Exact Algorithms

Homework 3

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1. (Exercise 5.1 from [1]) In the TRIANGLE PACKING problem, we are given an undirected graph G and a positive integer k, and the objective is to test whether G has k vertex-disjoint triangles. Using color coding show that the problem admits an algorithm with (expected) running time $2^{O(k)} \cdot n^{O(1)}$.

Hint: color the vertices with $3k$ colors. A *colorful* packing is a triangle packing in which any two vertices are colored by distinct colors. Design a dynamic programming which computes whether a set of vertices colored by $C \subseteq \{1, 2, \ldots, 3k\}$ with $|C| = 3i$ colors, for $i = 0, 1, \ldots, k$, contains a colorful packing. Use Stirling's approximation; $n! \approx (\frac{n}{e})^n$.

★2. (Exercise 5.3 from [1]) Consider the following problem (k, q)-SEPARATOR: given an undirected graph G and positive integers k and q, find a set at most k vertices such that $G - X$ has at least two components of size at least q.

2(a). Consider a random coloring of the vertices of G with two colors L and R. Evaluate the probability that all X vertices belong to L and q vertices of each of two "large" components belong to R.

2(b). Suppose that L and R is a partition of $V(G)$ as depicted in 2(a). Prove that there exists two vertices $s, t \in R$ and a set of vertices $Y \subseteq L$ of size at most k such that s and t belongs to distinct component in $G - Y$.

2(c). A minimum (s, t)-cut of an edge-weighted directed graph D (possibly with 2-cycles) is a set S of edges such that $D - S$ does not contain a directed path from s to t. Using a polynomial-time algorithm $\mathcal{A}$ solving finding a min (s, t)-cut as a black box, present a polynomial time algorithm for the following problem: given an undirected graph G and two distinct, non-adjacent vertices s and t, find a minimum-size set of vertices $Y \subseteq V(G) \setminus \{s, t\}$ such that s and t belongs to distinct components in $G - Y$.

2(d). Present an algorithm for solving (k, q)-SEPARATOR in expected running time $2^{O(q+k)} \cdot n^{O(1)}$.

3. (Exercise 5.10 from [1]) In the SET SPLITTING problem, we are given a family of sets $\mathcal{F}$ over a universe $\mathcal{U}$ and a positive integer k, and the goal is to test whether there exists a coloring of $\mathcal{U}$ with two colors such that at least k sets in $\mathcal{F}$ are not monochromatic (that is, they contain vertices of both colors). Obtain a randomized FPT-algorithm with running time $2^k \cdot (|\mathcal{U}| + |\mathcal{F}|)^{O(1)}$.
Hint: a set $A \subseteq \mathcal{U}$ of element is called a *witness* if there is a collection of k sets $\mathcal{F}' \subseteq \mathcal{F}$ and a coloring $c : \mathcal{U} \to \{1, 2\}$ such that for each set $f \in \mathcal{F}'$, there is an element of $A \cap f$ colored by 1 and an element of $A \cap f$ colored by 2.

4. (Exercise 5.11 from [1]) In the PARTIAL VERTEX COVER problem, we are given an undirected graph G and positive integers k and t, and the goal is to check whether there exists a set $X \subseteq V(G)$ of size at most k such that at least t edges of G are incident to vertices on X. Obtain an algorithm with running time

$2^{O(t)} \cdot n^{O(1)}$ for the problem.
Hint: Use color coding to color the edge set of G with t colors. Use dynamic programming to determine whether there exists at most i vertices to cover edges of colors in C. Inductively compute the table $P[C, i]$ for all $C \subseteq [t]$ and $i \in [k]$ which takes 1 if this holds, 0 otherwise. Note that $P[C, 1] = 1$ if and only if there exists a vertex $u \in V(G)$ such that u is incident with edges colored in C, especially $P[\emptyset, 1] = 1$ trivially. The rest of the table values shall be determined by the (incomplete) recursion for $i \geq 2$,

$$P[C, i] = \max_{C' \subseteq C} P[C \setminus C', i-1] \cdot P[???, ???]$$

5. Write a dynamic programming algorithm that solves the problem Longest Path on an acyclic digraph in time $O(|E|)$. Provide a correctness proof and running time analysis.

♣ Submit your solution via email (eunjungkim78@gmail.com) by 13 Feb 2019, midnight.
♣ Questions with ★ can be worked together with a colleague. But please write the solution by yourself.

References

[1] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015.