Vertex Cover Problem Parameterized Above and Below Tight Bounds

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Abstract. We study the well-known Vertex Cover problem parameterized above and below tight bounds. We show that two of the parameterizations (both were suggested by Mahajan, Raman and Sikdar, J. Computer and System Sciences, 75(2):137–153, 2009) are fixed-parameter tractable and two other parameterizations are W[1]-hard (one of them is, in fact, W[2]-hard).

1 Introduction

A parameterized problem Π can be considered as a set of pairs (I, k) where I is the *main part* and k (usually an integer) is the *parameter*. Π is called *fixed-parameter tractable* (FPT) if membership of (I, k) in Π can be decided by a *fixed-parameter* algorithm, i.e., an algorithm of running time $O(f(k)|I|^{O(1)})$, where $|I|$ denotes the size of I and $f(k)$ is a computable function (for further background and terminology on parameterized complexity we refer the reader to the monographs [8, 12, 19]). If the nonparameterized version of Π (where k is just a part of input) is NP-hard, then the function $f(k)$ must be superpolynomial provided P $\neq$ NP. Often $f(k)$ is "moderately exponential," which makes the problem practically feasible for small values of k. Thus, it is important to parameterize a problem in such a way that the instances with small values of k are of real interest.

For a graph $G = (V, E)$, a set $C \subseteq V$ is a *vertex cover* if for every edge $uv \in E$ at least one of the vertices u, v belongs to C. The well-known classical Vertex Cover problem is the problem of deciding whether a graph G has a vertex cover of size at most k. This problem is NP-complete and the *standard* parameterization considers k as the (only) parameter.

Vertex Cover was named the *Drosophila* of fixed-parameter algorithmics [8, 15, 19, 20] as (i) there is a long list of improvements on the exponential function in k of fixed-parameter algorithms with the currently best exponential bound being below 1.28^k [5], (ii) it has applications in various areas including bioinformatics and linear programming [4, 8, 12, 16, 19], (iii) Vertex Cover has been a benchmark for developing sophisticated data reduction and problem kernelization techniques [1], (iv) research on the problem has led us to new research directions within parameterized complexity such as counting [3], enumerating [6, 11] and parallel processing [2, 4].

2 Nonstandard Parameterizations of Vertex Cover

In the above-mentioned applications of Vertex Cover, k is relatively small, but this is not the case in some other applications. Indeed, let B be a positive integer and consider the family $\mathcal{G}_B$ of graphs with all degrees bounded from above by B [9, 13]. The vertex cover in a graph from $\mathcal{G}_B$ on m edges must have at least m/B vertices. However, m/B is not small for m large enough and, thus, the standard parameterization of Vertex Cover is of little interest for $\mathcal{G}_B$.

Mahajan, Raman and Sikdar [17] observed that m/B is a tight[3] lower bound on the minimum cardinality of a vertex cover of a graph in $\mathcal{G}_B$ and stated the following parameterized problem.

Vertex Cover Above Tight Lower Bound-1 (VCL1)
Instance: A positive integer B, a graph $G = (V, E) \in \mathcal{G}_B$, a positive integer k.
Parameters: k and B.
Question: Is there a vertex cover C of G with at most $m/B + k$ vertices?

In fact, Mahajan, Raman and Sikdar [17] stated VCL1 as a problem with one parameter, k, with B being a constant. They asked whether the one parameter version of VCL1 is fixed-parameter tractable. In Section 3 we prove that VCL1 is FPT, which implies that the one parameter version of VCL1 is FPT as well. Since VCL1 has two parameters rather than the usual one, our result can be viewed as a contribution towards Multivariate Algorithmics as outlined by Fellows [10].

A variation of Vertex Cover which has been studied extensively in the literature [15, 7] is Capacitated Vertex Cover, where every vertex of the graph has a given capacity which is a limit on the number of its incident edges it can cover. The m/B lower bound also applies in this case and we can define the following problem.

Capacitated Vertex Cover Above Tight Lower Bound-1 (CVCL1)
Instance: A positive integer B, a graph $G = (V, E) \in \mathcal{G}_B$, a capacity function $c : V \to \{0, \dots, B\}$ a positive integer k.
Parameter: k.
Question: Is there a vertex cover C of G with at most $m/B + k$ vertices such that each vertex $v \in C$ is used to cover at most c(v) of its incident edges?

