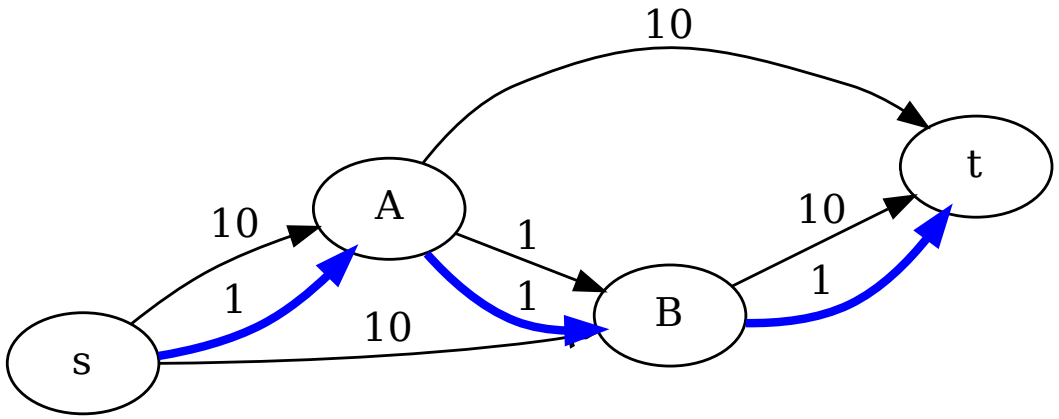

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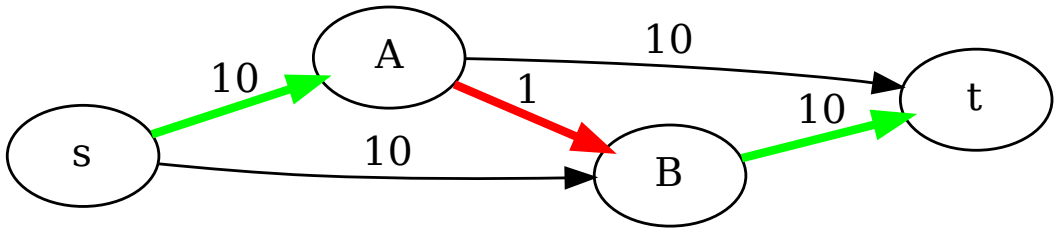

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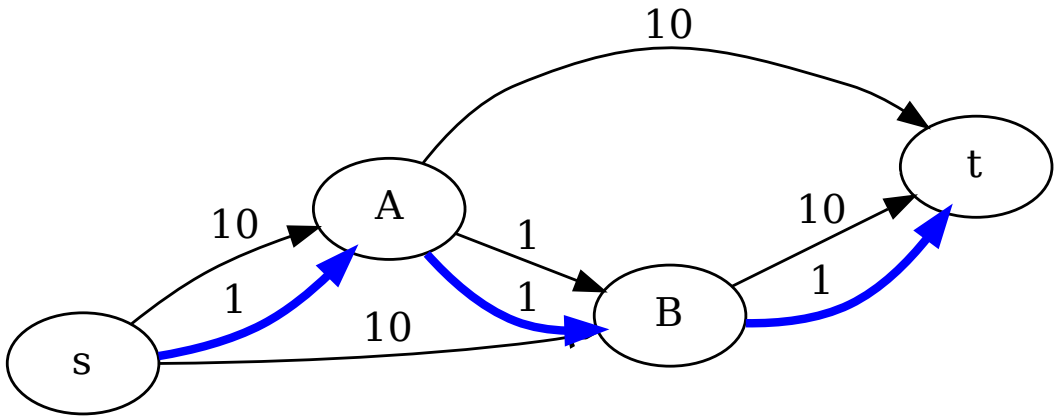
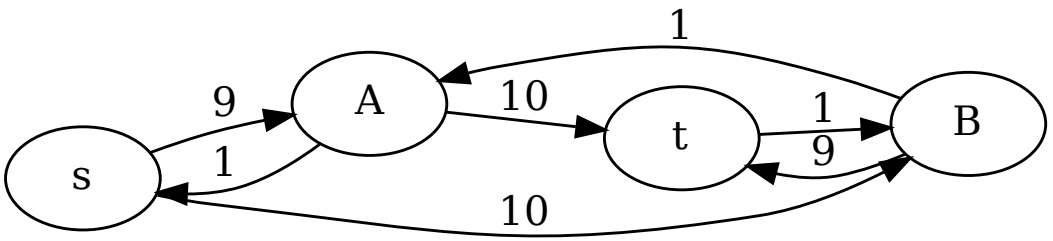


Update Capacities



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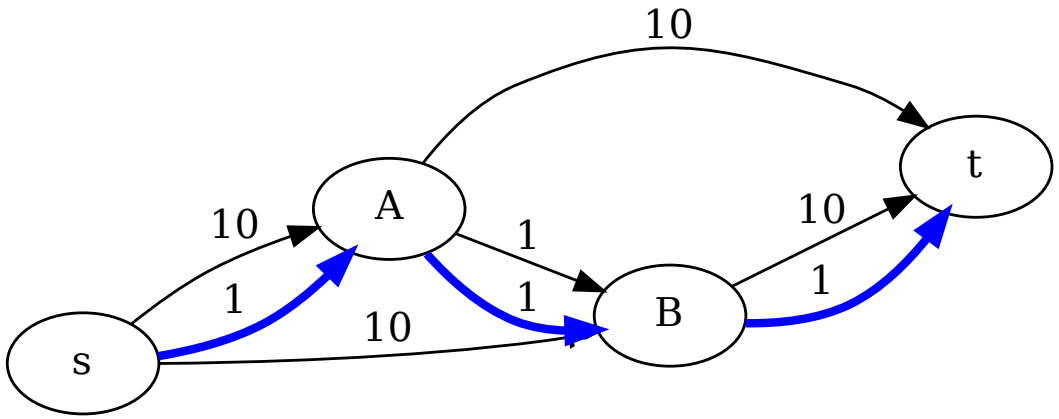

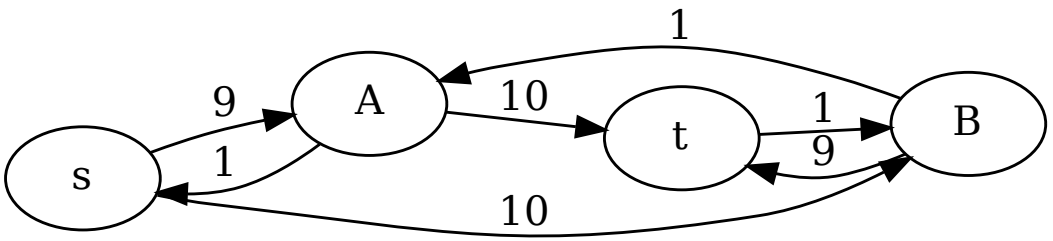

Now the residual network has a flow of value 19. Together with the path this gives a flow of value 20.

Correctness

Theorem

The Ford-Fulkerson algorithm calculates a maximum flow.

Correctness

Theorem

The Ford-Fulkerson algorithm calculates a maximum flow.

Lemma

The Ford-Fulkerson algorithm always terminates and outputs a valid flow.

Proof.

- Suppose we have a valid current flow **f**
- If the residual network G' has no $s \to t$ path, done.
- Otherwise, we compute a **larger** flow, by adding to **f** the minimum capacity of the computed path.
- New flow is feasible (why?).
- Flow cannot become ∞, so algorithm will eventually terminate.

Correctness

Theorem

The Ford-Fulkerson algorithm calculates a maximum flow.

Correctness

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The Ford-Fulkerson algorithm calculates a maximum flow.

Proof.

- Let $\mathbf{f}$ be the output valid flow of value f^*, G' the residual network where no $s \to t$ path exists.
- Let S be the set of vertices reachable from s in G'.
  - $\Rightarrow$ no arc comes out of S in G'.
- Claim: S gives a cut of weight f^*
  - Arcs coming into S have 0 flow (otherwise G' has outgoing arc from S)
  - Arcs coming out of S are removed in G'
  - $\Rightarrow$ for such arcs uv we have $\mathbf{f}(uv) = \mathbf{c}(uv)$
  - $f^* = \sum_{u \in S} \sum_{v \notin S} \mathbf{f}(uv) = \sum_{u \in S} \sum_{v \notin S} \mathbf{c}(uv)$
- $\Rightarrow$ no flow can have value higher than f^*, so current flow is maximum.

Consequences

Theorem

In any weighted digraph $G = (V, A)$, for any $s, t \in V$, the max-flow value from s to t is **equal** *to the min-cut value from s to t.*

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Proof.

- Max-Flow$\leq$Min-Cut is easy (as we saw).
- Ford-Fulkerson algorithm produces a flow **equal** to some cut, so the flow is maximum, the cut is minimum.

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Proof.

