

Structural Parameters, Tight Bounds, and Approximation for (k, r) -Center

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Abstract

In (k, r) -CENTER we are given a (possibly edge-weighted) graph and are asked to select at most k vertices (centers), so that all other vertices are at distance at most r from a center. In this paper we provide a number of tight fine-grained bounds on the complexity of this problem with respect to various standard graph parameters. Specifically:

► For any $r \geq 1$, we show an algorithm that solves the problem in $O^*((3r + 1)^{cw})$ time, where cw is the clique-width of the input graph, as well as a tight SETH lower bound matching this algorithm's performance. As a corollary, for $r = 1$, this closes the gap that previously existed on the complexity of DOMINATING SET parameterized by cw .

► We strengthen previously known FPT lower bounds, by showing that (k, r) -CENTER is W[1]-hard parameterized by the input graph's vertex cover (if edge weights are allowed), or feedback vertex set, even if k is an additional parameter. Our reductions imply tight ETH-based lower bounds. Finally, we devise an algorithm parameterized by vertex cover for unweighted graphs.

► We show that the complexity of the problem parameterized by tree-depth is $2^{\Theta(td^2)}$ by showing an algorithm of this complexity and a tight ETH-based lower bound.

We complement these mostly negative results by providing *FPT approximation schemes* parameterized by clique-width or treewidth which work efficiently independently of the values of k, r . In particular, we give algorithms which, for any $\epsilon > 0$, run in time $O^*((tw/\epsilon)^{O(tw)})$, $O^*((cw/\epsilon)^{O(cw)})$ and return a $(k, (1 + \epsilon)r)$ -center, if a (k, r) -center exists, thus circumventing the problem's W-hardness.

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1 Introduction

In this paper we study the (k, r) -CENTER problem: given a graph $G = (V, E)$ and a weight function $w : E \rightarrow \mathbb{N}^+$ which satisfies the triangle inequality and defines the length of each edge, we are asked if there exists a set K (the *center-set*) of at most k vertices of V , so that $\forall u \in V \setminus K$ we have $\min_{v \in K} d(v, u) \leq r$, where $d(v, u)$ denotes the shortest-path distance from v to u under weight function w . If w assigns weight 1 to all edges we say that we have an instance of un-weighted (k, r) -CENTER. (k, r) -CENTER is an extremely well-investigated optimization problem with numerous applications. It has a long history, especially from the point of view of approximation algorithms, where the objective is typically to minimize r for a given k [24, 45, 28, 19, 41, 30, 27, 1, 18]. The converse objective (minimizing k for a given r) has also been well-studied, with the problem being typically called r -DOMINATING SET in this case [11, 42, 35, 12].



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Because (k, r) -CENTER generalizes DOMINATING SET (which corresponds to the case $r = 1$), the problem can already be seen to be hard, even to approximate (under standard complexity assumptions). In particular, the optimal r cannot be approximated in polynomial time by a factor better than 2 (even on planar graphs [19]), while k cannot be approximated by a factor better than $\ln n$ [38]. Because of this hardness, we are strongly motivated to investigate the problem's complexity when the input graph has some restricted structure.

In this paper our goal is to perform a complete analysis of the complexity of (k, r) -CENTER that takes into account this input structure by using the framework of parameterized complexity. In particular, we provide *fine-grained* upper and lower bound results on the complexity of (k, r) -CENTER with respect to the most widely studied parameters that measure a graph's structure: treewidth **tw**, clique-width **cw**, tree-depth **td**, vertex cover **vc**, and feedback vertex set **fvs**. In addition to the intrinsic value of determining the precise complexity of (k, r) -CENTER, this approach is further motivated by the fact that FPT algorithms for this problem have often been used as building blocks for more elaborate approximation algorithms [16, 18]. Indeed, (some of) these questions have already been considered, but we provide a number of new results that build on and improve the current state of the art. Along the way, we also close a gap on the complexity of the flagship DOMINATING SET problem parameterized by clique-width. Specifically, we prove the following:

► (k, r) -CENTER can be solved (on unweighted graphs) in time $O^*((3r + 1)^{cw})$ (if a clique-width expression is supplied with the input), but it cannot be solved in time $O^*((3r + 1 - \epsilon)^{cw})$ for any (fixed) $r \geq 1$, unless the Strong Exponential Time Hypothesis (SETH) [25, 26] fails. The algorithmic result relies on standard techniques (dynamic programming on clique-width, fast subset convolution), as well as several problem-specific observations which are required to obtain the desired table size. The SETH lower bound follows from a direct reduction from SAT. A noteworthy consequence of our lower bound result is that, for the case of DOMINATING SET, it closes the gap between the complexity of the best known algorithm ($O^*(4^{cw})$ [9]) and the best previously known lower bound ($O^*((3 - \epsilon)^{cw})$ [34]).

► (k, r) -CENTER cannot be solved in time $n^{o(vc+k)}$ on edge-weighted graphs, or time $n^{o(fvs+k)}$ on unweighted graphs, unless the Exponential Time Hypothesis (ETH) is false. It was already known that an FPT algorithm parameterized just by tw (for unbounded r) is unlikely to be possible [10]. These results show that the same holds for the two more restrictive parameters fvs and vc , even if k is also added as a parameter. They are (asymptotically) tight, since it is easy to obtain $O^*(n^{fvs})$, $O^*(n^{vc})$, and $O^*(n^k)$ algorithms. We remark that (k, r) -CENTER is a rare example of a problem that turns out to be hard parameterized by vc . We complement these lower bounds by an FPT algorithm for the unweighted case, running in time $O^*(5^{vc})$.

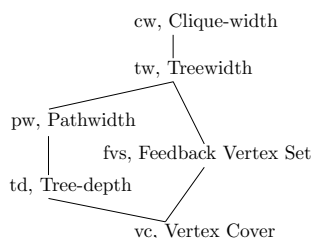
► (k, r) -CENTER can be solved in time $O^*(2^{O(td^2)})$ for unweighted graphs, but if it can be solved in time $O^*(2^{o(td^2)})$, then the ETH is false. Here the upper bound follows from known connections between a graph's tree-depth and its diameter, while the lower bound follows from a reduction from 3-SAT. We remark that this is a somewhat uncommon example of a parameterized problem whose parameter dependence turns out to be exponential in the square of the parameter.

These results, together with the recent work of [10] showing tight bounds of $O^*((2r + 1)^{tw})$ on the problem's complexity parameterized by tw , give a complete and often fine-grained, picture on (k, r) -CENTER for the most important graph parameters. One of the conclusions that can be drawn is that, as a consequence of the problem's hardness for vc (in the weighted case) and fvs , there are few cases where we can hope to obtain an FPT algorithm without bounding r : as r increases the complexity of exactly solving the problem quickly degenerates away from the case of DOMINATING SET, which is FPT for all considered parameters.

A further contribution of this paper is to complement this negative view by pointing out that it only applies if one insists on solving the problem *exactly*. If we allow algorithms that return a $(1 + \epsilon)$ -approximation to the optimal r , for arbitrarily small $\epsilon > 0$ and while respecting the given value of k , we obtain the following:

► *There exist algorithms which, for any $\epsilon > 0$, when given a graph that admits a (k, r) -center, return a $(k, (1 + \epsilon)r)$ -center in time $O^*((tw/\epsilon)^{O(tw)})$, or $O^*((cw/\epsilon)^{O(cw)})$, assuming a tree decomposition or clique-width expression is given in the input.*

The tw approximation algorithm is based on a technique introduced in [32], while the cw algorithm relies on a new extension of an idea from [23], which may be of independent interest. Thanks to these approximation algorithms, we arrive at an improved understanding of the complexity of (k, r) -CENTER by including the question of approximation, and obtain algorithms which continue to work efficiently even for large values of r . Figure 1 illustrates the relationships between parameters and Table 1 summarizes our results. Due to space restrictions, all definitions and related results are deferred to Appendix A, while all proofs of our results are deferred to Appendix B.



■ **Figure 1** Relationships of parameters. Algorithmic results are inherited downwards, hardness results upwards.

	cw	tw	fvs	td	vc
FPT exact	3 (w/u)	10 (w/u)		7 (u)	6 (u)
FPT-AS	16 (w/u)	13 (w/u)			
SETH LB	1 (u)				
ETH LB			5 (w/u)	8 (u)	4 (w)
W[1]-hard			5 (w/u)		4 (w)

■ **Table 1** A summary of our results (theorem numbers) for all considered parameters. Initials u/w denote the unweighted/weighted variants of the problem.

Related Work: Our work follows upon recent work by [10], which showed that (k, r) -CENTER can be solved in $O^*((2r + 1)^{tw})$, but not faster (under SETH), while its connected variant can be solved in $O^*((2r + 2)^{tw})$, but not faster. This paper in turn generalized previous results on DOMINATING SET for which a series of papers had culminated into an $O^*(3^{tw})$ algorithm [43, 2, 44], while on the other hand, [34] showed that an $O^*((3 - \epsilon)^{pw})$ algorithm would violate the SETH, where pw denotes the input graph's pathwidth. The complexity of (k, r) -CENTER by the related parameter branchwidth had previously been considered in [16] where an $O^*((2r + 1)^{\frac{3}{2}bw})$ algorithm is given. Moreover, [36] showed the problem parameterized by the number k of centers to be W[1]-hard in the L_∞ metric, in fact analysing COVERING POINTS WITH SQUARES, a geometric variant. It remains W[2]-hard for 2-degenerate graphs [22]. On clique-width, a $O^*(4^{cw})$ -time algorithm for DOMINATING SET was given in [9], while [40] notes that the lower bound of [34] for pathwidth/treewidth would also imply no $(3 - \epsilon)^{cw} \cdot n^{O(1)}$ -time algorithm exists for clique-width under SETH as well, since clique-width is at most 1 larger than pathwidth. For the edge-weighted variant, [20] shows that a $(2 - \epsilon)$ -approximation is W[2]-hard for parameter k and NP-hard for graphs of *highway dimension* $h = O(\log^2 n)$, while also offering a $3/2$ -approximation algorithm of running time $2^{O(kh \log(h))} \cdot n^{O(1)}$, exploiting the similarity of this problem with that of solving DOMINATING SET on graphs of bounded vc . Finally, for unweighted graphs, [33] provides efficient (linear/polynomial) algorithms computing $(r + O(\mu))$ -dominating sets and $+O(\mu)$ -approximations for (k, r) -CENTER, where μ is the *tree-breadth* or *cluster diameter* in a layering partition of the input graph.

2 Definitions and Preliminaries

We use standard graph-theoretic notation. For a graph $G = (V, E)$, $n = |V|$ denotes the number of vertices, $m = |E|$ the number of edges and for a subset $X \subseteq V$, $G[X]$ denotes the graph induced by X . Further, we assume the reader has some familiarity with standard definitions from parameterized complexity theory, such as the classes FPT, W[1] (see [15, 21, 17]). For a parameterized problem with parameter k , an FPT-AS is an algorithm which for any $\epsilon > 0$ runs in time $O^*(f(k, \frac{1}{\epsilon}))$ (i.e. FPT time when parameterized by $k + \frac{1}{\epsilon}$) and produces a solution at most a multiplicative factor $(1 + \epsilon)$ from the optimal (see [37]). We use $O^*(\cdot)$ to imply omission of factors polynomial in n .

In this paper we present approximation schemes with running times of the form $(\log n/\epsilon)^{O(k)}$. These can be seen to imply an FPT running time by a well-known win-win argument (see Lemma 17 in Appendix A): *If a parameterized problem with parameter k admits, for some $\epsilon > 0$, an algorithm running in time $O^*((\log n/\epsilon)^{O(k)})$, then it also admits an algorithm running in time $O^*((k/\epsilon)^{O(k)})$.*

Treewidth and *pathwidth* are standard notions in parameterized complexity which measure how close a graph is to being a tree or path (see Appendix A and [8, 5, 29]). We will also use the standard graph parameter of *clique-width*, which was introduced as a generalization of treewidth to dense graphs (see Appendix A and [13, 14]). Additionally, we will use the parameters *vertex cover number* and *feedback vertex set number* of a graph G , which are the sizes of the minimum vertex set whose deletion leaves the graph edgeless, or acyclic, respectively. Finally, we will consider the related notion of *tree-depth* [39], which is defined as the minimum height of a rooted forest whose completion (the graph obtained by connecting each node to all its ancestors) contains the input graph as a subgraph. We will denote these parameters for a graph G as $tw(G)$, $pw(G)$, $cw(G)$, $vc(G)$, $fvs(G)$, and $td(G)$, and will omit G if it is clear from the context. We recall the following well-known relations between these parameters [6, 14] which justify the hierarchy given in Figure 1 (Lemma 18 in Appendix A): *For any graph G we have $tw(G) \leq pw(G) \leq td(G) \leq vc(G)$, $tw(G) \leq fvs(G) \leq vc(G)$, $cw(G) \leq pw(G) + 1$, and $cw(G) \leq 2^{tw(G)+1} + 1$.*

We also recall here the two main complexity assumptions used in this paper [25, 26]. The Exponential Time Hypothesis (ETH) states that 3-SAT cannot be solved in time $2^{o(n+m)}$ on instances with n variables and m clauses. The Strong Exponential Time Hypothesis (SETH) states that for all $\epsilon > 0$, there exists an integer k such that k -SAT cannot be solved in time $(2 - \epsilon)^n$ on instances of k -SAT with n variables.

3 Clique-width

3.1 Lower bound based on SETH

The result of this section is that for any fixed constant $r \geq 1$, the existence of any algorithm for (k, r) -CENTER of running time $O^*((3r + 1 - \epsilon)^{cw})$, for some $\epsilon > 0$, would imply the existence of some algorithm for SAT of running time $O^*((2 - \delta)^n)$, for some $\delta > 0$.

Before we proceed, let us recall the high-level idea behind the SETH lower bound for DOMINATING SET given in [34], as well its generalization to (k, r) -CENTER given in [10]. In both cases the key to the reduction is the construction of long paths, which are conceptually divided into blocks of $2r + 1$ vertices. The intended solution consists of selecting, say, the i -th vertex of a block of a path, and repeating this selection in all blocks of this path. This allows us to encode $(2r + 1)^t$ choices, where t is the number of paths we make, which ends up being roughly equal to the treewidth of the construction. The reason this construction

works in the converse direction is that, even though the optimal (k, r) -CENTER solution may “cheat” by selecting the i -th vertex of a block, and then the j -th vertex of the next, one can see that we must have $j \leq i$. Hence, by making the paths that carry the solution’s encoding long enough we can ensure that the solution eventually settles into a pattern that encodes an assignment to the original formula (which can be “read” with appropriate gadgets).

In our lower bound construction for clique-width we need to be able to “pack” more information per unit of width: instead of encoding $(2r + 1)$ choices for each unit of treewidth, we need to encode $(3r + 1)$ choices for each label. Our high-level plan to achieve this is to use a *pair* of long paths for each label. Because we only want to invest one label for each pair of paths we are forced to periodically (every $2r + 1$ vertices) add cross edges between them, so that the connection between blocks can be performed with a single join operation (see the paths A_1, B_1 in Figure 2 for an illustration). Our plan now is to encode a solution by selecting a pair of vertices that will be repeated in each block, for example every i -th vertex of A_1 and every j -th vertex of B_1 . One may naively expect that this would allow us to encode $(2r + 1)^2$ choices for each label (which would lead to a SETH lower bound that would contradict the algorithm of Section 3.2). However, because of the cross edges, the optimal (k, r) -CENTER solution is not as well-behaved on a pair of cross-connected paths as it was on a path, and this makes it much harder to execute the converse direction of the reduction: a solution that takes every i -th vertex of A_1 could alternate repeatedly between various choices for B_1 , because the selected vertices of A_1 also cover parts of B_1 . Our strategy is therefore to identify $(3r + 1)$ ordered selection pairs and show that any valid solution must be well-behaved with respect to these pairs. An overview of our construction, omitting most technical details of the reduction’s inner mechanism follows.

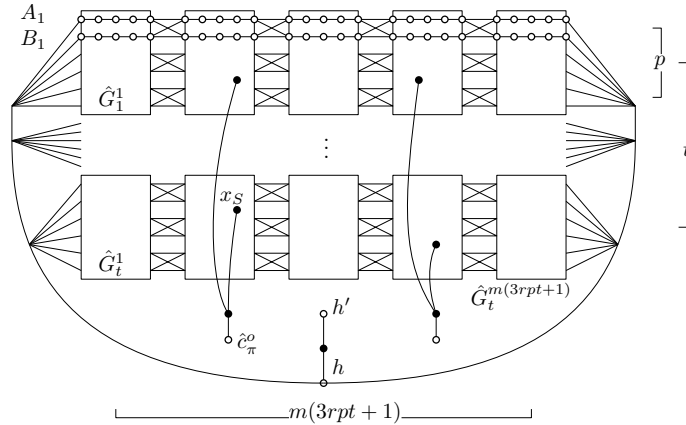
Construction overview: We construct a graph G , given some $\epsilon < 1$ and an instance ϕ of SAT with n variables and m clauses. We first choose an integer p , depending on ϵ and r (for technical reasons that become apparent in the proof of Theorem 1). Note that for the results of this section, both r and p are considered constants. We then group the variables of ϕ into $t = \lceil \frac{n}{\gamma} \rceil$ groups F_1, \dots, F_t , for $\gamma = \lfloor \log_2(3r + 1)^p \rfloor$, being also the maximum size of any such group. Our graph G will consist of t rows of $m(3rpt + 1)$ gadgets \hat{G} , each row corresponding to one such group of variables. Each gadget \hat{G} will contain p pairs of paths A_i, B_i and any selection of one vertex from each path will be associated with a specific partial assignment to the variables of the group. Gadgets of the same row will be connected in a path-like manner: for each $i \in [1, p]$ both final vertices of each pair A_i, B_i within each gadget will be connected to both first vertices of the corresponding pair A_i, B_i of the following gadget, with a global vertex h adjacent to all the first/last vertices of all such long paths, with an additional path of length r attached to h to ensure its selection in any minimum-sized center-set (and allowing for any selection in these first/last gadgets to be valid).

Furthermore, we will show that the possible selections of only one vertex from each path can be divided into $3r + 1$ equivalence classes: we define $3r + 1$ *canonical pairs* of numbers (α_y, β_y) , indexed (and ordered) by $y \in [1, 3r + 1]$, that give the indices of vertices from a pair of paths A_i, B_i (i.e. the α_y -th vertex of A_i and the β_y -th vertex of B_i) that would form the characteristic selection for each class, and show that any other selection within each class would be interchangeable (in terms of domination/coverage) with the characteristic selection, while if some pair with index y is used for selection of vertices from paths A_i, B_i in some gadget \hat{G}_τ^j , then any pair used for the paths of the following gadget \hat{G}_τ^{j+1} (on the same row) must be of index $y' \leq y$. Observe that, as the path selections from each column must be well behaved with respect to our canonical pairs, there is an upper bound of $3r$ on the number

of times the selection pattern can change on some pair of paths, giving $3rp$ for each row of gadgets and $3rpt$ times overall. In each gadget \hat{G} , we also make $3r + 1$ vertices u_i^y for each pair A_i, B_i that signify these canonical selections from each path and further, a group of $(3r + 1)^p$ vertices x_S for each set S that only contains one such u_i^y for each $i \in [1, p]$.

In this way, a selection of one vertex from each path A_i, B_i will correspond to a selection u_i^y , while all p such selections will correspond to one selection x_S that will in turn be associated with a partial assignment to the group of variables assigned to this row of gadgets (there are 2^γ partial assignments for each group and $(3r + 1)^p \geq 2^\gamma$ sets S). Further, each column of gadgets will correspond to one clause, with the first m columns assigned to one clause each and $3rpt + 1$ repetitions of this pattern giving the complete association. Our graph G will have one vertex \hat{c} for each such column of gadgets (representing the associated clause) at distance r from vertices x_S in the gadgets \hat{G} of its column that represent the partial assignments to the variables of the group associated with the gadget's row (and group F_τ) that would satisfy the clause (Figure 2 provides an illustration).

Thus, a satisfying assignment for ϕ will give a (k, r) -center for G by selecting in each gadget \hat{G} all vertices corresponding to the partial assignment for its associated group of variables from each pair of paths, as well as the matching u_i^y and x_S vertices (and h). For the converse direction, as the number of changes of selection pattern is $\leq 3rpt$ and the number of columns is $m(3rpt + 1)$, by the pigeonhole principle, there will always exist m consecutive columns for which the pattern does not change and thus we will be able to extract a consistent assignment for all clauses.



■ **Figure 2** A simplified picture of the complete construction. Note boxes indicate block gadgets \hat{G} , while there is no vertex anywhere between h and the first/last vertices of the long paths.

► **Theorem 1.** For any fixed $r \geq 1$, if (k, r) -CENTER can be solved in $O^*((3r + 1 - \epsilon)^{cw(G)})$ time for some $\epsilon > 0$, then SAT can be solved in $O^*((2 - \delta)^n)$ time for some $\delta > 0$.

► **Corollary 2.** If DOMINATING SET can be solved in $O^*((4 - \epsilon)^{cw(G)})$ time for some $\epsilon > 0$, then SAT can be solved in $O^*((2 - \delta)^n)$ time for some $\delta > 0$.

3.2 Dynamic programming algorithm

We next present an $O^*((3r + 1)^{cw})$ -time dynamic programming (DP) algorithm for unweighted (k, r) -CENTER, using a given clique-width expression T_G for G with at most cw labels. Even though the algorithm relies on standard techniques, there are several non-trivial, problem-specific observations that we need to make to reach a DP table size of $(3r + 1)^{cw}$.

Our first step is to re-cast the problem as a *distance-labeling* problem (not be confused with ‘label’/‘label-set’ for a clique-width expression), that is, to formulate the problem as that of deciding for each vertex what is its precise distance to the optimal solution K . This is helpful because it allows us to make the constraints of the problem local, and hence easier to verify: roughly speaking, we say that a vertex is satisfied if it has a neighbor with a smaller distance to K . It is now not hard to design a clique-width based DP algorithm for this version of the problem: for each label l we need to remember two numbers, namely the smallest distance value given to some vertex with label l , and the smallest distance value given to a *currently unsatisfied* vertex with label l , if it exists.

The above scheme directly leads to an algorithm running in time (roughly) $((r+1)^2)^{cw}$. In order to decrease the size of this table, we now make the following observation: if a label-set contains a vertex at distance i from K , performing a join operation will satisfy all vertices that expect to be at distance $\geq i+2$ from K , since all vertices of the label-set will now be at distance at most 2. This implies that, in a label-set where the minimum assigned value is i , states where the minimum unsatisfied value is between $i+2$ and r are effectively equivalent. With this observation we can bring down the size of the table to $(4r)^{cw}$, because (intuitively) there are four cases for the smallest unsatisfied value: $i, i+1, \geq i+2$, and the case where all values are satisfied.