The standard parameterization of Capacitated Vertex Cover is FPT [15, 7], but surprisingly CVCL1 turns out to be W[2]-hard as we prove in Section 4. Note also that m/B is a tight lower bound on the size of the solution even

[3] Indeed, consider the disjoint union of m/B stars $K_{1,B}$.

if only the *capacities* are bounded by B. The version of the problem where the capacities are bounded by B and the maximum degree is unbounded is of course more general than the variation we study here, so our hardness result extends also to this case.

Let $G = (V, E) \in \mathcal{G}_B$. Observe that the chromatic number $\chi(G)$ of G is at most $B+1$ (one can properly color the vertices of G in at most $B+1$ colors using the greedy algorithm [23]). Thus, G has an independent set I of size at least $n/(B+1)$, where $n = |V|$. Since $V \setminus I$ is a vertex cover, G has a vertex cover of size at most $nB/(B+1)$. Observe that $nB/(B+1)$ is a tight upper bound on the minimum size of a vertex cover of G since the disjoint collection of t copies of K_{B+1}'s must be covered by at least tB vertices. Mahajan, Raman and Sikdar [17] formulated the following problem (in fact, its one-parameter version with k being the parameter and B being a constant).

> **Vertex Cover Below Tight Upper Bound-1** (VCU1)
> *Instance:* A positive integer B, a graph $G = (V, E) \in \mathcal{G}_B$, and a positive integer k.
> *Parameter:* k and B.
> *Question:* Is there a vertex cover C of G with at most $nB/(B+1) - k$ vertices?

It is easy see that VCU1 is FPT using Brooks' Theorem [23]: $\chi(G) \leq B$ unless one of the connectivity components of G is K_{B+1} or, if $B = 2$ and one of the connectivity components of G is an odd cycle. If $\chi(G) \leq B$, we have that the vertex cover of G is at most $n(B-1)/B$. If $n(B-1)/B < nB/(B+1) - k$ we can trivially answer YES, otherwise we have $kB(B+1) \geq n$ which gives a problem kernel. Thus, to make VCU1 more interesting, one should replace $nB/(B+1) - k$ with $n(B-1)/B - k$. We do not know the parameterized complexity of this modification of VCU1.

Let us note in passing that if B is not considered a parameter or a constant but rather a part of the input (that is, if we study the problem on graphs of unbounded degree) and k is the only parameter, then this version of VCU1 is W[1]-hard. To see this, suppose that we are given an instance of the standard parameterization of INDEPENDENT SET consisting of a graph $G = (V, E)$ on n vertices containing at least one edge and the aim is to check the existence of an independent set of size k (or, a vertex cover of size $n - k$). Obtain a new graph H by adding to G a pair x, y of new vertices and connecting x to all vertices of G and to y. Observe that H has $N = n+2$ vertices and the maximum degree B of H is $N - 1$. Thus, the problem for H is whether H has a vertex cover of size $NB/(B+1) - k = n+1-k$. Since any minimum vertex cover of H must contain x and must not contain y, the problem for H is equivalent to asking whether G has a vertex cover of size $n - k$. A similar argument can be applied if we parameterize below $N(B-1)/B$, and, thus, these parameterizations of vertex cover are mainly of interest for graphs of bounded degree.

Mahajan, Raman and Sikdar [17] have also mentioned the following parameterization of VERTEX COVER whose parameterized complexity was determined recently. This problem is of interest since μ is a tight lower bound on the minimum size of a vertex cover.

Vertex Cover Above Tight Lower Bound-2 (VCL2)
Instance: A graph $G = (V, E)$ with a maximum matching of size μ and a positive integer k.
Parameter: k.
Question: Is there a vertex cover C of G with at most $\mu + k$ vertices?

The parameterized complexity of VCL2 remained open for quite some time until recently Razgon and O'Sullivan [21] proved that MIN 2-SAT DELETION is FPT and since VCL2 is fixed-parameter reducible to MIN 2-SAT DELETION [18], VCL2 is also FPT. (In MIN 2-SAT DELETION, we are given a CNF formula F with m clauses such that each clause has two literals and asked whether there is a truth assignment that satisfies at least $m - k$ clauses; k is the parameter.)