- Max-Flow$\leq$Min-Cut is easy (as we saw).
- Ford-Fulkerson algorithm produces a flow **equal** to some cut, so the flow is maximum, the cut is minimum.

□

- **Rare** duality theorem, considered one of the deepest in OR.
- For most optimization problems, no such duality is known.
  - Classes: NP vs coNP, P

Consequences

Corollary

If all capacities are integral, there exists an integral maximum flow.

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Corollary

If all capacities are integral, there exists an integral maximum flow.

- Statement is obvious for min-cuts.
- Non-obvious for flows, as we can partition flows into fractional pieces (this is allowed by the problem definition).

Complexity Analysis

1: Initialize flow $\mathbf{f} := 0$, residual graph $G' := G$.
2: **while** G' has an $s \to t$ path **do** ▷ How many iterations?
3: Find P, an $s \to t$ path in G' ▷ BFS/DFS: $O(m)$ time
4: Find c_p, min capacity of P in G' ▷ $O(n)$ time
5: For $uv \in P$ set $\mathbf{f}(uv) := \mathbf{f}(uv) + c_p$ ▷ $O(m)$ time
6: Update residual graph G'
7: **end while**
8: Output $\mathbf{f}$

- Each iteration takes $O(m)$ time, but how many times do we need to execute the main loop?
- Easy argument: each execution increases flow value, so if capacities are integer, complexity is $O(m \cdot f^*)$.

Complexity Analysis

Theorem

Ford-Fulkerson method runs in time $O(m \cdot f^*)$ and there exist instances on which it takes $\Omega(m \cdot f^*)$ time.

Complexity Analysis

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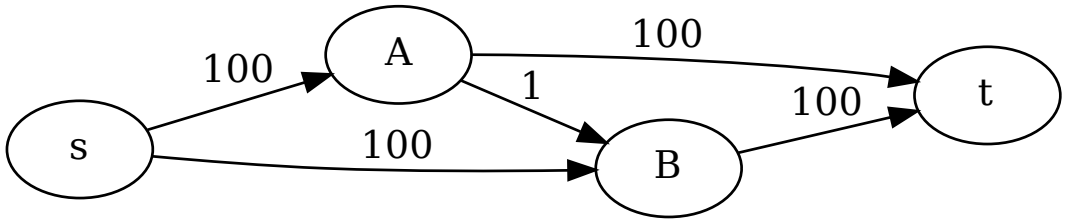
Ford-Fulkerson method runs in time $O(m \cdot f^*)$ and there exist instances on which it takes $\Omega(m \cdot f^*)$ time.

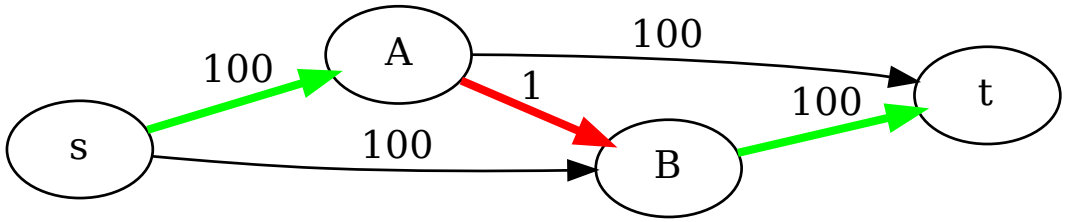
- **NB**: this means that FF method is an **exponential**-time algorithm!
  - More precisely, a **pseudopolynomial**-time algorithm.
  - **Pseudopolynomial**: polynomial-time if input numbers have $O(\log n)$ bits.
- Problem: we haven't specified **which** $s \to t$ path to pick in the residual graph.
- If bad paths are repeatedly picked, FF method takes a very long time...
- Generic method can still be useful if we know that f^* is small.

Bad Example



$f = 0$

Bad Example



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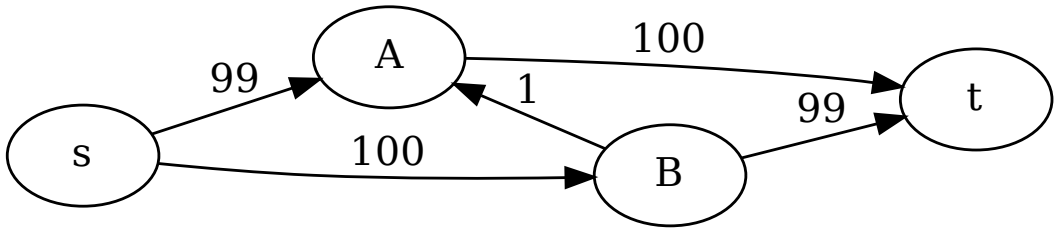
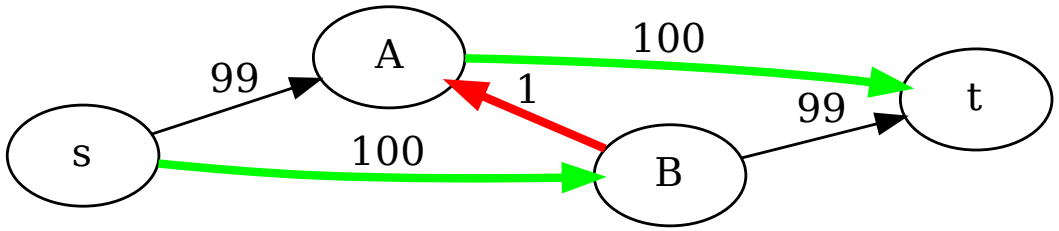


$f = 1$

Bad Example



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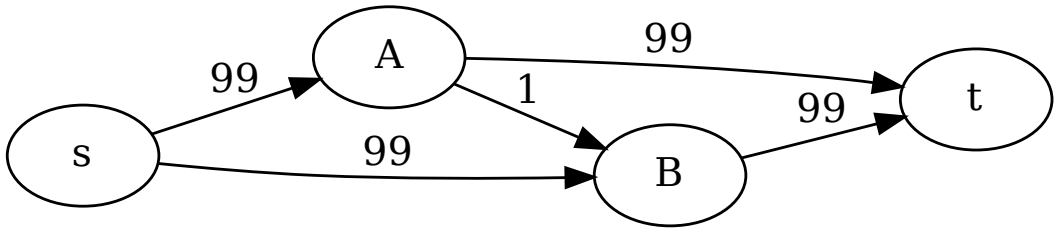
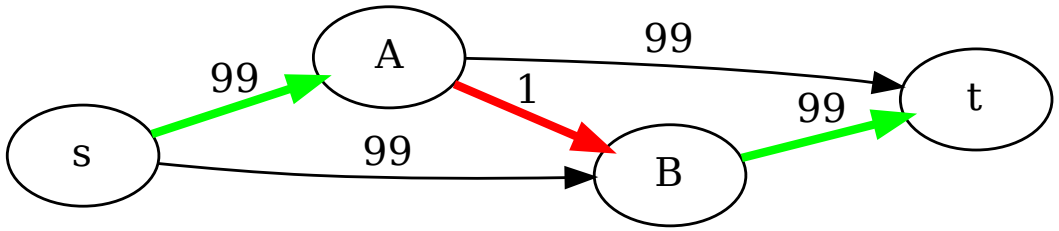


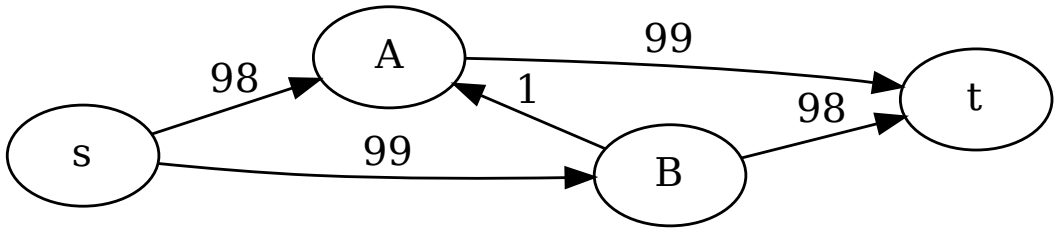
$f = 2$

Bad Example



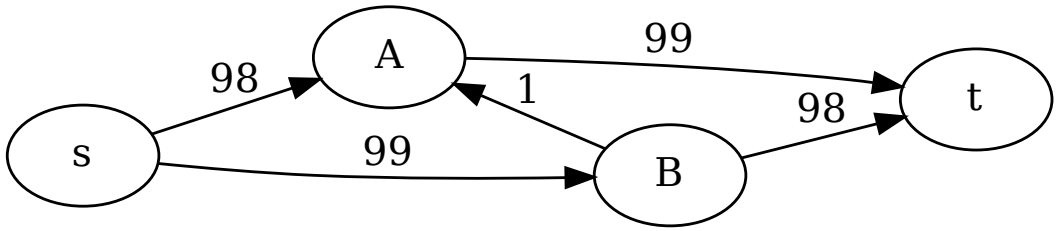
$f = 2$

Bad Example



$f = 3\ldots$

Bad Example



$f = 3\ldots$

Would you like to repeat this another 197 times?

