1. (Exercise 5.1 from [1]) In the TRIANGLE PACKING problem, we are given an undirected graph $G$ and a positive integer $k$, and the objective is to test whether $G$ has $k$ vertex-disjoint triangles. Using color coding show that the problem admits an algorithm with (expected) running time $2^{O(k)} \cdot n^{O(1)}$.

Hint: color the vertices with $3k$ colors. A colorful packing is a triangle packing in which any two vertices are colored by distinct colors. Design a dynamic programming which computes whether a set of vertices colored by $C \subseteq \{1, 2, \ldots, 3k\}$ with $|C| = 3i$ colors, for $i = 0, 1, \ldots, k$, contains a colorful packing. Use Stirling’s approximation; $n! \approx (ne)^n$.

2. (Exercise 5.3 from [1]) Consider the following problem $(k, q)$-SEPARATOR: given an undirected graph $G$ and positive integers $k$ and $q$, find a set at most $k$ vertices such that $G - X$ has at least two components of size at least $q$.

2(a). Consider a random coloring of the vertices of $G$ with two colors $L$ and $R$. Evaluate the probability that all $X$ vertices belong to $L$ and $q$ vertices of each of two “large” components belong to $R$.

2(b). Suppose that $L$ and $R$ is a partition of $V(G)$ as depicted in 2(a). Prove that there exists two vertices $s, t \in R$ and a set of vertices $Y \subseteq L$ of size at most $k$ such that $s$ and $t$ belongs to distinct component in $G - Y$.

2(c). A minimum $(s, t)$-cut of an edge-weighted directed graph $D$ (possibly with 2-cycles) is a set $S$ of edges such that $D - S$ does not contain a directed path from $s$ to $t$. Using a polynomial-time algorithm $A$ solving finding a min $(s, t)$-cut as a black box, present a polynomial time algorithm for the following problem: given an undirected graph $G$ and two distinct, non-adjacent vertices $s$ and $t$, find a minimum-size set of vertices $Y \subseteq V(G) \setminus \{s, t\}$ such that $s$ and $t$ belongs to distinct components in $G - Y$.

2(d). Present an algorithm for solving $(k, q)$-SEPARATOR in expected running time $2^{O(q+k)} \cdot n^{O(1)}$.

3. (Exercise 5.10 from [1]) In the SET SPLITTING problem, we are given a family of sets $F$ over a universe $U$ and a positive integer $k$, and the goal is to test whether there exists a coloring of $U$ with two colors such that at least $k$ sets in $F$ are not monochromatic (that is, they contain vertices of both colors). Obtain a randomized FPT-algorithm with running time $2^k \cdot (|U| + |F|)^{O(1)}$.

Hint: a set $A \subseteq U$ of element is called a witness if there is a collection of $k$ sets $F' \subseteq F$ and a coloring $c : U \to \{1, 2\}$ such that for each set $f \in F'$, there is an element of $A \cap f$ colored by 1 and an element of $A \cap f$ colored by 2.

4. (Exercise 5.11 from [1]) In the PARTIAL VERTEX COVER problem, we are given an undirected graph $G$ and positive integers $k$ and $t$, and the goal is to check whether there exists a set $X \subseteq V(G)$ of size at most $k$ such that at least $t$ edges of $G$ are incident to vertices on $X$. Obtain an algorithm with running time $2^{O(q+k)} \cdot n^{O(1)}$.
2^{O(t)} \cdot n^{O(1)} for the problem.

Hint: Use color coding to color the edge set of $G$ with $t$ colors. Use dynamic programming to determine whether there exists at most $i$ vertices to cover edges of colors in $C$. Inductively compute the table $P[C, i]$ for all $C \subseteq [t]$ and $i \in [k]$ which takes 1 if this holds, 0 otherwise. Note that $P[C, 1] = 1$ if and only if there exists a vertex $u \in V(G)$ such that $u$ is incident with edges colored in $C$, especially $P[\emptyset, 1] = 1$ trivially. The rest of the table values shall be determined by the (incomplete) recursion for $i \geq 2$,

$$P[C, i] = \max_{C' \subseteq C} P[C \setminus C', i - 1] \cdot P[???, ???]$$

5. Write a dynamic programming algorithm that solves the problem Longest Path on an acyclic digraph in time $O(|E|)$. Provide a correctness proof and running time analysis.

♣ Submit your solution via email (eunjungkim78@gmail.com) by 13 Feb 2019, midnight.
♣ Questions with ★ can be worked together with a colleague. But please write the solution by yourself.

References