A tournament is a directed graph containing exactly one arc (=edge with orientation) between every pair of vertices. The problem **Feedback Arc Set in Tournaments** is defined as below.

**Input:** a tournament $T = (V, A)$ on $n$ vertices.

**Parameter:** $k$.

**Task:** find a set of at most $k$ arcs whose reversal (=reversing the orientation of an arc) makes $T$ acyclic.

1. Consider the problem **Feedback Arc Set in Tournaments** for (a)-(c).

1(a). Establish that $F \subseteq A(T)$ is a feedback arc set of $T$ if and only if every directed triangle of $T$ contains at least one arc of $F$.

1(b). Use 1(a) to show that **Feedback Arc Set in Tournaments** admits a kernel with $O(k^3)$ vertices.

1(c). Devise reduction rules with safeness proof. Use them to establish that **Feedback Arc Set in Tournaments** admits a kernel on $O(k^2)$ vertices. (Hint: similar to $O(k^2)$ kernelization for Vertex Cover.)

2. We consider a polynomial-time algorithm for constructing a half-integral optimal solution to LP for **Vertex Cover** which does not solve LP directly. Let $G$ be an input instance to **Vertex Cover** and $G^*$ be an auxiliary bipartite graph so that:

- $\{v_1, v_2 : v \in V(G)\}$ is the vertex set, and
- for every edge $(u, v) \in E(G)$, $(u_1, v_2)$ and $(u_2, v_1)$ are edges of $G^*$.

Let $S^*$ be a minimum vertex cover of $G^*$. We define a solution $x^*$ to LP from $S^*$ as follows:

- $x^*_u = \begin{cases} 
0.5 & \text{if exactly one of } u_1 \text{ and } u_2 \text{ belong to } S^*. \\
1 & \text{if both of } u_1 \text{ and } u_2 \text{ belong to } S^*. \\
0 & \text{if none of } u_1 \text{ and } u_2 \text{ belongs to } S^*. 
\end{cases}$

2(a). Show that for any matching $M$ of a graph and any feasible solution $z$ to LP, the objective value of $z$ is at least $|M|$.

2(b). Show that for an arbitrary feasible solution $z$ to LP($G$), the solution $z'$ defined as $z'_u = z'_u = z_u$, for all $u \in V(G)$ is also feasible to LP($G^*$), where LP formulation for **Vertex Cover** of $G$ is denoted as LP($G$).
2(c). Show that \( x^* \) is an optimal solution to LP\((G)\). (Hint: Use König theorem which says that in a bipartite graph, the size of a maximum matching equals the size of a vertex cover.)

2(d). Neatly present a kernelization implied by this exercise and the size bound of the obtained kernel, and estimate the running time of the kernelization (search for existing literature if necessary).

★3. In the problem CLUSTER EDITING, we are given a graph \( G \) and a nonnegative integer \( k \) and want to find a set of at most \( k \) pairs of vertices \( F \subseteq \binom{V(G)}{2} \) so that the graph \( (V(G), E(G) \Delta F) \) obtained by editing \( G \) with \( F \) is a cluster graph. Here, \( X \Delta Y \) denotes the symmetric difference of the sets \( X \) and \( Y \). Hence, editing \( G \) with \( F \) is equivalent to adding a pair \( (u, v) \in F \) to \( G \) if \( (u, v) \) is a non-edge in \( G \), and removing \( (u, v) \in F \) if it is an edge in \( G \), thereby obtaining \( G = (V(G), (E(G) \setminus F) \cup (F \setminus E(G))) \).

We aim to establish that CLUSTER EDITING admits a kernel on \( O(k^2) \) vertices.

♣ Submit your solution via email (eunjungkim78@gmail.com) by 6 Feb 2019, midnight.
♣ Questions with ★ can be worked together with a colleague. But please write the solution by yourself.