The last trick that we need to achieve the promised running time departs slightly from the standard DP approach. We will say that a label-set is *live* in a node of the clique-width expression if there are still edges to be added to the graph that will be incident to its vertices. During the execution of the dynamic program, we perform a “fore-tracking” step, by checking the part of the graph that comes higher in the expression to determine if a label-set is live. If it is, we merge the case where the smallest unsatisfied value is $i+2$, with the case where all values are satisfied (since a join operation will eventually be performed). Otherwise, a partial solution that contains unsatisfied vertices in a non-live label-set can safely be discarded. This brings down the size of the DP table to $(3r+1)^{cw}$, and then we need to use some further techniques to make the total running time quasi-linear in the size of the table. This involves counting the number of solutions instead of directly computing a solution of minimum size, as well as a non-trivial extension of fast subset convolution from [4] for a $3 \times (r+1)$ -sized table (or *state-changes*, see [44, 9] and Chapter 11 of [15]).

► **Theorem 3.** *Given graph G , along with $k, r \in \mathbb{N}^+$ and clique-width expression T_G of clique-width cw for G , there exists an algorithm to solve the counting version of the (k, r) -CENTER problem in $O^*((3r+1)^{cw})$ time.*

4 Vertex Cover, Feedback Vertex Set and Tree-depth

In this section we first show that the edge-weighted variant of the (k, r) -CENTER problem parameterized by $vc+k$ is $W[1]$ -hard, and more precisely, that the problem does not admit a $n^{o(vc+k)}$ algorithm under the ETH. We give a reduction from k -MULTICOLORED INDEPENDENT SET (defined in Appendix A).

This is a well-known $W[1]$ -hard problem that cannot be solved in $n^{o(k)}$ under the ETH [15]. Using essentially the same reduction with that of Theorem 4, we obtain a similar hardness result for unweighted (k, r) -CENTER parameterized by fvs .

► **Theorem 4.** *The weighted (k, r) -CENTER problem is $W[1]$ -hard parameterized by $vc+k$. Furthermore, if there is an algorithm for weighted (k, r) -CENTER running in time $n^{o(vc+k)}$ then the ETH is false.*

► **Corollary 5.** *The (k, r) -CENTER problem is $W[1]$ -hard when parameterized by $fvs + k$. Furthermore, if there is an algorithm for weighted (k, r) -CENTER running in time $n^{o(fvs+k)}$, then the ETH is false.*

We next show that unweighted (k, r) -CENTER admits an algorithm running in time $O^*(5^{vc})$, in contrast to its weighted version (Theorem 4). We devise an algorithm that operates in two stages: first, it guesses the intersection of the optimal solution with the optimal vertex cover, and then it uses a reduction to SET COVER to complete the solution optimally.

► **Theorem 6.** *Given graph G , along with $k, r \in \mathbb{N}^+$ and a vertex cover of size vc of G , there exists an algorithm solving unweighted (k, r) -CENTER in $O^*(5^{vc})$ time.*

We next consider the un-weighted version of (k, r) -CENTER parameterized by td . Theorem 4 has established that weighted (k, r) -CENTER is $W[1]$ -hard parameterized by vc (and hence also by td by Lemma 18), but the complexity of unweighted (k, r) -CENTER parameterized by td does not follow from this theorem, since td is incomparable to fvs . Indeed, we show that (k, r) -CENTER is FPT parameterized by td and precisely determine its parameter dependence based on the ETH.

► **Theorem 7.** *Unweighted (k, r) -CENTER can be solved in time $O^*(2^{O(td^2)})$.*

► **Theorem 8.** *If (k, r) -CENTER can be solved in $2^{o(td^2)} \cdot n^{O(1)}$ time, then 3-SAT can be solved in $2^{o(n)}$ time.*

5 Treewidth: FPT approximation scheme

In this section we present an FPT approximation *scheme* (FPT-AS) for (k, r) -CENTER parameterized by tw . Given as input a weighted graph $G = (V, E)$, $k, r \in \mathbb{N}^+$ and an arbitrarily small error parameter $\epsilon > 0$, our algorithm is able to return a solution that uses a set of k centers K , such that all other vertices are at distance at most $(1 + \epsilon)r$ from K , or to correctly conclude that no (k, r) -center exists. The running time of the algorithm is $O^*((tw/\epsilon)^{O(tw)})$, which (for large r) significantly out-performs any *exact* algorithm for the problem (even for the unweighted case and more restricted parameters, as in Theorems 4 and 5), while only sacrificing a small ϵ error in the quality of the solution.

Our algorithm will rely heavily on a technique introduced in [32] (see also [3]) to approximate problems which are W -hard by treewidth. The idea is that, if the hardness of the problem is due to the fact that the DP table needs to store tw large numbers (in our case, the distances of the vertices in the bag from the closest center), we can significantly speed up the algorithm if we replace all entries by the closest integer power of $(1 + \delta)$, for some appropriately chosen δ . This will reduce the table size from (roughly) r^{tw} to $(\log_{(1+\delta)} r)^{tw}$.

The problem now is that a DP performing calculations on its entries will, in the course of its execution, create values which are not integer powers of $(1 + \delta)$, and will therefore have to be “rounded” to retain the table size. This runs the risk of accumulating rounding errors, but we manage to show that the error on any entry of the rounded table can be bounded by a function of the height of its corresponding bag, then using a theorem of [7] stating that any tree decomposition can be balanced so that its width remains almost unchanged, yet its total height becomes $O(\log n)$. Beyond these ideas, which are for the most part present in [32], we will also need a number of problem-specific observations, such as the fact that we can pre-process the input by taking the metric closure of each bag, and in this way avoid some error-prone arithmetic operations.

To obtain the promised algorithm we thus do the following: first we re-cast the problem as a distance-labeling problem (as in the proof of Theorem 3) and formulate an exact treewidth-based DP algorithm running in time $O^*(r^{O(tw)})$. We remark that the algorithm essentially reproduces the ideas of [10], and can be made to run in $O^*((2r+1)^{tw})$ if one uses fast subset convolution for the Join nodes (the naive implementation would need time $O^*((2r+1)^{2tw})$) but we give it here to ensure that we have a solid foundation upon which to build the approximation algorithm. We then apply the rounding procedure sketched above, and prove its approximation ratio by using the balancing theorem of [7] and indirectly comparing the value produced by the approximation algorithm with the value that would have been produced by the exact algorithm.

Distance-labeling: We give an equivalent formulation of (k, r) -CENTER that will be more convenient to work with in the remainder, similarly to Section 3.2. For an edge-weighted graph $G = (V, E)$, a distance-labeling function is a function $dl : V \rightarrow \{0, \dots, r\}$. We say that $u \in V$ is *satisfied* by dl , if $dl(u) = 0$, or if there exists $v \in N(u)$ such that $dl(u) \geq dl(v) + w((v, u))$. We say that dl is *valid* if all vertices of V are satisfied by dl , and we define the *cost* of dl as $|dl^{-1}(0)|$. The following lemma shows the equivalence between the two formulations:

► **Lemma 9.** *An edge-weighted graph $G = (V, E)$ admits a (k, r) -center if and only if it admits a valid distance-labeling function $dl : V \rightarrow \{0, \dots, r\}$ with cost k .*

► **Theorem 10.** *There is an algorithm which, given an edge-weighted graph $G = (V, E)$ and $r \in \mathbb{N}^+$, computes the minimum cost of any valid distance labeling of G in time $O^*(r^{O(tw)})$.*

We now describe an approximation algorithm based on the exact DP algorithm of Theorem 10. We make use of a result of [7] stating that: *There is an algorithm which, given a tree decomposition of width w of a graph on n nodes, produces a decomposition of the same graph with width at most $3w + 2$ and height $O(\log n)$ in polynomial time* and of the following lemma:

► **Lemma 11.** *Let $G = (V, E)$ be an edge-weighted graph, \mathcal{T} a tree decomposition of G , and $u, v \in V$ two vertices that appear together in a bag of \mathcal{T} . Let G' be the graph obtained from G by adding (u, v) to E (if it does not already exist) and setting $w((u, v)) = d_G(u, v)$. Then \mathcal{T} is a valid decomposition of G' , and $\forall k, r$, G' admits a (k, r) -center if and only if G does.*

Let us also give an approximate version of the distance labeling problem we defined above, for a given error parameter $\epsilon > 0$. Let $\delta > 0$ be some appropriately chosen secondary parameter (we will eventually set $\delta \approx \frac{\epsilon}{\log n}$). We define a δ -labeling function dl_δ as a function from V to $\{0\} \cup \{(1+\delta)^i \mid i \in \mathbb{N}, (1+\delta)^i \leq (1+\epsilon)r\}$. In words, such a function assigns (as previously) a distance label to each vertex, with the difference that now all values assigned are integer powers of $(1+\delta)$, and the maximum value is at most $(1+\epsilon)r$. We now say that a vertex u is ϵ -satisfied if $dl_\delta(u) = 0$ or, for some $v \in N(u)$ we have $dl_\delta(u) \geq dl_\delta(v) + \frac{w((v, u))}{1+\epsilon}$. As previously, we say that dl_δ is *valid* if all vertices are ϵ -satisfied, and define its cost as $|dl_\delta^{-1}(0)|$. The following Lemma 12 shows the equivalence of a valid δ -labeling function of cost k and a $(k, (1+\epsilon)^2r)$ -center for G and using it we conclude the proof of Theorem 13, stating the main result of this section.

► **Lemma 12.** *If for a weighted graph $G = (V, E)$ and any $k, r, \delta, \epsilon > 0$, there exists a valid δ -labeling function with cost k , then there exists a $(k, (1+\epsilon)^2r)$ -center for G .*

► **Theorem 13.** *There is an algorithm which, given a weighted instance of (k, r) -CENTER, $[G, k, r]$, a tree decomposition of G of width tw and a parameter $\epsilon > 0$, runs in time $O^*((tw/\epsilon)^{O(tw)})$ and either returns a $(k, (1+\epsilon))$ -center of G , or correctly concludes that G has no (k, r) -center.*

6 Clique-width revisited: FPT approximation scheme

We give here an FPT-AS for (k, r) -CENTER parameterized by cw , both for un-weighted and for weighted instances (for a weighted definition of cw which we explain below). Our algorithm builds on the algorithm of Section 5, and despite the added generality of the parameterization by cw , we are able to obtain an algorithm with similar performance: for any $\epsilon > 0$, our algorithm runs in time $O^*((\text{cw}/\epsilon)^{O(\text{cw})})$ and produces a $(k, (1 + \epsilon)r)$ -center if the input instance admits a (k, r) -center.

Our main strategy, which may be of independent interest, is to pre-process the input graph $G = (V, E)$ in such a way that the answer does not change, yet producing a graph whose tw is bounded by $O(\text{cw}(G))$. The main insight that we rely on, which was first observed by [23], is that a graph of low cw can be transformed into a graph of low tw if one removes all large bi-cliques. Unlike previous applications of this idea (e.g. [31]), we do not use the main theorem of [23] as a “black box”, but rather give an explicit construction of a tree decomposition, exploiting the fact that (k, r) -CENTER allows us to relatively easily eliminate complete bi-cliques. As a result, we obtain a tree decomposition of width not just bounded by some function of $\text{cw}(G)$, but actually $O(\text{cw}(G))$.

In the remainder we deal with the weighted version of (k, r) -CENTER. To allow clique-width expressions to handle weighted edges, we interpret the clique-width join operation η as taking three arguments. The interpretation is that $\eta(a, b, w)$ is an operation that adds (directed) edges from all vertices with label a to all vertices with label b and gives weight w to all these edges. It is not hard to see that if a graph has a (standard) clique-width expression with cw labels, it can also be constructed with cw labels in our context, if we replace every standard join operation $\eta(a, b)$ with $\eta(a, b, 1)$ followed by $\eta(b, a, 1)$. Hence, the algorithm we give also applies to un-weighted instances parameterized by (standard) clique-width. We will also deal with a generalization of (k, r) -CENTER, where we are also supplied, along with the input graph $G = (V, E)$, a subset $I \subseteq V$ of *irrelevant* vertices. In this version, a (k, r) -center is a set $K \subseteq V \setminus I$, with $|K| = k$, such that all vertices of $V \setminus I$ are at distance at most r from K . Clearly, the standard version of (k, r) -CENTER corresponds to $I = \emptyset$. As we explain in the proof of Theorem 16, this generalization does not make the problem significantly harder. In addition to the above, in this section we allow edge weights to be equal to 0. This does not significantly alter the problem, however, if we are interested in approximation and allow r to be unbounded, as the following lemma shows:

► **Lemma 14.** *There exists a polynomial algorithm which, for any $\epsilon > 0$, given an instance $I = [G, w, k, r]$ of (k, r) -CENTER, with weight function $w : V \rightarrow \mathbb{N}$, produces an instance $I' = [G, w', k, r']$ on the same graph with weight function $w' : V \rightarrow \mathbb{N}^+$, such that we have the following: for any $\rho \geq 1$, any $(k, \rho r')$ -center of I' is a $(k, \rho r)$ -center of I ; any $(k, \rho r)$ -center of I is a $(k, (1 + \epsilon)\rho r')$ -center of I' .*

Our main tool is the following lemma, whose strategy is to replace every large label-set by two “representative” vertices, in a way that retains the same distances among all vertices of the graph. Applying this transformation repeatedly results in a graph with small treewidth. The main theorem of this section then follows from the above.

► **Lemma 15.** *Given a (k, r) -CENTER instance $G = (V, E)$ along with a clique-width expression T for G on cw labels, we can in polynomial time obtain a (k, r) -CENTER instance $G' = (V', E')$ with $V \subseteq V'$, and a tree decomposition of G' of width $\text{tw} = O(\text{cw})$, with the following property: for all k, r , G has a (k, r) -center if and only if G' has a (k, r) -center.*

► **Theorem 16.** *Given $G = (V, E)$, $k, r \in \mathbb{N}^+$, clique-width expression T for G on cw labels and $\epsilon > 0$, there exists an algorithm that runs in time $O^*((cw/\epsilon)^{O(cw)})$ and either produces a $(k, (1 + \epsilon)r)$ -center, or correctly concludes that no (k, r) -center exists.*

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A Omitted Definitions and Preliminaries

This section of the appendix contains the formal definitions of notions omitted from Section 2, as well as the proof of Lemma 17.

We define the (k, r) -CENTER problem as follows: given a graph $G = (V, E)$ and a weight function $w : E \rightarrow \mathbb{N}^+$ which satisfies the triangle inequality and defines the length of each edge, we are asked if there exists a set K of at most k vertices of V , so that $\forall u \in V \setminus K$ we have $\min_{v \in K} d(v, u) \leq r$, where $d(v, u)$ denotes the shortest-path distance from v to u under weight function w . If w assigns weight 1 to all edges we say that we have an instance of un-weighted (k, r) -CENTER. Note that, dealing with (k, r) -CENTER we allow, in general, the function w to be non-symmetric. We also require edge weights to be strictly positive integers but, as we will see (Lemma 14 in Appendix B), this is not a significant restriction.

► **Lemma 17.** *If a parameterized problem with parameter k admits, for some $\epsilon > 0$, an algorithm running in time $O^*((\log n/\epsilon)^{O(k)})$, then it also admits an algorithm running in time $O^*((k/\epsilon)^{O(k)})$.*

Proof. We consider two cases: if $k \leq \sqrt{\log n}$ then $(\log n/\epsilon)^{O(k)} = (1/\epsilon)^{O(k)} (\log n)^{O(\sqrt{\log n})} = O^*((1/\epsilon)^{O(k)})$. If on the other hand, $k > \sqrt{\log n}$ we have $\log n \leq k^2$, so $O^*((\log n/\epsilon)^{O(k)}) = O^*((k/\epsilon)^{O(k)})$. ◀

A *tree (path) decomposition* of a graph $G = (V, E)$ is a pair (\mathcal{X}, T) with $T = (I, F)$ a tree (path) and $\mathcal{X} = \{X_i | i \in I\}$ a family of subsets of V (called *bags*), one for each node of T , with the following properties:

- 1) $\bigcup_{i \in I} X_i = V$;
- 2) for all edges $(v, w) \in E$, there exists an $i \in I$ with $v, w \in X_i$;
- 3) for all $i, j, k \in I$, if j is on the path from i to k in T , then $X_i \cap X_k \subseteq X_j$.

The *width* of a tree (path) decomposition $((I, F), \{X_i | i \in I\})$ is $\max_{i \in I} |X_i| - 1$. The *treewidth (pathwidth)* of a graph G is the minimum width over all tree (path) decompositions of G , denoted by $\text{tw}(G)$ ($\text{pw}(G)$).

Moreover, for rooted T , let $G_i = (V_i, E_i)$ denote the *terminal subgraph* defined by node $i \in I$, i.e. the induced subgraph of G on all vertices in bag i and its descendants in T . Also let $N_i(v)$ denote the neighborhood of vertex v in G_i and $d_i(u, v)$ denote the distance between vertices u and v in G_i , while $d(u, v)$ (absence of subscript) is the distance in G .

In addition, a tree decomposition can be converted to a *nice* tree decomposition of the same width (in $O(\text{tw}^2 \cdot n)$ time and with $O(\text{tw} \cdot n)$ nodes): the tree here is rooted and binary, while nodes can be of four types:

- a) Leaf nodes i are leaves of T and have $|X_i| = 1$;
- b) Introduce nodes i have one child j with $X_i = X_j \cup \{v\}$ for some vertex $v \in V$ and are said to *introduce* v ;
- c) Forget nodes i have one child j with $X_i = X_j \setminus \{v\}$ for some vertex $v \in V$ and are said to *forget* v ;
- d) Join nodes i have two children denoted by $i - 1$ and $i - 2$, with $X_i = X_{i-1} = X_{i-2}$.

The set of graphs of *clique-width* cw is the set of vertex-labelled graphs that can be inductively constructed by using the following operations:

- 1) Introduce: $i(l)$, for $l \in [1, \text{cw}]$ is the graph consisting of a single vertex with label l ;
- 2) Join: $\eta(G, a, b)$, for G having clique-width cw and $a, b \in [1, \text{cw}]$ is the graph obtained from G by adding all possible edges between vertices of label a and vertices of label b ;

- 3) Rename: $\rho(G, a, b)$, for G having clique-width cw and $a, b \in [1, \text{cw}]$ is the graph obtained from G by changing the label of all vertices of label a to b ;
- 4) Union: $G_1 \cup G_2$, for G_1, G_2 having clique-width cw is the disjoint union of graphs G_1, G_2 . Note we here assume the labels are integers in $[1, \text{cw}]$, for ease of exposition.

A *clique-width expression* of width cw for $G = (V, E)$ is a recipe for constructing a cw -labelled graph isomorphic to G . More formally, a clique-width expression is a rooted binary tree T_G , such that each node $t \in T_G$ has one of four possible types, corresponding to the operations given above. In addition, all leaves are introduce nodes, each introduce node has a label associated with it and each join or rename node has two labels associated with it. For each node t , the graph G_t is defined as the graph obtained by applying the operation of node t to the graph (or graphs) associated with its child (or children). All graphs G_t are subgraphs of G and for all leaves of label l , their associated graph is $i(l)$. For a graph G_t , the set of vertices with label l is called the *label-set* of l .

We next state a lemma giving the relationships between parameters used in this paper.

► **Lemma 18.** [6, 14] *For any graph G we have $\text{tw}(G) \leq \text{pw}(G) \leq \text{td}(G) \leq \text{vc}(G)$, $\text{tw}(G) \leq \text{fvs}(G) \leq \text{vc}(G)$, $\text{cw}(G) \leq \text{pw}(G) + 1$, and $\text{cw}(G) \leq 2^{\text{tw}(G)+1} + 1$.*

Finally, k -MULTICOLORED INDEPENDENT SET is defined as follows: we are given a graph $G = (V, E)$, with V partitioned into k cliques $V = V_1 \uplus \dots \uplus V_k$, $|V_i| = n, \forall i \in [1, k]$, and are asked to find an $S \subseteq V$, such that $G[S]$ forms an independent set and $|S \cap V_i| = 1, \forall i \in [1, k]$.

B Omitted constructions and proofs

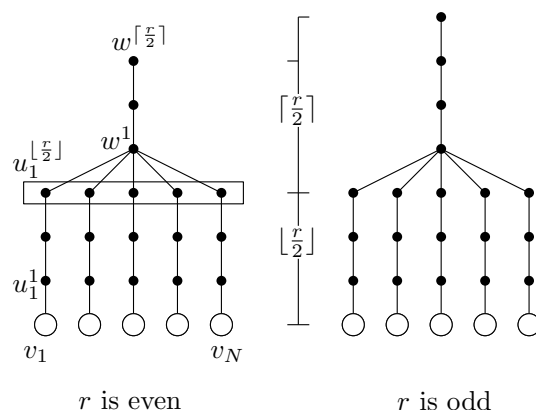
This section of the appendix contains all omitted constructions and proofs of statements.

B.1 From Section 3.1, Clique-width: Lower bound based on SETH

We describe the construction of a graph G , given some $\epsilon < 1$ and an instance ϕ of SAT with n variables and m clauses. We first choose an integer $p \geq \frac{1}{(1-\lambda)\log_2(3r+1)}$, for $\lambda = \log_{3r+1}(3r+1-\epsilon) < 1$, for reasons that become apparent in the proof of Theorem 1. Note that for the results of this section, both r and p are considered constants. We then group the variables of ϕ into $t = \lceil \frac{n}{\gamma} \rceil$ groups F_1, \dots, F_t , for $\gamma = \lfloor \log_2(3r+1)^p \rfloor$, being also the maximum size of any such group. Our construction uses a main Block gadget \hat{G} and three smaller gadgets \hat{T}_N, \hat{X}_N and \hat{U}_N as parts.

Guard gadget \hat{T}_N : This gadget has N *input* vertices and its purpose is to allow for any selection of a single input as a center to cover all vertices within the gadget, while offering no paths of length $\leq r$ between the inputs. Thus if two guard gadgets are attached to a set of inputs, any minimum-sized center-set will select one of them to cover all gadget vertices.

