Graph Theory Midterm Exam – 7/11/2025

Family name:

First name:

Student Number:

Guidelines

1. For all questions, dedicated answer boxes are provided. **Provide your answer inside these boxes**. Anything written outside the boxes will not be graded.
2. The sizes of the boxes give you a hint for the length of the answer expected.
3. **Write clearly!**
4. Extra pages are provided in the end to allow you to work out your answers before filling them in. Copies of the given graphs also appear there, to allow you to experiment with them, if necessary.
5. We use notation relying on Greek letters ($\Delta, \omega, \kappa, \ldots$) to signify concepts that have been seen in class (max degree, max clique, etc.). A reminder of these conventions is provided in the end for your convenience.
6. Exam duration: 90 minutes.
7. When you are finished, raise your hand and wait for your copy to be collected before (quietly) leaving the room.
8. **Don't panic!**
9. **Good luck!**

1 Lightning Round (4 points)

For each of the following statements, indicate whether the statement is true or false. Provide a justification for your answer (proof, example, counter-example, as appropriate).

1. If G is a tree, then $\text{mm}(G) = \text{vc}(G)$. Here mm, vc denote the size of the maximum matching and minimum vertex cover respectively.

2. There exists a graph where all vertices have odd degree.

3. There exists a graph G, such that G is isomorphic to its complement $\overline{G}$.

4. Every subgraph of a bipartite graph is bipartite.

5. For all graphs, if $\chi(G) \leq 5$, then $\Delta(G) \leq 6$.

6. For all graphs $\chi(G) \geq \alpha(G)$.

7. There exists a graph G where $\kappa(G), \kappa'(G), \delta(G)$ have three distinct values, where κ, κ', δ are respectively the vertex connectivity, edge connectivity, and minimum degree.

8. For all G, if $\chi(G) > 4$, then G contains an odd cycle subgraph.

2 Degree Sequences (3 points)

Prove or disprove the following statement. For all degree sequences $D = (d_1, d_2, \ldots, d_n)$ such that $d_i \geq 2$ for all $i \in \{1, \ldots, n\}$, we have the following: if there is a graph G with degree sequence D, then there is also a **connected** graph G' with degree sequence D.

3 Independent sets and Bipartiteness (3 points)

Prove or disprove: for all graphs G, if $\alpha(G) \leq 2$, then $\overline{G}$ is bipartite.

Prove or disprove the converse of the above: for all graphs G, if $\overline{G}$ is bipartite, then $\alpha(G) \leq 2$.

4 Coloring and Brooks (4 points)

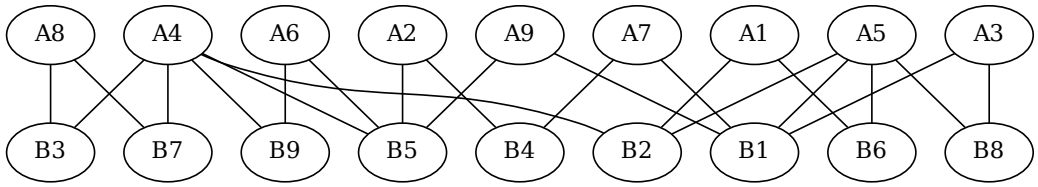
Let G be a graph with $\chi(G) = k$. We say that G is **critical** if all proper subgraphs G' of G have $\chi(G') \leq k-1$, that is, if removing any edge or vertex of G produces a graph with smaller chromatic number. Prove that if G is critical and $\chi(G) = k$, then $\delta(G) \geq k - 1$.

Furthermore, prove that if $\chi(G) = k$, for $k \geq 4$ and G is not a clique, then $\Delta(G) \geq k$.

Conclude, using the two previous questions, that if G is critical, G is not a clique, $\chi(G) = k \geq 4$, then the number of edges m of G satisfies $2m \geq n(k-1) + 1$.

5 Maximum Matching (3 points)

Consider the bipartite graph given below.



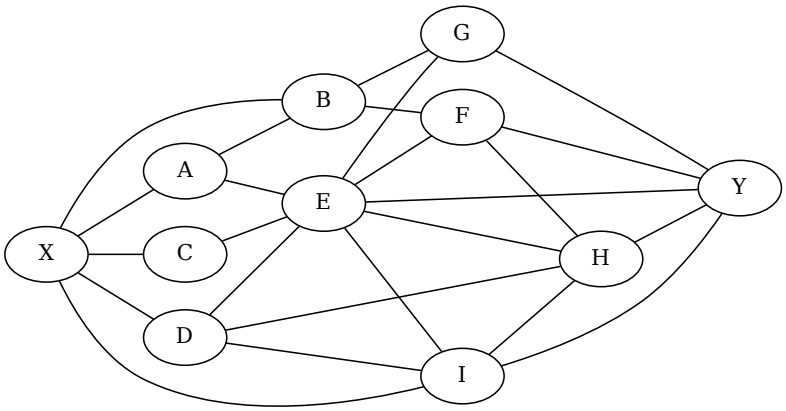
Prove or disprove: there exists a set $S \subseteq \{A1, A2, \ldots, A9\}$, such that $|N(S)| < |S|$.