Edmonds-Karp algorithm

Outline

- Take Ford-Fulkerson method but specify **which** path to select in each iteration.
- Natural choice: take the **shortest** path
  - **NB:** shortest in the sense of minimum number of arcs, disregarding weights.
- Key idea: shortest-path distances from s (in unweighted sense) can only increase for each iteration
- $\Rightarrow$ we can bound the number of iterations as a function of n, independently of f^*.

Analysis

Theorem

Edmonds-Karp algorithm runs in time $O(nm^2)$.

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Lemma

For all v, distance from s to v never decreases in the residual network.

Lemma

Every arc $ij \in A$ is saturated at most $\frac{n}{2}$ times.

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Proof.

- Let v be the first and closest to s vertex for which distance decreased between two iterations.
- Let G_1, G_2 be the two residual networks and $\mathrm{dist}_{G_2}(s, v) < \mathrm{dist}_{G_1}(s, v)$.
- Let u be the last vertex of a shortest $s \to v$ path in G_2.
- Claim: $uv \notin G_1$

Lemma

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- Claim: $uv \notin G_1$
  - Suppose for contradiction $uv \in G_1$:
  - $\mathrm{dist}_{G_1}(s, v) \leq \mathrm{dist}_{G_1}(s, u) + 1 \leq \mathrm{dist}_{G_2}(s, u) + 1 = \mathrm{dist}_{G_2}(s, v)$
  - So, distance to v actually did not decrease...

Lemma

For all v, distance from s to v never decreases in the residual network.

Proof.

(Continued...)

- $uv \notin G_1$ and $uv \in G_2$
- $\Rightarrow s \to t$ path in G_1 uses vu
- $\mathrm{dist}_{G_1}(s, v) = \mathrm{dist}_{G_1}(s, u) - 1 \leq \mathrm{dist}_{G_2}(s, u) - 1 = \mathrm{dist}_{G_2}(s, v) - 2$
- So, again distance to v increased, contradiction!

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Proof.

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- We claim, next time ij is saturated, $\mathrm{dist}_{G_3}(s, i) \geq d + 2$.

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- We claim, next time ij is saturated, $\text{dist}_{G_3}(s, i) \geq d + 2$.
- When ij first saturated, $\text{dist}_{G_1}(s, j) = \text{dist}_{G_1}(s, i) + 1 = d + 1$.
- Later, ji is used, so $\text{dist}_{G_2}(s, i) = \text{dist}_{G_2}(s, j) + 1 \geq d + 2$.
- $\text{dist}_{G_3}(s, i) \geq \text{dist}_{G_2}(s, i)$.

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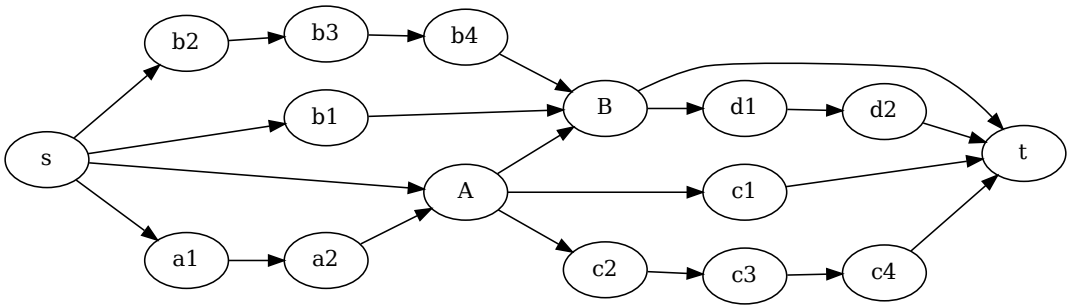
□

- Running time can be improved with better analysis.
- However, bad examples can force the algorithm to flip-flop an arc many times (see next slide).
- Best known algorithm: $m^{1+o(1)}$ time (almost-linear), with completely different techniques.

Bad Example



- Capacity 1 everywhere
- $s \to A$ paths: 1,3
- $s \to B$ paths: 2,4
- $A \to t$ paths: 2,4
- $B \to t$ paths: 1,3

Goal: make AB arc flip many times for Edmonds-Karp.

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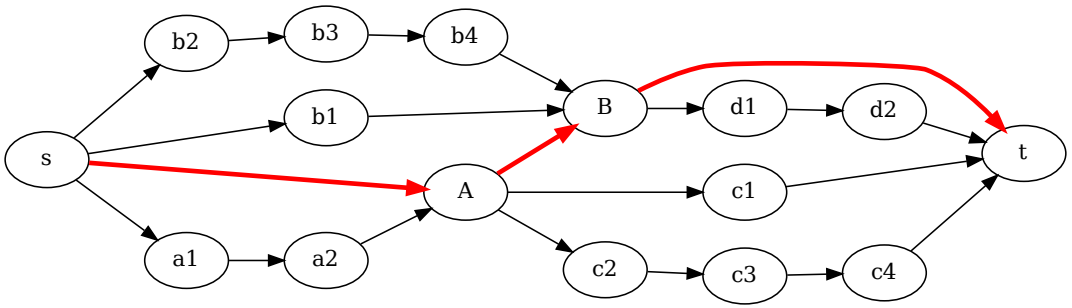

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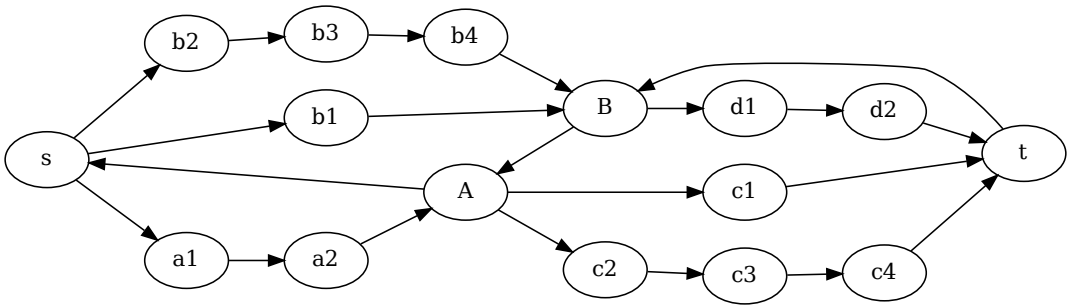

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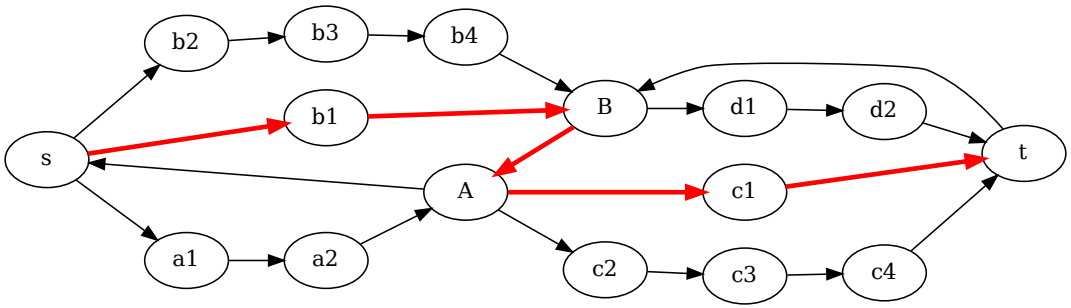

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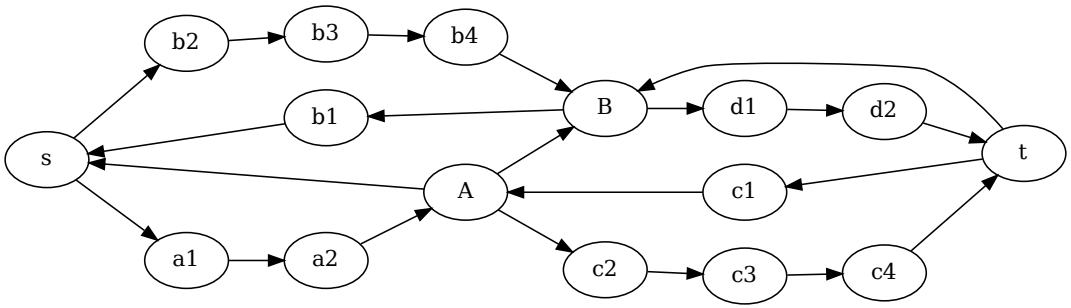

