Graph Theory Final Exam – 8/1/2025

Guidelines

1. **Write clearly!**
2. The maximum (theoretical) score in this test is 21/20.
3. Exam duration: 120 minutes.
4. Provide full justifications (proofs, examples, counter-examples, as needed) for all your answers.
5. **Don't panic!**
6. **Good luck!**

1 Graph Classes (5 points)

State which of the following statements are true or false, giving appropriate justification (proof, example, counter-example, depending on the statement).

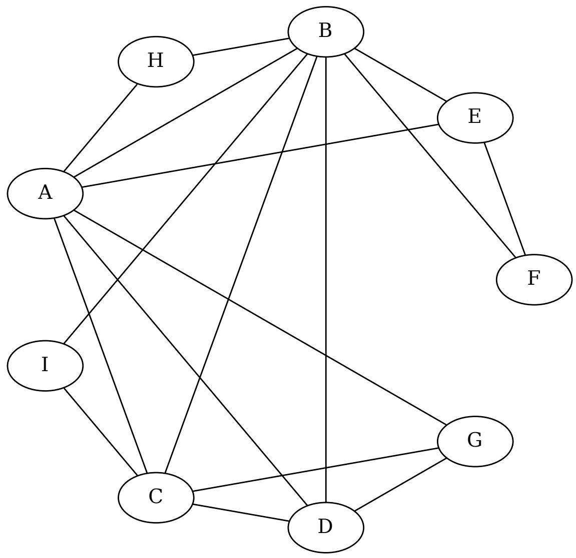
1. For all graphs G, if G is chordal, then $\overline{G}$ is chordal.
2. For all graphs G, if G is a split graph and it is connected, then its diameter is at most 3, that is, there is a path of length at most 3 between any two vertices in G.
3. For all graphs G, if G is bipartite, then G is planar.
4. There exists a graph G such that G is an interval graph but G is not a split graph.
5. Every connected cograph has a perfect matching.
6. There exists a cograph G that is not chordal.
7. If a graph G is bipartite and chordal, then G is a forest.
8. If $G = (V, E)$ is a chordal graph, and we add to G a new vertex u that is adjacent to all of V, then the new graph is also a chordal graph.
9. There exists a 3-regular graph G (i.e. a graph where all vertices have degree exactly 3) that has $n = 15$ vertices.
10. If G is a bipartite graph on n vertices and G has a perfect matching, then the maximum independent set size of G is $\alpha(G) = \frac{n}{2}$.

2 Coloring (4 points)

Let G be a graph with n vertices, m edges, and chromatic number $\chi(G) = k$ for some $k \geq 4$. Suppose that all subgraphs G' of G (except G itself) have chromatic number at most $k - 1$. In other words, G is critical, in the sense that removing any edge or vertex decreases the chromatic number.

Show that if G is not a clique, then $2m \geq n(k - 1) + 1$.

3 Algorithms for chordal graphs (4 points)



1. Prove that the graph given in the figure above is chordal, by providing an appropriate ordering of its vertices.
2. Prove that the graph is not an interval graph.
3. Give a coloring of the graph (that is, a partition of the vertices into independent sets) that uses as few colors as possible. Prove that your coloring is optimal by supplying a clique of appropriate size.

NB: Two more copies of the graph are given in the last page.

4 Planar Graphs (4 points)

Consider the graph G_n constructed as follows: we take a cycle C_n on $n \geq 10$ vertices and we add to the graph two new vertices x, y which are adjacent to each other and to all vertices of the cycle. (G_n has therefore $n + 2$ vertices.)

1. Prove that G_n is not planar.
2. Prove that if we remove from G_n the edge xy, then the graph becomes planar.
3. Prove that if we remove from G_n an edge of the cycle C_n, then the graph becomes planar.
4. Find a non-planar subgraph H of G_n that has as few edges as possible. (Hint: try to construct a sub-divided $K_{3,3}$ using few edges)

5 Threshold Graphs (4 points)

A graph $G = (V, E)$ is called a *threshold graph* if there exists a positive integer T (the threshold) and a weight function $w : V \to \mathbb{N}$ such that for all $u, v \in V$ we have $uv \in E \Leftrightarrow w(u) + w(v) \geq T$. In other words, the endpoints of edges have total sum at least T, while for any pair of non-neighbors the total sum is strictly less than T. If such a function w and threshold T exist for G, we say that G can be **represented** by w, T.

Examples:

- P_3 is a threshold graph. Given a path v_1, v_2, v_3 we can set $T = 5$ and $w(v_1) = w(v_3) = 2$, $w(v_2) = 3$ and this represents P_3. Observe that the only pairs of vertices whose total weight is at least T are exactly the endpoints of the edges of P_3.
- C_4 is **not** a threshold graph. Given a cycle x_1, x_2, x_3, x_4, if we were to represent it with a weight function w and threshold T, some vertex would have maximum weight, say without loss of generality x_1. Then, since $x_1x_3 \notin E$ it must be the case that $w(x_1) + w(x_3) < T$. But, $w(x_2) \leq w(x_1)$ (as x_1 has maximum weight), implying that $w(x_2)+w(x_3) \leq w(x_1)+w(x_3) < T$, which means that $x_2x_3 \notin E$, contradiction.

Prove the following properties of threshold graphs:

1. Show that if G is a threshold graph on n vertices, then G either contains a vertex of degree 0 or a vertex of degree $n - 1$.
2. Show that threshold graphs are split.
3. Show that threshold graphs are cographs.
4. Show that if G is both split and a cograph, then G is a threshold graph.

Extra space