Construction and size/cw bounds for \hat{T}_N : Given vertices v_1, \dots, v_N , we construct the gadget as follows: we first make $\lfloor \frac{r}{2} \rfloor$ vertices $u_i^1, \dots, u_i^{\lfloor \frac{r}{2} \rfloor}$ forming a path for each v_i and we connect each u_i^1 to each v_i . We then make another path on $\lceil \frac{r}{2} \rceil$ vertices, called $w^1, \dots, w^{\lceil \frac{r}{2} \rceil}$, and we make vertices $u_i^{\lfloor \frac{r}{2} \rfloor}$ adjacent to the starting vertex w^1 of this path for all $i \in [1, N]$. Finally, if r is even, we make all vertices $u_i^{\lfloor \frac{r}{2} \rfloor}$ into a clique, while for odd r the construction is already complete. Figure 3 provides an illustration. The number of vertices in the gadget is $|\hat{T}_N| = N \lfloor \frac{r}{2} \rfloor + \lceil \frac{r}{2} \rceil$, while the gadget can also be constructed by a clique-width expression using at most this number of labels, by handling each vertex as an individual label.



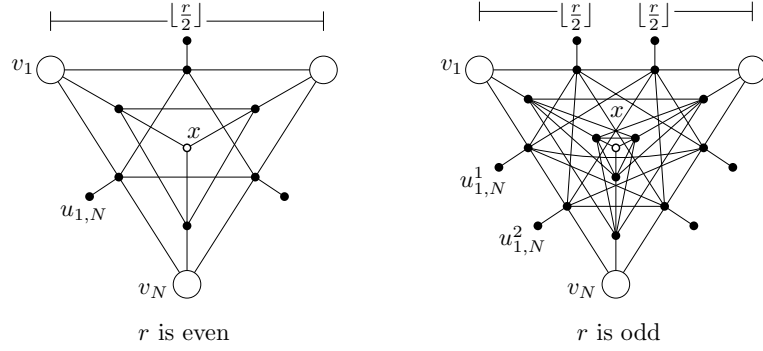
■ **Figure 3** A general picture of the guard gadget \hat{T}_N for even and odd r . Note the box indicating vertices forming a clique for the case of even r .

► **Lemma 19.** *All vertices of gadget \hat{T}_N are at distance $\leq r$ from any input vertex $v_i, i \in [1, N]$, while $d(v_i, v_j) = r + 1$, for any $i \neq j \in [1, N]$.*

Proof. The distance from any input vertex v_i , with $i \in [1, N]$, to vertex $u_i^{\lfloor \frac{r}{2} \rfloor}$ is $d(v_i, u_i^{\lfloor \frac{r}{2} \rfloor}) = \lfloor \frac{r}{2} \rfloor$, while the distance from $u_i^{\lfloor \frac{r}{2} \rfloor}$ to $w^{\lceil \frac{r}{2} \rceil}$ is $d(u_i^{\lfloor \frac{r}{2} \rfloor}, w^{\lceil \frac{r}{2} \rceil}) = \lceil \frac{r}{2} \rceil$, thus $d(v_i, w^{\lceil \frac{r}{2} \rceil}) = \lfloor \frac{r}{2} \rfloor + \lceil \frac{r}{2} \rceil = r$. Further, the distance from $u_i^{\lfloor \frac{r}{2} \rfloor}$ to any vertex u_j^1 , for $j \neq i$, is $d(u_i^{\lfloor \frac{r}{2} \rfloor}, u_j^1) = \lfloor \frac{r}{2} \rfloor$ for even r , and $d(u_i^{\lfloor \frac{r}{2} \rfloor}, u_j^1) = 1 + \lfloor \frac{r}{2} \rfloor$ for odd r . Thus $d(v_i, u_j^1) = \lfloor \frac{r}{2} \rfloor + \lfloor \frac{r}{2} \rfloor = r$ for even r , and $d(v_i, u_j^1) = \lfloor \frac{r}{2} \rfloor + 1 + \lfloor \frac{r}{2} \rfloor = r$ for odd r . Thus any vertex on these paths is at distance $\leq r$ from v_i , while any other input vertex v_j , being at distance 1 from its corresponding u_j^1 , is at distance $r + 1$ from v_i . ◀

Clique gadget \hat{X}_N : This gadget again has N input vertices and the aim now is to make sure that any selection of a single input vertex as a center will cover all other vertices of the gadget, that no selection of any single vertex that is not an input would suffice instead, while all paths connecting the inputs are of distance exactly r .

Construction and size/cw bounds for \hat{X}_N : Given vertices v_1, \dots, v_N , we first connect them to each other by paths on $r - 1$ new vertices, so that the distances between any two of them are exactly r . We then make all *middle* vertices that lie at distance $\lfloor \frac{r}{2} \rfloor$ from some vertex v_i on these paths adjacent to each other (into a clique): for even r the middle vertex of each path is at distance $r/2$ from the path's endpoint vertices v_i, v_j , while for odd r the middle vertices are the two vertices at distance $(r - 1)/2$ from one of the path's endpoints v_i, v_j . Thus all vertices on these paths are at distance at most r from any vertex v_i , as they lie within distance $< \lfloor \frac{r}{2} \rfloor$ from a middle vertex on their path, which is at distance exactly $\lfloor \frac{r}{2} \rfloor + 1$ from any input vertex v_i . Next, we make another vertex $u_{i,j}^l$ for each middle vertex of these paths between v_i, v_j (and $l \in [1, 2]$ for odd r) and we make it adjacent to its corresponding middle vertex. We then add another vertex x that we also connect to all v_i vertices by paths using $r - 1$ new vertices (thus at distance r from each v_i), making the middle vertices on these new paths adjacent to each other as well (into another, disjoint clique). See Figure 4 for an illustration. The size of the gadget is $|\hat{X}_N| = N + 1 + \binom{N}{2} \cdot r + N(r - 1)$ for even r and $|\hat{X}_N| = N + 1 + \binom{N}{2} \cdot (r + 1) + N(r - 1)$ for odd r . The gadget can also be constructed by a clique-width expression using at most this number of labels, by handling each vertex as an individual label.



■ **Figure 4** A general picture of the clique gadget \hat{X}_N for even and odd r .

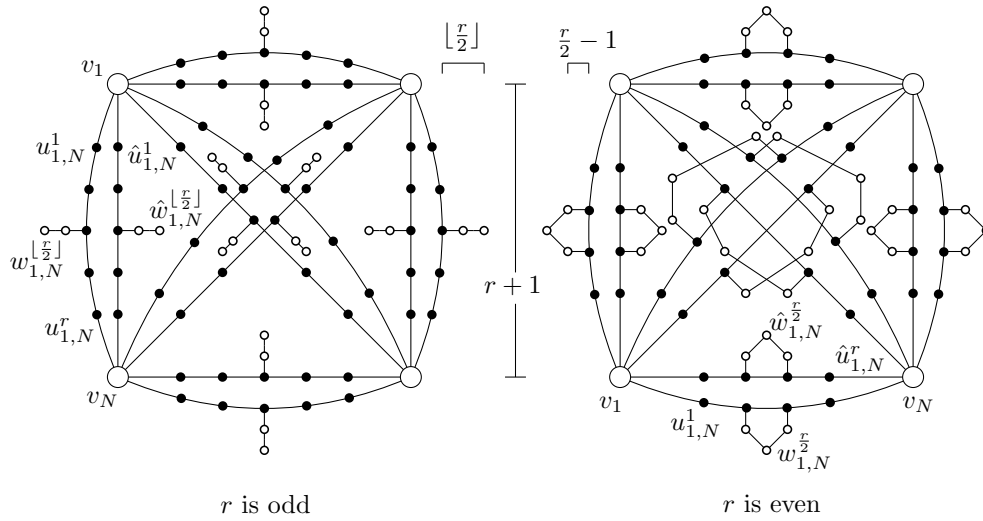
► **Lemma 20.** *Any minimum-sized center-set restricted to the vertices of \hat{X}_N , will select exactly one of the input vertices $v_i, i \in [1, N]$, to cover all other vertices in \hat{X}_N .*

Proof. Assume the selection of any single vertex from \hat{X}_N in any minimum-sized center-set: if the selected vertex is on one of the original paths, then vertex x is not covered, as its distance is at least $r + 1$ from any such vertex (via the closest v_i), while if the selected vertex, say w , is on one of the new paths between some v_i and x , there will be at least one vertex $u_{j,k}^l$ on a path between two input vertices v_j, v_k for $l \in [1, 2]$ that is not covered, as the distances from w to v_j, v_k are at least $1 + \lfloor \frac{r}{2} \rfloor$ and the distances from there to the $u_{j,k}^l$ are also $1 + \lfloor \frac{r}{2} \rfloor$. On the other hand, if the selected vertex is one of the inputs v_i , then all other vertices are covered: the distance from v_i to any other input v_j or x is exactly r , thus all vertices on these paths are covered (including vertices of the type $u_{i,j}^l$), while for vertices on paths not originating at v_i , the distance from v_i to some middle vertex on a path adjacent to it is $\lfloor \frac{r}{2} \rfloor$ and the distance from there to any vertex on some other path (or even adjacent to a middle vertex) is $\leq \lfloor \frac{r}{2} \rfloor$, giving an overall distance of $\leq r$. ◀

Assignment gadget \hat{U}_N : Once more, there are N input vertices v_1, \dots, v_N for this gadget, while the purpose here is to ensure that, assuming all input vertices have already been covered, any minimum-sized center-set will select exactly $N - 1$ of them to cover all other vertices in the gadget.

Construction and size/cw bound for \hat{U}_N : We first connect all input vertices to each other by two distinct paths each containing r new vertices, so that all distances between any pair of input vertices are exactly $r + 1$. Let the vertices on these paths between v_i, v_j be $u_{i,j}^l$ and $\hat{u}_{i,j}^l$ for $l \in [1, r]$. Then, for odd r , we also attach a path of $\lfloor \frac{r}{2} \rfloor$ vertices to the middle vertex of each path, that is, the vertices $u_{i,j}^{\lfloor \frac{r}{2} \rfloor + 1}$ and $\hat{u}_{i,j}^{\lfloor \frac{r}{2} \rfloor + 1}$ that are at distance $\lfloor \frac{r}{2} \rfloor + 1$ from both endpoints v_i, v_j of their paths. We call the vertices on these new paths $w_{i,j}^m$ and $\hat{w}_{i,j}^m$ for $m \in [1, \lfloor \frac{r}{2} \rfloor]$. For even r , we make a vertex $w_{i,j}^{r/2}$ (resp. $\hat{w}_{i,j}^{r/2}$) for each path and attach two paths of $r/2 - 1$ vertices to it, naming the vertices on these paths $w_{i,j}^{o,m}$ (resp. $\hat{w}_{i,j}^{o,m}$) for $o \in [1, 2]$ and $m \in [1, r/2 - 1]$, finally attaching the other endpoint vertex $w_{i,j}^{1,1}$ (resp. $\hat{w}_{i,j}^{1,1}$) to $u_{i,j}^{r/2}$ (resp. $\hat{u}_{i,j}^{r/2}$) and also $w_{i,j}^{2,1}$ (resp. $\hat{w}_{i,j}^{2,1}$) to $u_{i,j}^{r/2+1}$ (resp. $\hat{u}_{i,j}^{r/2+1}$), being the vertices at distance $r/2$ from one of the two endpoints v_i (and $r/2 + 1$ from the other v_j). Thus between any two inputs v_i, v_j , there are two vertices $w_{i,j}^{\lfloor \frac{r}{2} \rfloor}, \hat{w}_{i,j}^{\lfloor \frac{r}{2} \rfloor}$ at distance exactly r from both. See Figure 5 for an illustration. The size of the gadget is $|\hat{U}_N| = N + 2 \binom{N}{2} \cdot (r + \lfloor \frac{r}{2} \rfloor)$ for odd r and $|\hat{U}_N| = N + 2 \binom{N}{2} \cdot (2r - 1)$ for even r , while the gadget can also be constructed by a

clique-width expression using at most this number of labels, by handling each vertex as an individual label.



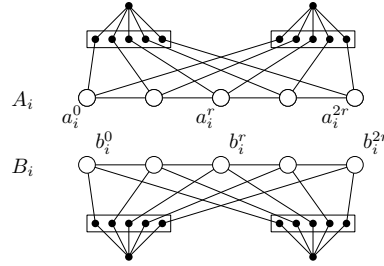
■ **Figure 5** A general picture of the assignment gadget \hat{U}_N for odd and even r .

► **Lemma 21.** *Assuming all input vertices v_1, \dots, v_N need not be covered by this selection, any minimum-sized center-set restricted to the vertices of \hat{U}_N will select exactly $N - 1$ of the input vertices to cover all other vertices in \hat{U}_N .*

Proof. The claim on the gadget’s function is shown by induction on N : for all $N \geq 2$ any minimum-sized center-set will select exactly $N - 1$ input vertices, while if any non-input vertex is selected there will be at least N vertices required. The base case is $N = 2$ and we have two input vertices v_1, v_2 and two paths between them, with selection of either v_1 or v_2 indeed covering all vertices on these paths. On the other hand, as on each path between v_1, v_2 there is a vertex $w_{1,2}^{\lfloor \frac{r}{2} \rfloor}$ or $\hat{w}_{1,2}^{\lfloor \frac{r}{2} \rfloor}$ at distance exactly r from both v_1, v_2 (and the distance between them is $2r$), selection of any single vertex on these paths will cover all vertices on its path but not all vertices on the other path and thus at least two selections will be required. For the induction step, assuming the claim holds for $N - 1$, we extend it for N : the gadget \hat{U}_N is constructed by the gadget \hat{U}_{N-1} by adding a new input vertex v_N and two paths from it to every other input vertex v_1, \dots, v_{N-1} . Let v_i with $i \in [1, N - 1]$ be the vertex that is not selected from the original $N - 1$ input vertices of the \hat{U}_{N-1} gadget. Then in \hat{U}_N , all vertices that were already in \hat{U}_{N-1} have been covered, as well as all vertices on the paths between vertices v_j with $j \in [1, N - 1], j \neq i$ and v_N . The only vertices that still need to be covered are the ones on the two paths between v_i, v_N . Again, as on each path between v_i, v_N there is a vertex $w_{i,N}^{\lfloor \frac{r}{2} \rfloor}$ or $\hat{w}_{i,N}^{\lfloor \frac{r}{2} \rfloor}$ at distance exactly r from both v_i, v_N , any selection of a single vertex from inside these two paths will be insufficient to cover both $w_{i,N}^{\lfloor \frac{r}{2} \rfloor}$ and $\hat{w}_{i,N}^{\lfloor \frac{r}{2} \rfloor}$, while selecting either v_i or v_N indeed covers all vertices. ◀

Block gadget \hat{G} : This gadget is the main building block of our construction and uses the above gadgets as inner components. We first make p pairs of paths $A_1, B_1, \dots, A_p, B_p$ consisting of $2r + 1$ vertices each, named a_i^0, \dots, a_i^{2r} and b_i^0, \dots, b_i^{2r} for every pair A_i, B_i with $i \in [1, p]$. We then make two copies of the guard gadget \hat{T}_N for A_i , where the $N = 2r + 1$ inputs are the vertices a_0, \dots, a_{2r} and repeat the same for B_i . We refer to all vertices in

these gadgets \hat{T}_{2r+1} as the *guards*.



■ **Figure 6** The paths A_i, B_i along with their attached guard gadgets for $r = 2$. Note the boxes indicating cliques (for even r).

Canonical pairs: We next define $3r+1$ *canonical pairs* of numbers $(\alpha_y \in [0, 2r], \beta_y \in [0, 2r])$, indexed and ordered by $y \in [1, 3r+1]$: for $1 \leq y \leq 2r$, the pair is given by $(\alpha_y = \lfloor \frac{y}{2} \rfloor, \beta_y = \lfloor \frac{y}{2} \rfloor)$ if y is odd, while if y is even the pair is given by $(\alpha_y = y/2 - 1, \beta_y = 2r - y/2 + 1)$. For $2r+1 \leq y \leq 3r+1$ the pair is given by $(\alpha_y = y - r - 1, \beta_y = y - r - 1)$. As an example, the pairs in the correct order for $r = 2$ are $(0,0), (0,4), (1,1), (1,3), (2,2), (3,3), (4,4)$.

In our construction, these pairs will correspond to the indices of the vertices of paths A_i (α_y) and B_i (β_y) that a (canonical) minimum-sized center-set can select. As already mentioned, in the final construction there will be a number of consecutive such pairs of paths A_i^j, B_i^j connected in a path-like manner. We first provide a substitution lemma showing that any minimum-sized center-set does not need to make a selection that does not correspond to one of the canonical pairs followed by a lemma showing that any center-set only making selections based on canonical pairs will have to respect the ordering of the pairs it uses, i.e. if some pair with index y is used for selection from paths A_i^j, B_i^j , then any pair used in the following paths A_i^{j+1}, B_i^{j+1} must be of index $y' \leq y$. This will form the basis of a crucial argument for showing that any solution will eventually settle into a specific pattern that encodes an assignment without alternations.

► **Lemma 22.** *In a series of M pairs of paths $A_i^j, B_i^j, j \in [1, M]$, where the last vertices of A_i^j, B_i^j are joined with the first vertices of $A_i^{j+1}, B_i^{j+1}, j \in [1, M-1]$, any center-set K of size $|K| = 2m$ that contains one vertex from each path A_i^j, B_i^j can be substituted by a center-set K' of size $|K'| = |K| = 2m$, where the indices of each pair of selected vertices is canonical.*

Proof. First, observe that due to the connection of both final vertices from each pair of paths A_i^j, B_i^j with both first vertices of the following pair A_i^{j+1}, B_i^{j+1} , any selection $a^l \in A_i^j$ and $b^o \in B_i^j$ is equivalent in terms of the vertices it covers with the opposite selection $a^o \in A_i^j$ and $b^l \in B_i^j$, i.e. the same vertices from preceding/succeeding pairs of paths $A_i^{j-1}, B_i^{j-1} / A_i^{j+1}, B_i^{j+1}$ are covered by both selections, while within A_i^j, B_i^j the vertices covered are the opposite. Due to this symmetry, any center-set K in which the index $l \in [0, 2r]$ of the selected vertex $a^l \in A_i^j$ is larger than the index $o \in [0, 2r]$ of the selected vertex $b^o \in B_i^j$, for any $j \in [1, M]$, can be substituted by a center-set K' of the same size in which $l \leq o$ (by replacing one with the other), without affecting the outcome.

Given such a center-set, we claim that any pair of selections $a_j^l \in A_i^j, b_j^o \in B_i^j$ can in fact be substituted by a canonical pair, by showing that *all* pairs can be partitioned in equivalence classes, each being represented by a canonical pair: the center-set can alternate between any

of the pairs within each class without any change in vertex coverage and we can thus replace any pair of selections by the canonical pair representing the class to which it belongs.

Consider the canonical pairs for $y \in [1, 2r]$ and odd: the pair is given by $(\alpha_y = \lfloor \frac{y}{2} \rfloor, \beta_y = \lfloor \frac{y}{2} \rfloor)$ and we define the corresponding class of non-canonical pairs to include all pairs where $(\alpha_y = \lfloor \frac{y}{2} \rfloor, \lfloor \frac{y}{2} \rfloor \leq \beta_y \leq 2r - \lceil \frac{y}{2} \rceil)$, i.e. any pair where α_y is the same as the canonical representative, yet β_y can now range from the same value $\lfloor \frac{y}{2} \rfloor$ up to $2r - \lceil \frac{y}{2} \rceil$. To see why any selections within the class are interchangeable, consider the two extreme cases: let $a_j^{\lfloor \frac{y}{2} \rfloor} \in A_i^j, b_j^{\lfloor \frac{y}{2} \rfloor} \in B_i^j$ be a selection followed by $a_{j+1}^{\lfloor \frac{y}{2} \rfloor} \in A_i^{j+1}, b_{j+1}^{2r - \lceil \frac{y}{2} \rceil}$ in the subsequent pair of paths. Both selections $a_j^{\lfloor \frac{y}{2} \rfloor}, b_j^{\lfloor \frac{y}{2} \rfloor}$ will be at distance $0 < d = r - \lfloor \frac{y}{2} \rfloor \leq r$ from the middle vertices a_j^r, b_j^r on their paths A_i^j, B_i^j and all vertices $a_j^{\lfloor \frac{y}{2} \rfloor + 1}, b_j^{\lfloor \frac{y}{2} \rfloor + 1}, \dots, a_j^{2r-d}, b_j^{2r-d}$ from the same paths will be covered by these selections. As the selection from A_i^j always matches the one from A_i^{j+1} , the remaining vertices on these paths will be covered by the subsequent selection $a_{j+1}^{\lfloor \frac{y}{2} \rfloor} \in A_i^{j+1}$, as well as all vertices on path B_i^{j+1} up to position $r - \lfloor \frac{y}{2} \rfloor - 2$, meaning the selection from this path can be up to distance $r + 1$ from this “last” covered vertex, giving the index of the furthest possible choice from B_i^{j+1} as $2r - \lfloor \frac{y}{2} \rfloor - 1 = 2r - \lceil \frac{y}{2} \rceil$, being exactly the extremal case for this class. Observe also that this selection will not cover more vertices of the subsequent paths A_i^{j+2}, B_i^{j+2} as it can reach up to vertices at position $r - \lfloor \frac{y}{2} \rfloor - 2$ in both these paths, which are exactly already covered by the selection of $a_{j+2}^{\lfloor \frac{y}{2} \rfloor} \in A_i^{j+2}$, meaning any other intermediate selections would indeed produce the same result as well.

Next, consider the canonical pairs for $y \in [2r + 2, 3r + 1]$: the pair is given by $(\alpha_y = y - r - 1, \beta_y = y - r - 1)$ and we define the corresponding class of non-canonical pairs to include all pairs where $(3r + 2 - y \leq \alpha_y \leq y - r - 1, \beta_y = y - r - 1)$, i.e. any pair where β_y is the same as the canonical representative, yet now α_y can range from the same value $y - r - 1$ down to $3r + 2 - y$. To see why any selections within the class are interchangeable, consider the two extreme cases, as before: let $a_j^{y-r-1} \in A_i^j, b_j^{y-r-1} \in B_i^j$ be a selection followed by $a_{j+1}^{3r+2-y} \in A_i^{j+1}, b_{j+1}^{y-r-1} \in B_i^{j+1}$ in the subsequent pair of paths. Both selections a_j^{y-r-1}, b_j^{y-r-1} will be at distance $0 \leq d = 3r + 1 - y \leq r$ from the final vertices a_j^{2r}, b_j^{2r} on their paths A_i^j, B_i^j and all vertices $a_{j+1}^0, b_{j+1}^0, \dots, a_{j+1}^{r-d-1}, b_{j+1}^{r-d-1}$ from the following paths A_i^{j+1}, B_i^{j+1} will be covered by these selections. As the selection from B_i^j always matches the one from B_i^{j+1} , all the remaining vertices of B_i^{j+1} are covered, as well as vertices $a_{j+1}^{d'}, \dots, a_{j+1}^{2r}$ from A_i^{j+1} , where $d' = (y - r - 1) - r + 2d + 2 = 4r + 3 - y$ is the maximum distance in A_i^{j+1} that selection b_{j+1}^{y-r-1} can reach, meaning the selection from this path A_i^{j+1} can be up to distance $r + 1$ from this “first” covered vertex (a_{j+1}^{4r+3-y}), giving the index of the nearest possible choice from A_i^{j+1} as $4r + 3 - y - (r + 1) = 3r + 2 - y$, being exactly the extremal case for this class.

