Graph Theory Midterm Exam – 8/11/2024

Family name:

First name:

Student Number:

Guidelines

1. For all questions, dedicated answer boxes are provided. **Provide your answer inside these boxes**. Anything written outside the boxes will not be graded.
2. The sizes of the boxes give you a hint for the length of the answer expected.
3. **Write clearly!**
4. Extra pages are provided in the end to allow you to work out your answers before filling them in. Copies of the given graphs also appear there, to allow you to experiment with them, if necessary.
5. The maximum (theoretical) score in this test is 21/20.
6. Exam duration: 90 minutes.
7. When you are finished, raise your hand and wait for your copy to be collected before (quietly) leaving the room.
8. **Don't panic!**
9. **Good luck!**

1 Lightning Round (4 points)

For each of the following statements, indicate whether the statement is true or false. Provide a justification for your answer (proof, example, counter-example, as appropriate).

1. There exists a tree of average degree 2.

2. There exists a 3-vertex-connected graph G (i.e. $\kappa(G) \geq 3$), such that G has average vertex degree 2.

3. There exists a graph G, where $\text{mm}(G) < \text{vc}(G)$, where mm, vc are the sizes of the maximum matching and minimum vertex cover respectively.

4. There exists a bipartite graph G that satisfies the condition of the previous question.

5. Every tree is bipartite.

6. Every subgraph of a tree is a tree.

7. There exists a graph G where $\kappa(G), \kappa'(G), \delta(G)$ have three distinct values, where κ, κ', δ are respectively the vertex connectivity, edge connectivity, and minimum degree.

8. In all bipartite graphs with at least 3 vertices, $\alpha(G) \geq \omega(G)$, where α, ω are respectively the size of the largest independent set and largest clique.

2 Complements and Connectivity (3 points)

Suppose that a graph G with $n \geq 4$ vertices contains no vertices of degree 0. Prove that if G is disconnected, then $\kappa(\overline{G}) \geq 2$, that is, $\overline{G}$ is at least 2-vertex-connected.

3 Edge Cuts (4 points)

Suppose a graph $G = (V, E)$ is connected and all vertices have even degree. Show that in this case G has no cut edge, that is, for all $e \in E$ we have that $G - e$ is connected.

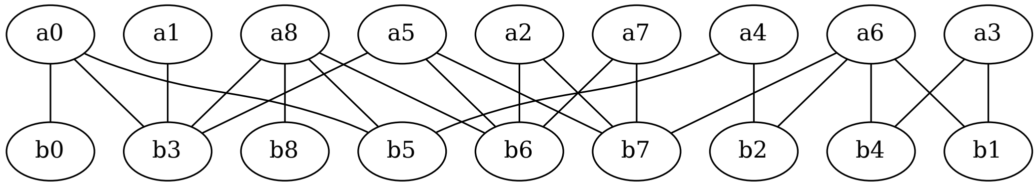
4 Degree Sequences (3 points)

Suppose a graph G has $n = 6$ vertices, $x_1, x_2, \ldots, x_6$. You are given the following information regarding their degrees: $\deg(x_1) = 5$, $\deg(x_2) = 4$, $\deg(x_3) = 3$, $\deg(x_4) = 2$, $\deg(x_5) = a$, $\deg(x_6) = b$, where a, b are unknown values.

Among all the values of (a, b) for which such a graph exists, determine the minimum possible value of $a + b$.

5 Maximum Matching (4 points)

Consider the bipartite graph given below.



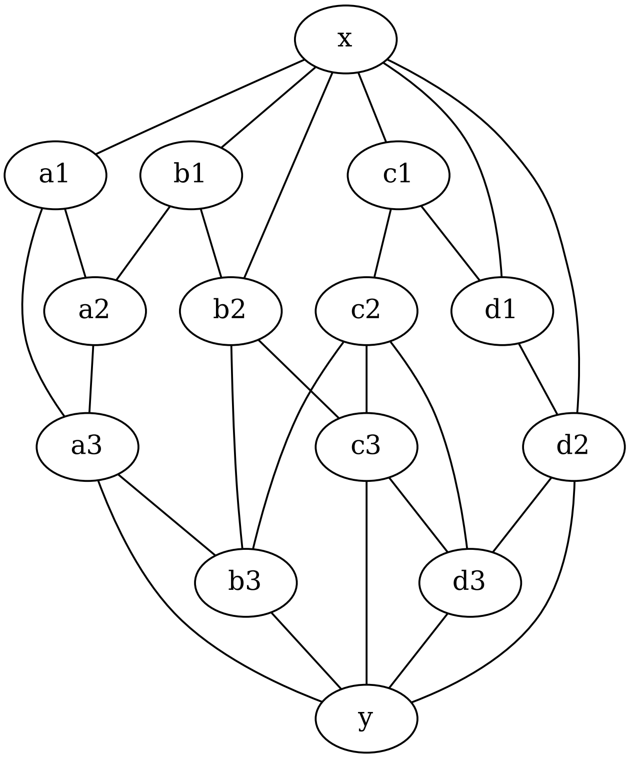
Give a matching of maximum size in the graph above. That is, give the list of edges that form your matching. (To ease presentation, you may also highlight the selected edges in the graph above. However, the list below will be considered your official answer.)