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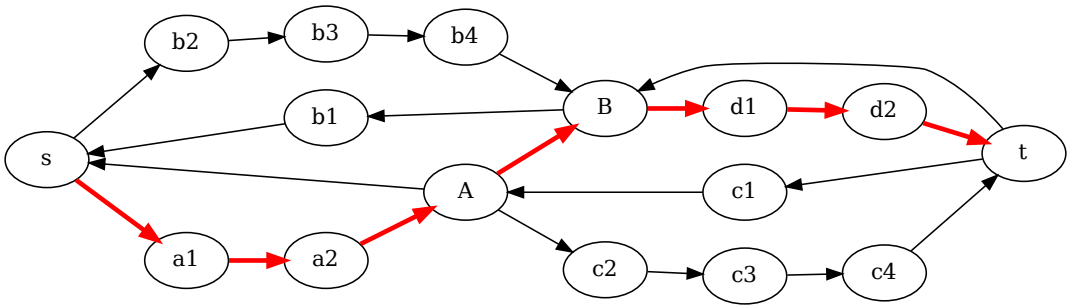

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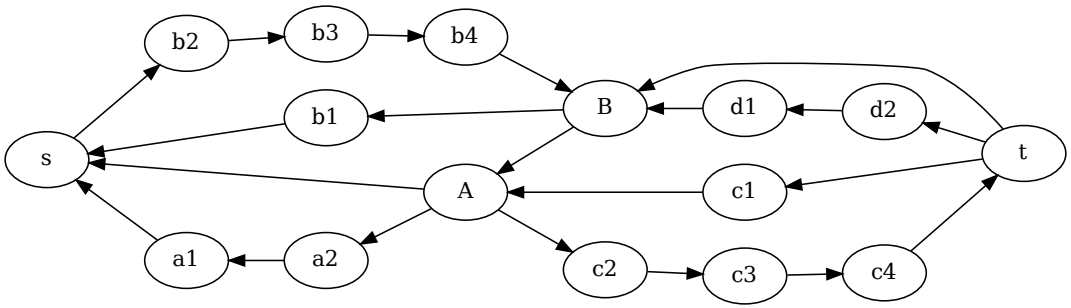

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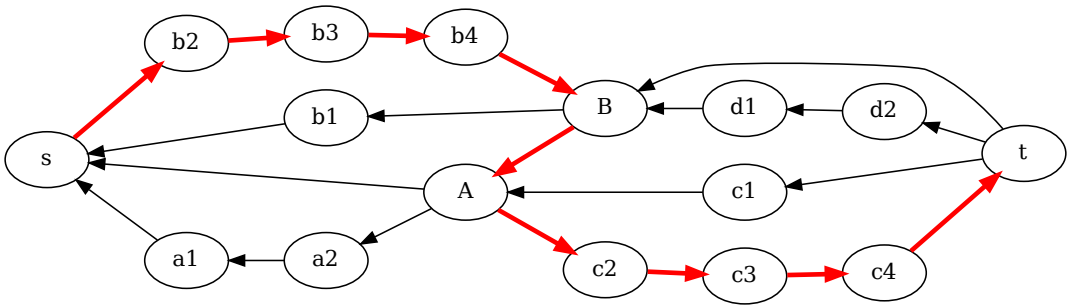

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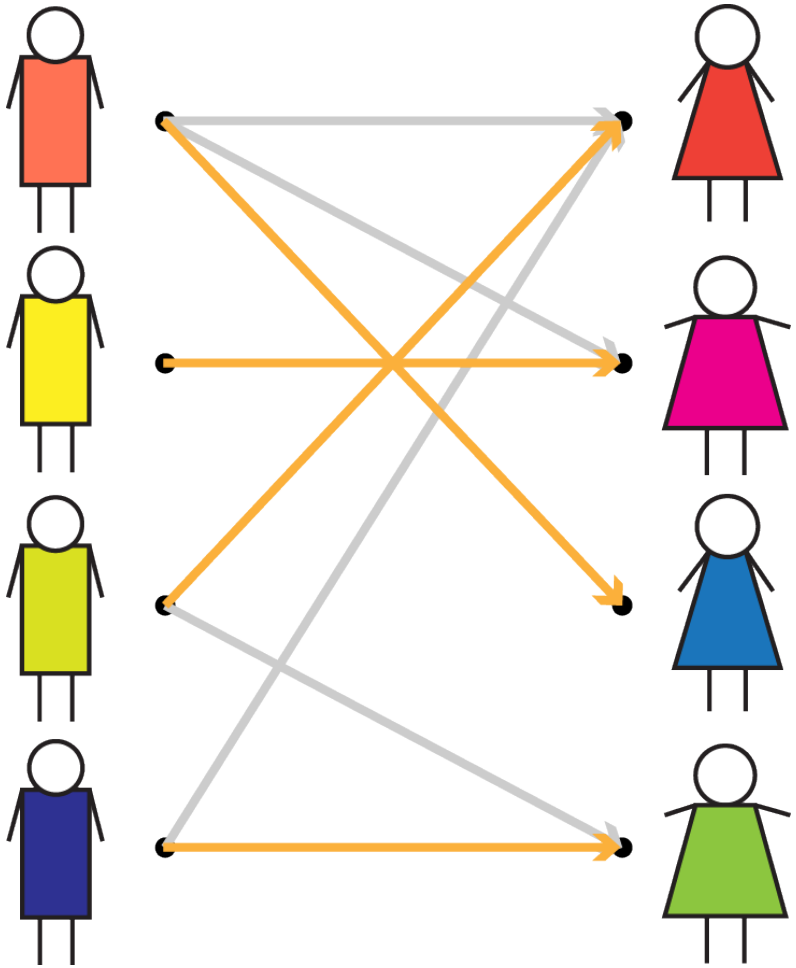
Bipartite Matching

Bipartite Matching

- We are given a graph $G = (V, E)$ with vertices partitioned into two parts A, B.
- **Goal:** We want to match elements of A to elements of B using edges of E.
- Each vertex should be matched to exactly one other vertex.

Motivation



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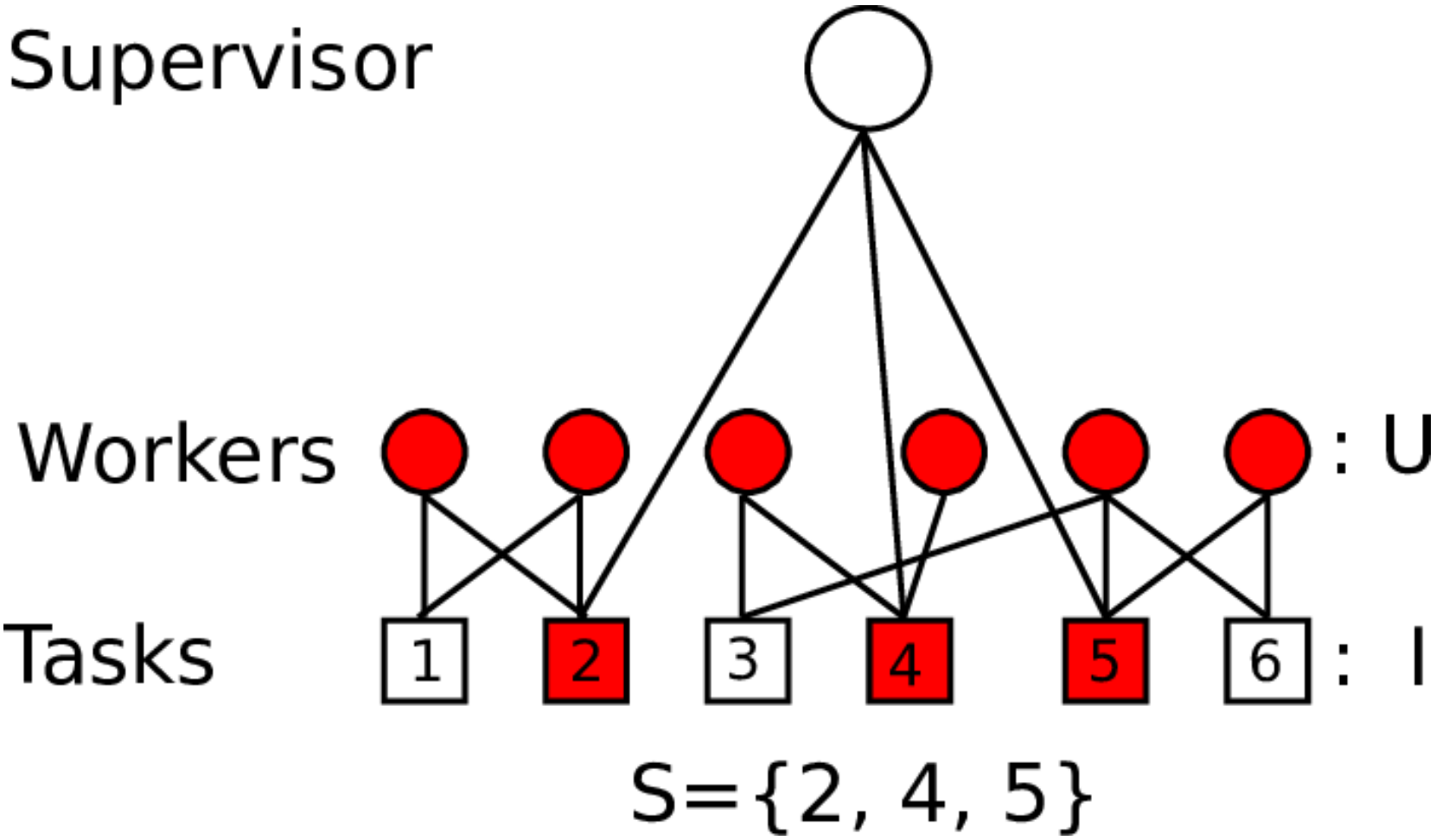

Matchings

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A **matching** in a graph $G = (V, E)$ is a set $M \subseteq E$ such that no two elements of M share a vertex.

