Grundy Distinguishes Treewidth from Pathwidth

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Acknowledgements

This is joint work with:

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Yota Otachi Nagoya U

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Full paper available at: `https://arxiv.org/abs/2008.07425`

What is this talk about?

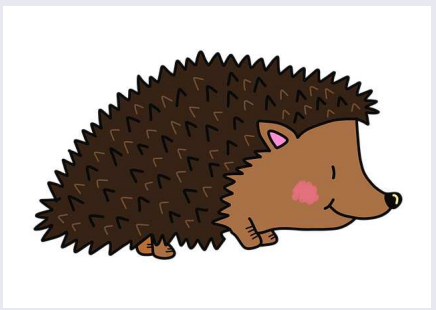
Two ways to look at this work

A talk about structural parameters

- Treewidth
- Pathwidth
- Treedepth, Cliquewidth, ...
- **Price of Generality**
  - Which problems are “easy” for pathwidth but “hard” for treewidth?

A talk about Grundy Coloring

- Well-known optimization problem
- MaxMin variant of Coloring
  - Find a proper coloring that uses the **max** number of colors but the color of no vertex can be decreased.



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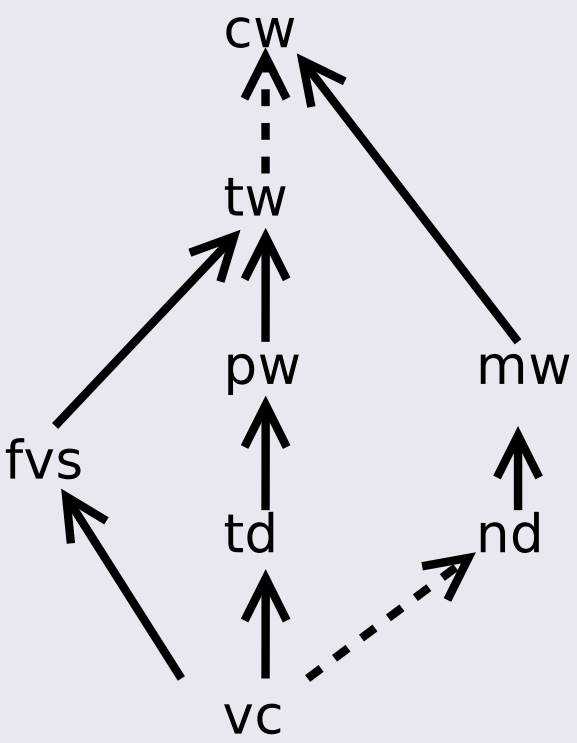
“The fox knows many things, but the hedgehog knows one big thing”, Aesop’s fables

What does the fox say?

Price of Generality – Structural Parameters

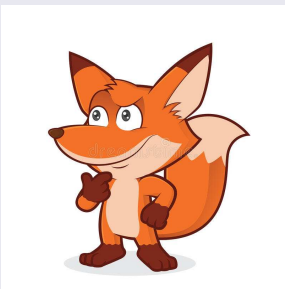


Each problem/parameter pair is typically either:

- FPT: solvable in $f(w)n^{O(1)}$
- XP and W-hard: solvable in $n^{g(w)}$, not FPT
- paraNP-hard: NP-hard for $w = O(1)$

- Tractability propagates "downwards", hardness "upwards"
- Big Picture Question: Which problems do we "lose" when we transition between parameters?

- **Price of Generality**
  - [Fomin, Golovach, Lokshtanov, Saurabh, SODA'09]
  - Showed EDS, MaxCut, Coloring, Hamiltonicity FPT for tw, W-hard for cw.



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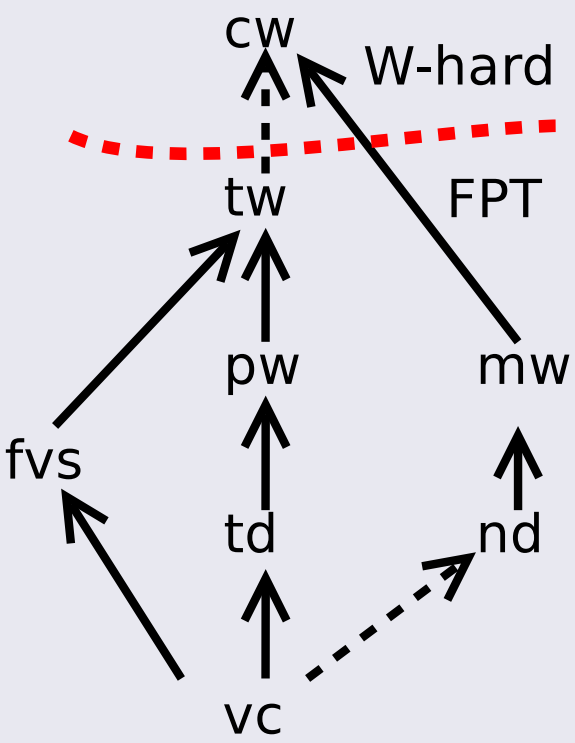

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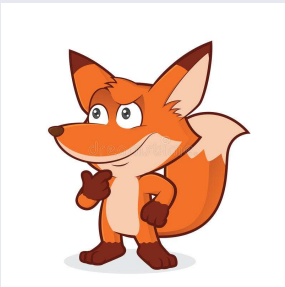

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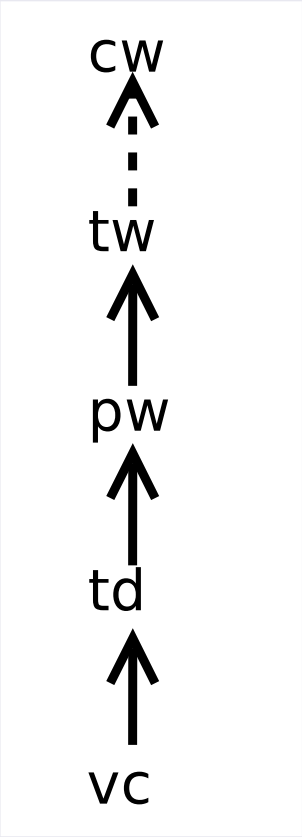
Price of Generality Continued

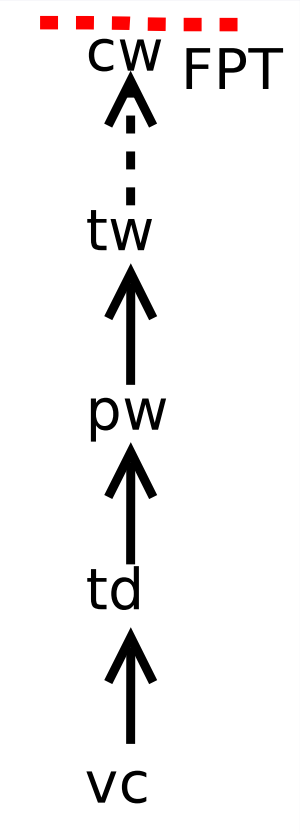


Comments

Price of Generality Examples

Clique-width	
Treewidth	
Pathwidth	
Tree-depth	
Vertex Cover	

Price of Generality Continued



Comments

Price of Generality Examples

	All MSO_1, Dominating Set, Vertex Cover
Clique-width	
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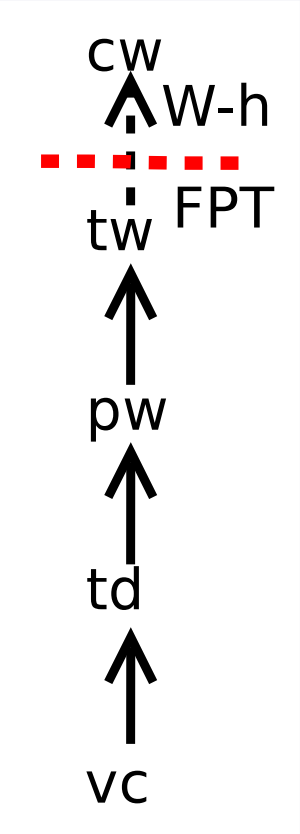
Vertex Cover	

Price of Generality Continued



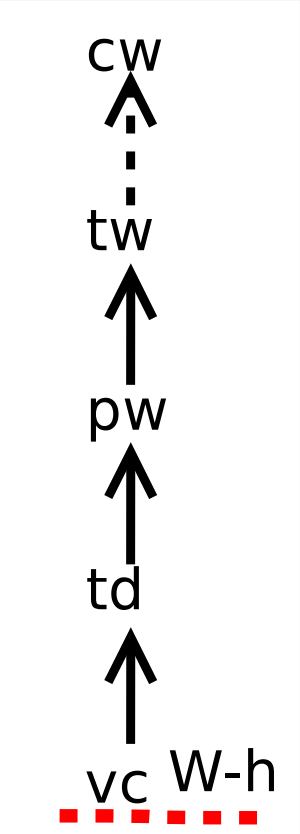
Price of Generality Examples

	All MSO_1, Dominating Set, Vertex Cover
Clique-width	
	Coloring, EDS, SAT, #Matching
Treewidth	
Pathwidth	
Tree-depth	
Vertex Cover	

Comments

- SAT: [Ordyniak, Paulusma, Szeider, TCS '13]
- #Matching: [Curticapean, Marx, SODA '16]

Price of Generality Continued



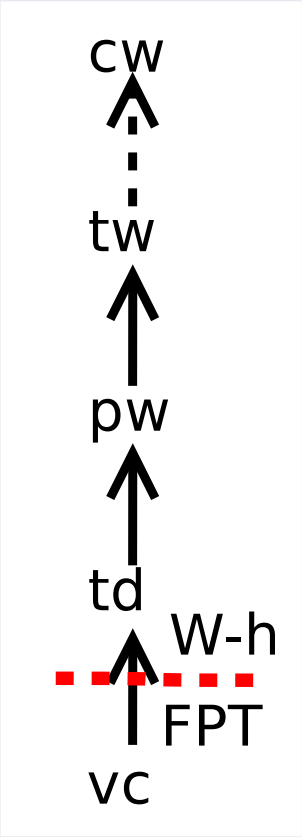
Price of Generality Examples

	All MSO_1, Dominating Set, Vertex Cover
Clique-width	
	Coloring, EDS, SAT, #Matching
Treewidth	
Pathwidth	
Tree-depth	
Vertex Cover	
	List Coloring, r-Dom Set, d-Ind Set

Comments

- List Coloring: [Fellows et al. Inf Comp '11]. First such problem!
- r-DS: [Katsikarelis, L., Paschos, DAM '19]
- Very few problems here!

Price of Generality Continued



Price of Generality Examples

	All MSO_1, Dominating Set, Vertex Cover
Clique-width	
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Treewidth	
Pathwidth	
Tree-depth	
	Capacitated DS/VC, BDD,...
Vertex Cover	
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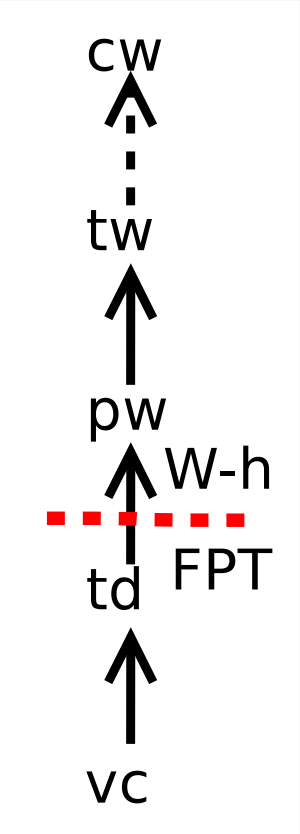
Comments

- Cap VC/DS: [Dom et al. IWPEC 2008]
- **Most problems W[1]-hard for tw are here**!

Price of Generality Continued



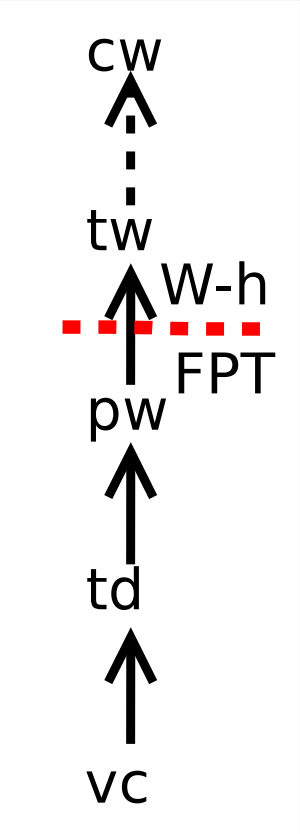
Price of Generality Examples

	All MSO_1, Dominating Set, Vertex Cover
Clique-width	
	Coloring, EDS, SAT, #Matching
Treewidth	
Pathwidth	
	Mixed Chinese Postman, r-DS
Tree-depth	
	Capacitated DS/VC, BDD,...
Vertex Cover	
	List Coloring, r-Dom Set, d-Ind Set

Comments

- MCP: [Gutin, Jones, Wahlström, SIDMA '16]. First of this type!
- Also: Bounded-Length Cut, Geodetic Set, ILP.

