Online Maximum Directed Cut

(A bird in the hand is worth $\sqrt{3}$ in the bush)

Amotz Bar-Noy and Michael Lampis

City University of New York

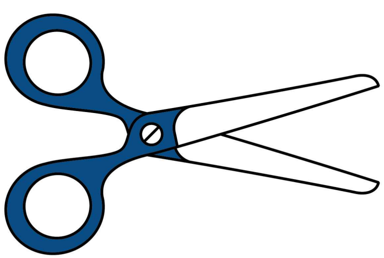
Outline

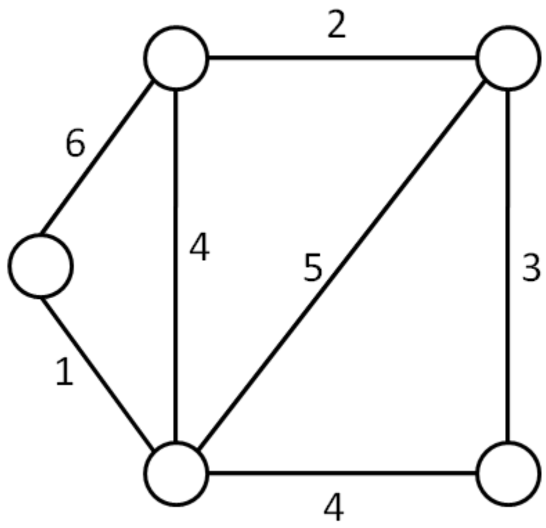
- Problem definition and motivation
- $\frac{3\sqrt{3}}{2}$-competitive algorithm for DAGs
- Lower bound
- 3-competitive algorithm for general graphs.

Note: only deterministic algorithms are considered.

Max (Di) Cut

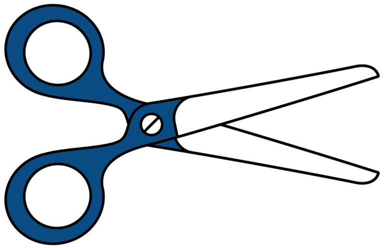
- Input: a (di)graph G with weights on the edges
- Goal: divide the vertices into two sets V_0,V_1 so as to maximize the total weight of edges going from V_0 to V_1.
- In directed version arcs going from V_1 to V_0 are not counted in the cut.

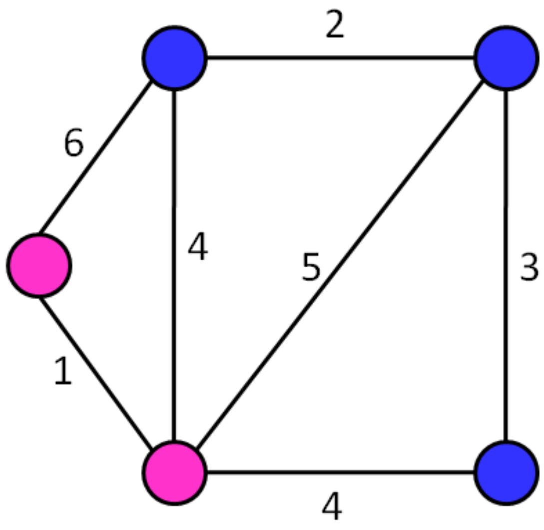




Max (Di) Cut

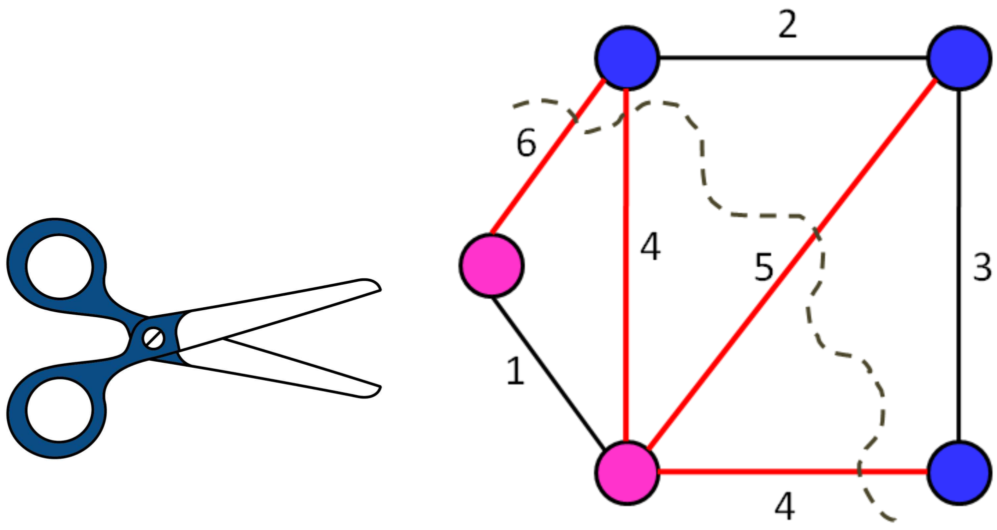
- Input: a (di)graph G with weights on the edges
- Goal: divide the vertices into two sets V_0,V_1 so as to maximize the total weight of edges going from V_0 to V_1.
- In directed version arcs going from V_1 to V_0 are not counted in the cut.