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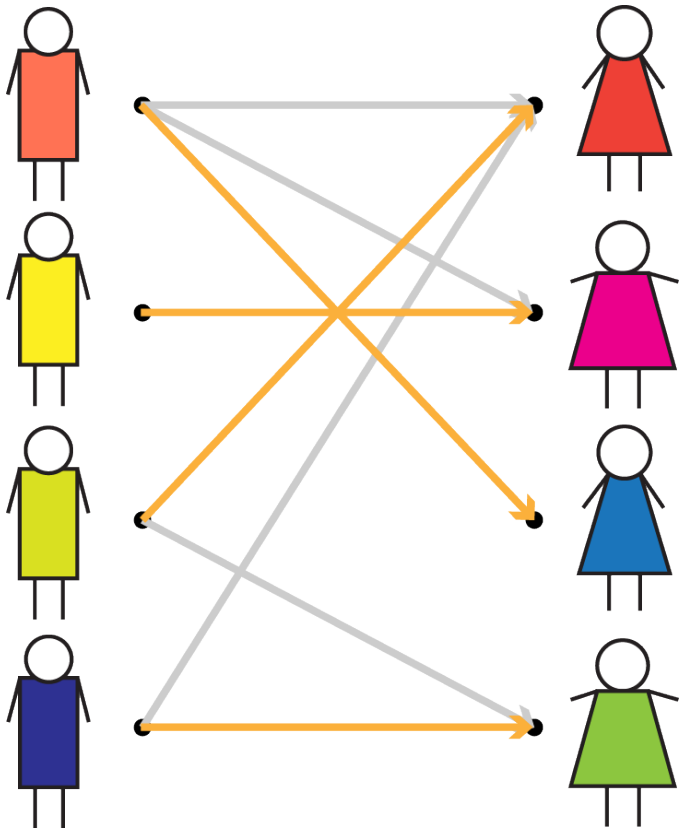


Matchings

Definition

A **matching** in a graph $G = (V, E)$ is a set $M \subseteq E$ such that no two elements of M share a vertex.

Definition

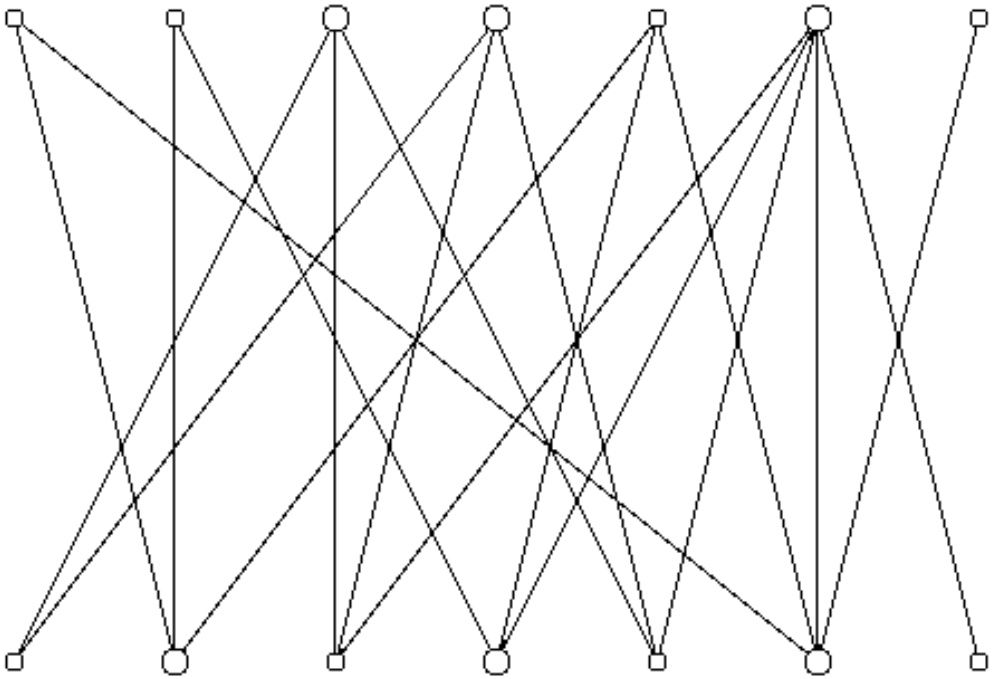
A matching M is **perfect** if all vertices are incident to an edge of M.

Definition

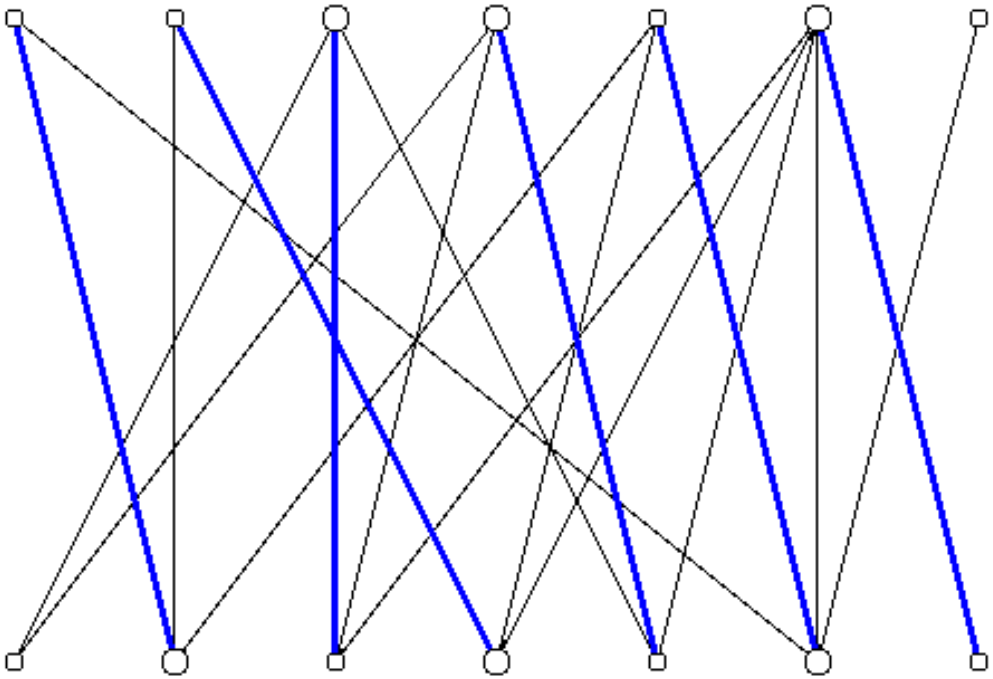
A matching M is **maximum** if all sets of edges of size $|M| + 1$ or more contain two edges incident on the same vertex.

Algorithmic problem: Given a graph, find the maximum matching.