The final observation required for the claim to be shown is that indeed all possible pairs $(l, o) \in [0, 2r]^2$, where $l \leq o$, either exactly match some canonical pair, or are contained in one of the classes given above. As any opposite selections are symmetrical and within each class the actual selections are interchangeable, any arbitrary center-set K can be substituted by a center-set K' of the same size, where all indices of each pair of selections is canonical. ◀

► **Lemma 23.** *In a series of M pairs of paths $A_i^j, B_i^j, j \in [1, M]$, where the last vertices of A_i^j, B_i^j are joined with the first vertices of A_i^{j+1}, B_i^{j+1} for $j \in [1, M - 1]$, in any center-set K that only selects vertices whose indices correspond to canonical pairs, the index y of any canonical pair selected in some pair of paths A_i^j, B_i^j must be larger than, or equal to the index y' of any canonical pair selected in its following pair of paths A_i^{j+1}, B_i^{j+1} .*

Proof. Consider two consecutive pairs of paths A_i^j, B_i^j and A_i^{j+1}, B_i^{j+1} and let $a_j^{\alpha_y}, b_j^{\beta_y}$ and $a_{j+1}^{\alpha_{y'}}, b_{j+1}^{\beta_{y'}}$ be the selections from A_i^j, B_i^j and A_i^{j+1}, B_i^{j+1} , respectively, with $y, y' \in [1, 3r + 1]$.

First, for any pair $(\alpha_y = y - r - 1, \beta_y = y - r - 1)$, where $y \in [2r + 1, 3r + 1]$, both selections $a_j^{\alpha_y}, b_j^{\beta_y}$ will be at distance $0 \leq d = 3r + 1 - y \leq r$ from the final vertices a_j^{2r}, b_j^{2r} on their paths A_i^j, B_i^j and all vertices $a_{j+1}^0, b_{j+1}^0, \dots, a_{j+1}^{r-d-1}, b_{j+1}^{r-d-1}$ from the following paths A_i^{j+1}, B_i^{j+1} will be covered by these selections. Thus for both paths A_i^{j+1}, B_i^{j+1} , the corresponding distance d' of the selections $a_{j+1}^{\alpha_{y'}}, b_{j+1}^{\beta_{y'}}$ from the final vertices $a_{j+1}^{2r}, b_{j+1}^{2r}$ will be the same for both and at least equal to d , which in turn implies both $\alpha_{y'} \leq \alpha_y$ and $\beta_{y'} \leq \beta_y$, that gives $y' \leq y$ (note that y' can be within $[1, 2r]$ as long as both inequalities hold).

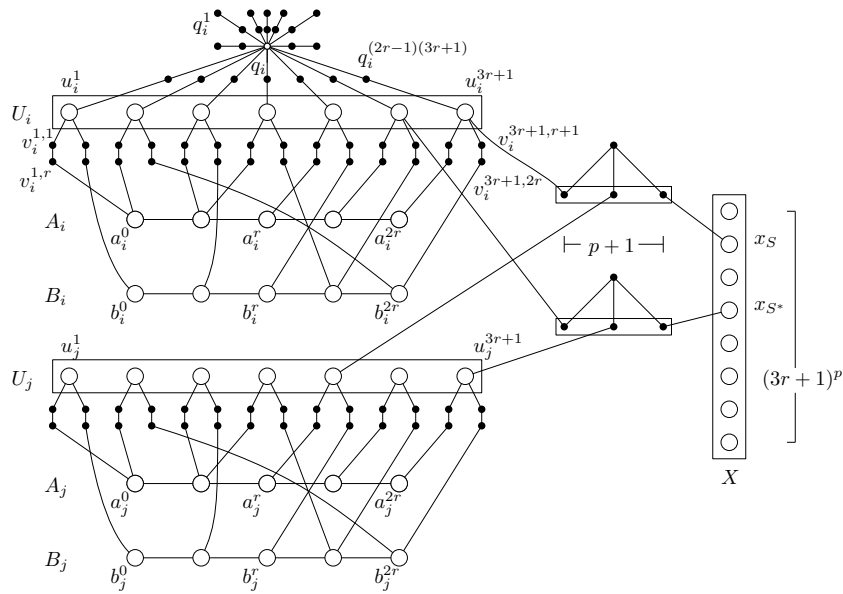
Next, for any pair $(\alpha_y = \lfloor \frac{y}{2} \rfloor, \beta_y = \lfloor \frac{y}{2} \rfloor)$, with $y \in [1, 2r]$ and odd, both selections $a_j^{\alpha_y}, b_j^{\beta_y}$ will be at distance $0 < d = r - \lfloor \frac{y}{2} \rfloor \leq r$ from the middle vertices a_j^r, b_j^r on their paths A_i^j, B_i^j and all vertices $a_j^{\alpha_y+1}, b_j^{\beta_y+1}, \dots, a_j^{2r-d}, b_j^{2r-d}$ from the same paths will be covered by these selections. Thus in at least one of the following paths A_i^{j+1}, B_i^{j+1} there must be some selection $a_{j+1}^{\alpha_{y'}}$ or $b_{j+1}^{\beta_{y'}}$ at distance d' , that is at least equal to d , from either a_{j+1}^r or b_{j+1}^r . As for all pairs (α_y, β_y) it is always $\beta_y \geq \alpha_y$, if this selection is from B_i^{j+1} , then it is $\beta_{y'} \leq \beta_y$ and $\alpha_{y'} \leq \alpha_y$ which gives $y' \leq y$, while if this selection is from A_i^{j+1} , we have $\alpha_{y'} \leq \alpha_y$, which means either also $\beta_{y'} \leq \beta_y$ and thus $y' \leq y$, or $\beta_{y'} = 2r - y'/2 + 1$ and $y' = y + 1$. In this case, observe that $b_{j+1}^{\beta_{y'}}$ covers all vertices $b_{j+1}^{r-y'/2+1}, \dots, b_{j+1}^{2r-y'/2}$, while $a_{j+1}^{\alpha_{y'}}$ can cover all vertices from b_{j+1}^0 (for $d' > 1$) up to $b_{j+1}^{r-y'/2-1}$ from B_i^{j+1} , thus leaving vertex $b_{j+1}^{r-y'/2}$ at distance $> r$ from any selected vertex.

Finally, for even $y \in [1, 2r]$ and any pair $(\alpha_y = y/2 - 1, \beta_y = 2r - y/2 + 1)$, selected vertex $a_j^{\alpha_y}$ covers all vertices $a_j^{\alpha_y+1}, \dots, a_j^{\alpha_y+r}$, while selected vertex $b_j^{\beta_y}$ can only cover all vertices from $a_j^{\alpha_y+r+2}$ to a_j^{2r} (for $y < 2r$), thus requiring at least one selection from the following pair of paths A_i^{j+1}, B_i^{j+1} , at distance at most $y/2 - 1$ from the first vertex on its path a_{j+1}^0 or b_{j+1}^0 . In either case, we have $\alpha_{y'} \leq \alpha_y$ and thus also $y' \leq y$. ◀

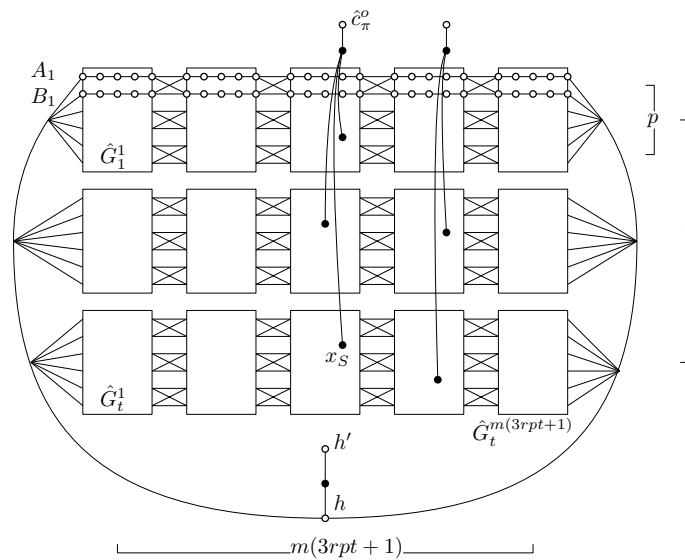
Now we can continue the description of the block gadget \hat{G} . For each pair of paths A_i, B_i with $i \in [1, p]$ we make $3r + 1$ vertices u_i^y , for $y \in [1, 3r + 1]$, that we connect to vertices $a_i^{\alpha_y}$ and $b_i^{\beta_y}$ by paths of length $r + 1$, i.e. each vertex u_i^y is at distance $r + 1$ from the vertices in A_i, B_i , whose indices match the numbers in the pair corresponding to its own index y (via one path for one such vertex in A_i or B_i). Let the r intermediate vertices on the path from each u_i^y to some vertex in A_i be called $v_i^{y,1}, \dots, v_i^{y,r}$ and the r intermediate vertices on the other path to some vertex in B_i be called $v_i^{y,r+1}, \dots, v_i^{y,2r}$.

Next, we add another vertex q_i that we attach to all vertices u_i^y by paths of length $r - 1$ (making the distances between them and q_i equal to r) and then we also attach $3r + 1$ paths of length r to q_i , naming the $(2r - 1)(3r + 1)$ vertices on these paths $q_i^1, \dots, q_i^{(2r-1)(3r+1)}$. Let U_i be the set of all u_i^y vertices for all $y \in [1, 3r + 1]$ and U be the union of all U_i for $i \in [1, p]$. We then make use of the assignment gadget \hat{U}_N described above by making a copy of \hat{U}_N for each U_i , where the $N = 3r + 1$ inputs are identified with the vertices u_i^y .

Then, for every set $S \subset U$ that contains exactly one vertex from each U_i (the number of such sets being $(3r + 1)^p$) we make a vertex x_S . Here we make use of the clique gadget \hat{X}_N described above, where these x_S vertices act as inputs and $N = (3r + 1)^p$. Let X be the set containing all vertices in the gadget, including all x_S vertices. Then, for every vertex x_S we make a copy of the guard gadget \hat{T}_N , where the $N = p + 1$ inputs are x_S and the vertices in U for which $u_i^y \in S$ (one from each U_i). See Figure 7 for an illustration. This concludes the construction of the Block gadget \hat{G} .



■ **Figure 7** Inside a block gadget \hat{G} : two pairs of paths A_i, B_i, A_j, B_j with attached U_i, U_j , clique gadget X and guard gadgets between U and X , while guard vertices and all q_j are omitted for clarity. Note boxes around vertices in U_i, U_j, X indicate a gadget, while boxes around the $p + 1$ vertices in the guard gadgets indicate cliques (for even r).



■ **Figure 8** A simplified picture of the complete construction. Note boxes indicate block gadgets \hat{G} , while there is no vertex anywhere between h and the first/last vertices of the long paths.

Global construction: Graph G is then constructed as follows. For every group F_τ of variables of ϕ with $\tau \in [1, t]$, we make $m(3rpt + 1)$ copies of the gadget \hat{G} that we call \hat{G}_τ^μ for $1 \leq \mu \leq m(3rpt + 1)$. Then, for each $\tau \in [1, t]$ we connect gadgets $\hat{G}_\tau^1, \hat{G}_\tau^2, \dots, \hat{G}_\tau^{m(3rpt+1)}$ in a path-like manner: for every $1 \leq \mu < m(3rpt + 1)$ and $1 \leq i \leq p$, we connect both vertices a_i^{2r}, b_i^{2r} in \hat{G}_τ^μ to both vertices a_i^0, b_i^0 in $\hat{G}_\tau^{\mu+1}$. We then make another vertex h that we connect to all vertices a_i^1, b_i^1 in \hat{G}_τ^1 for all $i \in [1, p]$ and $\tau \in [1, t]$, as well as to all vertices a_i^{2r}, b_i^{2r} in $\hat{G}_\tau^{m(3rpt+1)}$ for all $i \in [1, p]$ and $\tau \in [1, t]$, i.e. to all the first and last vertices on the long paths created upon connecting gadgets $\hat{G}_\tau^1, \hat{G}_\tau^2, \dots, \hat{G}_\tau^{m(3rpt+1)}$. We also attach a path of length r to vertex h , the final vertex of this path named h' .

Next, for every $1 \leq \tau \leq t$ we associate a set $S \subset U$ that contains exactly one vertex from each U_i with an assignment to the variables in group F_τ and as there are at most $2^\gamma = 2^{\lfloor \log_2(3r+1)^p \rfloor}$ assignments to the variables in F_τ and $(3r+1)^p \geq 2^\gamma$ sets S , the association can be unique for each τ . For each clause C_π of ϕ with $\pi \in [1, m]$, we make $3rpt + 1$ vertices \hat{c}_π^o for $0 \leq o < 3rpt + 1$. Then, to each \hat{c}_π^o we attach a path of length $r - 1$ and we consider every assignment to the variables of group F_τ for every $\tau \in [1, t]$ that satisfies the clause C_π : a vertex x_S in $\hat{G}_\tau^{mo+\pi}$, for every $0 \leq o < 3rpt + 1$, is adjacent to the endpoint of this path (thus being at distance r from \hat{c}_π^o), where S is the subset associated with this assignment within the gadget. This concludes our construction, while Figure 8 provides an illustration.

► **Lemma 24.** *If ϕ has a satisfying assignment, then G has a (k, r) -center of size $k = ((3r + 3)p + 1)m(3rpt + 1)t + 1$.*

Proof. Given a satisfying assignment for ϕ we show the existence of a (k, r) -center K of G of size $|K| = k = ((3r + 3)p + 1)m(3rpt + 1)t + 1$. Set K will include the vertex h and $(3r + 3)p + 1$ vertices from each gadget \hat{G}_τ^μ , for $\tau \in [1, t]$ and $\mu \in [1, m(3rpt + 1)]$. For each group F_τ of variables we consider the restriction of the assignment for ϕ to these variables and identify the set S associated with this restricted assignment. We first add vertex x_S to K and then, for every $i \in [1, p]$, we also add vertex q_i and all $u_i^z \in U_i \setminus S$, as well as the two vertices $a_i^{\alpha_y}$ and $b_i^{\beta_y}$ from the paths A_i, B_i that are connected to the only $u_i^y \in S$ for this i by paths of length $r + 1$, i.e. the vertices whose indices $\alpha_y, \beta_y \in [0, 2r]$ correspond to the pair (α_y, β_y) for the index $y \in [1, 3r + 1]$ of this vertex. In total, we have $3r$ vertices u_i^y , plus the three $a_i^{\alpha_y}, b_i^{\beta_y}$ and q_i for each $i \in [1, p]$, with the addition of vertex x_S completing the selection within each \hat{G}_τ^μ . Repeating the above for all $m(3rpt + 1)t$ gadgets completes the selection for G and what remains is to show that K is indeed a (k, r) -center of G .

First, our selection of one vertex from each path A_i, B_i ensures that all guard vertices are within distance r from some selected vertex. Next, for each $\tau \in [1, t]$ and $i \in [1, p]$, consider the “long paths” formed by joining both vertices a_i^{2r}, b_i^{2r} in \hat{G}_τ^μ to both a_i^0, b_i^0 in $\hat{G}_\tau^{\mu+1}$. Since the associations between sets S and partial assignments to variables of F_τ are consistent for each τ , the patterns of selection of vertices from each A_i and B_i are repeating, i.e. K contains every $(2r + 1)$ -th vertex on each path A_i and B_i , meaning all vertices on these paths are within distance r from some selected vertex, apart from the first/last $r - 1$ depending on the actual selection pattern, yet these vertices are within distance r from selected vertex h .

Next, within each gadget \hat{G}_τ^μ , our selection of vertices q_i for every $i \in [1, p]$ brings all $q_i^1, \dots, q_i^{(2r-1)(3r+1)}$ and u_i^j vertices within distance r from K . Further, our selection of $a_i^{\alpha_y}, b_i^{\beta_y}$ covers vertices $v_i^{y,1}, \dots, v_i^{y,2r}$ on the two paths from $a_i^{\alpha_y}, b_i^{\beta_y}$ to $u_i^y \in S$, while all other $v_i^{z,w}$ vertices are covered by our selection of each $u_i^z \in U_i \setminus S$. This selection of all $u_i^z \in U_i \setminus S$ also covers all vertices in the assignment gadgets as well as the guard gadgets \hat{T}_p between the u_i^z and $x_{S'}$ for all $S' \neq S$, while selection of x_S covers all vertices in X and all vertices in the guard gadget where x_S is an input.

Finally, concerning the clause vertices \hat{c}_π^o for $\pi \in [1, m]$ and $o \in [0, 3rpt]$, observe that if the given assignment for ϕ satisfies the clause C_π , there will be some literal contained therein that is set to true, that corresponds to a variable in a group F_τ for some $\tau \in [1, t]$ and a matching partial assignment for the variables in F_τ associated with some set S^* . Our set K contains vertex x_{S^*} in $\hat{G}_\tau^{mo+\pi}$ for every $o \in [0, 3rpt]$ and vertices \hat{c}_π^o are within distance r from x_{S^*} , through a path whose vertices are also covered by x_{S^*} . ◀

► **Lemma 25.** *If G has a (k, r) -center of size $k = ((3r + 3)p + 1)m(3rpt + 1)t + 1$, then ϕ has a satisfying assignment.*

Proof. Given a (k, r) -center K of G of size $|K| = k = ((3r + 3)p + 1)m(3rpt + 1)t + 1$, we show the existence of a satisfying assignment for ϕ . First, observe that K must contain vertex h , as all vertices on the path attached to it and h' must be within distance r from K . Next we require an averaging argument: as the remaining number of vertices in K is $((3r + 3)p + 1)m(3rpt + 1)t$ and there are $m(3rpt + 1)t$ gadgets \hat{G}_τ^μ , if there are more than $(3r + 3)p + 1$ vertices selected from some block gadget, then there will be less than these selected from some other block gadget. We will show that not all vertices within a block gadget can be covered by less than $(3r + 3)p + 1$ selected vertices, which implies that exactly this number is selected from each block gadget in any center-set of size k .

Consider a gadget \hat{G}_τ^μ . First, observe that at least one of the x_S vertices must be selected to cover all vertices in X and that any such selection indeed covers all vertices in X , as well as all vertices in the guard gadget of which it is an input. This leaves $(3r + 3)p$ vertices to cover all other p groups of vertices in the gadget. Observe also that for every $i \in [1, p]$, any minimum-sized center-set must contain one vertex from each path A_i, B_i to cover all the guards, as well as q_i to cover $q_i^1, \dots, q_i^{(2r-1)(3r+1)}$, as any single selection of some guard vertex will not be sufficient to cover all other guard vertices attached to the same path. This leaves $3rp$ vertices to cover the $v_i^{y,w}$ vertices on the paths between the a_i^j, b_i^l that have not been selected from each pair A_i, B_i , as well as all vertices in the guard gadgets in which each x_{S^*} that was not selected is an input. Due to the structure of the assignment gadgets \hat{U} , there must be $3r$ vertices selected from each U_i , thus completing the set K (and the averaging argument). We then claim that the selections from the paths A_i, B_i must match (complement) these selections from U_i , that in turn must match (also complement) the selection of x_S from X .

First, suppose that for some $i \in [1, p]$ the selections a_i^j, b_i^l with $j, l \in [0, 2r]$ from A_i, B_i do not correspond to some canonical pair for some $y \in [1, 3r + 1]$ with $(\alpha_y = j, \beta_y = l)$.¹ Vertex a_i^j will cover at most $2r$ vertices $v_i^{z,1}, \dots, v_i^{z,r}$ and $v_i^{z',1}, \dots, v_i^{z',r}$, for $z \neq z'$ (if index j happens to be included in two pairs and only the first r for inclusion in a single pair), but not vertices $v_i^{z,r+1}$ or $v_i^{z',r+1}$. Similarly, vertex b_i^l will also cover at most $2r$ vertices $v_i^{w,r+1}, \dots, v_i^{w,2r}$ and $v_i^{w',r+1}, \dots, v_i^{w',2r}$, again for $w \neq w'$, but not vertices $v_i^{w,1}$ or $v_i^{w',1}$. Note that since the two choices do not correspond to some canonical pair all these indices will be different: $z \neq z' \neq w \neq w'$. Now, as there are no more selections from A_i, B_i , all vertices $v_i^{y',1}$ and $v_i^{y',r+1}$ are definitely not covered for $y' \neq z, z', w, w'$ (due to any adjacent selections being at distance at least $r + 1$). In short, there is at least one vertex $v_i^{y,1}$ or $v_i^{y,r+1}$ (or both) that is not covered for each $y \in [1, 3r + 1]$. Since the number of selections from U_i is $3r$ and no other selection would reach these vertices (e.g. from some other $U_{i'}$, due to

¹ In fact, Lemma 22 already shows that any (k, r) -center that does not make selections based only on canonical pairs can always be substituted for a (k, r) -center that does. Nevertheless, we show here that this requirement is also enforced by the structure of the graph, mostly for completeness.

the guard gadgets employed anywhere in between sets U and X), there will be at least one such vertex that is not covered, implying the selections from every A_i, B_i must match some canonical pair y and the $3r$ selections from U_i must complement this y . Further, suppose the selection x_S from X , for $S = \{u_1^{y_1}, \dots, u_p^{y_p}\}$, does not match the $3r$ selections from each U_i , i.e. that for some $i \in [1, p]$, it is $u_i^{y_i} \in K \cap U_i$ and $u_i^z \notin K \cap U_i$ for all $z \neq y_i$. Then for set $S^* = S \setminus \{u_i^{y_i}\} \cup \{u_i^z\}$ we have that $x_{S^*} \notin K$ and also $S^* \not\subset K$. This means all vertices in the guard gadget attached to x_{S^*} are not covered.