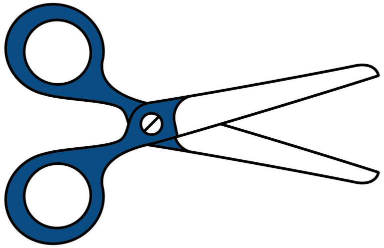
Max (Di) Cut

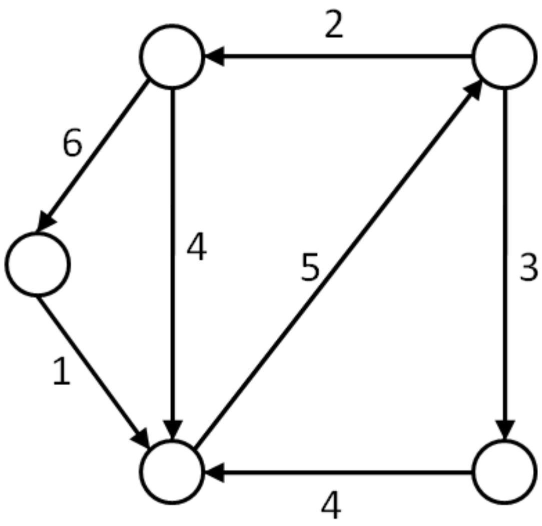
- Input: a (di)graph G with weights on the edges
- Goal: divide the vertices into two sets V_0,V_1 so as to maximize the total weight of edges going from V_0 to V_1.
- In directed version arcs going from V_1 to V_0 are not counted in the cut.



Max (Di) Cut

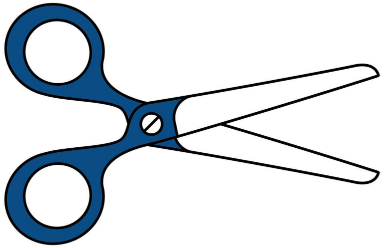
- Input: a (di)graph G with weights on the edges
- Goal: divide the vertices into two sets V_0,V_1 so as to maximize the total weight of edges going from V_0 to V_1.
- In directed version arcs going from V_1 to V_0 are not counted in the cut.

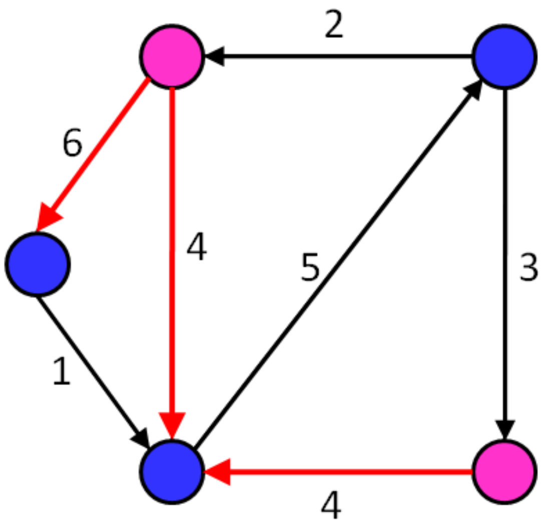




Max (Di) Cut

- Input: a (di)graph G with weights on the edges
- Goal: divide the vertices into two sets V_0,V_1 so as to maximize the total weight of edges going from V_0 to V_1.
- In directed version arcs going from V_1 to V_0 are not counted in the cut.