Example



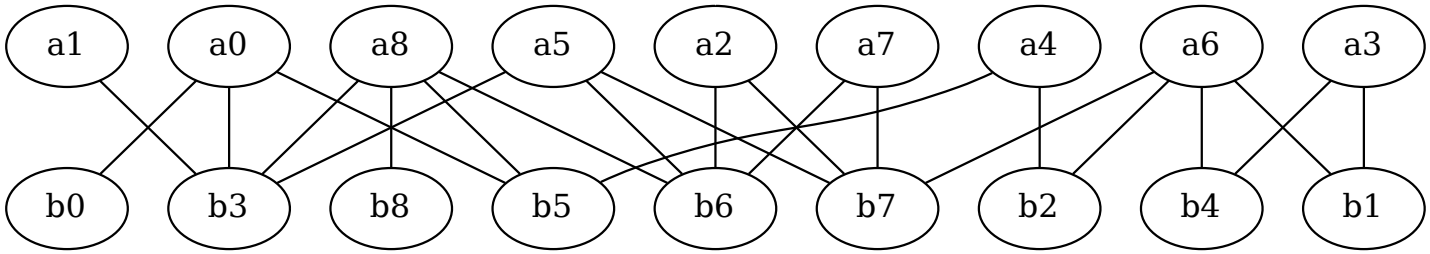
Example



Matchings to Flows

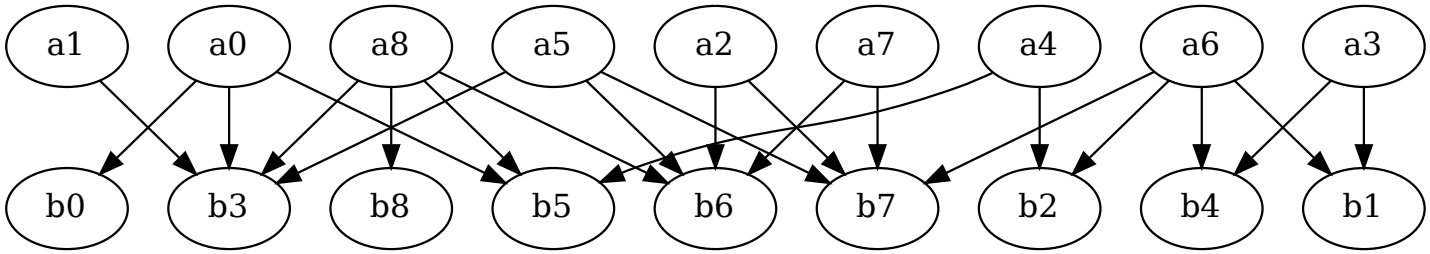
- Finding a maximum matching is a non-trivial problem
- Can be reduced to computing a maximum flow as follows:
  - Orient edges from A to B
  - Add a vertex s with arcs to A
  - Add a vertex t with arcs from B
  - Compute maximum $s \to t$ flow

Example



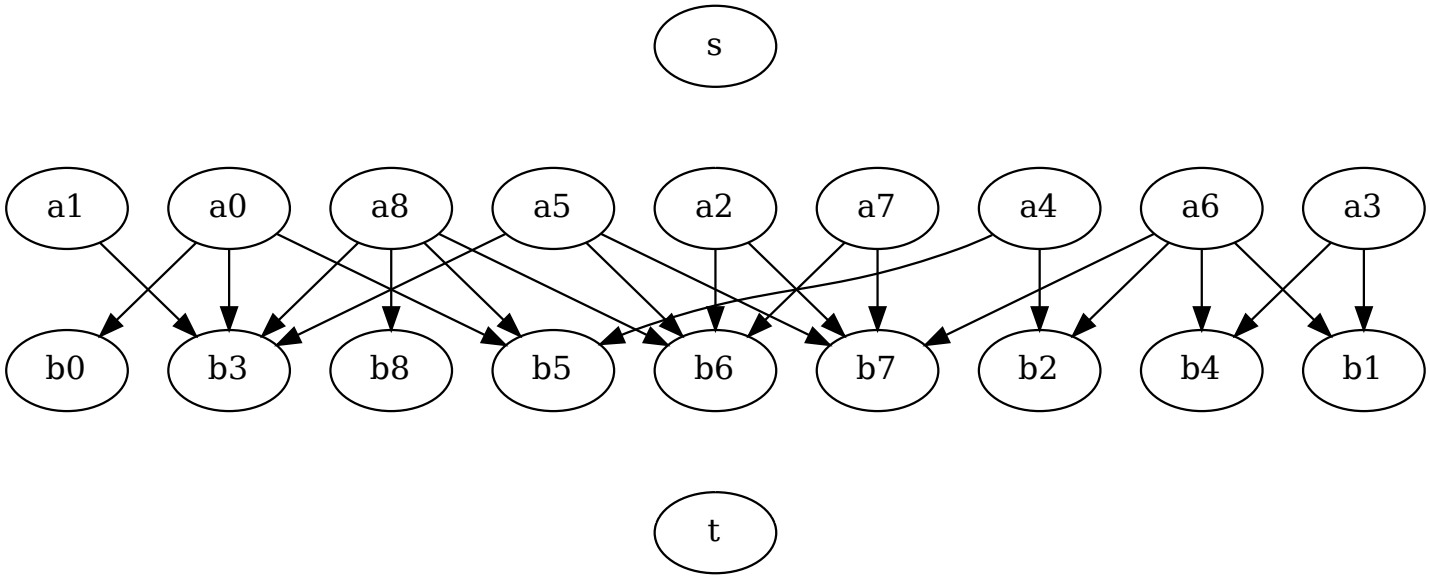
We want to calculate a maximum matching

Example



Orient graph, add s, t

Example



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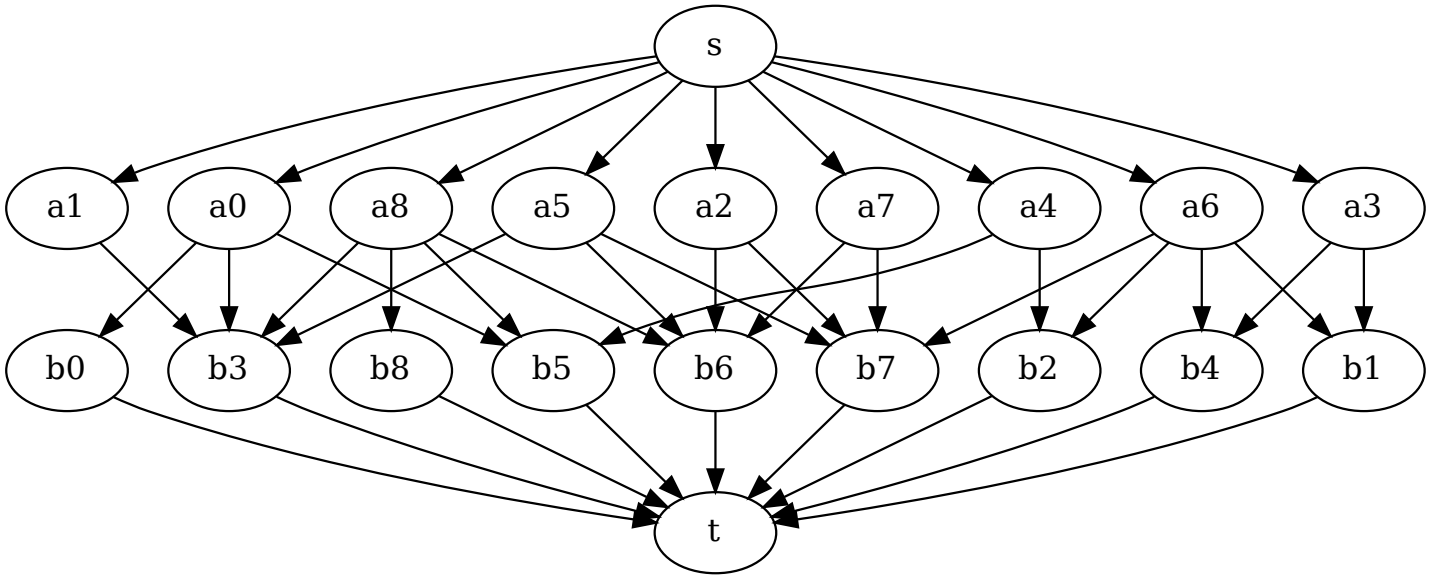