If μ' is the size of a *maximal* matching M of a graph G, G has a vertex cover with $2\mu'$ vertices, the set of vertices of M. Observe that $2\mu'$ is a tight upper bound on the minimum size of a vertex cover[4]. Thus, the following problem is another natural parameterization of VERTEX COVER.

Vertex Cover Below Tight Upper Bound-2 (VCU2)
Instance: A graph $G = (V, E)$, a maximal matching M of G and a positive integer k.
Parameter: k.
Question: Is there a vertex cover C of G with at most $2|M| - k$ vertices?

Unfortunately, VCU2 is W[1]-hard as we show in Section 4.

3 VCL1 is FPT

In this section we assume that the graph G under consideration is in $\mathcal{G}_B$, where B is a positive integral constant. The following proposition characterizes VCL1 instances $\{G = (V, E), k = 0, B\}$ for which the answer is YES. This proposition allows us to check whether G has a vertex cover of size m/B in polynomial time.

Proposition 1. *A graph G has a vertex cover of size exactly m/B if and only if G is a bipartite graph with one partite set of size m/B.*

Proof. If G is bipartite with one partite set of size m/B, then this partite set is clearly a vertex cover. Suppose $G = (V, E)$ has a vertex cover C of size m/B. Since a vertex can cover at most B edges, every vertex of C covers exactly B edges and no two of them cover the same edge. Therefore C forms an independent set and thus G is a bipartite graph with bipartite sets C and $V - C$. □

Now we will prove that VCL1 is fixed-parameter tractable.

[4] Consider the disjoint union of C_3's.

Lemma 1. *If the answer to a VCL1 instance is* YES*, then there exists a set D of at most kB edges whose deletion makes G bipartite.*

Proof. Suppose C is a vertex cover of G with at most $m/B + k$ vertices and let $D \subseteq E$ be the set of edges with both endvertices in C. Then we have $(m/B+k)B \geq m+|D|$ and thus $|D| \leq kB$. Since no two vertices of C cover the same edge in $G-D$, $G-D$ has no odd cycle implying that $G-D$ is bipartite. □

If deleting the edges of $D \subseteq E$ makes G bipartite, we say D is an *edge bipartization*. Due to Lemma 1, we may assume that the input graph G has an *edge bipartization* of size at most kB. The problem of deciding whether G has an edge bipartization of size at most p is known as the Edge Bipartization problem. This problem has been studied extensively and the best known fixed-parameter algorithm runs in time $O(2^p m^2)$, see [14].

Since we know that there is a small edge bipartization of G, we can solve Vertex Cover optimally as follows. Suppose D is an edge bipartization of G with $|D| \leq kB$, C is a vertex cover of G, and let $C^* = C \cap V_D$, where V_D is the set of endvertices of edges of D. Note that C^* is a vertex cover of $G[D]$ and let $C' \subseteq C^*$ be a minimal vertex cover of $G[D]$. Observe that $C \setminus C'$ is a vertex cover of the bipartite graph $G - C'$. Thus, a vertex cover of G is the union of a minimal vertex cover C' of $G[D]$ and a vertex cover of the bipartite graph $G - C'$.

Consider the following procedure Π to generate vertex covers of $G[D]$: pick either u or v from every edge $uv \in D$. We call all vertex covers generated by Π, *Π-vertex covers*. To see that the set of Π-vertex covers includes all minimal vertex covers of $G[D]$ observe that if $uv \in D$ and $\{u, v\} \subseteq C'$, where C' is a minimal vertex cover of $G[D]$, then a neighbor x of u is not in C'. Thus, u can be picked up by Π while considering the edge ux and v can be picked up by Π while considering the edge uv.

Our algorithm proceeds as follows. First, we find an edge bipartization D of G such that $|D| \leq kB$ using the algorithm of [14]. If such an edge bipartization D does not exist, the answer to VCL1 is NO. Otherwise, we use Π to generate all Π-vertex covers of $G[D]$ and for each such vertex cover C', we find a minimum-size vertex cover C'' of the bipartite graph $G-C'$ and check whether $|C'|+|C''| \leq m/B + k$.