Next, we require that there exists at least one $o \in [0, 3rpt]$ for every $\tau \in [1, t]$ for which $K \cap \{\bigcup_{i \in [1, p]} A_i \cup B_i\}$ is the same in all gadgets $\hat{G}_\tau^{mo+\pi}$ with $\pi \in [1, m]$, i.e. that there exists a number of successive copies of the gadget for which the pattern of selection of vertices from the paths A_i, B_i does not change. As noted above, set K must contain two vertices $a_i^{\alpha_y}, b_i^{\beta_y}$ from each A_i and B_i , such that the indices α_y, β_y of these two selections match the pair corresponding to the index y of some u_i^y . Consider the “long paths” consisting of paths A_i, B_i sequentially joined with their followers in the next gadget on the same row. Depending on the starting selection, observe that the pattern can “shift towards the left” a number of times in each pair of paths A_i, B_i , as the first and last $r - 1$ vertices will be covered by h . That is, a pattern can be selected on some pair A_i, B_i within some gadget \hat{G}_τ^μ and a different pattern can be selected on the pair A_i, B_i following it in gadget $\hat{G}_\tau^{\mu+1}$, without affecting whether all vertices on the long paths are covered. As shown by Lemma 23, this can only happen if the index y' that gives the pair of indices $(\alpha_{y'}, \beta_{y'})$ of the second pattern is smaller than or equal to the index y that gives the pair of indices (α_y, β_y) of the first pattern, or $y' \leq y$.

As there are $3r + 1$ different indices y and pairs (α_y, β_y) , the “shift to the left” can happen at most $3r$ times for each $i \in [1, p]$, thus at most $3rp$ times for each $\tau \in [1, t]$, or $3rpt$ times over all τ . By the pigeonhole principle, there must thus exist an $o \in [0, 3rpt]$ such that no such shift happens among the gadgets $\hat{G}_\tau^{mo+\pi}$, for all $\tau \in [1, t]$ and $\pi \in [1, m]$.

Our assignment for ϕ is then given by the selections for K in each gadget \hat{G}_τ^{mo+1} for this o : for every group F_τ we consider the selection of $x_S \in X$ that corresponds to a set $S \subset U$, that in turn is associated with a partial assignment for the variables in F_τ . In this way we get an assignment to all the variables of ϕ . To see why this also satisfies every clause C_π with $\pi \in [1, m]$, consider clause vertex \hat{c}_π^o : this vertex is at distance r from some selected vertex x_S in some gadget $\hat{G}_\tau^{mo+\pi}$. Since the pattern for selection from paths A_i, B_i remains the same in all gadgets $\hat{G}_\tau^{mo+1}, \dots, \hat{G}_\tau^{mo+\pi}$, so does the set U and also selection of vertices x_S , giving the same assignment for the variables of F_τ associated with S . ◀

► **Lemma 26.** *Graph G has clique-width $cw(G) \leq tp + f(r, \epsilon)$, for $f(r, \epsilon) = O(1)$.*

Proof. We show how to construct graph G using the clique-width operations introduce, join, relabel and at most $f(r, \epsilon)$ labels. We first introduce vertex h and all vertices on the path of length r from it to h' , using one label for each vertex, then consecutively join labels/vertices to form the path. The construction will then proceed in a vertical manner, successively constructing the gadgets \hat{G}_τ^μ for each $\tau \in [1, t]$, before proceeding to repeat the process for each of the $m(3rpt + 1)$ columns.

To construct gadget \hat{G}_τ^μ we do the following: we first introduce all guard vertices using one label for each vertex, subsequently applying the appropriate join operations. We do the same for all vertices $q_i, q_i^1, \dots, q_i^{(2r-1)(3r+1)}$, as well as vertices u_i^y in U_i and all vertices in gadgets \hat{U}_{3r+1} and $v_i^{y,1}, \dots, v_i^{y,2r}$, along with all vertices in X and the guard gadgets \hat{T} attached to each x_S . We have also introduced the clause vertex \hat{c}_π^o that corresponds to this column and all vertices on the path attached to it, the endpoint of which we now join with

matching vertices x_S (if any, for this F_τ). We have thus created all vertices (and appropriate edges) in gadget \hat{G}_τ^μ , apart from the paths A_i, B_i for $i \in [1, p]$, using one label per vertex. This accounts for the $f(r, \epsilon)$ labels, where function f is $O(r^p) = O(1)$.

Now, for $i = 1$, we introduce vertices a_1^j, b_1^j in turn for each $j \in [0, 2r]$, using one label for each vertex and appropriately join each with its previous in the path (for $j \geq 2$), the endpoints of the guard gadgets, as well as the corresponding vertices $v_1^{y,w}$, completing the construction of paths A_1, B_1 . We then relabel the vertex b_1^{2r} with the same label as a_1^{2r} (the last vertices of A_1, B_1) and repeat the process for $i = 2, \dots, p$. When the above has been carried out for all i , the construction of gadget \hat{G}_τ^μ has been completed and we can relabel all vertices to some “junk” label, apart from the two final vertices a_i^{2r}, b_i^{2r} for each $i \in [1, p]$ (that have the same label) and the endpoint of the path of length $r - 1$ attached to clause vertex \hat{c}_π^o . This relabelling with some junk label will enable us to reuse the same labels when constructing the same parts of other gadgets. We then repeat the above for the following gadget in the column, until the column is fully constructed. When the column has thus been constructed, we can also relabel the endpoint of the path from clause vertex \hat{c}_π^o to the junk label, thus reusing its former label for the endpoint of the path attached to the clause vertex of the subsequent column.

During construction of the following column, the first vertices a_i^0, b_i^0 on each path A_i, B_i will both be joined with the label that includes the last vertices a_i^{2r}, b_i^{2r} of the previous column’s corresponding path A'_i, B'_i and after each such join, we again relabel these two vertices with the junk label. The construction proceeds in this way until graph G has been fully constructed. Note that during construction of the first and last columns, the first/last vertices of each path have also been joined with vertex h .

In total, the number of labels used simultaneously by the above procedure are the $f(r, \epsilon)$ labels used each time for repeating constructions (also counting the constant number of “outside” labels for h , the clause vertices and the junk label), plus one label for each pair of paths A_i, B_i (containing the last vertices of these paths) in each of the t rows of the construction. The number of these being tp , the claimed bound follows. ◀

Theorem 1. *For any fixed $r \geq 1$, if (k, r) -CENTER can be solved in $O^*((3r + 1 - \epsilon)^{cw(G)})$ time for some $\epsilon > 0$, then SAT can be solved in $O^*((2 - \delta)^n)$ time for some $\delta > 0$.*

Proof. Assuming the existence of some algorithm of running time $O^*((3r + 1 - \epsilon)^{cw(G)}) = O^*((3r + 1)^\lambda)^{cw(G)}$ for (k, r) -CENTER, where $\lambda = \log_{3r+1}(3r + 1 - \epsilon)$, we construct an instance of (k, r) -CENTER, given a formula ϕ of SAT, using the above construction and then solve the problem using the $O^*((3r + 1 - \epsilon)^{cw(G)})$ -time algorithm. Correctness is given by Lemma 24 and Lemma 25, while Lemma 26 gives the upper bound on the running time:

$$O^*((3r+1)^{\lambda \text{cw}(G)}) \leq O^*((3r+1)^{\lambda(tp+f(r,\epsilon))}) \quad (1)$$

$$\leq O^*\left((3r+1)^{\lambda p \left\lceil \frac{n}{\lfloor \log_2(3r+1)^p \rfloor} \right\rceil}\right) \quad (2)$$

$$\leq O^*\left((3r+1)^{\lambda p \frac{n}{\lfloor \log_2(3r+1)^p \rfloor} + \lambda p}\right) \quad (3)$$

$$\leq O^*\left((3r+1)^{\lambda \frac{np}{\lfloor p \log_2(3r+1) \rfloor}}\right) \quad (4)$$

$$\leq O^*\left((3r+1)^{\delta' \frac{n}{\log_2(3r+1)}}\right) \quad (5)$$

$$\leq O^*(2^{\delta'' n}) = O((2-\delta)^n) \quad (6)$$

for some $\delta, \delta', \delta'' < 1$. Observe that in line (2) the function $f(r, \epsilon)$ is considered constant, as is λp in line (4), while in line (5) we used the fact that there always exists a $\delta' < 1$ such that $\lambda \frac{p}{\lfloor p \log_2(3r+1) \rfloor} = \frac{\delta'}{\log_2(3r+1)}$, as we have:

$$\begin{aligned} p \log_2(3r+1) - 1 &< \lfloor p \log_2(3r+1) \rfloor \\ \Leftrightarrow \frac{\lambda p \log_2(3r+1)}{p \log_2(3r+1) - 1} &> \frac{\lambda p \log_2(3r+1)}{\lfloor p \log_2(3r+1) \rfloor}, \end{aligned}$$

from which, by substitution, we get: $\frac{\lambda p \log_2(3r+1)}{p \log_2(3r+1) - 1} > \delta'$,

$$\text{now requiring: } \frac{\lambda p \log_2(3r+1)}{p \log_2(3r+1) - 1} \leq 1,$$

$$\text{or: } p \geq \frac{1}{(1-\lambda) \log_2(3r+1)},$$

that is precisely our definition of p . This concludes the proof. \blacktriangleleft

B.2 From Section 3.2, Clique-width: Dynamic programming algorithm

Distance labeling: We first require an alternative formulation of the problem, based on the existence of a function dl that assigns numbers $\text{dl}(v) \in [0, r]$ to all vertices $v \in V$.

Let $\text{dl} : V \mapsto [0, r]$ and $\text{dl}^{-1}(i)$ be the set of all vertices with assigned number $i \in [0, r]$ by dl . A function dl is then called *valid*, if for all labels $l \in [1, \text{cw}]$ and nodes t of the clique-width expression, at least one of the following conditions holds for all vertices $u \in V_l \cap G_t \setminus \text{dl}^{-1}(0)$:

- 1) There is a neighbor v of u in G_t with a strictly smaller number: $\exists v \in G_t : (u, v) \in E_t \wedge \text{dl}(v) < \text{dl}(u)$;
- 2) There is a vertex v in the same label l as u and at distance 2 from it in G_t , while their difference in numbers is at least 2: $\exists v \in G_t \cap V_l \wedge \text{dl}(v) \leq \text{dl}(u) - 2 \wedge d_t(u, v) = 2$;
- 3) There is a vertex v in the same label l as u in G_t with their difference in numbers at least 2 and some vertex w adjacent to it in the final graph G : $\exists v \in G_t \cap V_l \wedge \text{dl}(v) \leq \text{dl}(u) - 2 \wedge \exists w \in G : (u, w) \notin G_t \wedge (u, w) \in G$.

Note that in condition 3) above, vertex w will also be adjacent to u in G (and thus $d(u, v) = 2$), as both u and v belong to the same label. A vertex $u \in V_l \cap G_t$ with label $l \in [1, cw]$ is *satisfied* by dl for node t , if $dl(u) = 0$, or either of the first two conditions 1)2) above holds for u . Let $U_l^t(dl)$ be the set of vertices of label l that are not satisfied by dl for node t and $DL_r(t)$ be the set of all possible valid functions dl for given r , restricted to the vertices of G_t for node t of clique-width expression T_G , with disjoint sets $dl^{-1}(0)$. The following lemma shows the equivalence between the two formulations.

► **Lemma 27.** *A graph $G = (V, E)$ admits a (k, r) -center if and only if it admits a valid distance-labeling function $dl: V \rightarrow \{0, \dots, r\}$ with $|dl^{-1}(0)| = k$.*

Proof. To see why a valid function dl with $|dl^{-1}(0)| = k$ represents a solution to the (k, r) -CENTER problem consider the following: first, given a (k, r) -center of G , let dl be the function that assigns to each vertex $v \in V$ a number equal to its distance from the closest center, i.e. number 0 to the centers, 1 to their immediate neighbors and so on. This function is valid as for every vertex u there always exists some neighbor v with $dl(v) < dl(u)$, being the neighbor that lies on the path between u and its closest center, while also $|dl^{-1}(0)| = k$. On the other hand, given such a valid function dl , the set $dl^{-1}(0)$ is indeed a (k, r) -center: we have $|dl^{-1}(0)| = k$ and first, vertices in $dl^{-1}(1)$ must have a neighbor in the center-set, while vertices in $dl^{-1}(2)$ are at distance at most 2 from some center. Then, for $i \in [3, r]$, vertices u in $dl^{-1}(i)$ either have a neighbor in $dl^{-1}(j)$ with $j < i$, or are at distance at most 2 from some vertex in $dl^{-1}(j)$ with $j \leq i - 2$, that by induction must in both cases be at distance at most j from some center, making the distance between u and some vertex in $dl^{-1}(0)$ at most $i \leq r$. ◀

Table description: There is a table D_t associated with every node t of the clique-width expression, while each table entry $D_t[\kappa, s_1, \dots, s_w]$, with $w \leq cw$, is indexed by a number $\kappa \in [0, k]$ and a w -sized tuple (s_1, \dots, s_w) of *label-states*, assigning a state $s_l = (v_l, u_l)$ to each label $l \in [1, w]$, where $v_l = \min_{x \in V_l} dl(x) \in [0, r]$ is the minimum number assigned to any vertex x in label l , while $u_l \in \{0, 1, 2\}$ is the difference between v_l and the minimum number assigned by dl to any vertex $y \in U_l^t(dl)$ that is not satisfied by dl for this node: $u_l = 0$ when $\min_{y \in U_l^t(dl)} (dl(y) - v_l) = 0$, $u_l = 1$ when $\min_{y \in U_l^t(dl)} (dl(y) - v_l) = 1$ and $u_l = 2$ when $\min_{y \in U_l^t(dl)} (dl(y) - v_l) \geq 2$, or when $U_l^t(dl) = \emptyset$. Note that states $(0, 0)$ and $(r, 1)$ do not signify any valid situation and are therefore not used.

There are thus $3r + 1$ possible states for each label, each being a pair signifying the minimum number assigned to any vertex in the label and whether the difference between this and the minimum number of any vertex in the label that is not yet satisfied by dl is either exactly 0,1, or greater than 1, with absence of unsatisfied vertices also considered in the latter case. For a node t with w involved labels, each table entry $D_t[\kappa, s_1, \dots, s_w]$ contains the number $|DL_r(t)|$ of valid functions dl restricted to the vertices of G_t with disjoint sets $dl^{-1}(0)$ and $|dl^{-1}(0)| = \kappa$, such that for each label $l \in [1, w]$, its state in the tuple gives the conditions that must be satisfied for this label by any such function dl that is to be counted in the entry's value. In particular, we have $\forall t \in T, D_t[\kappa, s_1, \dots, s_w] : \{\kappa \in [0, k]\} \times \{(1, 0), \dots, (r, 0), (0, 1), \dots, (r - 1, 1), (0, 2), \dots, (r, 2)\}^w \mapsto \mathbb{N}^0$, where $w \in [1, cw]$.

The inductive computations of table entries for each type of node follows.

Introduce node: For node t with operation $i(l)$ and $l \in [1, cw]$, we have:

$$D_t[\kappa, s_l] := \begin{cases} 1, & \text{if } v_l = 0, u_l = 2, \kappa = 1; \\ 1, & \text{if } v_l \neq 0, u_l = 0, \kappa = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Join node: For node t with operation $\eta(a, b)$, child node $t - 1$ and $a, b \in [1, w]$, let $Q(s'_a = (v'_a, u'_a), s'_b = (v'_b, u'_b)) := \{(s_a = (v_a, u_a), s_b = (v_b, u_b)) | [v_a = v'_a \wedge v_b = v'_b] \wedge [((u'_a = 2 \wedge v_a + u_a > v_b) \vee (u_a = u'_a < 2 \wedge v_a + u_a \leq v_b)) \wedge ((u'_b = 2 \wedge v_b + u_b > v_a) \vee (u_b = u'_b < 2 \wedge v_b + u_b \leq v_a))]\}$. In words, $Q(s'_a, s'_b)$ is the set of all pairs of label states (s_a, s_b) , such that if label a is joined with label b , their new states could be s'_a, s'_b , i.e. all pairs of states where the v values remain the same for both a, b , as no new numbers are introduced within any label by a join operation, yet some vertices may become satisfied through the addition of new edges and thus the u values of their label might change to 2. We then have:

$$D_t[\kappa, s_1, \dots, s'_a, s'_b, \dots, s_w] := \sum_{(s_a, s_b) \in Q(s'_a, s'_b)} D_{t-1}[\kappa, s_1, \dots, s_a, s_b, \dots, s_w].$$

Rename node: For node t with operation $\rho(w + 1, w)$ and child node $t - 1$ (we assume without loss of generality that the last label is renamed into the one preceding it), let $M(s = (v, u)) := \{(s_a = (v_a, u_a), s_b = (v_b, u_b)) | [v = \min\{v_a, v_b\}] \wedge [((u = 0) \wedge ((v_a = v \wedge u_a = 0 \wedge v_b \geq v \wedge u_b \geq 0) \vee (v_b = v \wedge u_b = 0 \wedge v_a \geq v \wedge u_a \geq 0)) \vee ((u = 1) \wedge ((v_a = v \wedge u_a = 1 \wedge v_b = v \wedge u_b \geq 1) \vee (v_a = v \wedge u_a \geq 1 \wedge v_b = v \wedge u_b = 1) \vee (v_a = v \wedge u_a = 1 \wedge v_b > v \wedge u_b \geq 0) \vee (v_b = v \wedge u_b = 1 \wedge v_a > v \wedge u_a \geq 0) \vee (v_a = v \wedge u_a = 2 \wedge v_b = v + 1 \wedge u_b = 0) \vee (v_b = v \wedge u_b = 2 \wedge v_a = v + 1 \wedge u_a \geq 0)) \vee (((u = 2) \wedge (u_a = u_b = 2) \vee (v < v_a \wedge u_b = 2 \wedge ((v_a - v \geq 2 \wedge u_a = 0) \vee (v_a - v \geq 1 \wedge u_a = 1))) \vee (v < v_b \wedge u_a = 2 \wedge ((v_b - v \geq 2 \wedge u_b = 0) \vee (v_b - v \geq 1 \wedge u_b = 1)))]]\}$. In words, $M(s)$ is the set of all pairs of labels (s_a, s_b) for two labels a, b , such that renaming one into the other could produce state s for the resulting label, i.e. the pairs of states where the resulting v value is the minimum of v_a, v_b , while u comes from any of the appropriate combinations of u_a, u_b for each case. We then have:

$$D_t[\kappa, s_1, \dots, s'_w] := \sum_{(s_w, s_{w+1}) \in M(s'_w)} D_{t-1}[\kappa, s_1, \dots, s_w, s_{w+1}].$$

Union node: For node t with operation $G_{t-1} \cup G_{t-2}$ and children nodes $t - 1, t - 2$, we assume (again, without loss of generality) that all labels in $[1, y \leq w]$ are involved in $t - 1$ and all labels in $[1, z \leq w]$ are involved in $t - 2$, such that for some $i \in [1, w]$, all labels $1 \leq j \leq i$ are involved in both nodes $t - 1, t - 2$, i.e. labels $1, \dots, i \leq y, z \leq w$ are common to both preceding nodes. Also observe that for any resulting state s'_j of label j , following application of the union operation on two partial solutions where its states are s_j and \bar{s}_j , the pair (s_j, \bar{s}_j) would be included in $M(s'_j)$, similarly to renaming j from one partial solution with state s_j to the same label from the other partial solution with state \bar{s}_j . We then have:

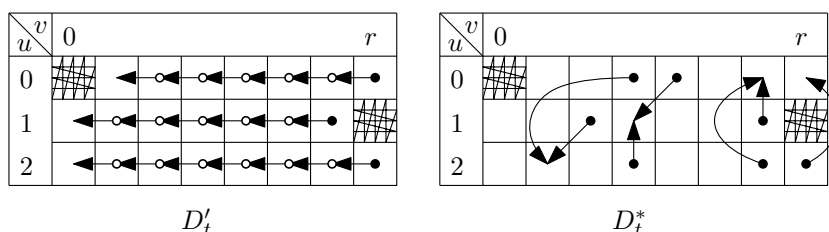
$$D_t[\kappa, s'_1, \dots, s'_i, \dots, s_w] := \sum_{\substack{(s_j, \bar{s}_j) \in M(s'_j), j \in [1, i] \\ \kappa_1 \in [0, \kappa]}} (D_{t-1}[\kappa_1, s_1, \dots, s_i, \dots, s_y] \cdot D_{t-2}[\kappa - \kappa_1, \bar{s}_1, \dots, \bar{s}_i, \dots, s_z]).$$

State changes: Since the number of entries involved in the above computation of a union node's table can be exponential in the clique-width of the graph (due to the number of possible combinations of state pairs in the sum), in order to efficiently compute the tables of union nodes we will use *state changes*: for a union node t we will first transform the tables D_{t-1}, D_{t-2} of its children into tables D_{t-1}^*, D_{t-2}^* of a new type that employs a different state representation, for which the union operation can be efficiently performed to produce table D_t^* , that we finally will transform back to table D_t , thus progressing with our dynamic programming algorithm.

In particular, each entry of table $D_t^*[\kappa, s_1, \dots, s_w]$ of a node t for $\kappa \in [0, k]$ and $w \in [1, cw]$ will be an aggregate of entries from $D_t[\kappa, s_1, \dots, s_w]$, with its value equal to the sum of the appropriate values of the original table. For label l , its state $s_l^* = (v_l^*, u_l^*)$ in the new state signification for table D_t^* will correspond to all states where $v_l^* \geq v_l + i$ and $u_l^* \geq u_l - i$ in the previous state signification for all (applicable) $i \in \{0, 1, 2\}$. Observe that these correspondences exactly parallel the states described in the definition of set $M(s_l^*)$ given above for the computation of a rename node, i.e. all states that could combine to produce the resulting state s^* .

First, let D'_t be a copy of table D_t . The transformation then works in two stages, of w steps each, label-wise: we first produce the intermediate table D'_t from D_t and then table D_t^* from D'_t . For table D'_t we require that all entries $D'_t[\kappa, s'_1, \dots, s'_w]$ contain the sum of all entries of D_t where $s'_l = (v'_l \geq v_l, u'_l = u_l)$ and all other label-states and κ are fixed: at step l , we first add the entry where $s_l = (r, 0)$ to the entry where $s_l = (r - 1, 0)$ and then the entry with $s_l = (r - 2, 0)$ to the previous result and so on until $s_l = (1, 0)$. We then repeat this process for $u = 1$ from $s_l = (r - 1, 1)$ until $s_l = (0, 1)$ and then for $u = 2$ from $s_l = (r, 2)$ until $s_l = (0, 2)$. We then proceed to the next label until table D'_t is computed.