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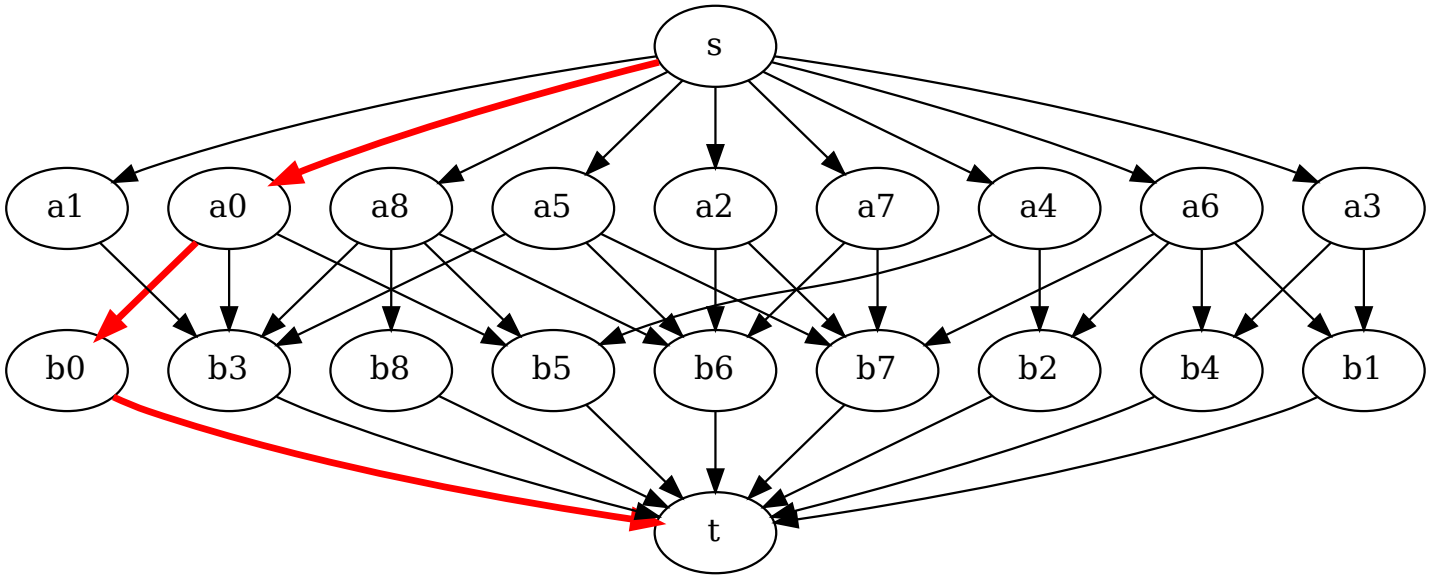
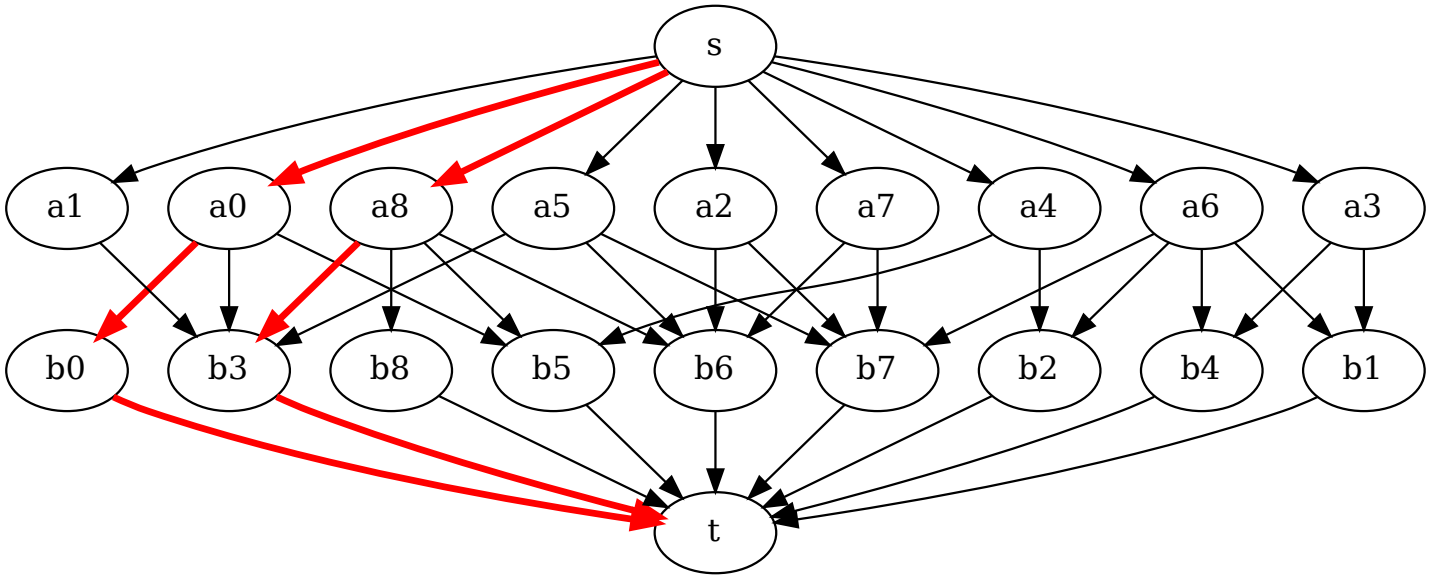


Run some max flow algorithm (some details skipped)

Example



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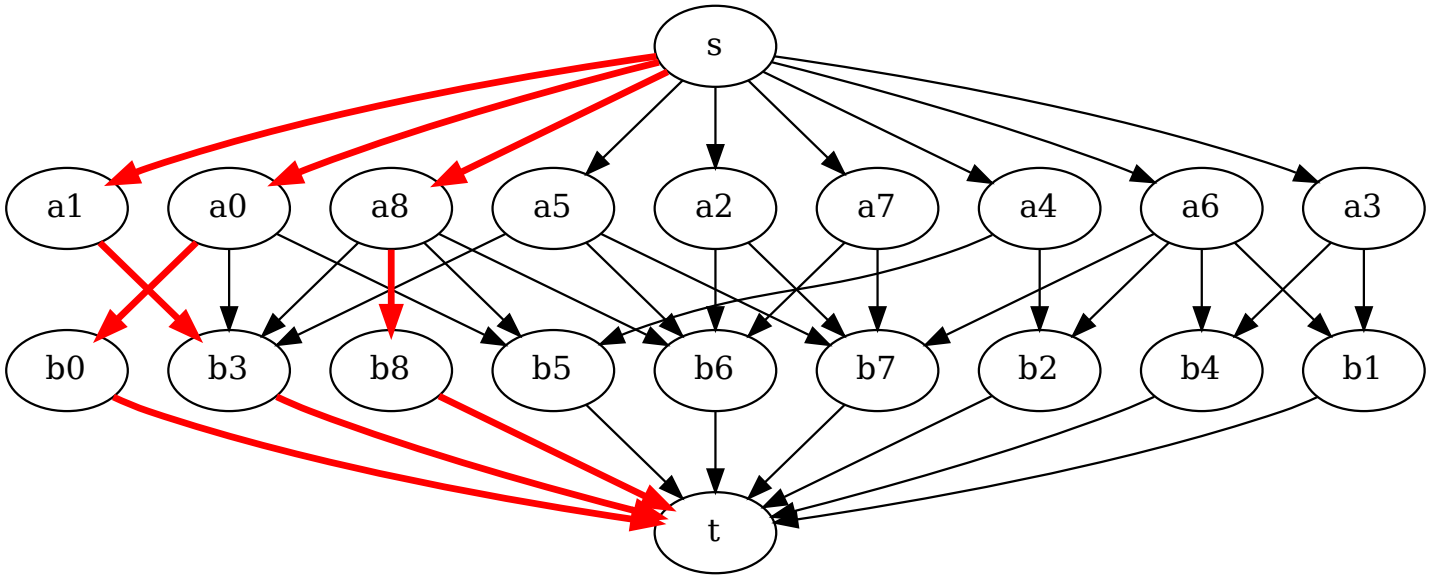