Price of Generality Continued



Price of Generality Examples

	All MSO_1, Dominating Set, Vertex Cover
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	Coloring, EDS, SAT, #Matching
Treewidth	
	???
Pathwidth	
	Mixed Chinese Postman, r-DS
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	Capacitated DS/VC, BDD,...
Vertex Cover	
	List Coloring, r-Dom Set, d-Ind Set

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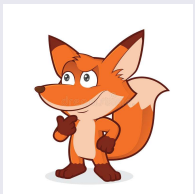
No **natural** problem known??



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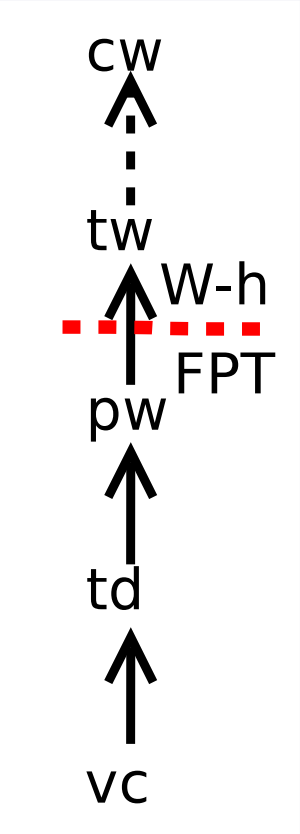

Price of Generality Examples

	All MSO_1, Dominating Set, Vertex Cover
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Treewidth	
	Grundy Coloring!
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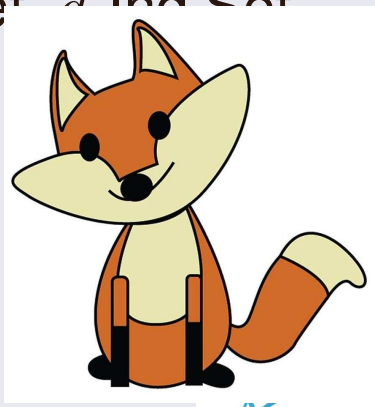
Main result of this talk:

Grundy Coloring is such a problem!



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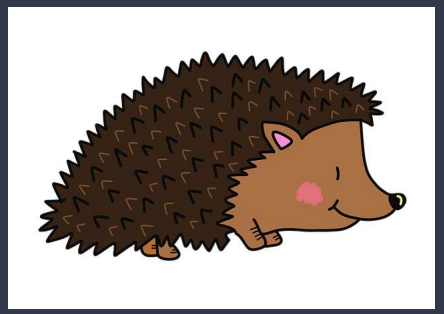
A Lesson from the fox

Price of Generality and Combinatorics

- Sometimes, the reason a problem becomes FPT for a more restricted parameter is more combinatorial than algorithmic.
- Example:
  - Coloring is FPT for tw, W-hard for cw.
  - But algorithm runs in k^{tw}. Is this FPT?
  - Yes! Because in all graphs $\chi(G) \leq tw(G)$.
  - This bound makes all the difference: Coloring is FPT by $cw + k$.
- Example:
  - r-Dom Set is FPT for td, W-hard for pw.
  - Why W-hard for pw? DP runs in $r^{O(pw)}$. But r could be large!
  - Why FPT for td? Graphs of tree-depth t have no simple path of length $> 2^t$, so $r \leq 2^{td}$.
  - Again saved by combinatorial bound on optimal!

Let’s nail this problem!



Grundy Coloring

- Input: Graph $G = (V, E)$ on n vertices
- Repeat n times
  - Select an uncolored vertex u of G
  - Assign u the smallest color that is not currently used in any of its neighbors (**First-Fit**)
- Goal: Order the vertices in such a way that number of colors used is **maximized**.

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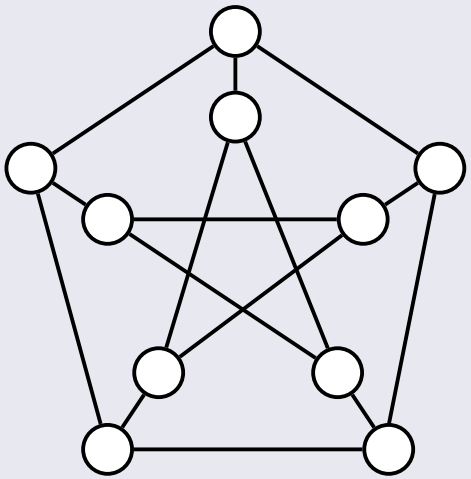

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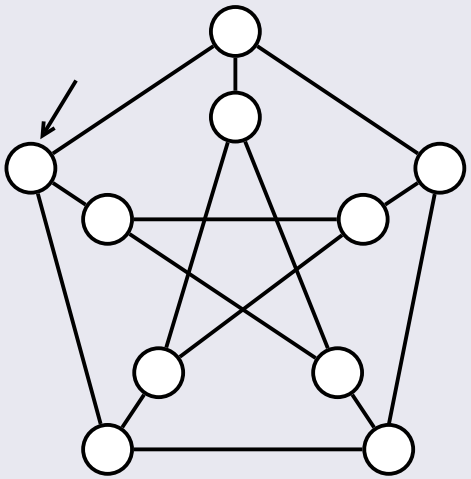

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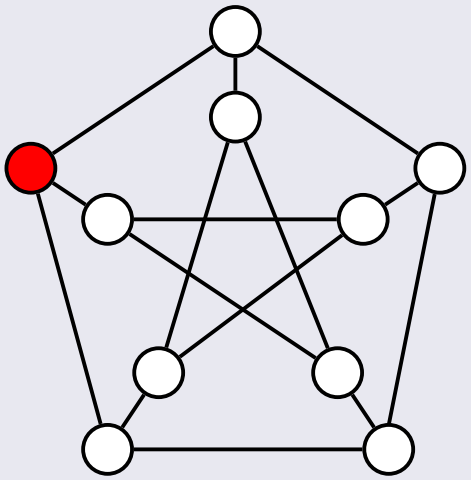

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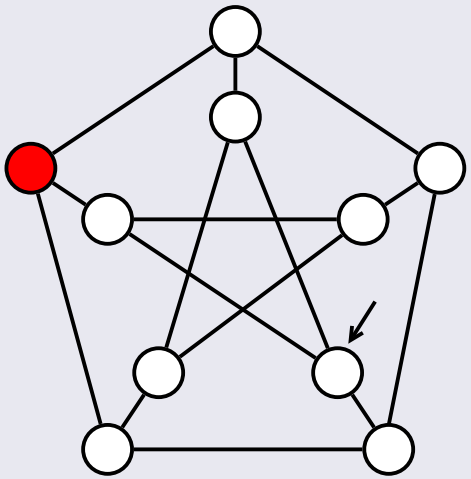

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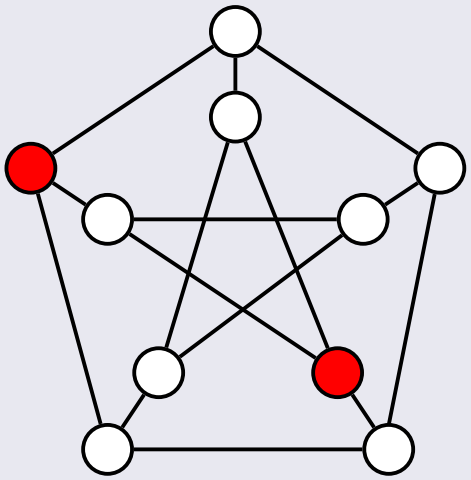

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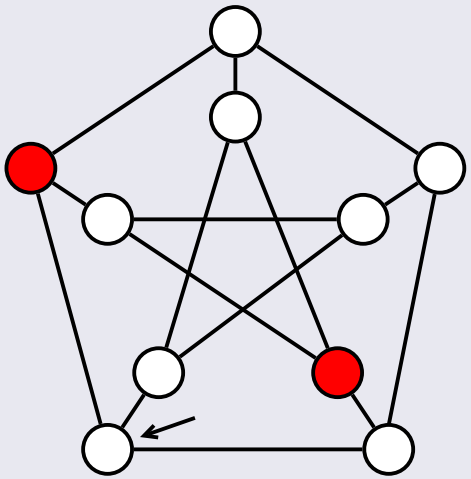

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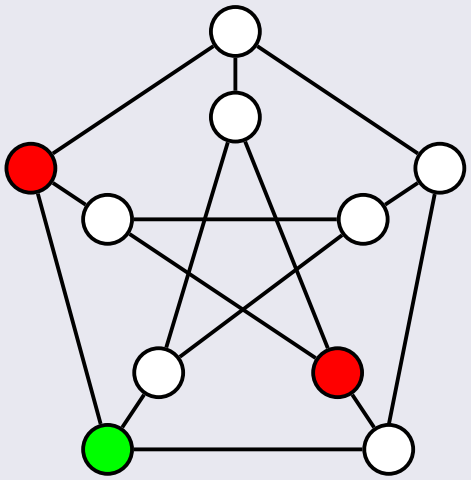

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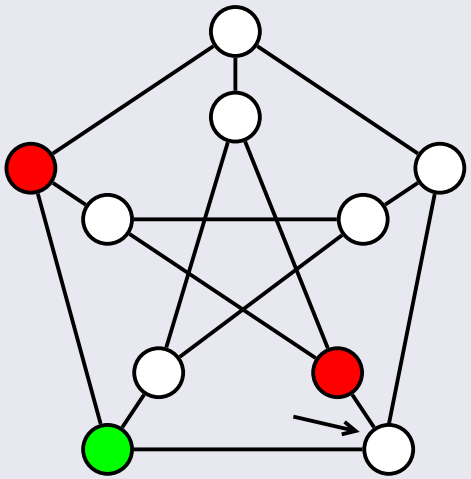

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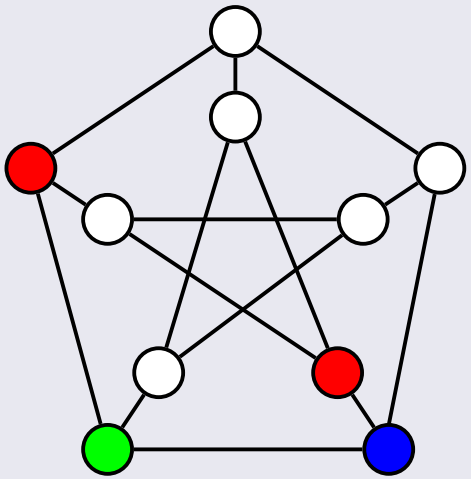

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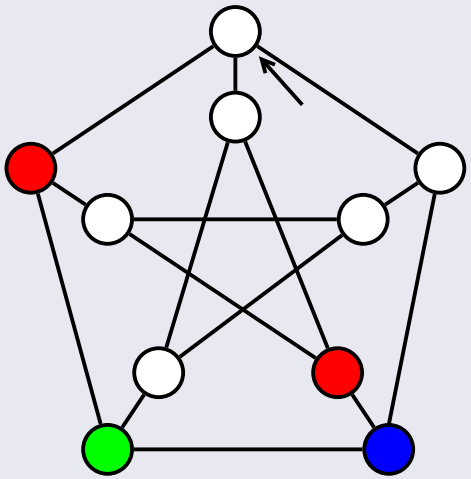

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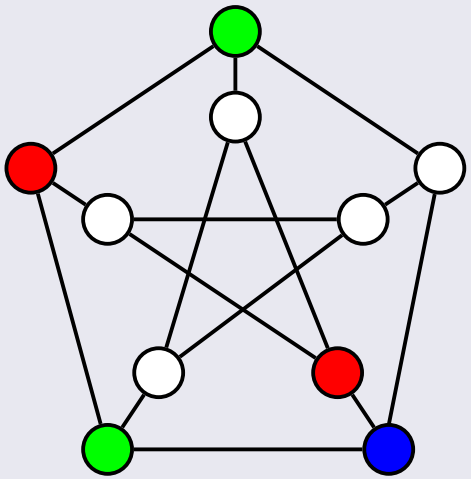