For the next stage, we initialize D_t^* as a copy of D'_t and also work on w steps, again label-wise, fixing all other label-states and κ : at step l , we first add the entries where $s_l = (v_l, 1)$ and $s_l = (v_l, 2)$ from D'_t to entries where $s_l = (v_l, 0)$ from D_t^* , for each v_l from $r - 1$ to 1, in turn (and only the one with $s_l = (r, 2)$ from D'_t for the one where $s_l = (r, 0)$ from D_t^*). We then add the entries where $s_l = (v_l + 1, 0)$ and $s_l = (v_l, 2)$ from D'_t to entries where $s_l = (v_l, 1)$ from D_t^* , for each v_l from r to 0, in turn. Finally, we add the entries where $s_l = (v_l + 1, 1)$ and the entries where $s_l = (v_l + 2, 0)$ from D'_t to entries where $s_l = (v_l, 2)$ from D_t^* , for each v_l from r to 0, in turn. We then proceed to the next step for the following label until table D_t^* is computed. See Figure 9 for an illustration of the relationships between states/entries of these tables.



■ **Figure 9** A conceptual representation of the relationships between states/entries to be added for the computation of the alternative table types D', D^* . Arrows imply the transfer of values (addition), black endpoints indicate corresponding entries from a previous table type and white endpoints indicate previously computed entries in the same table.

Note that the above procedure is fully reversible,² as for each label l , entries where $s_l = (r, 2)$ are the same in all tables D_t, D'_t, D_t^* and thus, to obtain D'_t from D_t^* we again work label-wise, fixing all other label-states and κ : at step l , we first compute the entry of D'_t where $s_l = (r, 0)$, by subtracting from its value the value of the entry where $s_l = (r, 2)$ and we then do the same for the entry where $s_l = (r - 1, 2)$, moving to $s_l = (r - 1, 1)$ by subtracting the ones where $s_l = (r - 1, 2)$ and $s_l = (r, 0)$, then to $s_l = (r - 1, 0)$ by subtracting the ones where $s_l = (r - 1, 1)$ and $s_l = (r - 1, 2)$ and so on, for each v_l from $r - 1$ to 0 , in turn. After all w steps we have obtained table D'_t and from this we can similarly obtain table D_t , as again, for each label l the entries where $s_l = (r, 0), (r - 1, 1), (r, 2)$ are the same: at step l , we compute the entries where $s_l = (v_l, i)$ by subtracting from the corresponding entries of D'_t all entries where $s_l = (v_l + 1, i)$, for v_l from $r - 1$ to 0 and $i \in \{0, 1, 2\}$ in turn. For both transformations and directions, we perform at most two additions/subtractions per entry for $\kappa \cdot (3r + 1)^{cw}$ entries, for each step $l \in [1, w \leq cw]$.

Thus we can compute table D_t^* by simply multiplying the values of the two corresponding entries from D_{t-1}^*, D_{t-2}^* , as they now contain all required information for this state representation, with the inverse transformation of the result giving table D_t :

$$D_t^*[\kappa, s_1, \dots, s_w] := \sum_{\kappa_1=0}^{\kappa_1=\kappa} D_{t-1}^*[\kappa_1, s_1, \dots, s_w] \cdot D_{t-2}^*[\kappa - \kappa_1, s_1, \dots, s_w].$$

Fore-tracking: As is already apparent, our algorithm actually solves the counting version of (k, r) -CENTER, by reading the values of entries from table D_z of the final node z , where all labels l are of state $s_l = (v_l, u_l = 2)$. Now, as already noted, these states actually correspond to both situations where either all unsatisfied vertices in the label are of number ≥ 2 than the label's v value, or where there are no unsatisfied vertices in the label for this dl. To ensure our algorithm only counts valid functions dl, we employ a “fore-tracking” policy: whenever some entry is being computed for the table of some rename or union node t , where some label l has state $s_l = (v_l, u_l = 2)$, we establish whether condition 3) given in the definition of a valid function dl is also satisfied by all counted functions dl for all vertices in l , by verifying that some join operation $\eta(l, m)$ is applied between l and another label m in some subsequent node t' of the clique-width expression (even after l is potentially renamed to some other label). If such a join operation is indeed to be applied (and the label-set is live), there will be a vertex $w \in V_m$ that is adjacent to all vertices $u, v \in V_l$ in the final graph G and as $u_l = 2$, there must also be some vertex $v \in V_l$ with $dl(v) \leq dl(u) - 2$, for any unsatisfied vertex u . Since it will be $d(u, v) = 2$, all such vertices u will be satisfied in G for all such dl, that will in fact be valid.

On the other hand, in the absence of such a join operation and the case where t is a rename node, we consider the definition of set $M(s)$ above, where there are three options for a resulting state $s_l = (v_l, u_l)$ with $u_l = 2$ from two preceding states $s_a = (v_a, u_a), s_b = (v_b, u_b)$ (corresponding to the last bracket of clauses in the set's definition): we must have either $u_a = u_b = 2$, i.e. that both states have a difference of at least 2 between the minimum number of any vertex and the minimum of any unsatisfied vertex, or $v_l < v_a$ and $u_b = 2$, while either $v_a - v_l \geq 2$ and $u_a = 0$, or $v_a - v_l \geq 1$ and $u_a = 1$ (or vice-versa), i.e. that one label must have state $(> v_l, 0)$ and the other can have any state from $(\geq v_l + 2, 0)$ or $(\geq v_l + 1, 1)$, their

² This is the reason for *counting* the number of solutions for each κ , instead of finding the *minimum* κ for which at least one solution exists: there is no additive inverse operation for the min-sum semiring, yet the sum-product ring is indeed equipped with subtraction.

combined numbers giving $(v_l, 2)$. Now, if no join operation follows the rename node t for this label, we simply disregard any options in the above computation (from the last two) that consider states where one of the preceding labels a, b had $u_a, u_b < 2$.

Similarly, for a union node t we consider the state changes given above: if no join node follows t for this label, we simply disregard the additions (and subsequently their corresponding subtractions) for this label in the table's transformation from D'_t to D_t^* in step l , where entries with $s_l = (v_l + 1, 1)$ and $s_l = (v_l + 2, 0)$ from D'_t are added to entries with $s_l = (v_l, 2)$ from D_t^* and keep all such entries as they are (direct copies from D'_t). In this way, as any state where $u = 2$ for some label must be “the direct result” of some join operation (validating condition 1), or there will be some subsequent join operation satisfying all vertices of number ≥ 2 than the minimum (condition 2), no non-valid functions dl for which unsatisfied vertices are infused in the label producing some “false” state where $u = 2$ can be counted, as these vertices are not to be satisfied by some following join node (3).

Correctness: To show correctness of our algorithm we need to establish that for every node $t \in T_G$, each table entry $D_t[\kappa, s_1, \dots, s_w]$ contains the number of partial solutions to the sub-problem restricted to the graph G_t , i.e. the number of valid functions $dl \in DL_r(t)$ with $|dl^{-1}(0)| = \kappa$ (and disjoint sets $dl^{-1}(0)$), such that for each label $l \in [1, w]$ (being the labels involved in t) the conditions imposed by its state $s_l = (v_l, u_l)$ are satisfied by all such dl , i.e. $\min_{x \in V_l} dl(x) = v_l$ and $\min_{y \in U_l^t(dl)} (dl(y) - v_l) = 0, 1$ when $u_l = 0, 1$, while $\min_{y \in U_l^t(dl)} (dl(y) - v_l) \geq 2$, or $U_l^t(dl) = \emptyset$ when $u_l = 2$. That is:

$$\forall t \in T_G, w \in [1, cw], \forall (\kappa, s_1, \dots, s_w) \in \quad (7)$$

$$\in \{\kappa \in [0, k]\} \times \{(1, 0), \dots, (r, 0), (0, 1), \dots, (r-1, 1), (0, 2), \dots, (r, 2)\}^w : \quad (8)$$

$$\left\{ D_t[\kappa, s_1, \dots, s_w] = |DL_r(t)| : \forall dl \in DL_r(t), \forall l \in [1, w] : \quad (9)$$

$$\left\{ \left(\min_{x \in V_l} dl(x) = v_l \right) \wedge \quad (10)$$

$$\wedge \left[(u_l = 0 \wedge \min_{y \in U_l^t(dl)} (dl(y) - v_l) = 0) \vee \quad (11)$$

$$\vee (u_l = 1 \wedge \min_{y \in U_l^t(dl)} (dl(y) - v_l) = 1) \vee \quad (12)$$

$$\vee (u_l = 2 \wedge \left(\min_{y \in U_l^t(dl)} (dl(y) - v_l) \geq 2 \vee U_l^t(dl) = \emptyset \right) \right\} \wedge \quad (13)$$

$$\wedge \{ |dl^{-1}(0)| = \kappa \}. \quad (14)$$

This is shown by induction on the nodes $t \in T_G$:

- Introduce nodes: This is the base case of our induction. For node t with operation $i(l)$ and $l \in [1, cw]$, all entries are properly initialized as there is one function dl that includes the introduced vertex in the center-set and one that does not, thus $|DL_r(t)| = 1$ for $s_l = (0, 2), \kappa = 1$ and $s_l = (> 0, 0), \kappa = 0$, while for any other configuration $DL_r(t) = \emptyset$. In the following cases, we assume (our induction hypothesis) that all entries of D_{t-1} (and D_{t-2} for union nodes) satisfy the above statement (7-14).
- Join nodes: For node t with operation $\eta(a, b)$ and $a, b \in [1, w]$, all edges are added between vertices of labels a and b . Thus for each entry $D_t[\kappa, s_1, \dots, s'_a, s'_b, \dots, s_w]$, all valid functions dl counted in the entries of the previous table $D_{t-1}[\kappa, s_1, \dots, s_a, s_b, \dots, s_w]$ remain valid if $v'_a = v_a, v'_b = v_b$ (10) and $u'_a = u_a, u'_b = u_b$ (11-13), while for all entries in D_t where $v'_a = v_a, v'_b = v_b$ and $u'_a = 2$ (resp. $u'_b = 2$), all functions counted for entries of

D_{t-1} where $v_a + u_a > v_b$ (resp. $v_b + u_b > v_a$), are valid as well (13), since in the resulting graph G_t of node t , vertices in a (resp. b) can become satisfied by the addition of such edges (and validation of condition 1). This is precisely the definition of set $Q(s'_a, s'_b)$.

- **Rename nodes:** For node t with operation $\rho(w+1, w)$, all vertices of label $w+1$ will now be included in label w . Thus for each entry $D_t[\kappa, s_1, \dots, s'_w]$, we require all valid functions dl counted in the entries of the previous table $D_{t-1}[\kappa, s_1, \dots, s_w, s_{w+1}]$, where states s_w, s_{w+1} describe the possible situations for labels $w, w+1$ that could combine to give the new state s'_w for label w : first, value v'_w denotes the minimum allowed number in any vertex that was previously in w or $w+1$, yet at least one of these labels must have had some vertex with exactly this number and thus $v'_w = \min\{v_w, v_{w+1}\}$ (10). Next, value u'_w denotes whether the difference between this number and the minimum number of any unsatisfied vertex is 0, 1, or greater (if any) and the possible combinations of states s_w, s_{w+1} are fully described in the definition of set $M(s'_w)$: for u'_w to be 0 we must have $v_w = v'_w$ and $u_w = 0$, while $v_{w+1} \geq v_w$ and $u_{w+1} \geq 0$ (or vice-versa), i.e. at least one of u_w, u_{w+1} must be 0 with its corresponding value v_w, v_{w+1} being the minimum, while the state of the other can have any at least equal values, as only the minimum is retained (11). For u'_w to be 1 we must have either $v_w = v'_w$ and $u_w = 1$, while also $v_{w+1} = v'_w$ and $u_{w+1} \geq 1$, or $v_w = v'_w$ and $u_w = 1$, while $v_{w+1} > v'_w$ and $u_{w+1} \geq 0$ (or vice-versa), i.e. one label must have the same target state, while the other must also have the same v with a u greater than 1, or a greater v and any u so as not to make u'_w smaller than the target 1 (12). Finally, for u'_w to be 2 we must have either $u_w = u_{w+1} = 2$, i.e. that both states have a difference of at least 2 between the minimum number of any vertex and the minimum of any unsatisfied vertex, or $v'_w < v_w$ and $u_{w+1} = 2$, while either $v_w - v'_w \geq 2$ and $u_w = 0$, or $v_w - v'_w \geq 1$ and $u_w = 1$ (or vice-versa), i.e. that one label must have state $(> v'_w, 0)$ and the other can have any state from $(\geq v'_w + 2, 0)$ or $(\geq v'_w + 1, 1)$, their combined numbers inducing $(v'_w, 2)$ (13).
- **Union nodes:** For node t with operation $G_{t-1} \cup G_{t-2}$ and children nodes $t-1, t-2$, the resulting graph G_t is the disjoint union of graphs G_{t-1}, G_{t-2} , where all common labels $j \in [1, i]$ involved in both $t-1, t-2$ will now include all vertices that were included in label j in one of the previous graphs, while nothing changes for all other labels. This can be seen to be similar to the result of a rename operation between the label j from graph G_{t-1} and the same label from G_{t-2} . Thus for each entry $D_t[\kappa, s'_1, \dots, s'_i, \dots, s_w]$, we require the *product* of the numbers of all valid functions dl counted in entries $D_{t-1}[\kappa_1, s_1, \dots, s_i, \dots, s_y]$ for $t-1$ and those counted in entries $D_{t-2}[\kappa - \kappa_1, \bar{s}_1, \dots, \bar{s}_i, \dots, s_z]$ for $t-2$, summed over all possible combinations of states s_j, \bar{s}_j that belong to some pair in $M(s'_j)$ for every $j \in [1, i]$, as well as over all values of κ_1 from 0 to κ , since the number of valid functions in G_t respecting the conditions of the target entry is equal to a number of valid functions in G_{t-1} (counted in entries of the first type) for each valid function that complements each of them in G_{t-2} (counted in entries of the second type): the sizes of $dl^{-1}(0)$ always add up to κ (14), while the states s_j, \bar{s}_j of all common labels j in any pair of entries from D_{t-1}, D_{t-2} would result in the desired target state s_j for the same label in the entry from D_t in question, as they belong in the set $M(s'_j)$ (10-13).

Theorem 3. *Given graph G , along with $k, r \in \mathbb{N}^+$ and clique-width expression T_G of clique-width cw for G , there exists an algorithm to solve the counting version of the (k, r) -CENTER problem in $O^*((3r+1)^{cw})$ time.*

Proof. Correctness of the dynamic programming algorithm is given above, while for the final computation at the root z of T_G , all entries $D_z[k, s_1, \dots, s_w]$ with $u_l = 2, \forall l \in [1, w]$

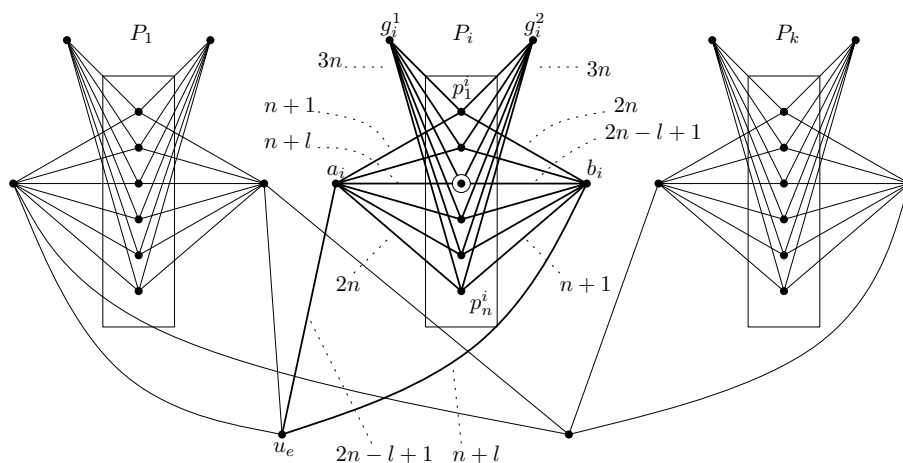
and $w \in [1, cw]$ can be considered for identification of the number of valid functions dl, or (k, r) -centers of graph $G_t = G$. For the decision version of the problem, the algorithm can output YES if any of these entries' value is > 0 and NO otherwise.

For the algorithm's complexity, there are at most $k(3r + 1)^{cw}$ entries in each table D_t of any node t and any entry of the $O(n \cdot cw^2)$ join/rename nodes can be computed in $O(r)$ time, while for the $O(n)$ union nodes, table transformations require $O^*(k \cdot cw \cdot (3r + 1)^{cw})$ time and any entry of the transformed tables can be computed in $O(k)$ time. ◀

B.3 From Section 4, Vertex Cover, Feedback Vertex Set and Tree-depth

Construction: Given an instance $[G = (V, E), k]$ of k -MULTICOLORED INDEPENDENT SET, we will construct an instance $[G' = (V', E'), k]$ of edge-weighted (k, r) -CENTER, where $r = 3n$. First, for every set $V_i \subseteq V$, we create a set $P_i \subseteq V'$ of n vertices $p_l^i, \forall l \in [1, n], \forall i \in [1, k]$ (that directly correspond to the vertices of V_i) and two *guard* vertices g_i^1, g_i^2 , attaching them to all vertices in P_i by edges of weight $r = 3n$. Next, for each $i \in [1, k]$, we create another pair of vertices a_i, b_i and connect a_i to each vertex p_l^i by an edge of weight $n + l$, while b_i is connected to each vertex p_l^i by an edge of weight $2n - l + 1$. Now each P_i contains all vertices a_i, b_i, g_i^1, g_i^2 and each $p_l^i, \forall l \in [1, n]$.

Finally, for each edge $e \in E$ with endpoints in V_{i_1}, V_{i_2} and $i_1 \neq i_2$ (not part of a clique), we create a vertex u_e that we connect to vertices a_{i_1}, b_{i_1} and a_{i_2}, b_{i_2} . We set the weights of these edges as follows: suppose that e connects the j_1 -th vertex of V_{i_1} to the j_2 -th vertex of V_{i_2} . Then we set $w(u_e, a_{i_1}) = 2n - j_1 + 1, w(u_e, b_{i_1}) = n + j_1, w(u_e, a_{i_2}) = 2n - j_2 + 1, w(u_e, b_{i_2}) = n + j_2$.



■ **Figure 10** A general picture of graph G' . The circled vertex is p_l^i , while dotted lines match edges to weights.

► **Lemma 28.** *If G has a k -multicolored independent set, then G' has a $(k, 3n)$ -center.*

Proof. Let $I \subseteq V$ be a multicolored independent set in G of size k and $v_{l_i}^i$ denote the vertex selected from each V_i , or $I := \{v_{l_1}^1, \dots, v_{l_i}^i, \dots, v_{l_k}^k\}$. Let $S \subseteq V'$ be the set of vertices $p_{l_i}^i$ in G' that correspond to each $v_{l_i}^i$. We claim S is a (k, r) -center of G' : since one vertex is selected from each P_i in G' , all the guards g_i^1, g_i^2 and vertices a_i, b_i are within distance $r = 3n$ from selected vertex $p_{l_i}^i$ for some $l_i \in [1, n]$ and every $i \in [1, k]$. For vertices u_e , observe that

selected vertex $p_{l_i}^i$ is at distance $n + l_i + 2n - l_i + 1 = 3n + 1$ from u_e , through either a_i or b_i , if its corresponding vertex in G is an endpoint of e , or $e = (v_{l_i}^i, w)$ for some $w \in V \setminus V_i$. As I is an independent set of G , however, there can be no edge between any two selected vertices $v_{l_i}^i, v_{l_j}^j \in I$ with $i \neq j$ and thus, we know that for every u_e at least one of the vertices $p_{l_i}^i, p_{l_j}^j$ selected for S from the two components P_i, P_j to which u_e is connected will not be one of the vertices corresponding to the endpoints of edge e , or $e \neq (v_{l_i}^i, v_{l_j}^j)$. Let $e := (v_x^i, v_y^j)$, with $x \neq l_i$ and/or $y \neq l_j$. Assuming without loss of generality that $x \neq l_i$ is the case, we have that the distances from u_e to $p_{l_i}^i$ are $n + l_i + 2n - x + 1$ via a_i and $2n - l_i + n + x$ via b_i . If $x > l_i$ then distance via a_i is $3n + 1 + l_i - x \leq 3n$, while if $x < l_i$ the distance via b_i is $3n + x - l_i < 3n$. ◀

► **Lemma 29.** *If G' has a $(k, 3n)$ -center, then G has a k -multicolored independent set.*

Proof. Let $S \subseteq V'$ be the $(k, 3n)$ -center in G' . As $|S| = k$ and all guard vertices $g_i^1, g_i^2, \forall i \in [1, k]$ must be within distance $3n$ from some selected vertex, set S must contain exactly one vertex from each P_i , or $S = \{p_{l_1}^1, \dots, p_{l_k}^k\}$ for some $l_i \in [1, n]$ for each $i \in [1, k]$. We let $I \subseteq V$ be the set of vertices $v_{l_i}^i \in V_i, i \in [1, k]$ that correspond to each $p_{l_i}^i$ and claim that I is an independent set in G : suppose there is an edge $e = (v_{l_i}^i, v_{l_j}^j) \in E, i \neq j$ and $v_{l_i}^i, v_{l_j}^j \in I$. Then there must be a vertex $u_e \in G'$ with edges to $a_i, b_i \in P_i$ and $a_j, b_j \in P_j$, where we have $p_{l_i}^i, p_{l_j}^j \in S$. The distance from u_e to $p_{l_i}^i$ is $2n - l_i + 1 + n + l_i > 3n$, via either a_i or b_i , while the distance from u_e to $p_{l_j}^j$ is also $2n - l_j + 1 + n + l_j > 3n$, via either a_j, b_j , meaning u_e is not covered by S , giving a contradiction. ◀

Theorem 4. *The weighted (k, r) -CENTER problem is $W[1]$ -hard parameterized by $vc + k$. Furthermore, if there is an algorithm for weighted (k, r) -CENTER running in time $n^{o(vc+k)}$ then the ETH is false.*

Proof. Observe that the set $Q \subset V'$ that includes all guard vertices g_i^1, g_i^2 and $a_i, b_i, \forall i \in [1, k]$, is a minimum vertex cover of G' , as all edges have exactly one endpoint in Q . This means $vc(G') \leq 4k$. In addition, parameter k remains the same in both the instances of k -MULTICOLORED INDEPENDENT SET and (k, r) -CENTER. Thus, the construction along with lemmas 28 and 29, indeed imply the statement. ◀

Corollary 5. *The (k, r) -CENTER problem is $W[1]$ -hard when parameterized by $fvs + k$. Furthermore, if there is an algorithm for weighted (k, r) -CENTER running in time $n^{o(fvs+k)}$, then the ETH is false.*

Proof. We use the same reduction as in Theorem 4, except that we replace all weighted edges by unweighted paths through new vertices in such a way that distances between original vertices are preserved. It is not hard to see that any set that was a vertex cover of the previous graph is a feedback vertex set of the new graph, hence $fvs = O(k)$. Lemma 28 goes through unchanged, while for Lemma 29 it suffices to observe that, because of the guard vertices, no valid solution can be using one of the new vertices as a center. ◀

Theorem 6. *Given graph G , along with $k, r \in \mathbb{N}^+$ and a vertex cover of size vc of G , there exists an algorithm solving unweighted (k, r) -CENTER in $O^*(5^{vc})$ time.*

Proof. Let C be the given vertex cover of G and $I = V \setminus C$ be the remaining independent set. We assume without loss of generality that the graph is connected (otherwise each component

can be handled separately). We also assume that $r \geq 2$, otherwise the problem reduces to DOMINATING SET which is already known to be solvable faster than $O^*(5^{vc})$.