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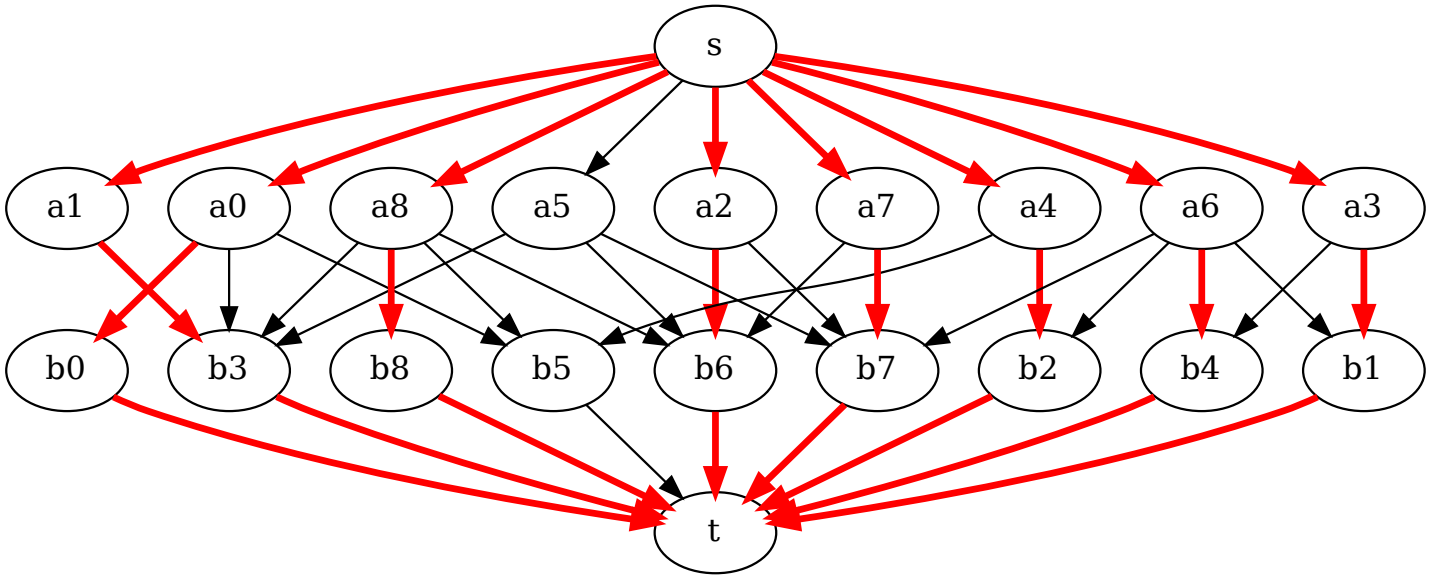
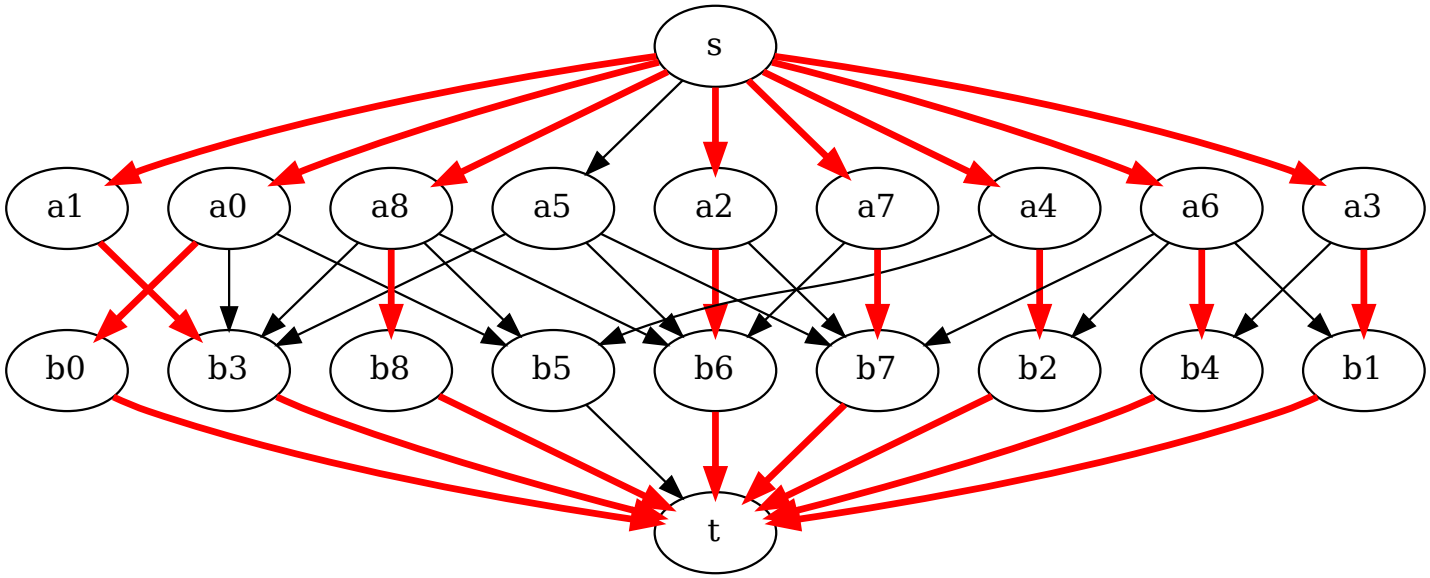


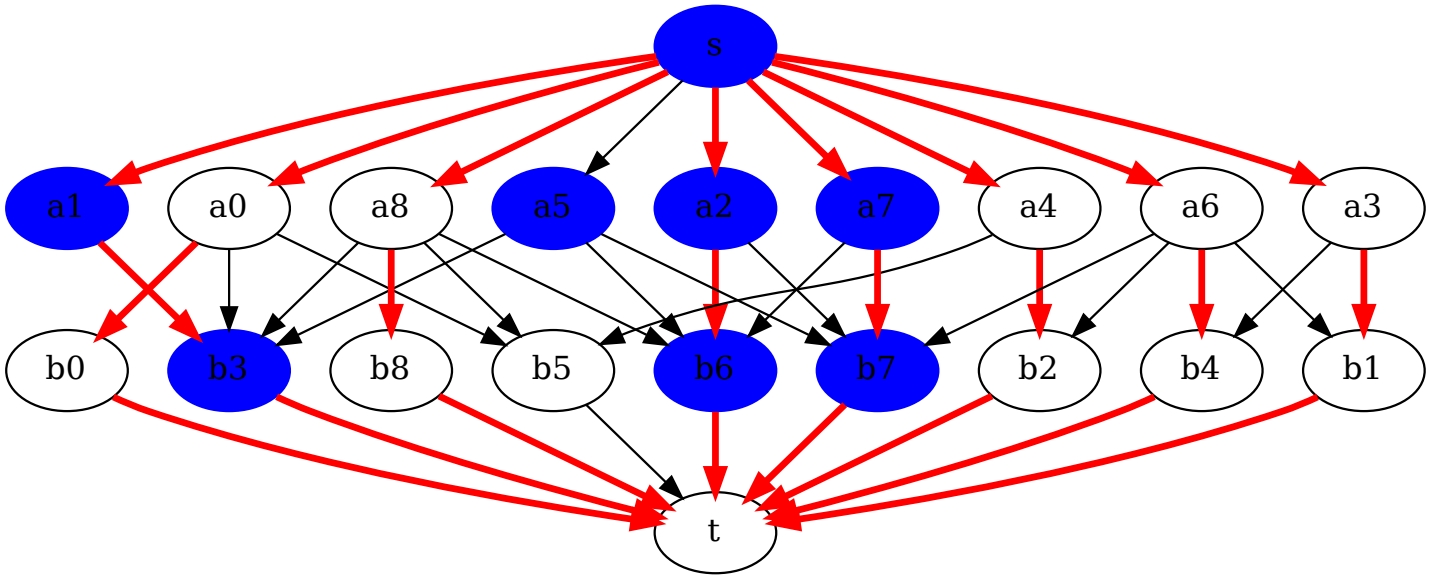
Arcs used in flow $\leftrightarrow$ Matching edges

Example



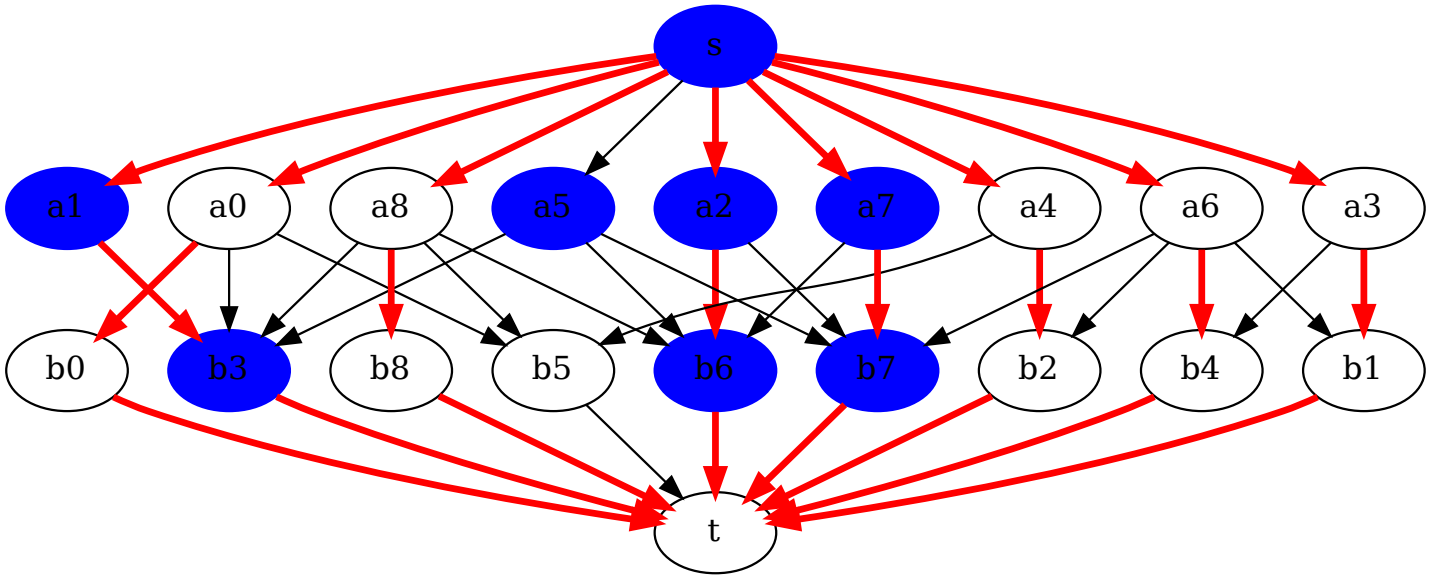
Flow is maximum $\leftrightarrow$ Matching is maximum

Example



Flow is maximum as blue set has cut size 8 (8 arcs coming out)

Example



$\Rightarrow$ maximum matching has size 8

Analysis

- Given $G = (V, E)$ with $V = A \cup B$ and $|E| = m$ we run Ford-Fulkerson
- Observe that $f^* \leq n$
- Running time: $O(mn)$
- (As mentioned, a much more complicated almost-linear time algorithm exists.)