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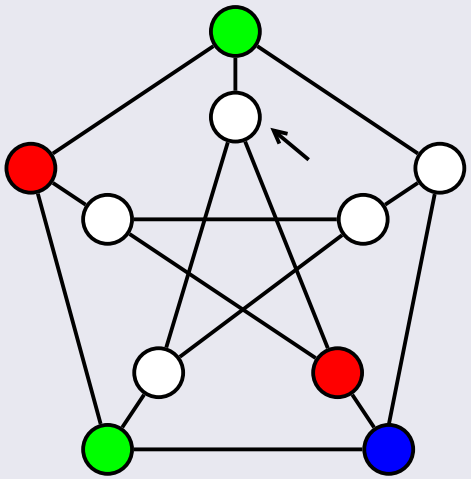

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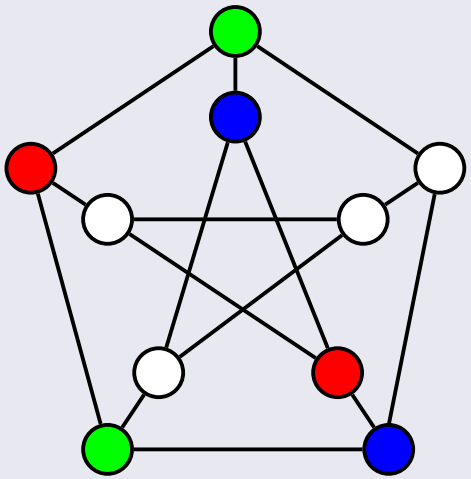

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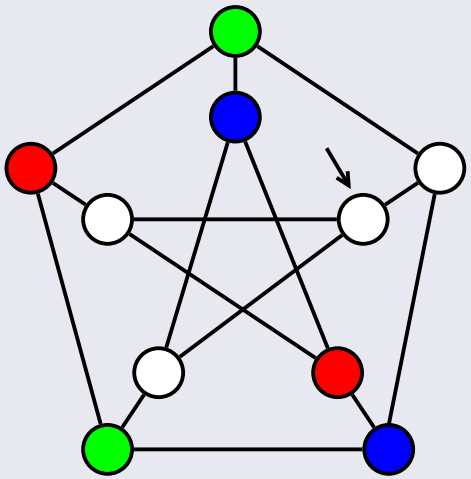

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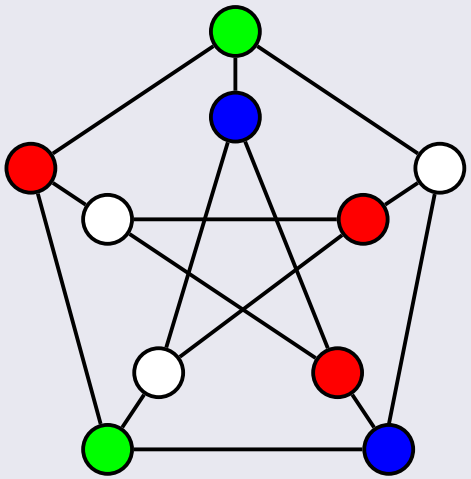

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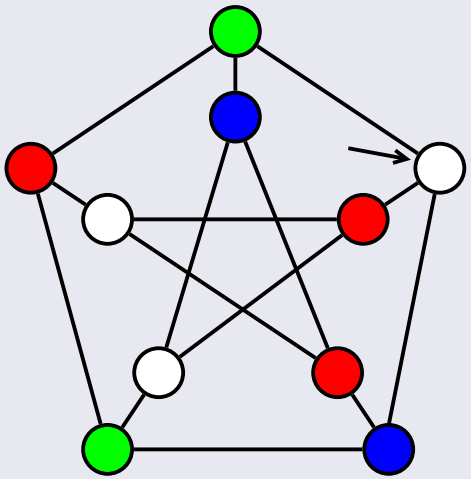

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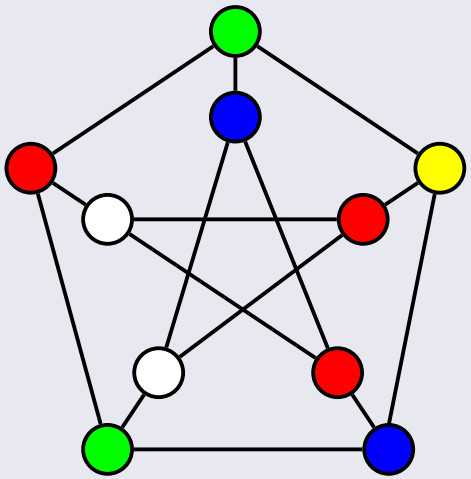
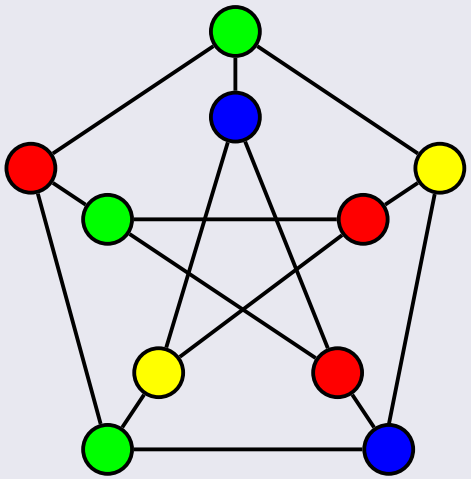


Red	1
Green	2
Blue	3
Yellow	4

Grundy Coloring

- Input: Graph $G = (V, E)$ on n vertices
- Repeat n times
  - Select an uncolored vertex u of G
  - Assign u the smallest color that is not currently used in any of its neighbors (**First-Fit**)
- Goal: Order the vertices in such a way that number of colors used is **maximized**.



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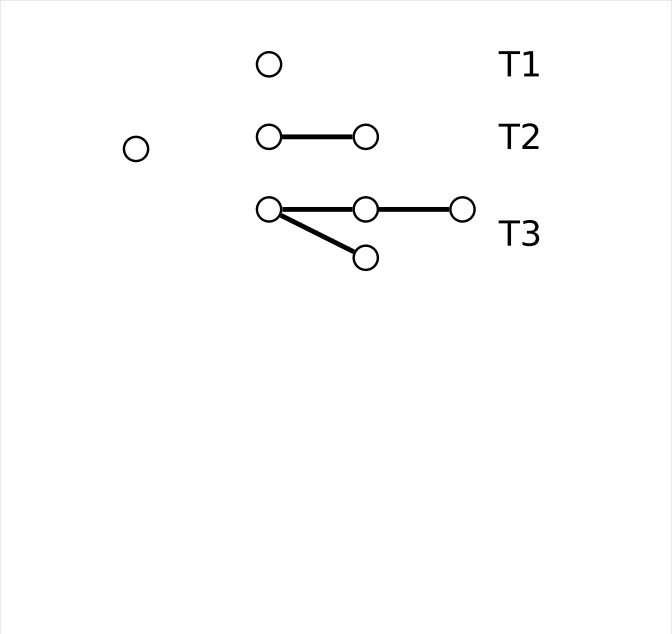

- Two recursive constructions
- T_1 is a vertex.
- T_k is a new root connected to $T_{k-1}, T_{k-2}, \ldots, T_1$.

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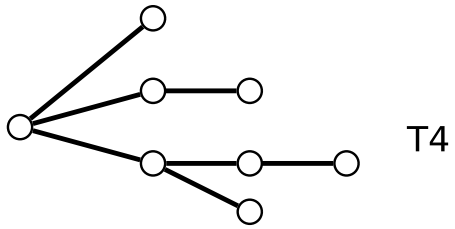

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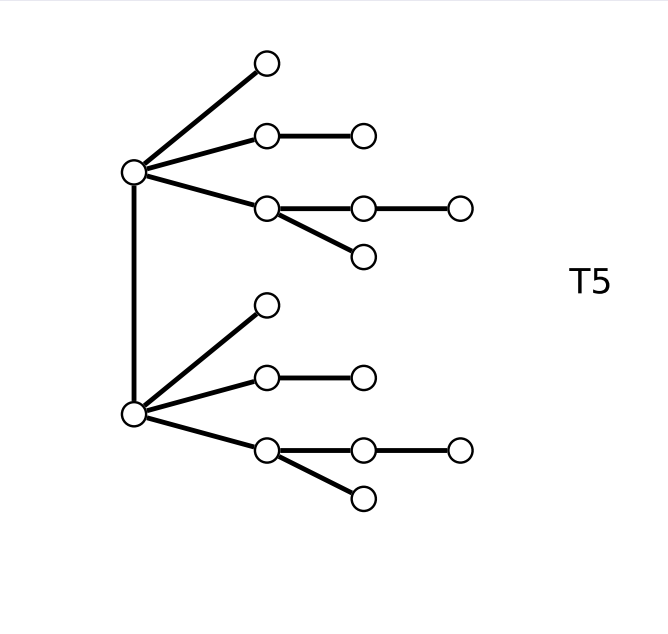

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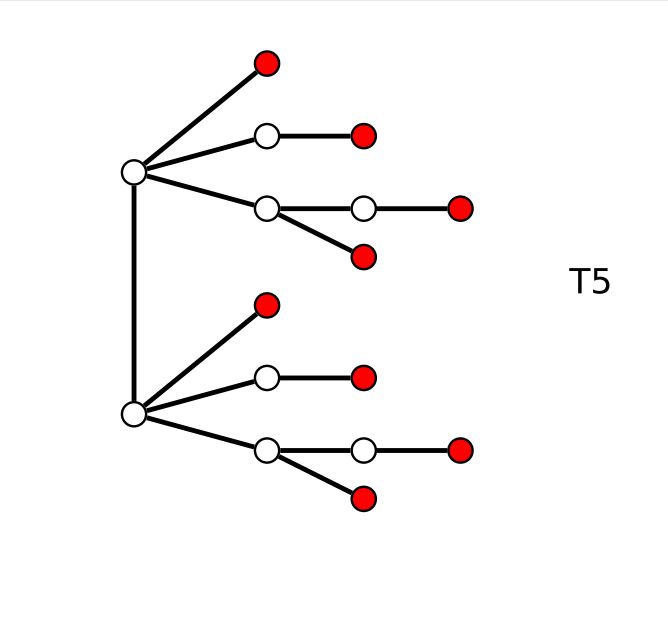

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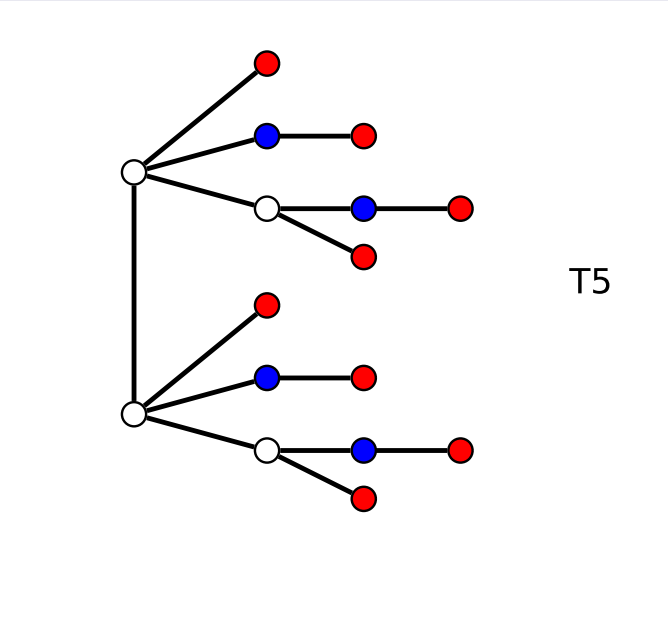

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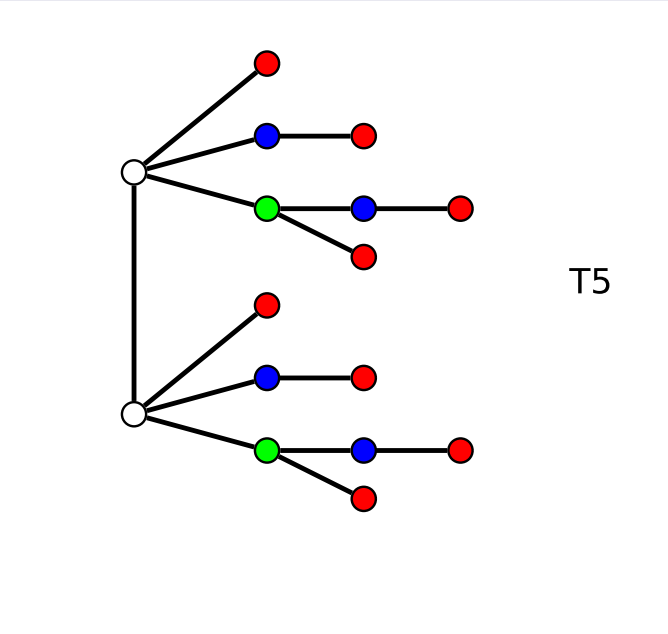

