Graph Theory Midterm Exam – 7/11/2025

Family name:

First name:

Student Number:

Guidelines

1. For all questions, dedicated answer boxes are provided. **Provide your answer inside these boxes**. Anything written outside the boxes will not be graded.
2. The sizes of the boxes give you a hint for the length of the answer expected.
3. **Write clearly!**
4. Extra pages are provided in the end to allow you to work out your answers before filling them in. Copies of the given graphs also appear there, to allow you to experiment with them, if necessary.
5. We use notation relying on Greek letters ($\Delta, \omega, \kappa, \ldots$) to signify concepts that have been seen in class (max degree, max clique, etc.). A reminder of these conventions is provided in the end for your convenience.
6. Exam duration: 90 minutes.
7. When you are finished, raise your hand and wait for your copy to be collected before (quietly) leaving the room.
8. **Don't panic!**
9. **Good luck!**

1 Lightning Round (4 points)

For each of the following statements, indicate whether the statement is true or false. Provide a justification for your answer (proof, example, counter-example, as appropriate).

1. If G is a tree, then $\mathrm{mm}(G) = \mathrm{vc}(G)$. Here mm, vc denote the size of the maximum matching and minimum vertex cover respectively.

   True, because trees are bipartite so Konig's theorem applies.

2. There exists a graph where all vertices have odd degree.

   True. For example, K_4.

3. There exists a graph G, such that G is isomorphic to its complement $\overline{G}$.

   True. For example P_4.

4. Every subgraph of a bipartite graph is bipartite.

   True, because if the subgraph contained an odd cycle, the original graph would as well.

5. For all graphs, if $\chi(G) \leq 5$, then $\Delta(G) \leq 6$.

   False, for example $K_{1,10}$.

6. For all graphs $\chi(G) \geq \alpha(G)$.

   False, for example $K_{10,10}$.

7. There exists a graph G where $\kappa(G), \kappa'(G), \delta(G)$ have three distinct values, where κ, κ', δ are respectively the vertex connectivity, edge connectivity, and minimum degree.

   True. Take two copies of a K_5. Place two edges from one vertex of the first copy to two vertices of the other copy. We have $\delta = 4$, $\kappa = 1$, and $\kappa' = 2$.

8. For all G, if $\chi(G) > 4$, then G contains an odd cycle subgraph.

   True. If G did not contains such a subgraph, it would be bipartite and $\chi(G) \leq 2$.

2 Degree Sequences (3 points)

Prove or disprove the following statement. For all degree sequences $D = (d_1, d_2, \ldots, d_n)$ such that $d_i \geq 2$ for all $i \in \{1, \ldots, n\}$, we have the following: if there is a graph G with degree sequence D, then there is also a **connected** graph G' with degree sequence D.

Take G to be a graph with degree sequence D and suppose it contains several connected components. Let C_1, C_2 be such components. We will construct a graph G' where the vertices of $C_1 \cup C_2$ form a single component, while the degree sequence stays the same. Repeating this we get a connected graph.

Observe that because the minimum degree is at least 2, $G[C_1]$ and $G[C_2]$ contain cycles, so let $e_1 = x_1y_1$ be an edge of a cycle of C_1 and $e_2 = x_2y_2$ an edge of a cycle of C_2. Remove these edges from the graph and add the edges x_1x_2, y_1y_2. This keeps all degrees constant and merges the components, because removing an edge from a cycle does not impact connectivity.

3 Independent sets and Bipartiteness (3 points)

Prove or disprove: for all graphs G, if $\alpha(G) \leq 2$, then $\overline{G}$ is bipartite.

False, for example C_5

Prove or disprove the converse of the above: for all graphs G, if $\overline{G}$ is bipartite, then $\alpha(G) \leq 2$.

True. If $\overline{G}$ is bipartite, then G can be partitioned into two cliques. Each independent set can take at most one vertex from each clique, so $\alpha(G) \leq 2$.

4 Coloring and Brooks (4 points)

Let G be a graph with $\chi(G) = k$. We say that G is **critical** if all proper subgraphs G' of G have $\chi(G') \leq k-1$, that is, if removing any edge or vertex of G produces a graph with smaller chromatic number. Prove that if G is critical and $\chi(G) = k$, then $\delta(G) \geq k - 1$.

