Exact Algorithms
Homework 6

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The number of ★s in front of each exercise is an indicator of difficulty: (1=easy, 3=hard, >3=research level). Each student should submit his/her own solutions in person or by email to (michail.lampis@dauphine.fr) by 3/3. Discussions and collaborations are allowed (and encouraged), but in your solutions you should include a short acknowledgement mentioning the names of people with whom you collaborated.

1. (★) Consider the following problem: Given a SAT formula ϕ where no variable appears negated (that is, a formula that does not contain $\neg$), decide if there exists a satisfying assignment which sets at most k variables to value 1. This problem is known as WEIGHTED SAT.

- Show that this problem can be solved in roughly n^k time, where n is the number of variables.
- Show an FPT reduction from k-Dominating Set to this problem.
- Conclude that under the SETH this problem cannot be solved in $n^{0.99k}$ time.

Solution: The problem is solvable in $\binom{n}{k}$ time, by trying out all assignments that set exactly k variables to True (setting fewer variables to True does not make sense, since the formula does not contain negations). A reduction from k-DS works as follows: given a graph $G = (V, E)$, we make a variable for each vertex $u \in V$, and a clause for each vertex containing all variables of $N[u]$. This proves that this problem cannot be solved in $n^{0.99k}$ under SETH, since the reduction from k-DS preserves the values of n and k.

2. (★) Consider the following problem: Given a 3-SAT formula ϕ, decide if there exists a satisfying assignment that sets at most k variables to value 1. Show that this problem is FPT, and there exists an algorithm to decide it running in time roughly 3^k. This problem is known as WEIGHTED 3-SAT.

Solution: We start we the assignment that sets all variables to 0. If this satisfies the formula, we are done. Suppose then that there exists an unsatisfied clause. We consider each variable that appears positively in this clause and branch: in each branch we set each such variable to 1 (we know it was currently set to 0, since the clause was not satisfied). This increases the number of variables set to 1 by 1, so this branching cannot happen more than k times. Since we have at most 3 branches, and we obviously stop if the current assignment has $> k$ true variables, the time complexity is 3^k.

3. (★) Consider again the WEIGHTED 3-SAT problem, but with the difference that now we are asking if there exists a satisfying assignment that sets **exactly** k variables to 1. Observe that there exists an n^k algorithm for this problem. Give an FPT reduction from k-Clique to this problem, establishing that, if the ETH is true, there is no $n^{o(k)}$ algorithm that solves it.

Solution: The n^k algorithm comes from considering all $\binom{n}{k}$ possible solutions. A reduction from k-Clique is the following: given a graph $G = (V, E)$, we make a variable for each vertex. For all $u, v \in V$ such that $(u, v) \notin E$ we make the clause $(\neg u \vee \neg v)$. This ensure that no satisfying assignment can set both u and v to True. Hence, satisfying assignments of weight k correspond to k-cliques. Note that this is an instance of 2-SAT, but we may use the standard reduction if we wish to obtain clauses of size exactly 3.

4. (★★) In the k-INDUCED PATH problem we are given a graph G and we seek a set of k vertices S such that $G[S]$ induces a path. Show that this problem has no $n^{o(k)}$ algorithm, assuming the ETH.

See the paper "Time-Approximation Trade-offs for Inapproximable Problems" by Bonnet et al., JCSS 2018

5. (★★★) In the HALL SET problem we are given a bipartite graph $G(A, B, E)$. We are asked if there exists $S \subseteq A$ such that $|S| \leq k$ and $|N(S)| < |S|$. Show that there is no FPT (in k) algorithm for this problem, assuming the ETH. Hint: try a reduction from k-Clique, where B contains the vertices of the original graph and A the edges of the original graph.

See the paper "Don't Be Strict in Local Search!", by Gaspers et al., AAAI 2012