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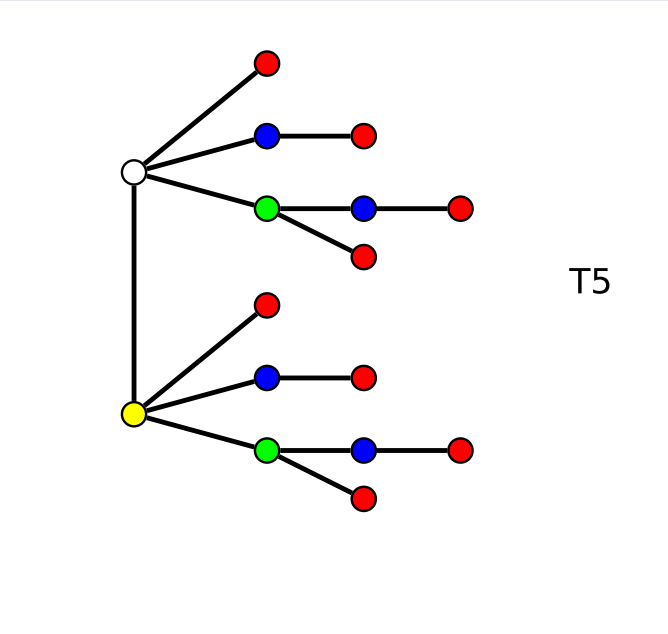

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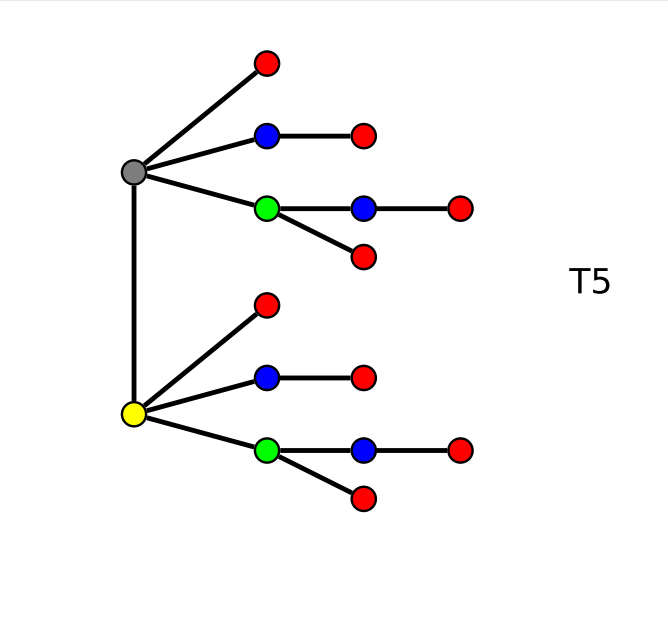

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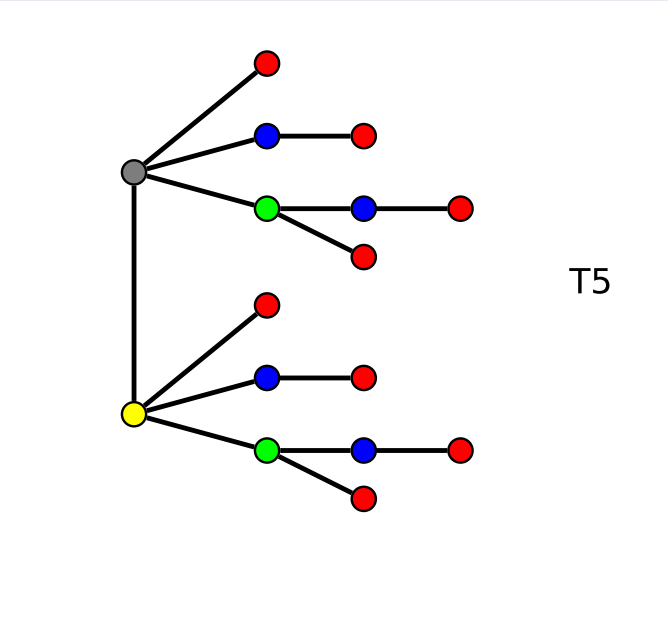

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- We have $\Gamma(T_k) = k$ but $\chi(T_k) = 2$.
- $|T_k| = 2^{k-1}$.
- This is tight: for all trees $\Gamma(T) \leq \log n$.

- **More generally:** for all graphs $\Gamma(G) \leq tw(G) \log n$.

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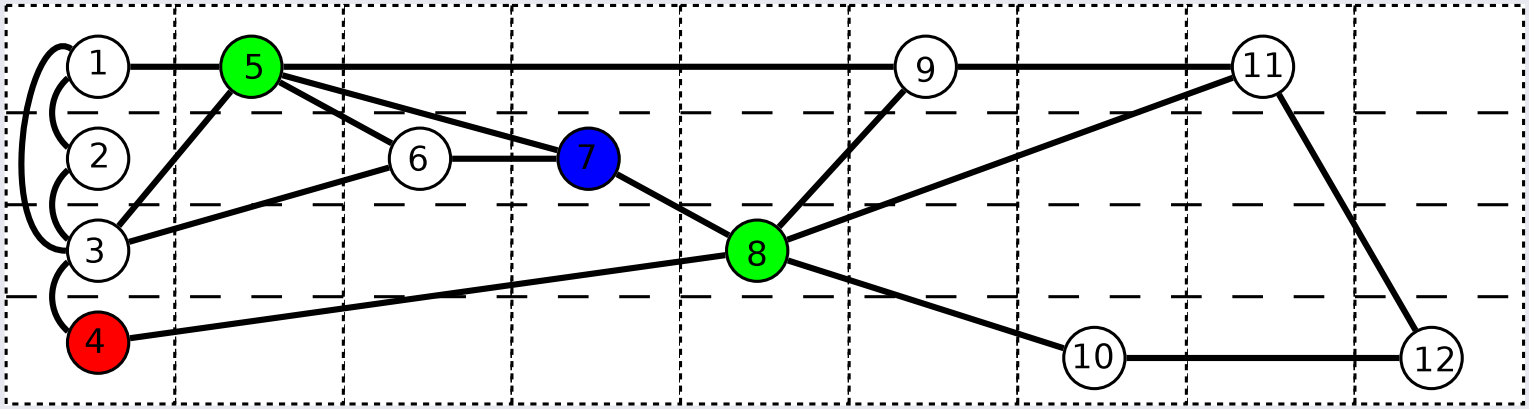
Algorithm for Grundy and Treewidth

- XP algorithm due to [Telle&Proskurowski SIDMA'97]



- Standard Coloring DP: recall color of each vertex in bag
- Reminder: Bags are separators
- Only need to remember which colorings of the bag can be extended to the left.
  - Complexity: $\rightarrow k^{tw}$

Algorithm for Grundy and Treewidth

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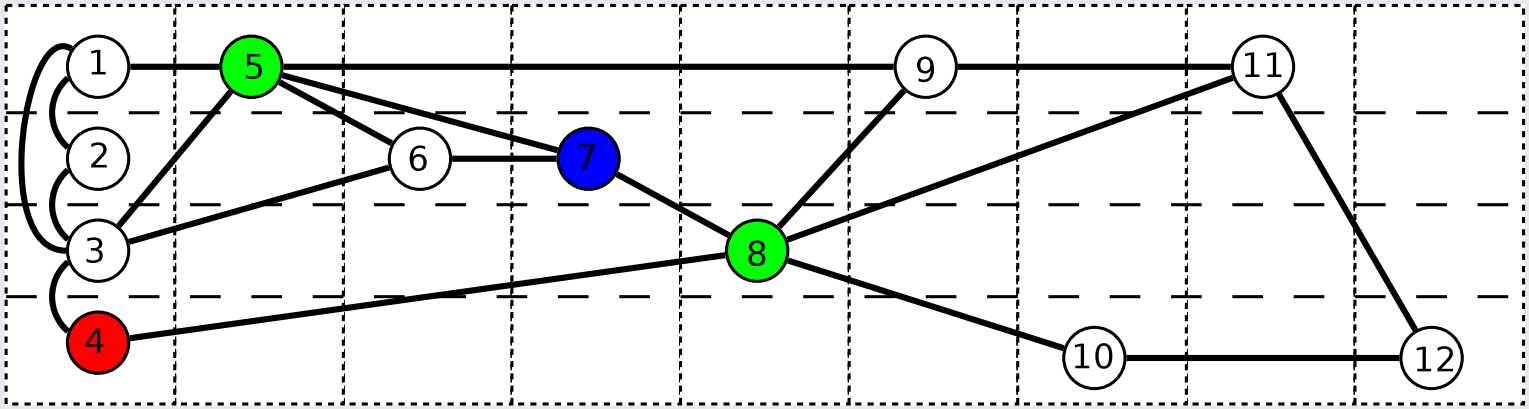

- Grundy: for each vertex we also need to make sure that it is dominated by **all** lower colors
  - In this example, this coloring is only valid if 6 takes color Red
- Need to remember for each vertex **the subset** of colors it has seen in its neighborhood
  - $\rightarrow (2^k)^{tw}$

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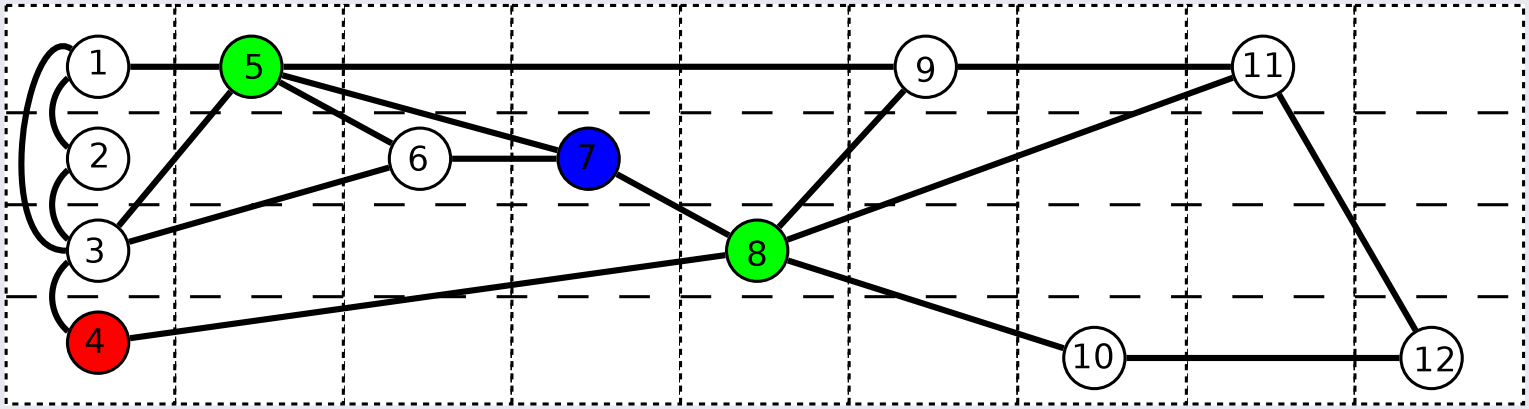
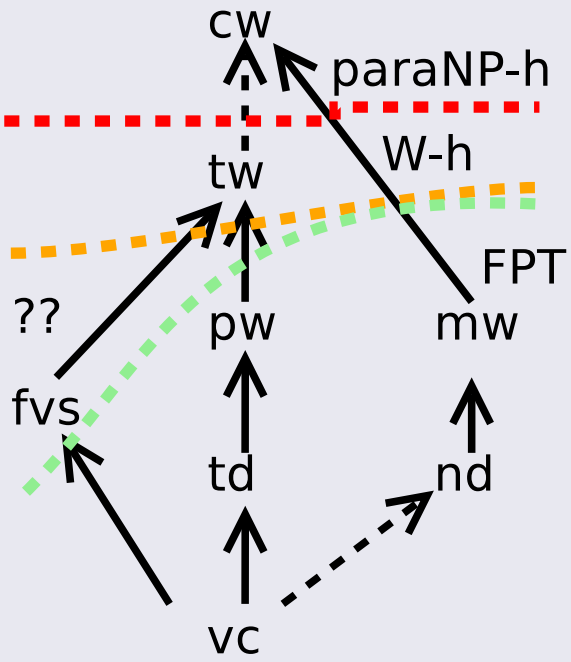


- Overall running time $O^*((k2^k)^{tw})$.
- Is this XP?
- Yes, if we use that $k \leq tw \log n$
- Running time: $n^{O(tw^2)}$

Our results



Main results:

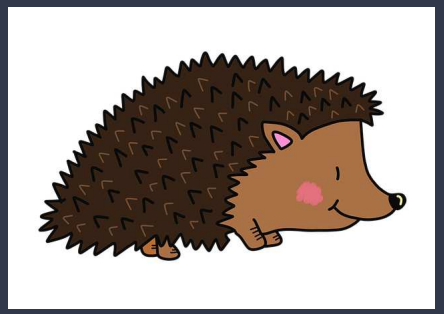
- Grundy Coloring is W[1]-hard by treewidth
- Grundy Coloring is FPT by pathwidth

Also:

- Grundy Coloring is NP-h for clique-width$= 6$
- Grundy Coloring is FPT for modular width

- Key insight: ability to bound $\Gamma(G)$ is crucial
  - For bounded pw we have bounded Γ
  - For bounded tw we have $\Gamma \leq tw \log n$
  - No upper bound on Γ for bounded cw

W-hardness for treewidth



Proof Outline

- Desired result: Grundy Coloring is W[1]-hard by treewidth
- Proof: Reduction from k-MCC
  - k-MCC: given properly k-colored graph, decide if exists k-Clique.