Known results

- Max Cut was one of Karp's 21 original NP-complete problems.
- Very good SDP-based approximations (1.383 (Goemans and Williamson) and 1.165 (Feige and Goemans))
- Trivial 2 and 4-approximate combinatorial algorithms.
- For Max Di Cut combinatorial 2-approximation (Halperin and Zwick)
- Max Di Cut is NP-hard even if restricted to DAGs (last year's ISAAC!)

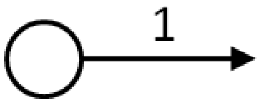
Why online?

- Disclaimer: No real application is known!
- For Max Di Cut it is natural to consider a class of greedy heuristics
  - A vertex of high out-degree should be more likely to be placed in V_0.
  - Specifically, in DAGs, sources should always be placed in V_0.
  - For subsequent vertices we have a choice between a certain profit and a potential profit.
- This won't work (the problem is NP-hard). But how bad is it?

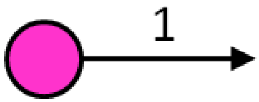
The online model

- Online model: algorithms make local choices based on the past and vertex degree.
- The adversary reveals with each vertex its connections to previous vertices and its total in and out-degree.
- The algorithm then places the vertex in V_0 or V_1.
- For DAGs the adversary must reveal vertices respecting the topological ordering of the DAG.

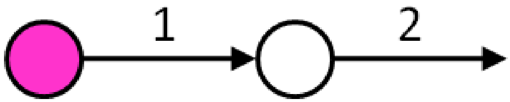
Example – Naive Greedy



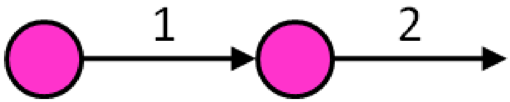
Example – Naive Greedy



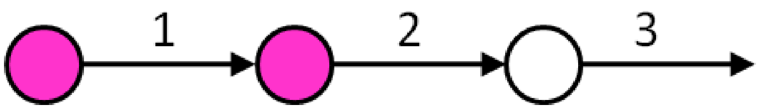
Example – Naive Greedy



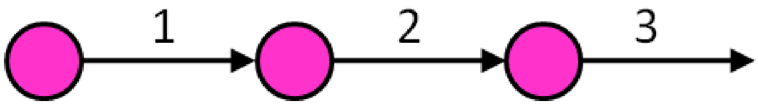
Example – Naive Greedy



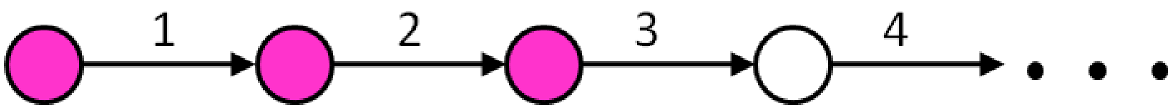
Example – Naive Greedy



Example – Naive Greedy



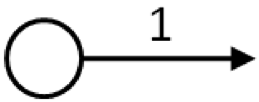
Example – Naive Greedy



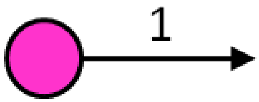
A bird in the hand is worth two in the bush

- Naive greedy is bad because it's too optimistic.
- Better to weigh the risks we take. Place a vertex in V_0 only if the promised potential profit is at least twice as much as the certain profit of V_1.
  - Now harder to fool the algorithm to assign a long string of 0s. The edge weights must increase exponentially.
  - Easy to see algorithm is no better that 2-competitive.