Solution:

Suppose for contradiction that there exists a vertex v of degree at most $k - 2$. Remove v from G to obtain G' which is $(k-1)$-colorable. Inserting v into this coloring is always possible, as v has at most $k-2$ neighbors, so at least one of the colors $\{1, \dots, k - 1\}$ is available. This produces a $(k - 1)$-coloring of G, contradiction.

Furthermore, prove that if $\chi(G) = k$, for $k \geq 4$ and G is not a clique, then $\Delta(G) \geq k$.

Solution:

Suppose that $\Delta(G) \leq k - 1$. We can apply Brooks' theorem, because G is not a clique, and it is not an odd cycle (as $\chi(G) \geq 4$, but odd cycles can be 3-colored). But then, Brooks' theorem implies that $\chi(G) \leq \Delta(G) \leq k - 1$, contradiction.

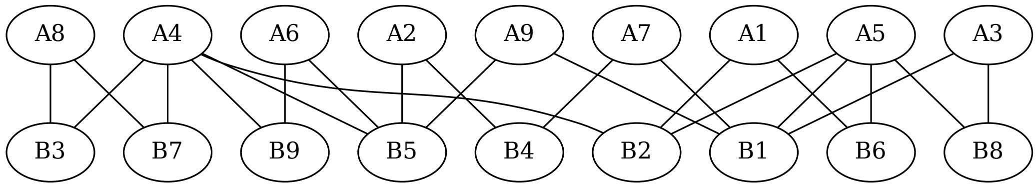
Conclude, using the two previous questions, that if G is critical, G is not a clique, $\chi(G) = k \geq 4$, then the number of edges m of G satisfies $2m \geq n(k - 1) + 1$.

Solution:

All vertices have degree at least $k - 1$ and there exists at least one vertex of degree at least k, so the sum of all degrees is at least $n(k - 1) + 1$.

5 Maximum Matching (3 points)

Consider the bipartite graph given below.



Prove or disprove: there exists a set $S \subseteq \{A1, A2, \ldots, A9\}$, such that $|N(S)| < |S|$.

The set $(A8, B3), (A4, B7), (A6, B9), (A2, B5), (A9, B1), (A7, B4), (A1, B2), (A5, B6), (A3, B8)$ is a perfect matching, so no set S as described exists, by Hall's theorem.

Suppose we remove from the graph the edge between $A3$ and $B8$. Find the maximum matching of the new graph and prove that your matching is maximum.

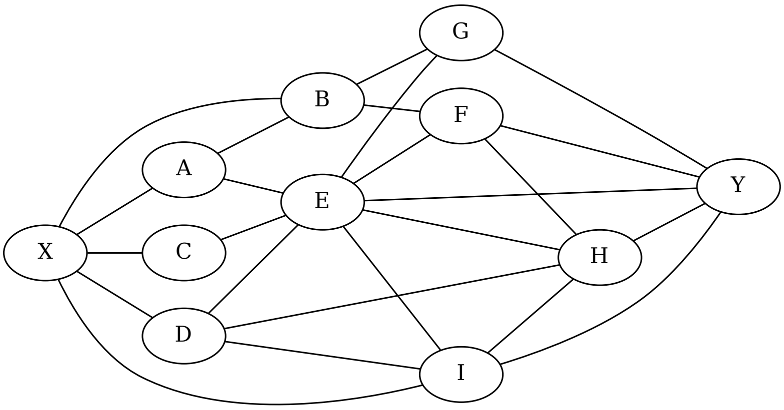
The previous matching minus the removed edge is now maximum. To see this, it suffices to prove that the new graph has a set S with $|N(S)| < |S|$ and use Hall's theorem. Such a set is $A3, A9, A2, A7$, which has neighbors $B1, B4, B5$.

Give a minimum vertex cover of the new graph (i.e. without the removed edge) and explain why your solution is minimum.

$\{B1, B4, B5, A1, A4, A5, A6, A8\}$. This is minimum as there is a matching of size 8.

6 Cuts and Menger (3 points)

Consider the graph G given below.



Find an XY-vertex separator of size 4 in G.
I, D, E, B.

Prove that no XY-separator has size at most 3.