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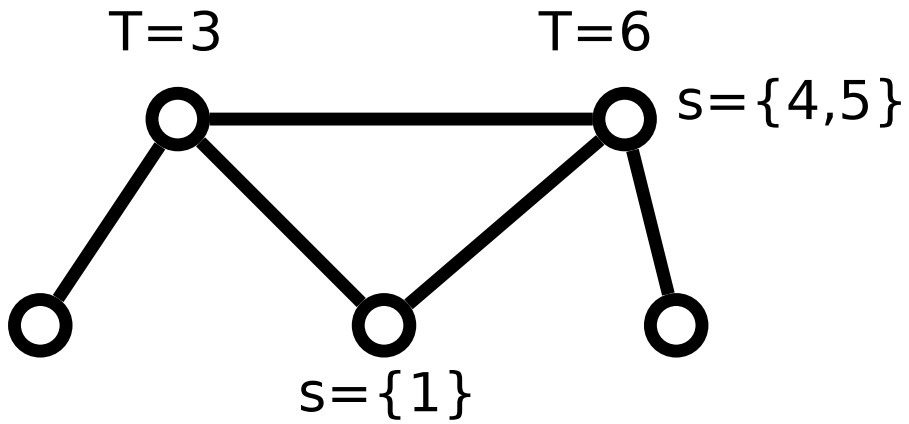
Steps:

- Define more general “Grundy with Targets and Supports”
- Show that GwTS is W[1]-hard parameterized by **pathwidth**
  - Not a typo! More info later...
- Use binomial trees to reduce GwTS/pw to Grundy/tw

Grundy with Supports and Targets – Example



- Example of generalized problem instance.
- Two vertices have a target we want to achieve.
- Some vertices have a support set: we don’t need to assign them neighbors of these colors to obtain a higher color.

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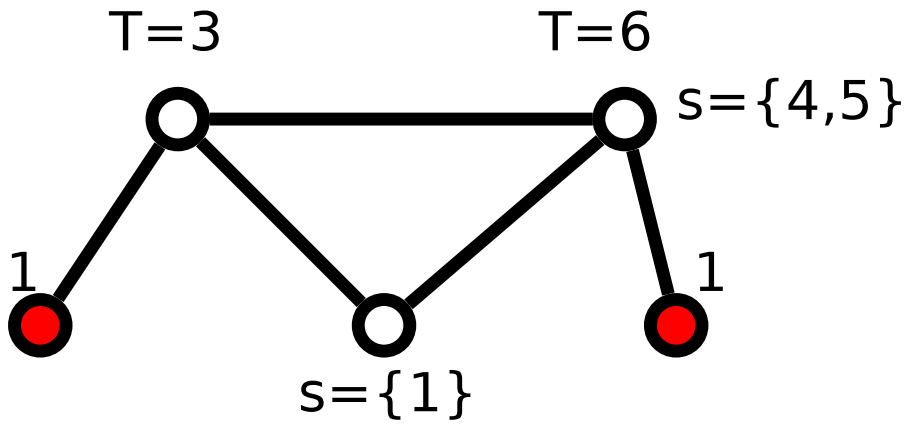

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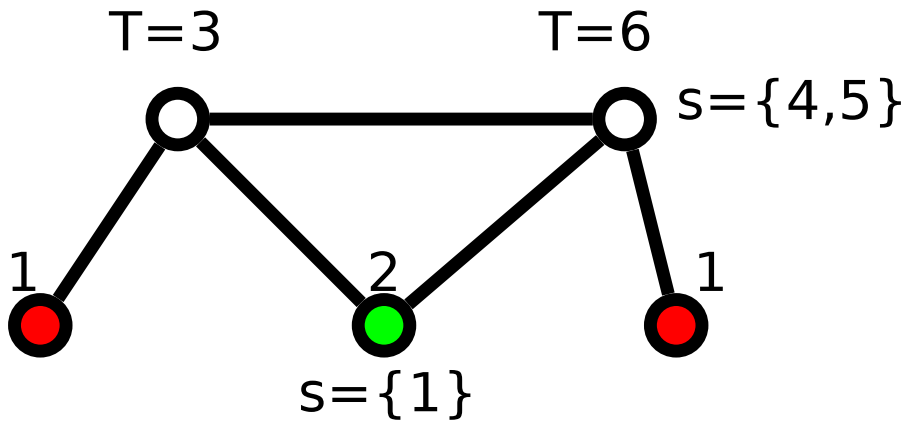

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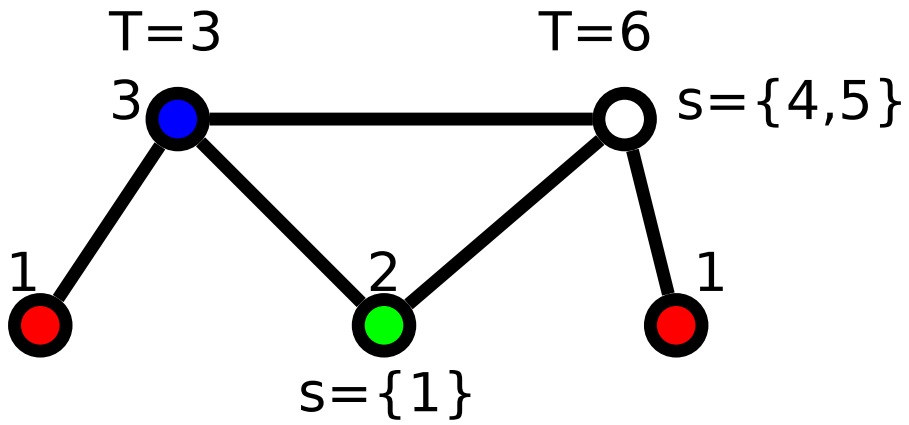

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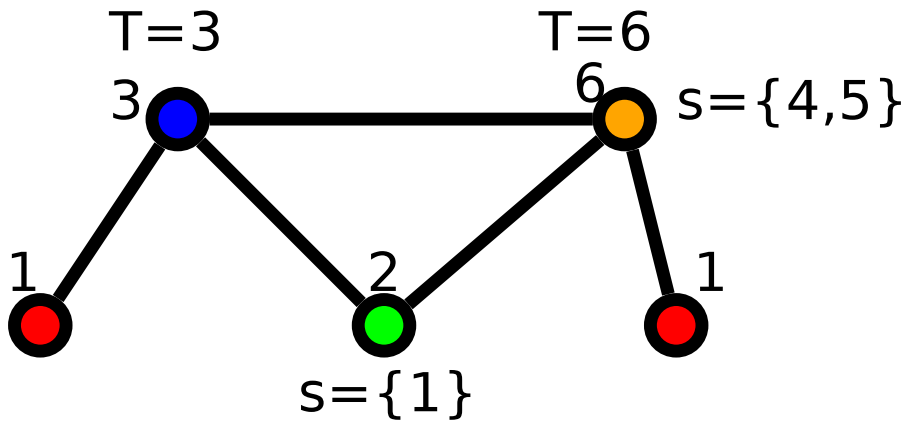

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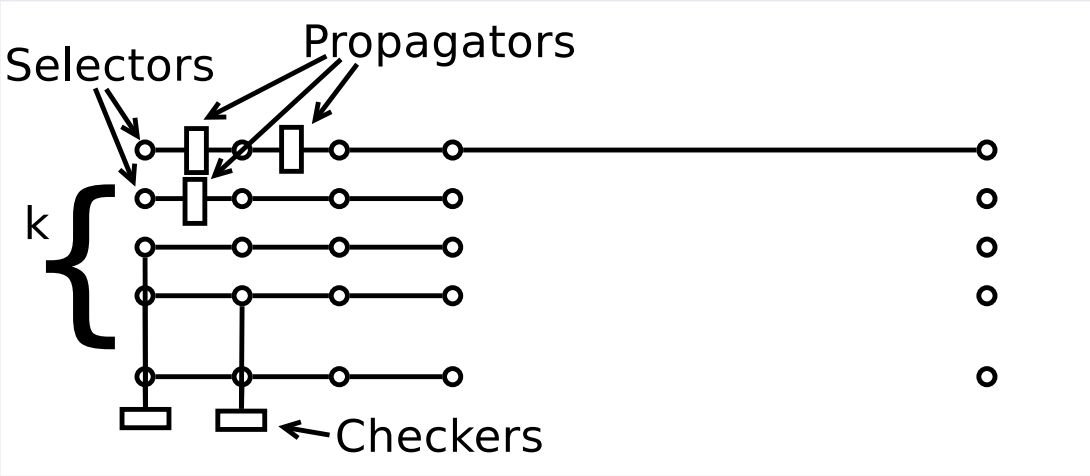

- $k \times m$ “grid” where each row represents a color class

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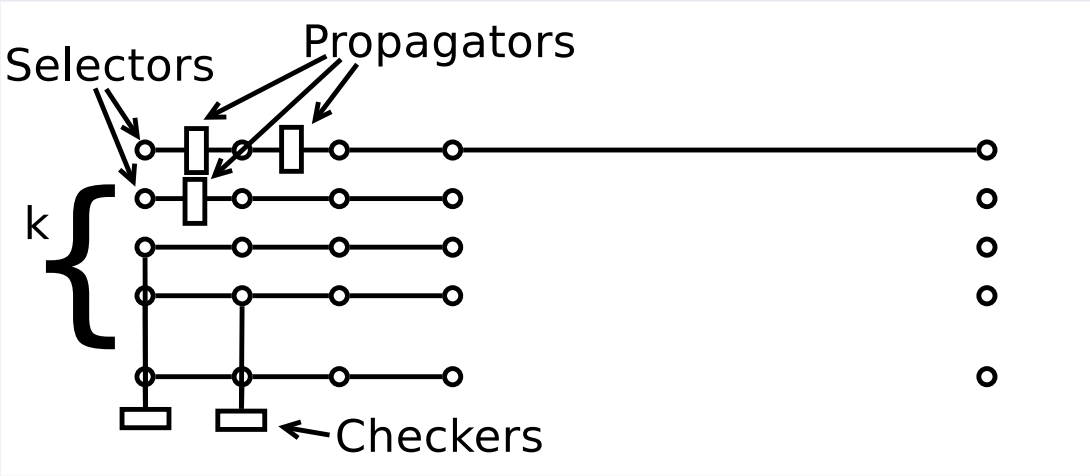

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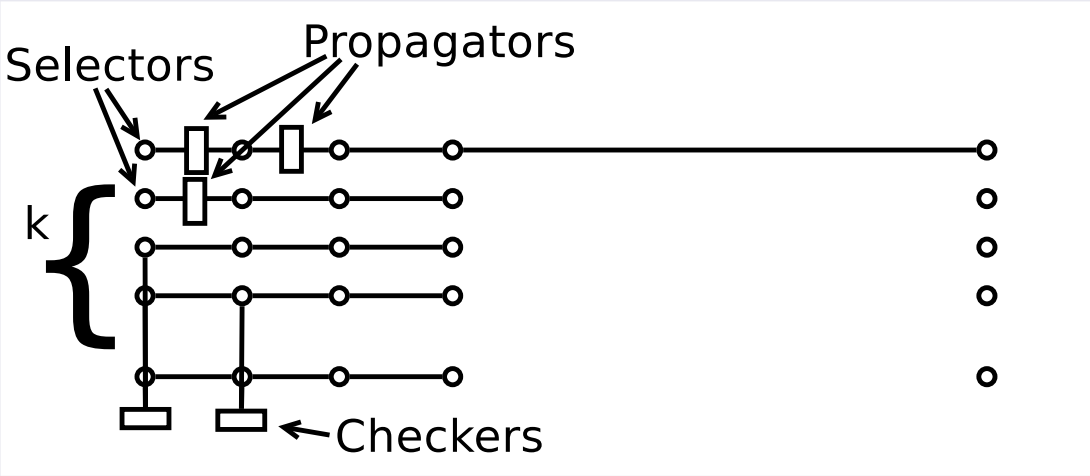