Prove that the graph does not contain a matching larger than the one you supplied.

Give a minimum vertex cover of the given graph and explain why it is minimum.

6 Cuts and Menger (3 points)

Consider the graph G given below.

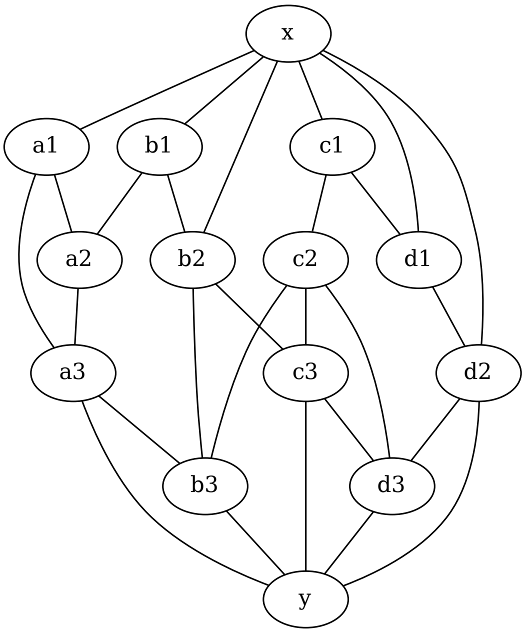


Find a minimum xy vertex cut in G. Explain why your cut is minimum. (1 point)

Suppose we add to G the edge $(a2, b3)$. Find a minimum xy vertex cut in the new graph. Explain why your cut is minimum. (2 points)

Extra Space

Use this space for notes or to play (this part will not be read). For convenience, we also give you two copies of each of the graphs supplied in previous exercises.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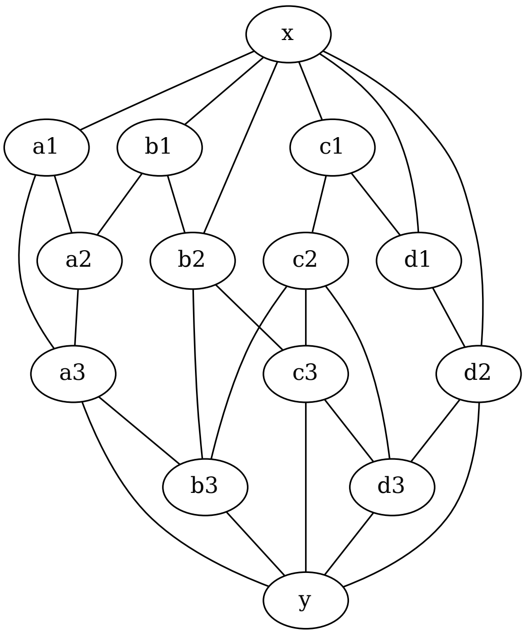
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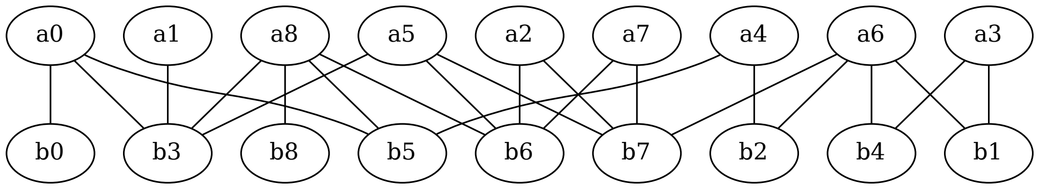
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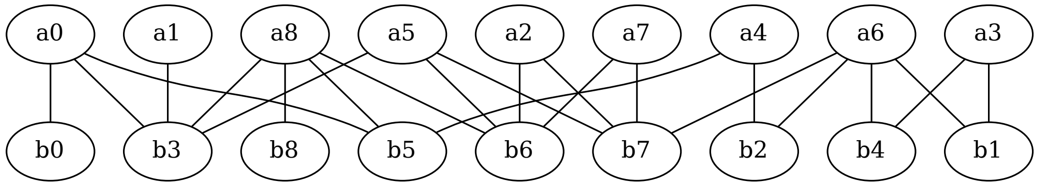
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