Exact Algorithms

Homework 2

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A tournament is a directed graph containing exactly one arc (=edge with orientation) between every pair of vertices. The problem FEEDBACK ARC SET IN TOURNAMENTS is defined as below.

Input: a tournament $T = (V, A)$ on n vertices.

Parameter: k.

Task: find a set of at most k arcs whose reversal (=reversing the orientation of an arc) makes T acyclic.

1. Consider the problem FEEDBACK ARC SET IN TOURNAMENTS for (a)-(c).

1(a). Establish that $F \subseteq A(T)$ is a feedback arc set of T if and only if every directed triangle of T contains at least one arc of F.

1(b). Use 1(a) to show that FEEDBACK ARC SET IN TOURNAMENTS admits a kernel with $O(k^3)$ vertices.

1(c). Devise reduction rules with safeness proof. Use them to establish that FEEDBACK ARC SET IN TOURNAMENTS admits a kernel on $O(k^2)$ vertices. (Hint: similar to $O(k^2)$ kernelization for VERTEX COVER.)

2. We consider a polynomial-time algorithm for constructing a half-integral optimal solution to LP for VERTEX COVER which does not solve LP directly. Let G be an input instance to VERTEX COVER and G^* be an auxiliary bipartite graph so that:

- $\{v_1, v_2 : v \in V(G)\}$ is the vertex set, and
- for every edge $(u, v) \in E(G)$, (u_1, v_2) and (u_2, v_1) are edges of G^*.

Let S^* be a minimum vertex cover of G^*. We define a solution x^* to LP from S^* as follows:

$$x_u^* = \begin{cases} 0.5 & \text{if exactly one of } u_1 \text{ and } u_2 \text{ belong to } S^*. \\ 1 & \text{if both of } u_1 \text{ and } u_2 \text{ belong to } S^*. \\ 0 & \text{if none of } u_1 \text{ and } u_2 \text{ belongs to } S^*. \end{cases}$$

2(a). Show that for any matching M of a graph and any feasible solution z to LP, the objective value of z is at least $|M|$.

2(b). Show that for an arbitrary feasible solution z to LP(G), the solution z' defined as $z'_{u_1} = z'_{u_2} = z_u$, for all $u \in V(G)$ is also feasible to LP(G^*), where LP formulation for VERTEX COVER of G is denoted as LP(G).

2(c). Show that x^* is an optimal solution to LP(G). (Hint: Use König theorem which says that in a bipartite graph, the size of a maximum matching equals the size of a vertex cover.)

2(d). Neatly present a kernelization implied by this exercise and the size bound of the obtained kernel, and estimate the running time of the kernelization (search for existing literature if necessary).

★3. In the problem CLUSTER EDITING, we are given a graph G and a nonnegative integer k and want to find a set of at most k pairs of vertices $F \subseteq \binom{V(G)}{2}$ so that the graph $(V(G), E(G) \triangle F)$ obtained by editing G with F is a cluster graph. Here, $X \triangle Y$ denotes the symmetric difference of the sets X and Y. Hence, editing G with F is equivalent to adding a pair $(u, v) \in F$ to G if (u, v) is a non-edge in G, and removing $(u, v) \in F$ if it is an edge in G, thereby obtaining $G = (V(G), (E(G) \setminus F) \cup (F \setminus E(G))$.
We aim to establish that CLUSTER EDITING admits a kernel on $O(k^2)$ vertices.

♣ Submit your solution via email (eunjungkim78@gmail.com) by 6 Feb 2019, midnight.
♣ Questions with ★ can be worked together with a colleague. But please write the solution by yourself.