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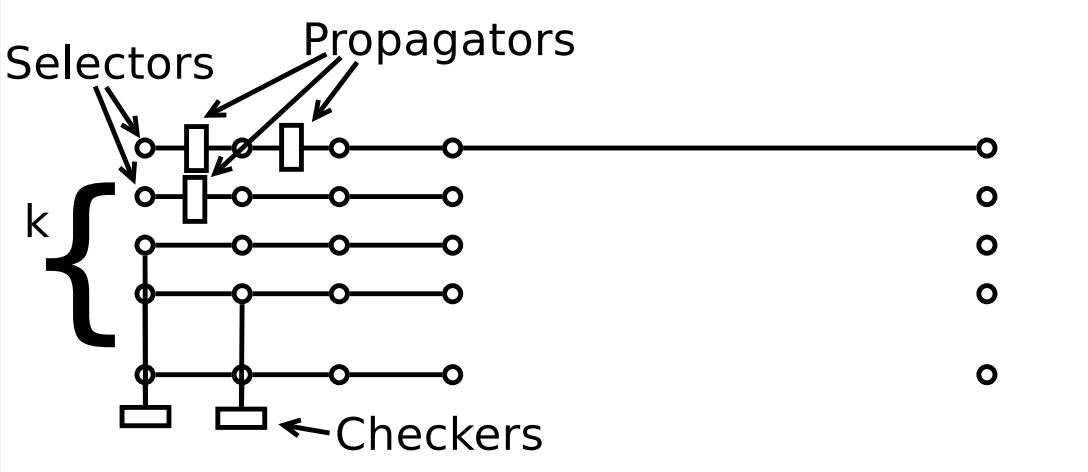

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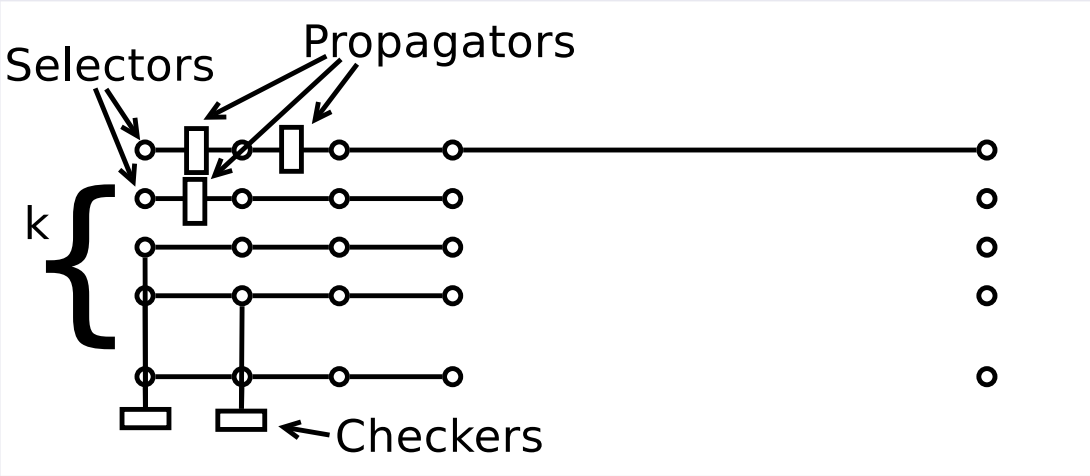

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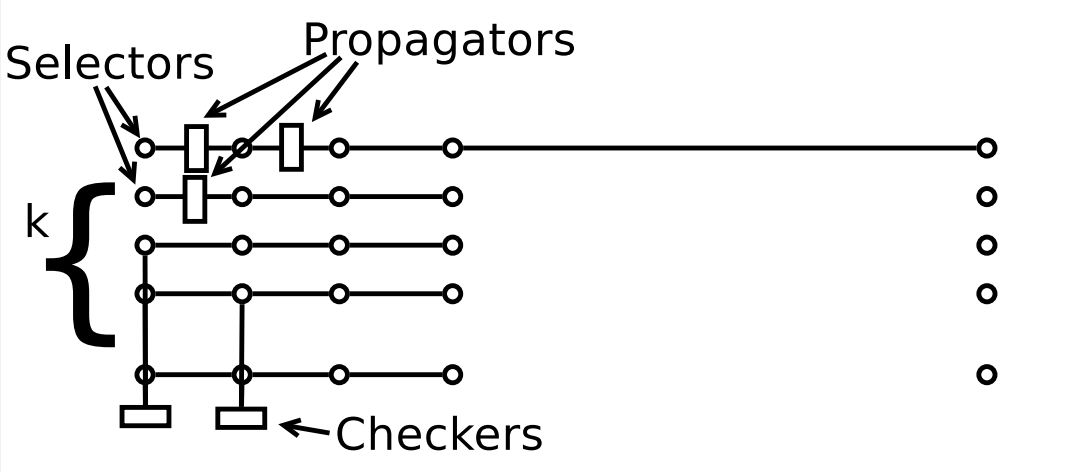

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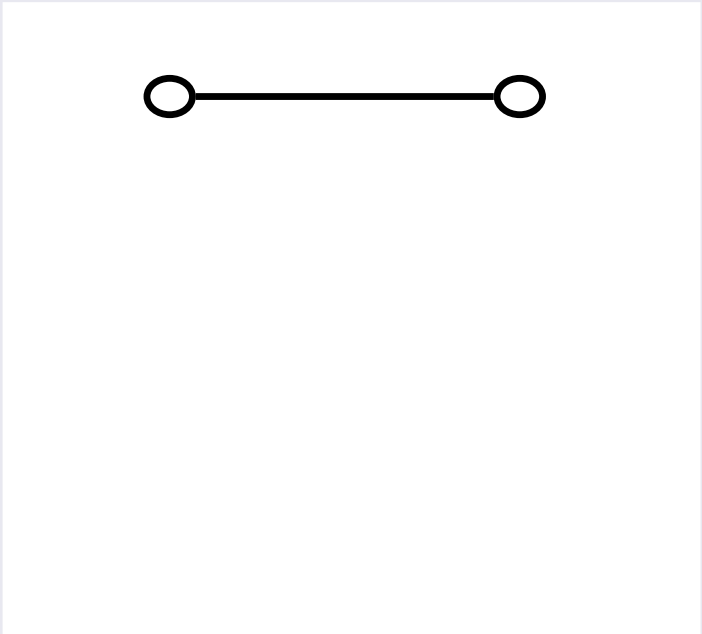
- Main difficulty: selectors and propagators

Selector Gadget



Intuition:

- We construct $\log n$ independent edges, numbered $1 \ldots \log n$.
- Endpoints of edge i get support $[1 \ldots 2i-2]$.
- $\rightarrow$ they can be colored with $2i-1, 2i$.
- For each edge we have a choice to put the larger color left or right.
- $2^{\log n} = n$ choices can be encoded.

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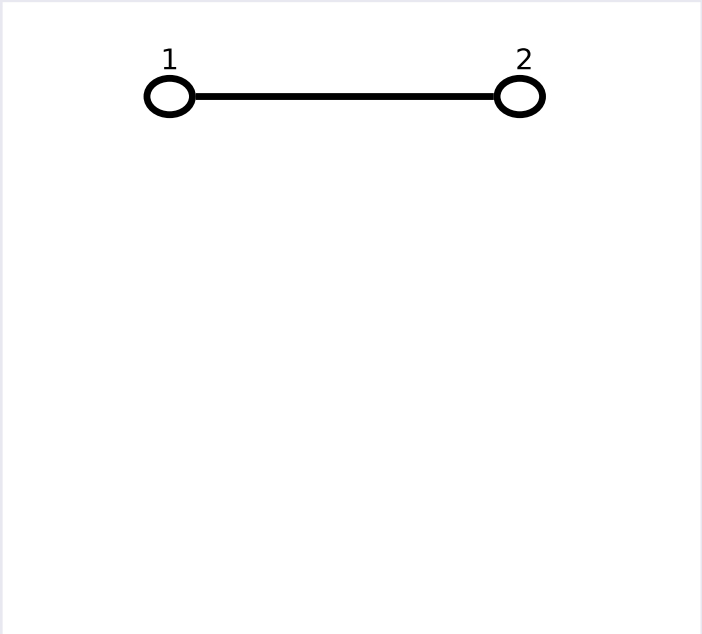

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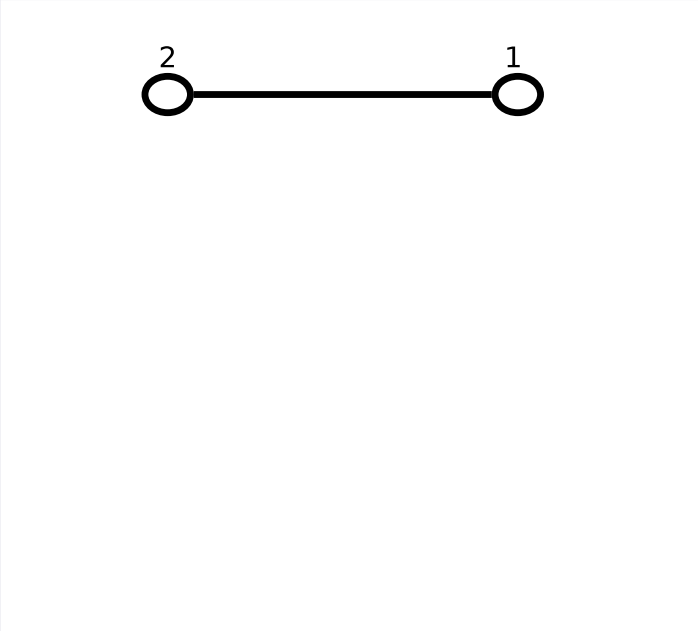

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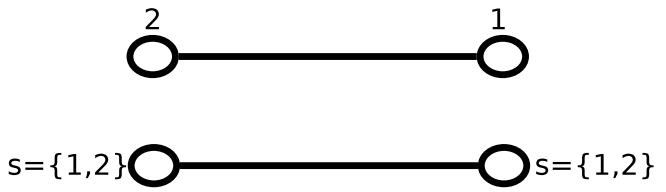

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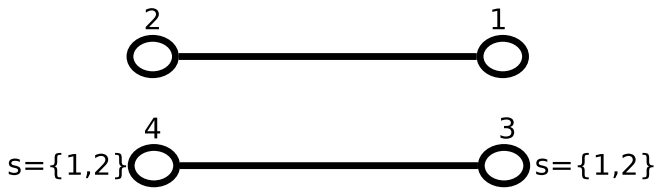

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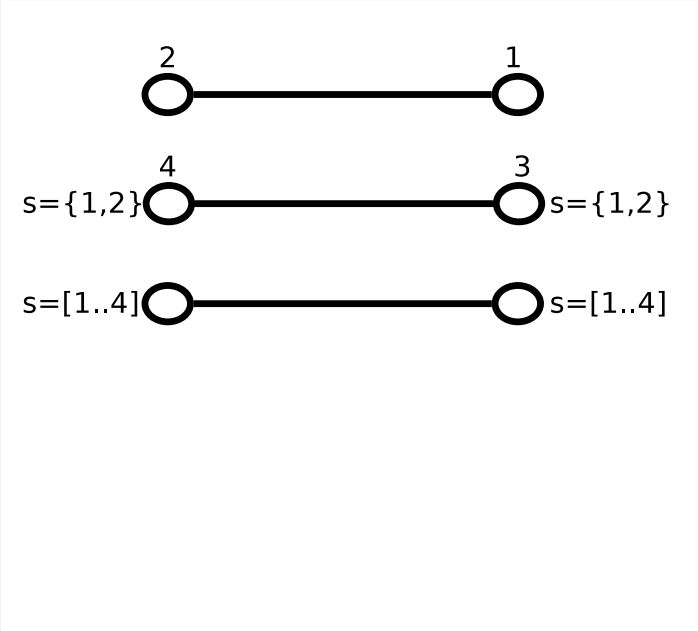