Let us evaluate the running time of this algorithm. We can find D in time $O(2^{kB} m^2)$. Clearly, all Π-vertex covers of $G[D]$ can be generated in $O(2^{kB} n^{O(1)})$ time. For a vertex cover C', we can find a minimum-size vertex cover C'' of the bipartite graph $G - C'$ in time $O(m\sqrt{n})$. Thus, the total running time of our algorithm is $O(2^{kB} n^{O(1)})$.

Thus, we have obtained the following result.

Theorem 1. *The problem VCL1 can be solved in time $O(2^{kB} n^{O(1)})$.*

4 W[2] and W[1]-hardness Results

Theorem 2. *CVCL1 is W[2]-hard.*

Proof. We give a parameterized reduction from Dominating Set, a well-known W[2]-hard problem [8]: Given a graph $G = (V, E)$ with $|V| = n$ and maximum degree B, we are asked whether it has a dominating set of size k. We will construct a graph $G' = (V', E')$ with maximum degree $B + 2$ such that G has a dominating set of size k if and only if G' has a vertex cover of size $|E'|/(B+2)+k$. Let us also note again that in this case we consider B a part of the input, not a parameter or a constant.

For every vertex $v \in V$ construct a choice gadget which will be a complete bipartite graph $K_{B+1,B+2}$. Also, for every vertex $v \in V$ construct a domination gadget, which is simply a vertex v_d with $B - d(v) + 1$ leaves attached to it, where $d(v)$ is the degree of v in G. Finally, for each vertex v and every $u \in N[v]$, where $N[v]$ is the closed neighborhood of v in G, add an edge from a different vertex of the larger partite set of v's choice gadget to the vertex u_d in u's domination gadget. This completes the construction of G'.

The capacities of all the vertices of G' are equal to their degrees, except for the vertices v_d of the domination gadgets, to which we give capacities $B + 1$, which is one less than their degree.

First, let us calculate the lower bound for this graph. In every choice gadget we have $(B + 1)(B + 2)$ edges. Also, there are $d(v) + 1$ edges connecting every v_d with the choice gadgets and $B - d(v) + 1$ edges connecting it to the attached leaves, so there are $B + 2$ edges incident on each v_d. Therefore, in total we have $|E'| = n(B+1)(B+2)+n(B+2) = n(B+2)^2$. The maximum degree is $B+2$, therefore we are looking for a vertex cover of size at most $n(B + 2) + k$.

Suppose that the original graph has a dominating set D of size k. In G' we select the following vertices in the vertex cover: from each choice gadget corresponding to a vertex in D we select the larger partite set of the bipartite graph ($B+2$ vertices) and we select the smaller partite set of the bipartite graph ($B + 1$ vertices) from all the other choice gadgets. We also select all vertices v_d. In total we have selected $n(B + 1) + k + n = n(B + 2) + k$ vertices. It is not hard to see that this is indeed a vertex cover. It is also a capacitated vertex cover because the only vertices constrained by the capacities are the vertices v_d. However, because D is a dominating set every v_d is connected to a choice gadget from which we picked the larger partite set, thus for every v_d one of its incident edges is covered from our selection in the choice gadgets and v_d has enough capacity to cover all of its remaining incident edges.

Now for the converse, suppose that G' has a capacitated vertex cover of size $n(B + 2) + k$. First, note that without loss of generality we may assume that this vertex cover includes all vertices v_d, because if such a vertex is not in the cover at least one leaf is in the cover and we can simply exchange the two. Also, without loss of generality no leaves are in the cover, because we have already argued that the neighbors of the leaves are in the cover. Therefore, the only reason to include a leaf may be that the cover includes some v_d but none of its neighbors in the choice gadgets, thus exceeding v_d's capacity. However, in such a case we can exchange the leaf with one of v_d's neighbors from the choice gadgets, thus covering at least as many edges without exceeding v_d's capacity.