Let K be some (unknown) optimal solution. Our algorithm first guesses a partition of C into three sets $C = S \cup R \cup Q$ such that $S = K \cap C$, Q contains the vertices of C which are at distance exactly r from K , and R the rest (which are at distance $< r$ from K). Our algorithm first guesses this partition by trying out all 3^{vc} possibilities.

Suppose we are given a partition $C = S \cup R \cup Q$ as above. We would like to check if this is the correct partition and then find a set $Z \subseteq I$ such that $S \cup Z$ is an optimal solution. First, we verify that all vertices of Q are at distance $\geq r$ from S (otherwise we already know this is not a correct partition). Second, we check if there exists $v \in I$ such that $N(v) \subseteq Q$. In such a case, we again know that this is not a correct partition, since such a v would need to be included in K , which would imply that its neighbors in Q are at distance $< r$ from K . We can therefore assume that all vertices in I have some neighbor in $S \cup R$.

We now formulate an instance of SET COVER as follows: the universe contains all vertices of $R \cup Q$ which are not already covered, that is, all vertices of R which are at distance $\geq r$ from S and all vertices of Q which are at distance $> r$ from S . We construct a set for each vertex of $v \in I$ and we place in it all vertices $u \in R$ such that $d(v, u) < r$ and all vertices $u \in Q$ such that $d(v, u) \leq r$. We solve this SET COVER instance in time $O^*(2^{|R \cup Q|})$ using dynamic programming, and this gives us a set $Z \subseteq I$. We output $S \cup Z$ as a solution to (k, r) -CENTER. We observe that this is always a valid solution because by construction all vertices of R are at distance $< r$ from $S \cup Z$, and all vertices of I have a neighbor in $S \cup R$. If we started with the correct partition of C into S, R, Q then this solution is optimal because $K \cap I$ must give a feasible set cover of the instance we constructed.

To analyze the running time, observe that if $|S| = i$, then the algorithm goes through 2^{vc-i} possible partitions of $C \setminus S$ into R, Q , and then for each partition spends $2^{vc-i} n^{O(1)}$ to solve SET COVER. Hence, the algorithm runs in time $\sum_{i=0}^{vc} \binom{vc}{i} 4^{vc-i} n^{O(1)} = \sum_{i=0}^{vc} \binom{vc}{i} 4^i n^{O(1)} = 5^{vc} n^{O(1)}$. \blacktriangleleft

We next consider the un-weighted version of (k, r) -CENTER parameterized by td . We begin with a simple upper bound argument and then make use of the following known fact on tree-depth, while the algorithm then follows from the dynamic programming procedure of [10] and the relationship between r, td and tw :

► **Lemma 30.** *For any graph $G = (V, E)$ we have $d(G) \leq 2^{td+1} - 2$, where $d(G)$ denotes the graph's diameter.*

Proof. We use the following equivalent inductive definition of tree-depth: $td(K_1) = 0$ while for any other graph $G = (V, E)$ we set $td(G) = 1 + \min_{u \in V} td(G \setminus u)$ if G is connected, and $td(G) = \max_C td(G[C])$ if G is disconnected, where the maximum ranges over all connected components of G .

We prove the claim by induction. The inequality is true for K_1 , whose diameter is 0. For the inductive step, the interesting case is when $G = (V, E)$ is connected, since otherwise we can assume that the claim has been shown for each connected component and we are done. Let $u \in V$ be such that $td(G) = 1 + td(G \setminus u)$. Consider two vertices $v_1, v_2 \in V \setminus \{u\}$ which are at maximum distance in G . If v_1, v_2 are in the same connected component of $G' := G \setminus u$, then $d_G(v_1, v_2) \leq d_{G'}(v_1, v_2) \leq d(G') \leq 2^{td(G')+1} - 2 \leq 2^{td(G)+1} - 2$, where we have used the inductive hypothesis on G' . So, suppose that v_1, v_2 are in different connected components of G' . It must be the case that u has a neighbor in the component of v_1 (call it v'_1) and in the component of v_2 (call it v'_2), because G is connected. We have

$$d_G(v_1, v_2) \leq d_G(v_1, v'_1) + 2 + d_G(v_2, v'_2) \leq d_{G'}(v_1, v'_1) + 2 + d_{G'}(v_2, v'_2) \leq 2d(G') + 2 \leq 2 \cdot 2^{\text{td}(G')+1} - 2 = 2^{\text{td}(G)+1} - 2. \blacktriangleleft$$

Theorem 7. *Unweighted (k, r) -CENTER can be solved in time $O^*(2^{O(\text{td}^2)})$.*

Proof. The main observation is that we can assume that $r \leq d(G)$, because otherwise the problem is trivial. Hence, by Lemma 30 we have $r \leq 2^{\text{td}+1}$. We can now rely on Lemma 18 to get $\text{tw} \leq \text{td}$, and the algorithm of [10] which runs in time $O^*((2r+1)^{\text{tw}})$ gives the desired running time. \blacktriangleleft

The remainder of this section is devoted to the ETH-based lower bound for tree-depth.

Construction: Given an instance ϕ of 3-SAT on n variables and m clauses, where we can assume that $m = O(n)$ (by the Sparsification Lemma, see [26]), we will create an instance $[G = (V, E), k]$ of the unweighted (k, r) -CENTER problem, where $k = \sqrt{n}$ and $r = 3 \cdot c\sqrt{n}$ for an appropriate constant c . (To simplify notation, we assume without loss of generality that \sqrt{n} is an integer).

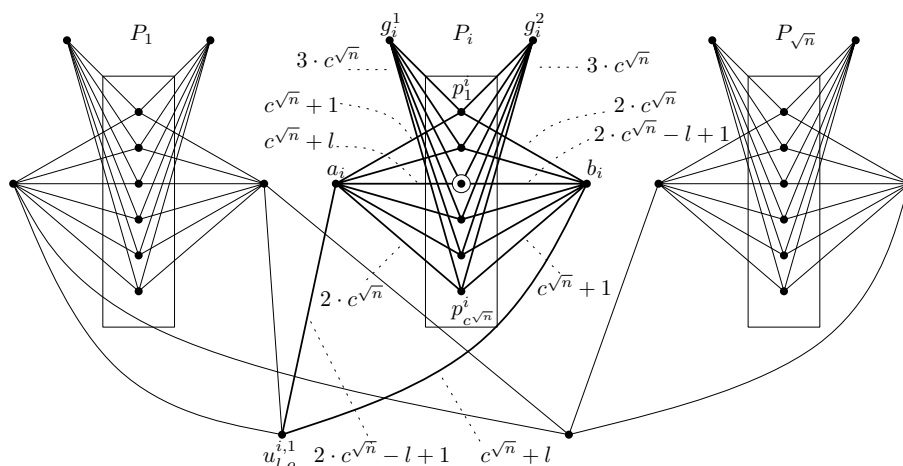
We first group the clauses of ϕ into \sqrt{n} equal-sized groups $F_1, \dots, F_{\sqrt{n}}$. As a result, each group involves $O(\sqrt{n})$ variables, so there are $2^{O(\sqrt{n})}$ possible assignments to the variables of each group. Select c appropriately so that each group F_i has at most $c\sqrt{n}$ possible partial assignments ϕ_i^j for the variables of clauses in F_i .

We then create for each $i \in \{1, \dots, \sqrt{n}\}$, a set P_i of at most $c\sqrt{n}$ vertices $p_1^i, \dots, p_{c\sqrt{n}}^i$, such that each vertex of P_i represents a partial assignment to the variables of F_i that satisfies all clauses of F_i . We add two *guard* vertices g_i^1, g_i^2 , attaching them to all vertices of P_i by paths of length $r = 3 \cdot c\sqrt{n}$. Next, for each $i \in \{1, \dots, \sqrt{n}\}$, we create another pair of vertices a_i, b_i , connecting a_i to each vertex p_l^i by a path of length $c\sqrt{n} + l$, while b_i is connected to each vertex p_l^i by a path of length $2 \cdot c\sqrt{n} - l + 1$. Now each P_i contains all vertices a_i, b_i, g_i^1, g_i^2 and each $p_l^i, \forall l \in \{1, \dots, c\sqrt{n}\}$.

Finally, for any two *conflicting* partial assignments ϕ_l^i, ϕ_o^j , with $l, o \in \{1, c\sqrt{n}\}$ and $i, j \in \{1, \sqrt{n}\}$, i.e. two partial assignments that assign conflicting values to at least one variable, we create a vertex $u_{l,o}^{i,j}$ that we connect to vertices a_i, b_i and a_j, b_j : if $p_l^i \in P_i$ is the vertex corresponding to ϕ_l^i and $p_o^j \in P_j$ is the vertex corresponding to ϕ_o^j , then vertex $u_{l,o}^{i,j}$ is connected to a_i by a path of length $2 \cdot c\sqrt{n} - l + 1$ and to b_i by a path of length $c\sqrt{n} + l$, as well as to a_j by a path of length $2 \cdot c\sqrt{n} - o + 1$ and b_j by a path of length $c\sqrt{n} + o$. See Figure 11 for an illustration.

► **Lemma 31.** *If ϕ has a satisfying assignment, then there exists a (k, r) -center in G of size $k = \sqrt{n}$ and $r = 3 \cdot c\sqrt{n}$.*

Proof. Consider the satisfying assignment for ϕ and let $\phi_{l_i}^i$, with $l_i \in \{1, \dots, c\sqrt{n}\}$ and $i \in \{1, \dots, \sqrt{n}\}$, be the restriction of that assignment for all variables appearing in clauses of group F_i . We claim the set K , consisting of all vertices $p_{l_i}^i$ corresponding to $\phi_{l_i}^i$, is a (k, r) -center for G with $k = \sqrt{n}$ and $r = 3 \cdot c\sqrt{n}$. Since one vertex is selected from each P_i in G , all the guards g_i^1, g_i^2 and vertices a_i, b_i are within distance $3 \cdot c\sqrt{n}$ from selected vertex $p_{l_i}^i$. For vertices $u_{l_i, l_j}^{i,j}$, observe that selected vertex $p_{l_i}^i$ is at distance $c\sqrt{n} + l_i + 2 \cdot c\sqrt{n} - l_i + 1 = 3 \cdot c\sqrt{n} + 1$ from $u_{l_i, l_j}^{i,j}$, through either a_i or b_i , while vertex $p_{l_j}^j$ is at distance $c\sqrt{n} + l_j + 2 \cdot c\sqrt{n} - l_j + 1 = 3 \cdot c\sqrt{n} + 1$ from $u_{l_i, l_j}^{i,j}$, through either a_j or b_j , only if the corresponding partial assignments $\phi_{l_i}^i$ and $\phi_{l_j}^j$ are conflicting. As $\phi_{l_i}^i, \phi_{l_j}^j$ are parts of a satisfying assignment for ϕ , however, this will not be the case and at least one path from $u_{l_i, l_j}^{i,j}$ to either $p_{l_i}^i$, or $p_{l_j}^j$, will be of length $\leq 3 \cdot c\sqrt{n}$. \blacktriangleleft



■ **Figure 11** A general picture of graph G . Note all straight lines denote paths of length equal to the number shown by dotted lines, while the circled vertex is p_i^i .

► **Lemma 32.** *If there exists a (k, r) -center in G of size $k = \sqrt{n}$ and $r = 3 \cdot c\sqrt{n}$, then ϕ has a satisfying assignment.*

Proof. Let $K \subset V$ be the (k, r) -center in G , with $r = 3 \cdot c\sqrt{n}$ and $k = \sqrt{n}$. As $|K| = k = \sqrt{n}$ and all guard vertices $g_i^1, g_i^2, \forall i \in \{1, \dots, \sqrt{n}\}$, must be within distance at most $3 \cdot c\sqrt{n}$ from some vertex selected in K , the set K must contain exactly one vertex from each P_i , or $K = \{p_{l_1}^1, \dots, p_{l_{\sqrt{n}}}^{\sqrt{n}}\}$, for some $l_i \in \{1, \dots, c\sqrt{n}\}$ and for each $i \in \{1, \dots, \sqrt{n}\}$. We set the assignment for ϕ to consist of all partial assignments $\phi_{l_i}^i$, with $i \in \{1, \dots, \sqrt{n}\}$, corresponding to vertices $p_{l_i}^i \in K$ and claim that this is a valid assignment that satisfies ϕ . It is not hard to see that, if the assignment is valid, then it satisfies the clause, as we have only listed partial assignments that satisfy all clauses of each group. To see that the assignment does not assign conflicting values to any variable, suppose for contradiction that there are two conflicting partial assignments $\phi_{l_i}^i, \phi_{l_j}^j$ and a vertex $u_{l_i, l_j}^{i, j} \in G$ with paths to $a_i, b_i \in P_i$ and $a_j, b_j \in P_j$, where we have $p_{l_i}^i, p_{l_j}^j \in K$. The distance from $u_{l_i, l_j}^{i, j}$ to $p_{l_i}^i$ is $2 \cdot c\sqrt{n} - l_i + 1 + c\sqrt{n} + l_i > 3 \cdot c\sqrt{n}$, via either a_i or b_i , while its distance to $p_{l_j}^j$ is also $2 \cdot c\sqrt{n} - l_j + 1 + c\sqrt{n} + l_j > 3 \cdot c\sqrt{n}$, via either a_j, b_j , meaning $u_{l_i, l_j}^{i, j}$ is not covered by K , giving a contradiction. ◀

► **Lemma 33.** *The tree-depth of G is $4\sqrt{n} + \lceil \log(3 \cdot c\sqrt{n}) \rceil + 1 = O(\sqrt{n})$.*

Proof. Again we use the definition of tree-depth used in Lemma 30. Consider the graph G after removal of all guard vertices g_i^1, g_i^2 and all $a_i, b_i, \forall i \in [1, \sqrt{n}]$. The graph now consists of a number of sub-divided stars, centered either on some vertex $u_{l,o}^{i, j}$, or some $p_{l_i}^i$, while the maximum distance from each such center to a leaf vertex in the star is $3 \cdot c\sqrt{n} - 1$, for a path connecting $p_{l_i}^i$ to a guard g_i^1, g_i^2 , omitting the final edge due to removal of g_i^1, g_i^2 . The claim follows since paths of length n have treedepth exactly $\lceil \log(n + 1) \rceil$ (this can be shown by repeatedly removing the middle vertex of a path). By the definition of treedepth, after removal of $4\sqrt{n}$ vertices from G , the maximum treedepth of each resulting disconnected component is $\lceil \log(3 \cdot c\sqrt{n}) \rceil = \lceil \sqrt{n} \cdot \log(c) + \log(3) \rceil$. ◀

Theorem 8. *If (k, r) -CENTER can be solved in $2^{o(td^2)} \cdot n^{O(1)}$ time, then 3-SAT can be solved in $2^{o(n)}$ time.*

Proof. Suppose there is an algorithm for the (k, r) -CENTER problem with running time $2^{o(td^2)}$. Given an instance ϕ of 3-SAT, we use the above construction to create an instance $[G, k, r]$ of (k, r) -CENTER, with $k = \sqrt{n}$ and $r = 3 \cdot c\sqrt{n}$, in time $O(\sqrt{n} \cdot c\sqrt{n} + c^2\sqrt{n})$. As, by Lemma 33, we have $td(G) = O(\sqrt{n})$, using the supposed algorithm for (k, r) -CENTER, we can decide whether ϕ has a satisfying assignment in time $2^{o(td^2)} \cdot n^{O(1)} = 2^{o(n)}$. ◀

B.4 From Section 5, Treewidth: FPT approximation scheme

Lemma 9. *An edge-weighted graph $G = (V, E)$ admits a (k, r) -center if and only if it admits a valid distance-labeling function $dl: V \rightarrow \{0, \dots, r\}$ with cost k .*

Proof. For one direction, if there is a (k, r) -center K , we construct a function dl by assigning as value to each vertex $v \in V$ the distance from the closest vertex of K to v . This function is valid and has a cost of $|K| = k$. For the converse direction, if such a function exists, we set $K = dl^{-1}(0)$. It is now not hard to verify that all vertices v are at distance at most $dl(v)$ from a vertex of K . ◀

Exact-algorithm: We formulate an exact algorithm which, given an edge-weighted graph $G = (V, E)$, finds the minimum cost of a distance-labeling function for G . By Lemma 9, this is equivalent to (k, r) -CENTER.

Theorem 10. *There is an algorithm which, given an edge-weighted graph $G = (V, E)$ and $r \in \mathbb{N}^+$, computes the minimum cost of any valid distance labeling of G in time $O^*(r^{O(tw)})$.*

Proof. The algorithm uses standard techniques, so we sketch some of the details. The idea is that for any bag B_t of a tree decomposition of G we maintain a table $D_t: ((B_t \rightarrow \{0, \dots, r\}) \times (B_t \rightarrow \{0, 1\})) \rightarrow \{0, \dots, n\} \cup \{\infty\}$. Let B_t^\downarrow denote the vertices of G contained in bags in the sub-tree rooted at B_t . Then, the meaning of D_t is that for each distance labeling $dl_t: B_t \rightarrow \{0, \dots, r\}$ of the vertices of B_t and for each subset $S \subseteq B_t$ it contains the minimum cost of any distance labeling of $G[B_t^\downarrow]$ which satisfies all vertices of B_t^\downarrow , except perhaps those of $B_t \setminus S$, and which agrees with dl_t on B_t .

Using this table we can now formulate a DP algorithm as follows:

- For a Leaf node B_t that only contains a single vertex u , we set $D_t(dl, \{u\}) = 1$, if $dl(u) = 0$ and $D_t(dl, \emptyset) = \infty$, otherwise. We set $D_t(dl, \emptyset) = 0$ for any other distance labeling of u .
- For a Forget node B_t that forgets a vertex u and has a child node $B_{t'}$, we set $D_t(dl, S) = \min_{dl'} D_{t'}(dl', S \cup \{u\})$, where the minimum ranges over all labeling functions dl' on $B_{t'}$ that agree with dl on $B_t = B_{t'} \setminus \{u\}$. In other words, we only retain partial solutions which have already satisfied the vertex u that we forget.
- For an Introduce node B_t that introduces a vertex u and has a child node $B_{t'}$ (so $B_t = B_{t'} \cup \{u\}$), we consider every labeling dl' of $B_{t'}$ and every $S' \subseteq B_{t'}$ such that $D_{t'}(dl', S') \neq \infty$. For each such dl', S' , and for each $i \in \{0, \dots, r\}$, we construct a dl function for B_t which agrees with dl' on $B_{t'}$ and sets $dl(u) = i$. We compute S to be the set that contains S' as well as all vertices of $B_{t'}$ satisfied by $dl(u)$, and u if it is satisfied by a vertex of $B_{t'}$ or $dl(u) = 0$. More precisely, we add $v \in B_{t'}$ to S if $dl(v) \geq dl(u) + w((u, v))$, and we add u to S if $dl(u) = 0$, or if there exists $v \in B_{t'}$ for which $dl(u) \geq dl(v) + w((v, u))$. We set $D_t(dl, S) = D_{t'}(dl', S') + 1$ if $dl(u) = 0$, and $D_t(dl, S) = D_{t'}(dl', S') + 1$ otherwise. All other entries of D_t are set to ∞ .

- For a Join node B_t with two children B_{t_1}, B_{t_2} , for each labeling function dl on B_t and each $S \subseteq B_t$ we set $D_t(dl, S) = \min_{S_1 \subseteq S} D_{t_1}(dl, S_1) + D_{t_2}(dl, S \setminus S_1) - |dl^{-1}(0) \cap B_t|$. Note that we subtract $|dl^{-1}(0) \cap B_t|$ to avoid double-counting the vertices of B_t . ◀

Lemma 11. *Let $G = (V, E)$ be an edge-weighted graph, \mathcal{T} a tree decomposition of G , and $u, v \in V$ two vertices that appear together in a bag of \mathcal{T} . Let G' be the graph obtained from G by adding (u, v) to E (if it does not already exist) and setting $w((u, v)) = d_G(u, v)$. Then \mathcal{T} is a valid decomposition of G' , and $\forall k, r$, G' admits a (k, r) -center if and only if G does.*

Proof. To see that G' admits a (k, r) -center if and only if G does, observe that adding the weighted edge (u, v) did not change the distances between any two vertices. To see that \mathcal{T} remains a valid decomposition we recall that u, v appear together in a bag of \mathcal{T} . ◀

Lemma 12. *If for a weighted graph $G = (V, E)$ and any $k, r, \delta, \epsilon > 0$, there exists a valid δ -labeling function with cost k , then there exists a $(k, (1 + \epsilon)^2 r)$ -center for G .*

Proof. Again we set $K = dl_\delta^{-1}(0)$, where dl_δ is a valid δ -labeling function. Recall that dl_δ assigns value $(1 + \delta)^i$, for $i \in \mathbb{N}$ to all vertices of $V \setminus K$. We prove by induction on i , that a vertex u for which $dl_\delta(u) = (1 + \delta)^i$ is at distance at most $(1 + \epsilon)dl_\delta(u)$ from K . First, if $dl_\delta(u) = (1 + \delta)$ and u is satisfied, then u has a neighbor v with $dl_\delta(u) \geq dl_\delta(v) + \frac{w((v, u))}{1 + \epsilon}$. Because $w((v, u)) > 0$, we must have $v \in K$, so we conclude that $d(K, u) \leq (1 + \epsilon)dl_\delta(u)$. Similarly, if we have $dl_\delta(u) = (1 + \delta)^{i+1}$ and u is satisfied, there must exist $v \in N(u)$ with $dl_\delta(u) \geq dl_\delta(v) + \frac{w((v, u))}{1 + \epsilon}$, so $dl_\delta(v) \leq (1 + \delta)^i$. By the inductive hypothesis, it is $d(K, v) \leq (1 + \epsilon)dl_\delta(v)$. So $d(K, u) \leq w((v, u)) + d(K, v) \leq (1 + \epsilon) \left(dl_\delta(v) + \frac{w((v, u))}{1 + \epsilon} \right) \leq (1 + \epsilon)dl_\delta(u)$. Since for all $u \in V$ we have $dl_\delta(u) \leq (1 + \epsilon)r$, all vertices are at distance at most $(1 + \epsilon)^2 r$ from K . ◀

Theorem 13. *There is an algorithm which, given a weighted instance of (k, r) -CENTER, $[G = (V, E), k, r]$, a tree decomposition of G of width tw and a parameter $\epsilon > 0$, runs in time $O^*((tw/\epsilon)^{O(tw)})$ and either returns a $(k, (1 + \epsilon))$ -center of G , or correctly concludes that G has no (k, r) -center.*

Proof. Our algorithm will follow along the same lines as the algorithm of Theorem 10, with the major difference that the labeling functions we consider are δ -labeling functions, for a value of δ that we define further below. More precisely, if $\Sigma_r := \{0\} \cup \{(1 + \delta)^i \mid i \in \mathbb{N}, (1 + \delta)^i \leq (1 + \epsilon)r\}$, then for each bag B_t of \mathcal{T} we define a table as a function $D_t^\delta : ((B_t \rightarrow \Sigma_r) \times (B_t \rightarrow \{0, 1\})) \rightarrow \{0, \dots, n\} \cup \{\infty\}$. Let B_t^\downarrow denote the vertices of G contained in bags in the sub-tree rooted at B_t . The meaning of D_t^δ here is similar to that of D_t in the algorithm of Theorem 10. For each δ -distance labeling $dl_\delta : B_t \rightarrow \Sigma_r$, and for each $S \subseteq B_t$, D_t^δ tells us what is the minimum cost of any δ -distance labeling of B_t^\downarrow which satisfies all vertices B_t^\downarrow , except perhaps those of $B_t \setminus S$, and which agrees with dl_δ on B_t .

