TP 2: Diameter, BFS, and Expander graphs

Summary

This TP has two parts: in the first part we perform some theoretical analysis, discuss the relation between the maximum degree Δ and the diameter of a graph, and observe that in all graphs with $\Delta = O(1)$ we have diameter $\Omega(\log n)$. We also discuss the concept of expander graphs, which are well-connected constant-degree graphs and show that such graphs have diameter $\Theta(\log n)$. In the second part we program a family of expander graphs and use BFS to experimentally verify that the constructed graphs do indeed have logarithmic diameter.

1 Theoretical Part: Degree, Diameter, Eccentricity

1.1 Degree vs Diameter

We will first observe that graphs with "small" degree must have "large" diameter.

Recall that we use Δ to denote the maximum degree of an undirected graph $G = (V, E)$. For $x, y \in V$ we use $\mathrm{dist}(x, y)$ to denote the shortest-path distance from x to y. Then, the **diameter** of a graph is $\mathrm{diam}(G) = \max_{x,y \in V} \mathrm{dist}(x, y)$, that is, the largest (shortest-path) distance between any two vertices.

- Show that in any graph of maximum degree $\Delta \geq 3$ we have $\mathrm{diam}(G) + 1 \geq \frac{\log n}{\log \Delta}$. Hence, if $\Delta = O(1)$, then $\mathrm{diam}(G) = \Omega(\log n)$.

1.2 Eccentricity

In general, computing the diameter of a graph requires computing the shortest path distance for any pair of vertices (and taking the maximum). However, an approximation to the diameter can be found by computing the distances from a single vertex (SSSP).

We define the eccentricity of a vertex $x \in V$ as $\mathrm{ecc}(x) = \max_{y \in V} \mathrm{dist}(x, y)$, that is, the distance from x to the vertex which is farthest from x.

- Show that for all $x \in V$ we have $\mathrm{ecc}(x) \leq \mathrm{diam}(G) \leq 2\mathrm{ecc}(x)$.

1.3 Expander Graphs

A graph is called an expander graph if it satisfies two properties (i) $\Delta = O(1)$ (ii) for every set $S \subseteq V$ with $|S| \leq n/2$ there are $\Omega(|S|)$ edges with exactly one endpoint in S.

Informal explanation: a graph is called an expander graph if it is at the same time **very sparse** and **very well-connected**. Observe that these requirements are contradictory, and it is therefore not obvious how one could construct an expander graph family[1]. You can convince yourself that constructing an expander is hard by taking a few examples: are cliques expanders? are trees expanders? are grid graphs expanders? (The answer is No in all cases). Despite the seeming difficulty in constructing an expander, such graphs are of major importance in theoretical computer science, as both properties (sparsity and connectivity) are generally very desirable.[2]

In this exercise we ask you to prove one notable property of expander graphs.

[1] Indeed, constructing expanders was for a long time an open problem in discrete mathematics.

[2] `https://en.wikipedia.org/wiki/Expander_graph`

- Show that if G is an expander, then $\mathrm{diam}(G) = O(\log n)$.

Observe that by the previous exercise, this implies that $\mathrm{diam}(G) = \Theta(\log n)$ for expander graphs.

2 BFS and the Gabber-Galil construction

The goal of this exercise is to program a classical expander graph construction and verify experimentally that the graphs we construct have logarithmic diameter. We will **not** verify that the graphs have the expansion property (this is not a polynomial-time solvable problem).

2.1 Gabber-Galil expanders

Fix a positive integer n. We define the Gabber-Galil expander graphs as follows. The vertex set is $V = \{0, \ldots, n-1\} \times \{0, \ldots, n-1\}$, that is, each vertex is a pair (x, y) of integers between 0 and $n-1$; while the edge set is as follows (where all operations are modulo n):

- Vertex (x, y) is adjacent to vertices $(x+1, y), (x-1, y), (x, y+1), (x, y-1)$. Observe that these edges essentially form a two-dimensional grid.
- Vertex (x, y) is adjacent to vertices $(x+y, y), (x, x+y), (x-y, y), (x, y-x)$.

Observe that the graph we have described above is undirected: if the rules state that (a, b) is adjacent to (c, d), then they also state that (c, d) is adjacent to (a, b). Therefore, the graph we constructed has maximum degree 8. Furthermore, note that the graph may have some self-loops or parallel edges. In the remainder you may choose to remove or keep these extra edges (they do not affect the rest of the analysis).

We will **not** prove this, but the graph we have constructed has the expander property: any set S of vertices with size at most $n^2/2$ has at least $c|S|$ edges connecting it to the rest of the graph, for some constant c not depending on n.

- Write a function that takes an argument n and returns the adjacency list representation of a Gabber-Galil expander with n^2 vertices.

2.2 Testing diameters through eccentricities

We will now try to verify experimentally that the graphs of the previous exercise have diameter $O(\log n)$. However, computing the diameter of a graph generally cannot be done in linear time (we have to solve APSP). We will therefore prefer to show that the eccentricity of vertex $(0, 0)$ grows logarithmically with n. Note that this implies that the diameter also grows at most logarithmically (because of the relation we proved between eccentricity and diameter).

- Program BFS and execute it on the graphs of the previous exercise, starting from vertex $(0, 0)$ for values of n up to a few thousand vertices (therefore, for graphs of a few million vertices). Verify that the eccentricity seems to increase logarithmically in n. In particular, even for n around 3000 (so around 9 million vertices) the eccentricity of 0 should be around 30. **NB:** the execution time of BFS should increase quadratically in n (linearly in the graph size).