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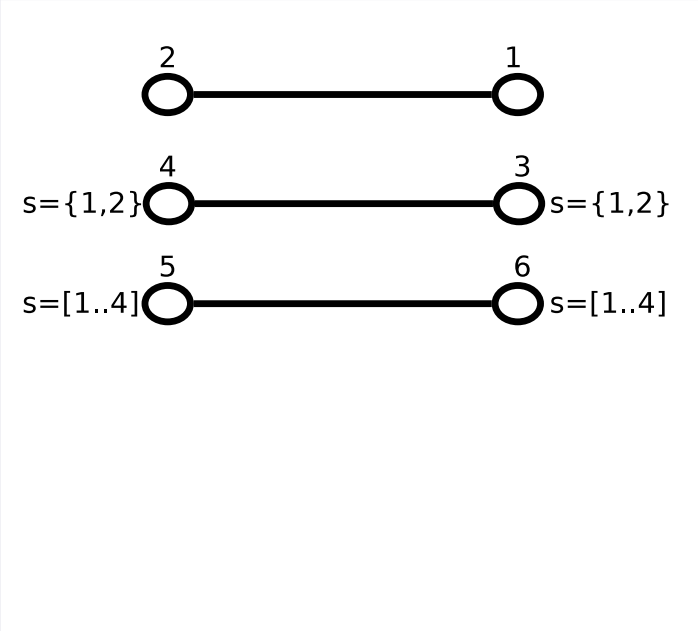

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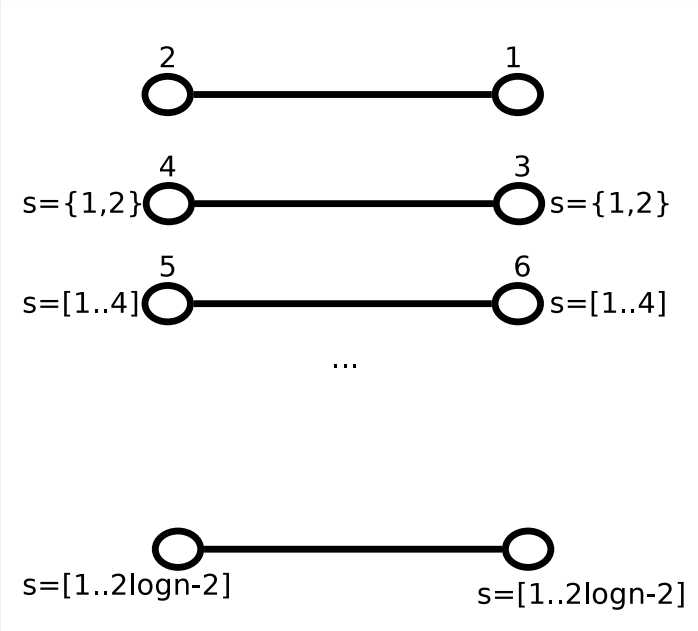

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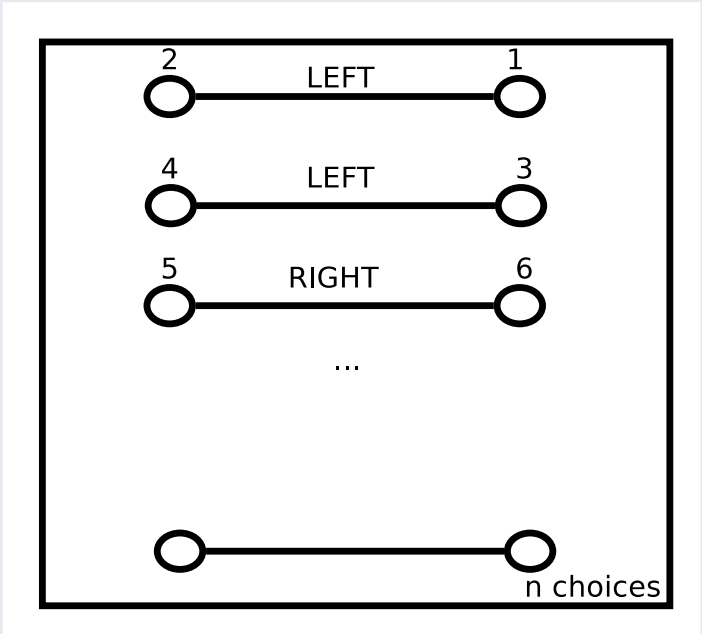
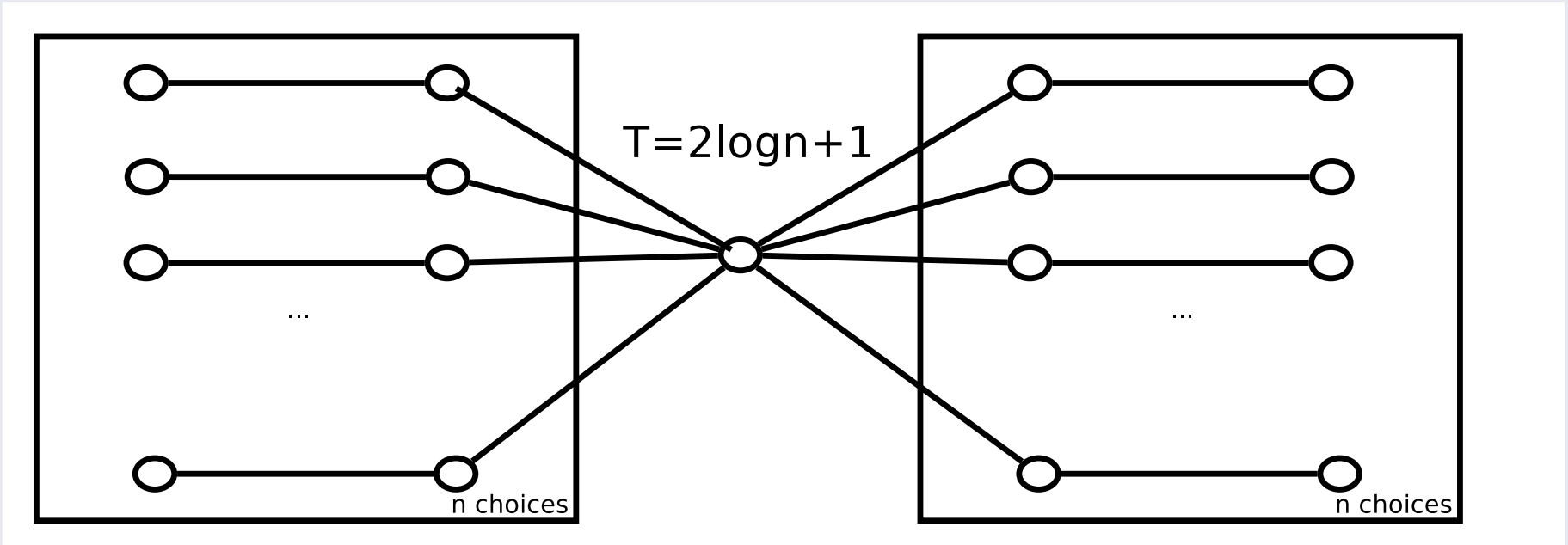


Intuition:

- We construct $\log n$ independent edges, numbered $1 \ldots \log n$.
- Endpoints of edge i get support $[1 \ldots 2i-2]$.
- $\rightarrow$ they can be colored with $2i-1, 2i$.
- For each edge we have a choice to put the larger color left or right.
- $2^{\log n} = n$ choices can be encoded.

Propagator Gadget



Intuition:

- A propagator is a vertex with target $2 \log n + 1$ connected to different sides of consecutive selectors.
- Its neighborhood must cover all colors in $\{1, \ldots, 2 \log n\}$.
- For each (starting from largest) colors $2i - 1, 2i$ can only be found on i-th edge.
- Therefore, assignment must remain consistent.

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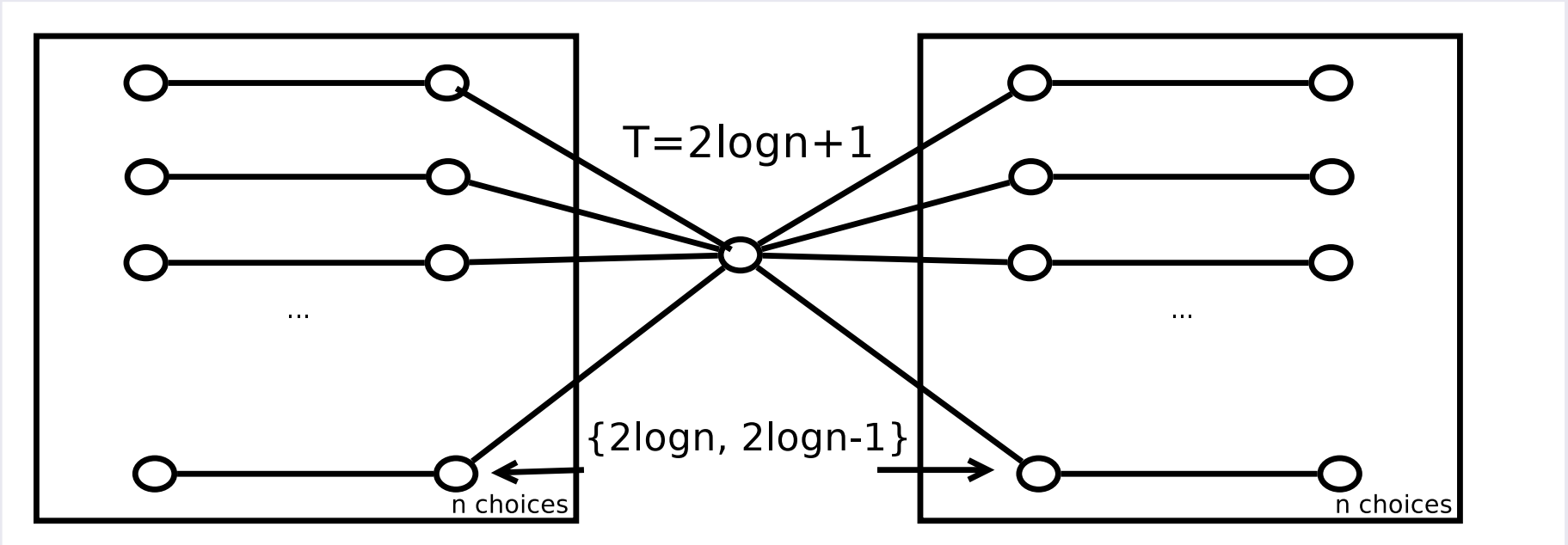

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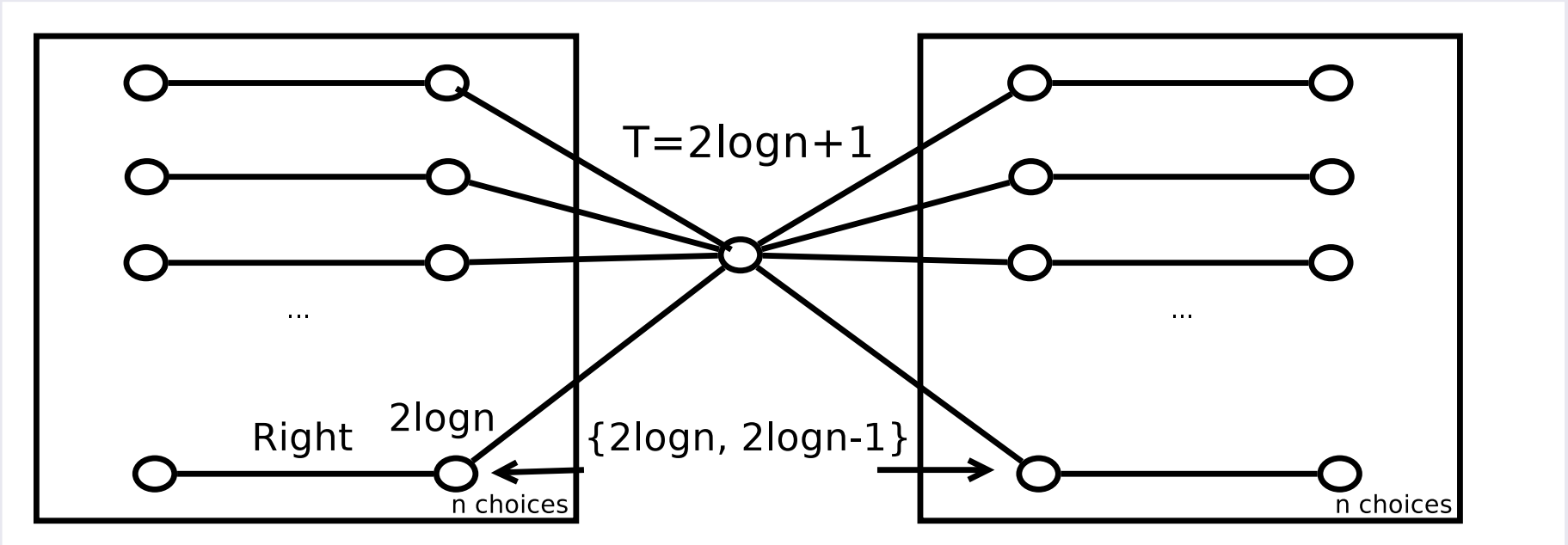