Finally, in each choice gadget the cover includes either the $B+1$ vertices of the smaller partite set or the $B+2$ vertices of the larger partite set, because if a single vertex of the larger partite set is in the cover it is clearly optimal to take all vertices of the larger partite set and none from the smaller partite set. So we can conclude that there exist k choice gadget where we have picked all of the larger partite set and we have picked the smaller partite set from all the others. Now, if this is indeed a capacitated vertex cover, every v_d has a neighbor in the cover so that its capacity is not exceeded and as argued previously this neighbor is from the choice gadgets. Therefore, if we select in G the k vertices which correspond to the k gadgets where we picked the larger partite set they must form a dominating set. □

Theorem 3. *VCU2 is W[1]-hard.*

Proof. We give a parameterized reduction from Independent Set to VCU2. Let $G = (V, E)$ with parameter k be an instance of Independent Set. Let $n = |V|$ and $m = |E|$. We shall construct a new graph $G' = (V', E')$ as follows. We replace each vertex $u \in V$ by a path $u_1u_2u_3$ of length two. For each edge $e = uv \in E$, we build a gadget graph $G_e = (V_e, E_e)$ with $V_e = \{e_u, e_v, e_w, e_z\}$ and $E_e = \{e_ue_v, e_ve_w, e_we_u, e_we_z\}$. Finally let E' have edges u_3e_u and v_3e_v for each edge $e = uv \in E$ plus all the edges mentioned above. We set $M = \{u_2u_3 : u \in V\} \cup \{e_ue_w : e = uv \in E\}$. Observe that M is a maximal matching of G'. (In fact, it is a maximal matching of G' with minimum number of edges.) In the following, we show that there is a vertex cover of size at most $n - k$ in G (i.e., G has an independent set with at least k vertices) if and only if there is a vertex cover of size at most $2|M| - k$ in G'. Note that $|M| = n + m$ and $2|M| - k = 2n + 2m - k$.

First, suppose C is a vertex cover in G with at most $n - k$ vertices. Then for $C' \subseteq V'$ we pick up the following vertices: (1) u_2 for every $u \in V$, (2) e_w for every $e = uv \in E$, (3) if $u \in C$ and $v \notin C$ for edge $e = uv \in E$, choose u_3 and e_v, (4) if $u, v \in C$ for edge $uv \in E$, choose u_3, v_3 and e_u. Observe that C' covers every edge of G' and contains exactly $n + m + m + |C| \leq 2|M| - k$ vertices.

Second, if we have a vertex cover C' of G', then we know a vertex cover of size at most $|C'|$ can be obtained by replacing a vertex of degree 1 by its neighbor. So we may assume that $\{u_2 : u \in V\} \cup \{e_w : e = uv \in E\}$ belong to C'. The edges of G' not covered by these vertices are a path u_3e_u, e_ue_v, e_vv_3 of length two for every edge $e = uv \in E$; we denote the subgraph of G induced by all such paths by G''. Suppose $C' \leq 2|M| - k = 2n + 2m - k$ and let $C'' = C''_m \cup C''_v$ be a vertex cover in G'' of size at most $n + m - k$, where $C''_m = C'' \cap \{e_u, e_v : e = uv \in E\}$ and $C''_v = C'' \cap \{u_3 : u \in V\}$. Since at least one of e_u and e_v must be in C'' in order to cover the edge e_ue_v, we have $|C''_m| \geq m$ and thus $|C''_v| \leq n - k$. Moreover, edges covered by $\{e_u, e_v\}$ can be also covered by either $\{u_3, e_v\}$ or $\{e_u, v_3\}$. Hence we may assume that $|C''_m| = m$. Now observe that C''_v includes at least one of u_3 and v_3 for every edge $e = uv \in E$ so as to cover those edges in $G'' - C''_m$. Therefore $\{u \in V : u_3 \in C''_v\}$ is a vertex cover of G. □

5 Further Research

We could not determine the parameterized complexity of the next problem mentioned in Section 2. It is likely that VCU is FPT. At least, this follows for the special case of the problem when the size of a maximum clique in G is sufficiently smaller than B, see [22].

> **Vertex Cover Below Tight Upper Bound** (VCU)
> *Instance:* A positive integer B, a graph $G = (V, E) \in \mathcal{G}_B$, and a positive integer k.
> *Parameter:* k and B.
> *Question:* Is there a vertex cover C of G with at most $n(B-1)/B - k$ vertices?

It's also interesting to consider the difference between VCL1, which is FPT, and CVCL1, which we showed to be W[1]-hard. CVCL1 is harder than VCL1 because of two changes: the introduction of capacities and the unbounding of the maximum degree. Thus, another problem left open here is what happens when we make only one of these two changes, that is, consider VCL1 parameterized by k alone or CVCL1 parameterized by B and k.

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