Before we proceed, we pre-process the input using the result of [7]: *There is an algorithm which, given a tree decomposition of width w of a graph on n nodes, produces a decomposition of the same graph with width at most $3w + 2$ and height $O(\log n)$ in polynomial time and Lemma 11. In particular, we construct a tree decomposition \mathcal{T}' of width at most $3tw + 2$ and height at most $H \leq c \log n$ for some constant c , and then for any two vertices u, v which appear together in a bag of \mathcal{T}' , we add the edge (u, v) to G , as in Lemma 11. We define the height of a bag B_t of the decomposition recursively as follows: the height of a Leaf bag is 1,*

for any other bag B_t its height is 1 plus the maximum of the heights of its children. Clearly, under this definition the root bag has height H and all other bags have height $< H$. We now set $\delta = \frac{\epsilon}{2H} = \Theta\left(\frac{\epsilon}{\log n}\right)$. Observe that because of this setting we have for all $h \leq H$ that $(1 + \delta)^h \leq (1 + \frac{\epsilon}{2H})^H \leq e^{\epsilon/2} \leq (1 + \epsilon)$ for sufficiently small ϵ (e.g. it suffices to assume without loss of generality $\epsilon < 1/4$). Our goal will be to return a $(k, (1 + \epsilon)^2 r)$ -center, if a (k, r) -center exists by producing a δ -labeling and invoking Lemma 12. The approximation ratio can then be brought down to $(1 + \epsilon)$ by adjusting ϵ appropriately.

Our algorithm now follows the algorithm of Theorem 10, with the only difference that the satisfaction of a vertex follows the definition of ϵ -satisfaction (that is, we effectively divide edge weights by $(1 + \epsilon)$). Specifically, we have the following:

- For a Leaf node B_t that only contains a single vertex u , we set $D_t^\delta(\text{dl}_\delta, \{u\}) = 1$, if $\text{dl}_\delta(u) = 0$ and $D_t^\delta(\text{dl}_\delta, \{u\}) = \infty$, otherwise. We set $D_t^\delta(\text{dl}_\delta, \emptyset) = 0$ for any other distance labeling of u .
- For a Forget node B_t that forgets a vertex u and has a child node $B_{t'}$, we set $D_t^\delta(\text{dl}_\delta, S) = \min_{\text{dl}'_\delta} D_{t'}^\delta(\text{dl}'_\delta, S \cup \{u\})$ where the minimum ranges over all labeling functions dl'_δ on $B_{t'}$ that agree with dl_δ on $B_t = B_{t'} \setminus \{u\}$. In other words, we only retain partial solutions which have already satisfied the vertex u that we forget.
- For an Introduce node B_t that introduces a vertex u and has a child node $B_{t'}$ (so $B_t = B_{t'} \cup \{u\}$), we consider every labeling dl'_δ of $B_{t'}$ and every $S' \subseteq B_{t'}$ such that $D_{t'}^\delta(\text{dl}'_\delta, S') \neq \infty$. For each such dl'_δ, S' , and for each $i \in \Sigma_r$ we construct a dl_δ function for B_t which agrees with dl'_δ on $B_{t'}$ and sets $\text{dl}_\delta(u) = i$. We compute S to be the set that contains S' as well as all vertices of $B_{t'}$ satisfied by $\text{dl}(u)$, and u if it is satisfied by a vertex of $B_{t'}$ or $\text{dl}_\delta(u) = 0$. More precisely, we add $v \in B_{t'}$ in S if $\text{dl}_\delta(v) \geq \text{dl}_\delta(u) + \frac{w((u,v))}{1+\epsilon}$, and we add u to S if $\text{dl}_\delta(u) = 0$, or if there exists $v \in B_{t'}$ for which $\text{dl}_\delta(u) \geq \text{dl}_\delta(v) + \frac{w((v,u))}{1+\epsilon}$. We set $D_t^\delta(\text{dl}_\delta, S) = D_{t'}^\delta(\text{dl}'_\delta, S') + 1$ if $\text{dl}(u) = 0$, and $D_t^\delta(\text{dl}_\delta, S) = D_{t'}^\delta(\text{dl}'_\delta, S') + 1$ otherwise. All other entries of D_t are set to ∞ .
- For a Join node B_t with two children B_{t_1}, B_{t_2} , for each labeling function dl_δ on B_t and each $S \subseteq B_t$, we set $D_t(\text{dl}_\delta, S) = \min_{S_1, S_2 \subseteq S: S_1 \cup S_2 = S} D_{t_1}(\text{dl}_\delta, S_1) + D_{t_2}(\text{dl}_\delta, S_2) - |\text{dl}_\delta^{-1}(0) \cap B_t|$.

To establish correctness of the above algorithm we proceed in two directions. First, we show that for any bag B_t we have $D_t^\delta(\text{dl}_\delta, S) \leq i$, if and only if there exists a δ -labeling of B_t^\downarrow which satisfies all vertices of B_t^\downarrow , except perhaps those of $B_t \setminus S$, and has cost i . This can be shown by a standard inductive argument: the statement is clearly true on Leaf bags, and it can be shown to be true on other bags if we assume it to be true on their children. As a result, by examining the table of the root bag at the end of the algorithm's execution, we can correctly compute the minimum cost of a valid δ -labeling. If that cost is at most k , by Lemma 12 we obtain a $(k, (1 + \epsilon)^2 r)$ -center of G .

It is now more interesting to establish the converse direction: we would like to show that if a (k, r) -center exists then there exists a δ -labeling of G , which will therefore be found by the algorithm. The reason that this direction is more difficult is that the converse of Lemma 12 does not hold for any δ ; indeed if δ is too small it could be the case that a (k, r) -center exists, but the graph does not admit a δ -labeling. We will thus need to prove that the δ we have selected is sufficient. In the remainder, suppose there exists a $\text{dl} : V \rightarrow \{0, \dots, r\}$ which encodes a (k, r) -center of G .

We will establish by induction the following property: for any bag B_t of the decomposition such that the height of B_t is h , let dl_t denote the restriction of dl to B_t and S the vertices of B_t which are satisfied in $G[B_t^\downarrow]$ by dl . We will show that there always exists $\text{dl}_\delta : B_t \rightarrow \Sigma_r$,

$S_\delta \supseteq S$ such that $D_t^\delta(\text{dl}_\delta, S_\delta) \leq |\text{dl}^{-1}(0) \cap B_t^\downarrow|$ and for all $u \in B_t$ we have $\frac{\text{dl}(u)}{(1+\delta)^h} \leq \text{dl}_\delta(u) \leq (1+\delta)^h \text{dl}(u)$.

We observe that the property holds for Leaf bags $B_t = \{u\}$, because the algorithm considers both the case where $u \in K$ (in which $\text{dl}(u) = \text{dl}_\delta(u) = 0$), as well as every possible labeling of u with an integer power of $(1+\delta)$, hence it considers a labeling such that $\text{dl}(u), \text{dl}_\delta(u)$ are at most a multiplicative factor $(1+\delta)$ apart. The property can easily be shown inductively for Forget and Join bags, if we assume the property for their children, because whenever $\text{dl}_\delta(u) \in [\frac{\text{dl}(u)}{(1+\delta)^h}, (1+\delta)^h \text{dl}(u)]$, this implies $\text{dl}_\delta(u) \in [\frac{\text{dl}(u)}{(1+\delta)^{h+1}}, (1+\delta)^{h+1} \text{dl}(u)]$. Hence, the interesting case is Introduce bags.

Consider an Introduce bag B_t with child $B_{t'}$, so that $B_t = B_{t'} \cup \{u\}$. Let $S \subseteq B_t$ be the set of vertices satisfied by dl in $G[B_t^\downarrow]$, and similarly, let $S' \subseteq B_{t'}$ be the corresponding set in $B_{t'}$. Clearly, $S' \subseteq S$. We now claim that at least one of the following three must be true: (i) $\text{dl}(u) = 0$, (ii) $S = S' \cup \{u\}$, (iii) $u \notin S$. To see this, suppose for contradiction that $\text{dl}(u) \neq 0$ and S contains all of S' , u and some vertex $v_1 \in B_t \setminus S'$. Then $\text{dl}(v_1) \geq \text{dl}(u) + w((u, v_1))$, because $v_1 \in S$ but $v_1 \notin S'$. Furthermore, there exists $v_2 \in B_{t'}$ such that $\text{dl}(u) \geq \text{dl}(v_2) + w((v_2, u))$, because $u \in S$ and $\text{dl}(u) \neq 0$. Hence, $\text{dl}(v_1) \geq \text{dl}(v_2) + w((v_2, u)) + w((u, v_1)) \geq \text{dl}(v_2) + w((v_2, v_1))$, where the last inequality follows because we have used Lemma 11 to take the metric closure of $B_{t'}$. But the last inequality implies $v_1 \in S'$, contradiction.

We therefore need to establish that, for each of the three cases above, the algorithm will produce an entry $D_t^\delta(\text{dl}_\delta, S_\delta)$ with $S \subseteq S_\delta$ and $\text{dl}_\delta(u)$ which is at most a factor of $(1+\delta)^h$ apart from $\text{dl}(u)$. Assume by the inductive hypothesis that $D_{t'}^\delta(\text{dl}'_\delta, S'_\delta) \leq i$, for some $S'_\delta \supseteq S'$, dl'_δ which has $\forall v \in B_{t'}, \text{dl}'_\delta(v) \in [\frac{\text{dl}(v)}{(1+\delta)^{h-1}}, (1+\delta)^{h-1} \text{dl}(v)]$, and for $i = |\text{dl}^{-1}(0) \cap B_{t'}^\downarrow|$.

We have:

- (i) If $\text{dl}(u) = 0$, the algorithm will consider a dl_δ which sets $\text{dl}_\delta(u) = 0$ and agrees with dl'_δ on $B_{t'}$. From the entry $D_{t'}^\delta(\text{dl}'_\delta, S'_\delta) \leq i$ it will construct an entry $D_t(\text{dl}_\delta, S_\delta) \leq i+1$, with $S \subseteq S_\delta$, because for each $v \in S \setminus S'$, we have $\text{dl}(v) \geq w((u, v))$, therefore, $\text{dl}_\delta(v) \geq \frac{\text{dl}(v)}{(1+\delta)^{h-1}} \geq \frac{w((u, v))}{1+\epsilon}$, where we have used the fact that $(1+\delta)^h \leq 1+\epsilon$.
- (ii) Here we assume $\text{dl}(u) \neq 0$ and $u \in S$. There exists therefore $v \in B_{t'}$ with $\text{dl}(u) \geq \text{dl}(v) + w((v, u))$. Let $r := (1+\delta)^{h-1} \text{dl}(u)$. The algorithm will consider the function dl_δ which agrees with dl'_δ on $B_{t'}$ and sets $\text{dl}_\delta(u) = (1+\delta)^{\lceil \log_{(1+\delta)} r \rceil}$. We have $\text{dl}_\delta(u) \geq (1+\delta)^{h-1} \text{dl}(u) \geq (1+\delta)^{h-1} (\text{dl}(v) + w((v, u))) \geq \text{dl}_\delta(v) + \frac{w((v, u))}{1+\epsilon}$, thus the algorithm will (correctly) add u to S'_δ to obtain $S_\delta \supseteq S$. Furthermore, $(1+\delta)^{h-1} \text{dl}(u) \leq \text{dl}_\delta(u) \leq (1+\delta)^h \text{dl}(u)$.
- (iii) Here the interesting case is if $S \setminus S' \neq \emptyset$. Consider a $v \in S \setminus S'$. We have $\text{dl}(v) \geq \text{dl}(u) + w((u, v))$, since $v \in S \setminus S'$. Let $r := \frac{\text{dl}(v)}{(1+\delta)^h}$. The algorithm will consider the dl_δ which agrees with dl'_δ on $B_{t'}$ and sets $\text{dl}_\delta(u) = (1+\delta)^{\lceil \log_{(1+\delta)} r \rceil}$. We have $\text{dl}_\delta(v) \geq \frac{\text{dl}(v)}{(1+\delta)^{h-1}} \geq \frac{\text{dl}(u)}{(1+\delta)^{h-1}} + \frac{w((u, v))}{1+\epsilon} \geq \text{dl}_\delta(u) + \frac{w((u, v))}{1+\epsilon}$. Hence, the algorithm will extend S'_δ by adding to it all elements of $S \setminus S'$. Furthermore, $(1+\delta)^{-h+1} \text{dl}(u) \geq \text{dl}_\delta(u) \geq (1+\delta)^{-h} \text{dl}(u)$.

We can therefore conclude that whenever a (k, r) -center exists, we will be able to recover from the root bag of the algorithm's DP table a $(k, (1+\epsilon)^2 r)$ -center with the same or lower cost. In particular, by the property we established above, if B_r is the root bag, there exists dl_δ such that $D_r^\delta(\text{dl}_\delta, B_r) \leq k$, and for all $u \in B_r$ we have $\text{dl}_\delta(u) \leq (1+\delta)^H \text{dl}(u) \leq (1+\epsilon)r$.

Finally, let us consider the algorithm's running time. Observe that $|\Sigma_r| = O(\log_{(1+\delta)} r) = O(\frac{\log r}{\log(1+\delta)}) = O(\frac{\log r}{\delta})$, where we have used that fact that $\ln(1+\delta) \approx \delta$ for sufficiently small δ (that is, sufficiently large n). By recalling that $\delta = \Theta(\log n/\epsilon)$ we get a running time of $O(\log n/\epsilon)^{O(\text{tw})}$, which implies the promised running time by Lemma 17. \blacktriangleleft

B.5 From Section 6, Clique-width revisited

Lemma 14. *There exists a polynomial algorithm which, for any $\epsilon > 0$, given an instance $I = [G, w, k, r]$ of (k, r) -CENTER, with weight function $w : V \rightarrow \mathbb{N}$, produces an instance $I' = [G, w', k, r']$ on the same graph with weight function $w' : V \rightarrow \mathbb{N}^+$, such that we have the following: for any $\rho \geq 1$, any $(k, \rho r')$ -center of I' is a $(k, \rho r)$ -center of I ; any $(k, \rho r)$ -center of I is a $(k, (1 + \epsilon)\rho r')$ -center of I' .*

Proof. We define a scaling factor $B := \frac{n}{\epsilon}$. We set $w'((u, v)) = B \cdot w((u, v)) + 1$, for all $(u, v) \in E$. We set $r' = B \cdot r$. Suppose that we have a $(k, \rho r)$ -center K for w . We use the same solution for w' and the maximum distance from K to any vertex is at most $B\rho r + n = \rho r' + n$. However, $n \leq \epsilon r'$, hence this solution has maximum distance at most $(1 + \epsilon)\rho r'$. For the converse direction, suppose that we have a solution for the function w' with maximum distance $\rho r'$. We use the same solution and now the cost is at most $\rho r'/B \leq \rho r$, since $w((u, v)) \leq w'((u, v))/B$. ◀

Lemma 15. *Given a (k, r) -CENTER instance $G = (V, E)$ along with a clique-width expression T for G on cw labels, we can in polynomial time obtain a (k, r) -CENTER instance $G' = (V', E')$ with $V \subseteq V'$, and a tree decomposition of G' of width $tw = O(cw)$, with the following property: for all k, r , G has a (k, r) -center if and only if G' has a (k, r) -center.*

Proof. Suppose we are given a clique-width expression of G , represented as a binary tree T , using cw labels. We will first construct a new clique-width expression T' , using $cw + 3$ labels in a way that preserves all (k, r) -centers. The end result will be that T' does not contain any join operations involving large label-sets. We will use this property to construct a tree decomposition of the resulting graph.

Let c be an appropriate small constant (for concreteness, let $c = 10$). We will say that a label $l \in \{1, \dots, cw\}$ is *big* in a node $x \in T$, if the graph described by the sub-expression rooted at x has at least c vertices with label l . We say that l is *newly big* in $x \in T$, if l is big in x but it is not big in any of the children of x in T . Finally, we say that l is *active inside* (respectively *active outside*) in $x \in T$ if l is newly big in x and, furthermore, there exists an arc $e \in E$ that is not yet present in x , whose head (respectively tail) is incident on a vertex with label l in x . In other words, a label is active (inside or outside) in a sub-expression, if it contains many vertices and there is a join operation (of the respective direction) that is applied to its vertices somewhere higher in T .

Suppose that the nodes of T are ordered in some arbitrary way, so that a child is ordered before its parent. Our transformation is the following: as long as there exists $l \in \{1, \dots, cw\}$ and $x \in T$ such that l is active (inside or outside) in x , we select the first such x in the ordering and let l be an active label in x . We insert directly above x in T the following operations: if l is active inside we introduce a vertex s with label $cw + 1$, and if l is active outside we introduce a vertex t with label $cw + 2$; we then add the operations $\eta(cw + 1, l, 0)$, $\eta(l, cw + 2, 0)$, followed by the rename operations $\rho(l, cw + 3)$, $\rho(cw + 1, l)$, $\rho(cw + 2, l)$. In words, this transformation (potentially) adds a “source” s , or a “sink” t to the graph (or both), gives a junk label $cw + 3$ to all vertices that previously had label l in x , and uses s, t as the new representatives of this class of vertices. We repeat this process until no $x \in T$ contains an active label; this finishes in polynomial time because once we eliminate all active labels from a node x , we do not re-visit it. If the original instance had a set of irrelevant vertices $I \subseteq V$, we construct a set of irrelevant vertices I' for G' by adding to I all the s, t vertices created in the above procedure.

Let us first argue that any (k, r) -center of the original instance can be transformed into a (k, r) -center of the new instance. Consider one step of the above procedure applied on a node $x \in T$. The main observation is that such a step does not change the distance between any two vertices $u, v \in V$ in the final graph. Suppose u has label l in x . Any edge incident on u in G that was removed as a result of this operation does not yet appear in x and hence can be replaced by a path of the same cost through s or t . Similarly, any u, v path in the new graph that uses s or t in the new graph, must also use an edge (that is not present in x) which was removed by the operation. Such a path can be replaced by this edge in G .

As a result of the above argument, all distances between vertices of V are preserved after the end of the procedure. Since all vertices of $V' \setminus V$ are irrelevant in G' , any (k, r) -center of G is also a (k, r) -center of G' and vice-versa, as any solution can ignore these vertices and no solution can contain any of them.

To see that G' has small treewidth, observe that we now have a clique-width expression for G' that uses $cw + 3$ labels and has the following property: for any join operation $\eta(a, b, w)$ applied on a node x , there are at most $2c$ vertices with labels a or b in x . To see this, observe that this operation is either an operation that was originally in T , or an operation added during the transformation procedure. In the former case, neither label is big in x , otherwise, there would be a descendant of x where a label is active, and we would have applied the operation. In the latter case, there is a single vertex with label b , and a was newly active, hence it cannot contain more than $2c - 2$ vertices (if it was the result of a rename or union operation on two previously small label-sets).

We can therefore construct a tree decomposition of G' by taking its clique-width expression T' and making a bag for each of its nodes. For the bag B_x that corresponds to node x of T' , we find all labels $l \in \{1, \dots, cw + 3\}$ which are small in x (that is, there are at most $2c$ vertices of this label) and place all vertices with that label in x in B_x . This is a valid tree decomposition because: (i) as argued above all join operations involve two small label-sets, hence any edge of the graph is contained in some bag, (ii) if $u \in B_x$ but $u \notin B_y$, for B_y being the parent of B_x , we know that u also does not appear in any ancestor of B_y , because once a vertex is part of a large label, it remains so and it does not appear in the other child of B_y (if it exists), or its descendants, because if B_y has two children it must correspond to a disjoint union operation in T' . To complete the proof we observe that we have placed at most $2c(cw + 3)$ vertices in each bag. ◀

Theorem 16. *Given graph $G = (V, E)$, along with $k, r \in \mathbb{N}^+$, clique-width expression T for G on cw labels and error parameter $\epsilon > 0$, there exists an algorithm that runs in time $O^*((cw/\epsilon)^{O(cw)})$ and either produces a $(k, (1 + \epsilon)r)$ -center, or correctly concludes that no (k, r) -center exists.*

Proof. Given the (k, r) -CENTER instance, we use Lemma 15 to obtain an instance G' and its tree decomposition of width $O(cw)$. We then invoke Lemma 14 to make all edge weights strictly positive, and then use the algorithm of Section 5. We remark that this algorithm can be easily adjusted to handle the more general case of the problem where some vertices are irrelevant: we simply need to modify the computation on Forget nodes to allow the algorithm to retain a solution that has not satisfied a vertex u if u is irrelevant and the computation on Introduce nodes to disregard a solution that considers u for inclusion in the center-set. ◀