Suppose we remove from the graph the edge between $A3$ and $B8$. Find the maximum matching of the new graph and prove that your matching is maximum.

Give a minimum vertex cover of the new graph (i.e. without the removed edge) and explain why your solution is minimum.

6 Cuts and Menger (3 points)

Consider the graph G given below.



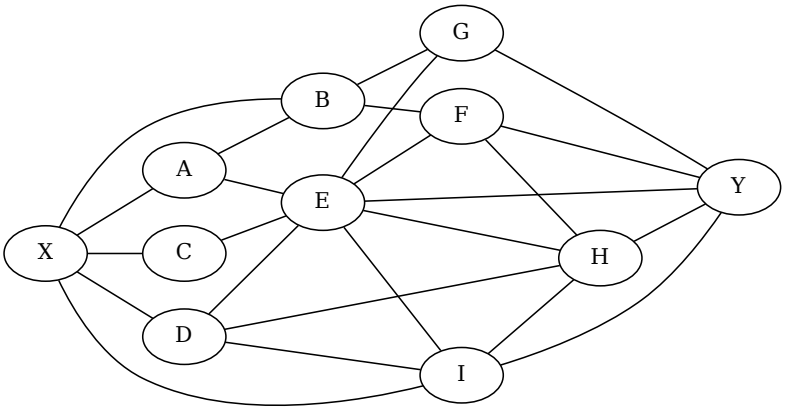
Find an XY-vertex separator of size 4 in G.

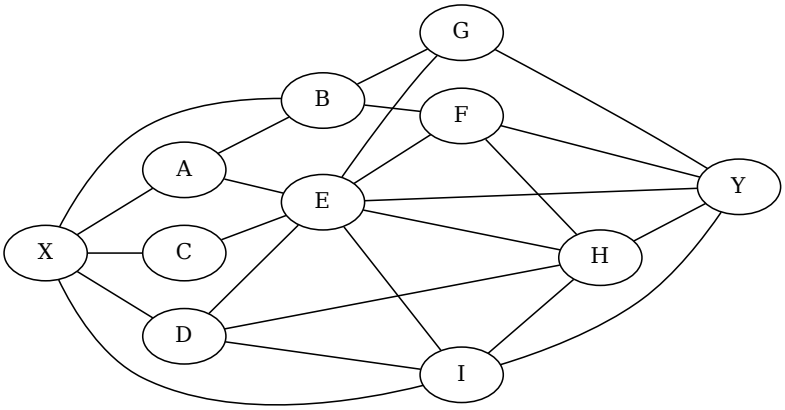
Prove that no XY-separator has size at most 3.

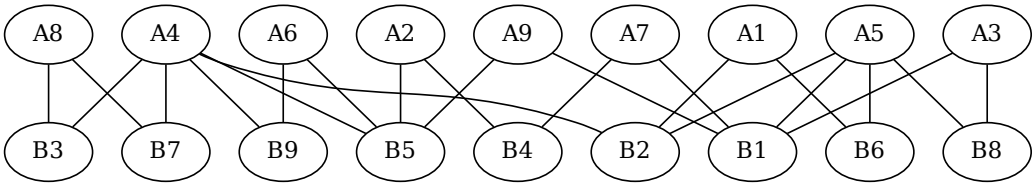
Determine whether there exists a pair of vertices u, v in the graph (other than the pair $\{X, Y\}$), such that if we add the edge uv to the graph, then every XY-vertex separator must have size at least 5.

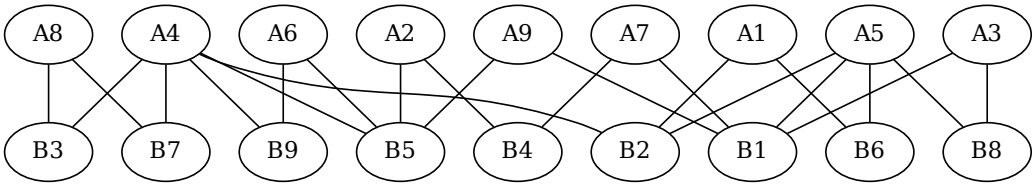
Extra Space

Use this space for notes or to play (this part will not be read). For convenience, we also give you two copies of each of the graphs supplied in previous exercises.






A8
A4
A6
A2
A9
A7
A1
A5
A3
B3
B7
B9
B5
B4
B2
B1
B6
B8


A8
A4
A6
A2
A9
A7
A1
A5
A3
B3
B7
B9
B5
B4
B2
B1
B6
B8

Notation Reminder

- $\Delta(G)$: maximum degree of G.
- $\delta(G)$: minimum degree of G.
- $\kappa(G)$: minimum size of a vertex seperator of G.
- $\kappa_{xy}(G)$: minimum size of a vertex separator separating x from y.
- $\kappa'(G), \kappa'_{xy}(G)$: same as above but for edge separators.
- $\omega(G)$: maximum clique size of G.
- $\alpha(G)$: maximum independent set size of G.
- $\chi(G)$: chromatic number of G.
- $\text{mm}(G)$: maximum matching size in G.
- $\text{vc}(G)$: minimum vertex cover size in G.