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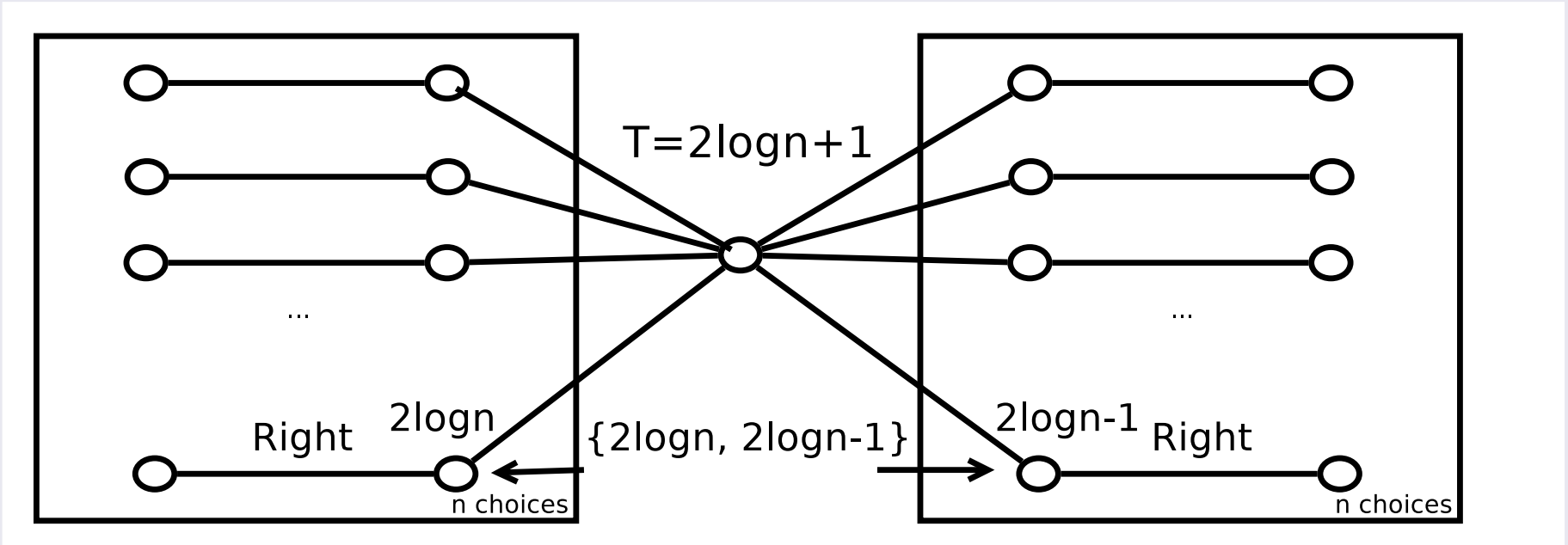

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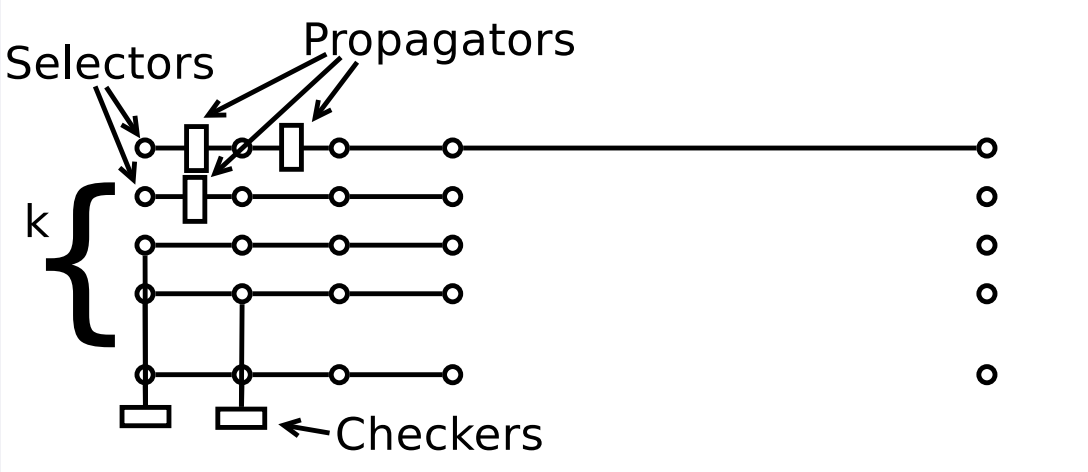

How is this reduction going?

- Graph will have pathwidth $\approx k$
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  - Bottleneck of DP: must remember set of colors seen
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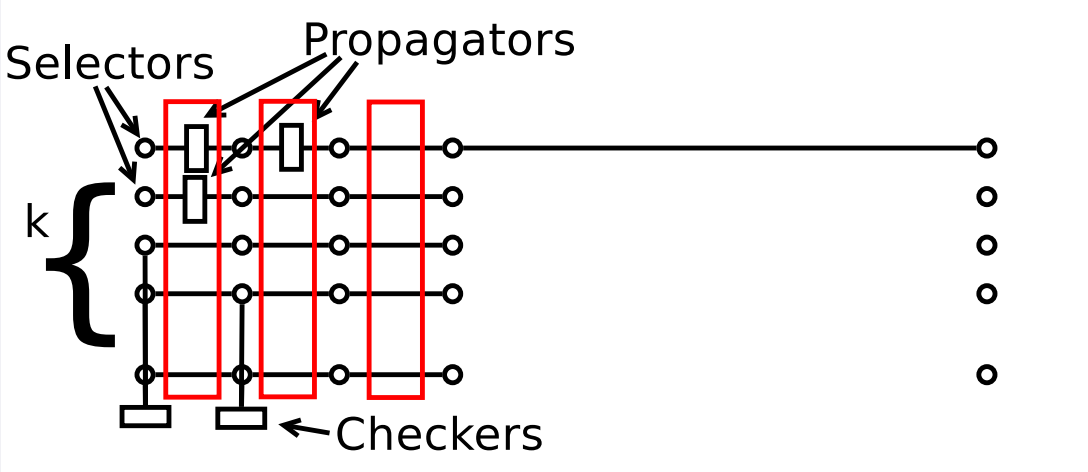

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Regular Grundy

- To implement supports we attach binomial trees to supported vertices.
  - Does not increase treewidth.
  - Crucial: all supports are $O(\log n)$, so binomial trees have polynomial size.
- To implement targets we add a huge binomial tree $T_{10 \log n}$.
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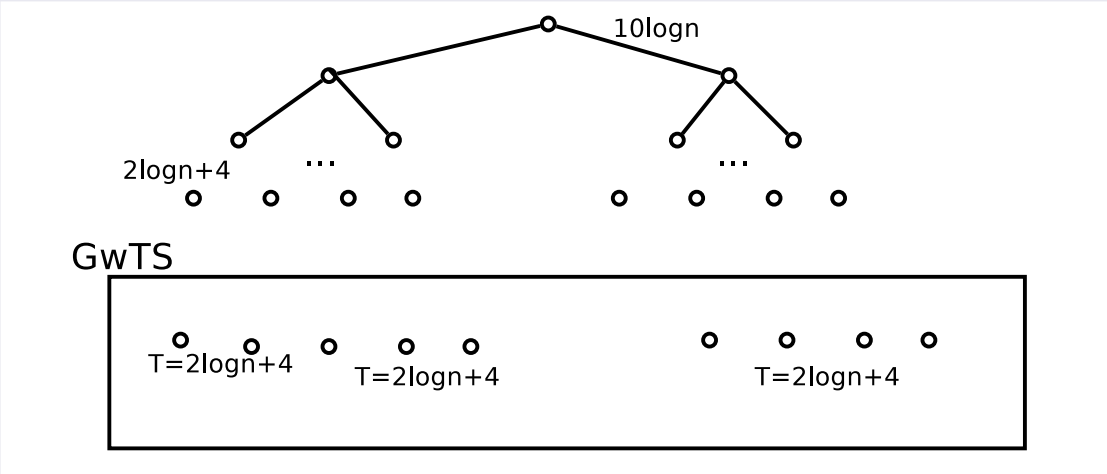

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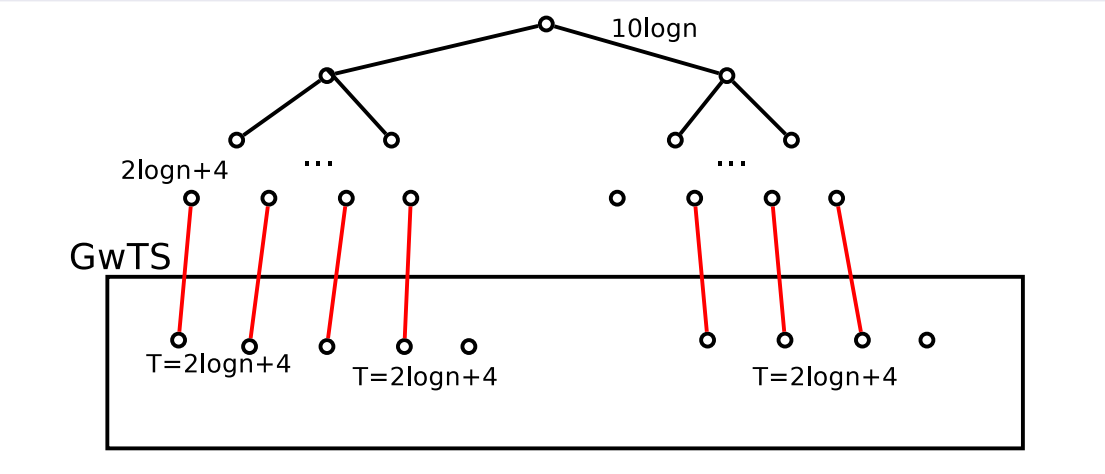
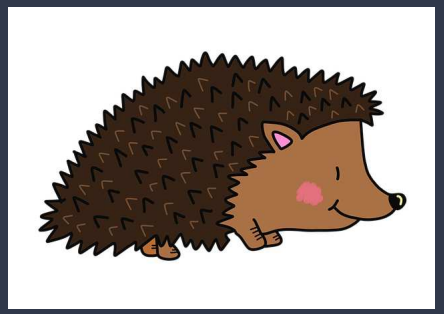


Interesting Trick:
Graph of pathwidth k + Tree
$\Rightarrow$
Graph of treewidth k

Summary

- Grundy is W[1]-hard by treewidth
- Reduction shows Grundy with Targets and Supports is W[1]-hard by pathwidth!
- Key reason why this doesn't work for regular Grundy: we need binomial trees
- Binomial trees have large pathwidth ($\Theta(\log n)$)

FPT for pathwidth



Cmbinatorics to Algorithms

Two ingredients for FPT algorithm by pathwidth:

- DP algorithm running in $2^{k \cdot tw}$ we saw
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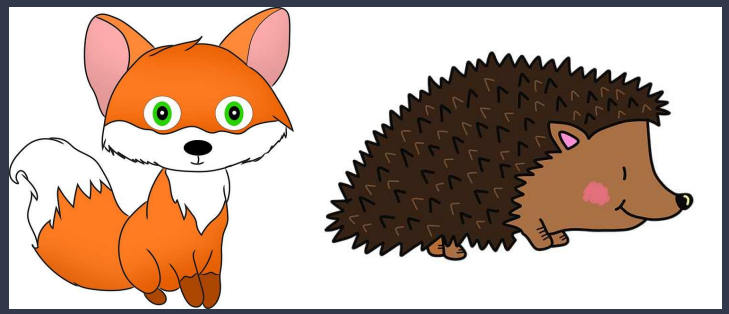
  - Uses connection pathwidth$\leftrightarrow$interval graphs
- Plugging in the bound and using $tw \leq pw$ we get

Thm: Grundy Coloring can be solved in $O^*(2^{O(pw^2)})$

Conclusions



Conclusions – Open Questions

- Grundy Coloring is first (?) natural problem to be FPT for pathwidth, W-hard for treewidth

Open questions:

- Other such problems separating tw/pw?
- Problems separatings them for other reasons?
- FPT by fvs?
- Gap between $n^{o(\sqrt{tw})}$ LB and n^{tw^2} algorithm?

Thank you!

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Questions?