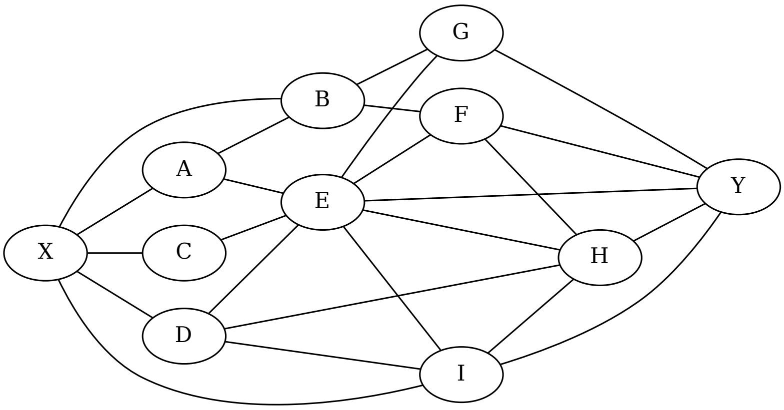
We have four disjoint paths from X to Y: X, I, Y, X, D, H, Y, X, C, E, F, Y, X, A, B, G, Y.

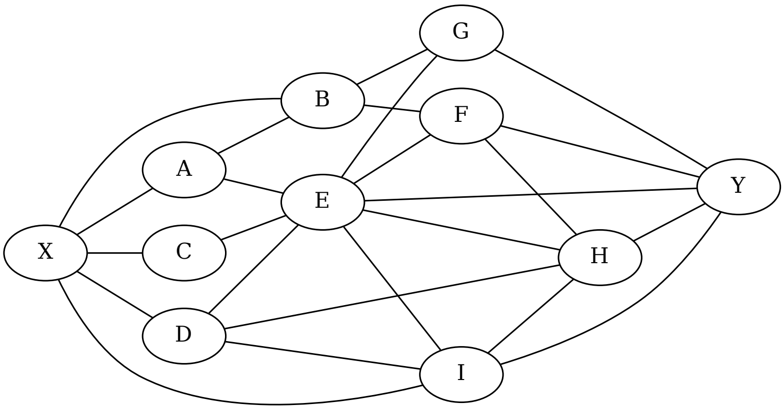
Determine whether there exists a pair of vertices u, v in the graph (other than the pair $\{X, Y\}$), such that if we add the edge uv to the graph, then every XY-vertex separator must have size at least 5.

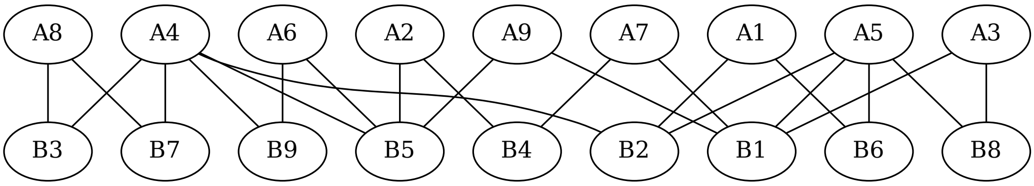
The edge AY will do the trick. We then have the five disjoint paths: X, I, Y, X, D, H, Y, X, C, E, F, Y, X, B, G, Y, X, A, Y.

Extra Space

Use this space for notes or to play (this part will not be read). For convenience, we also give you two copies of each of the graphs supplied in previous exercises.






A8
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A9
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A5
A3
B3
B7
B9
B5
B4
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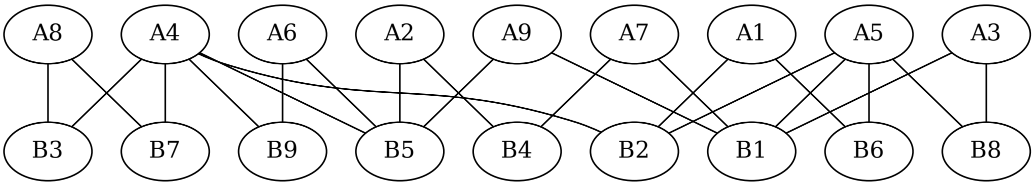
B8


A8
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B3
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B5
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B1
B6
B8

Notation Reminder

- $\Delta(G)$: maximum degree of G.
- $\delta(G)$: minimum degree of G.
- $\kappa(G)$: minimum size of a vertex seperator of G.
- $\kappa_{xy}(G)$: minimum size of a vertex separator separating x from y.
- $\kappa'(G), \kappa'_{xy}(G)$: same as above but for edge separators.
- $\omega(G)$: maximum clique size of G.
- $\alpha(G)$: maximum independent set size of G.
- $\chi(G)$: chromatic number of G.
- $\mathrm{mm}(G)$: maximum matching size in G.
- $\mathrm{vc}(G)$: minimum vertex cover size in G.