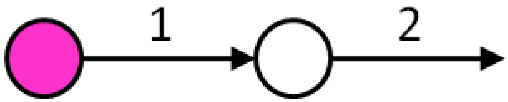
Worst case example



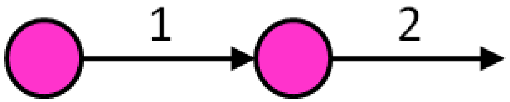
Worst case example



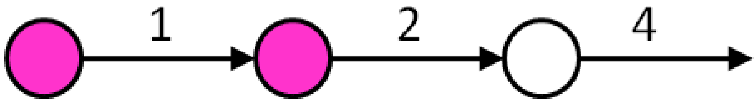
Worst case example



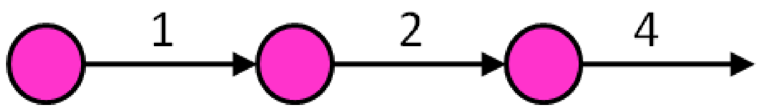
Worst case example



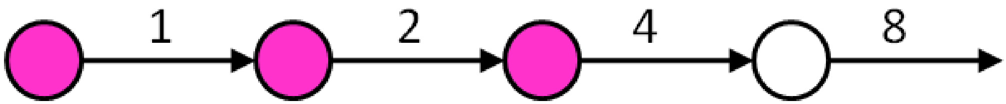
Worst case example



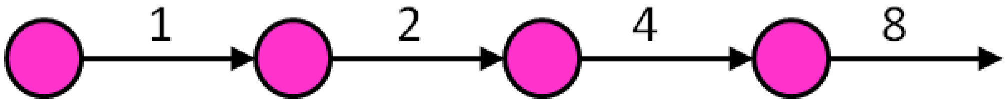
Worst case example



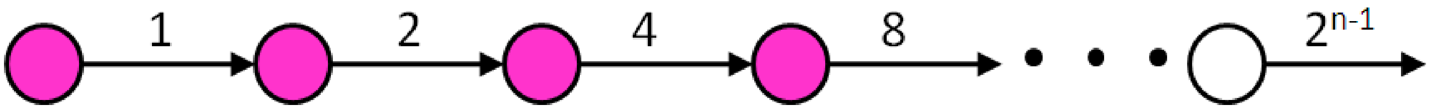
Worst case example



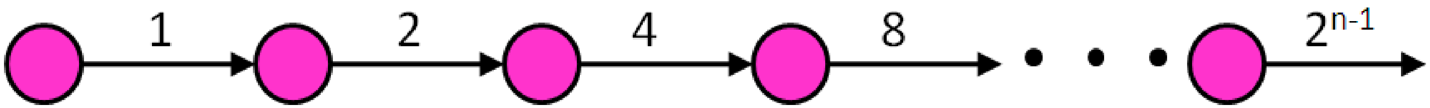
Worst case example



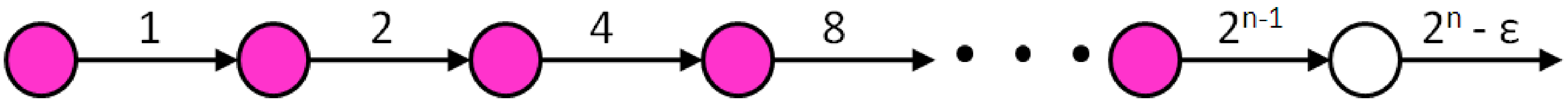
Worst case example



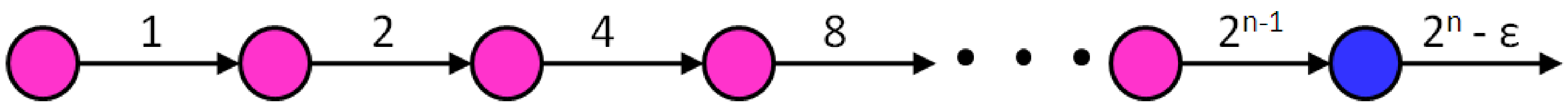
Worst case example



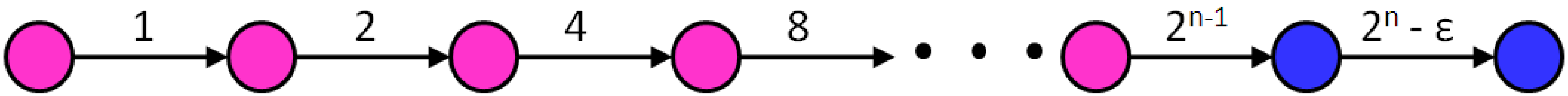
Worst case example



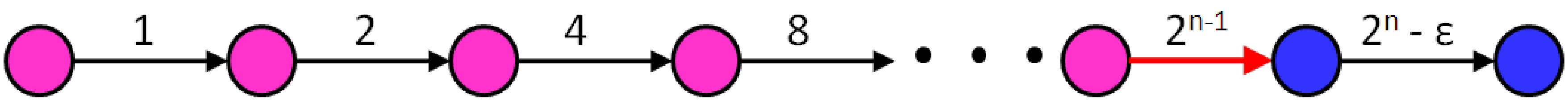
Worst case example



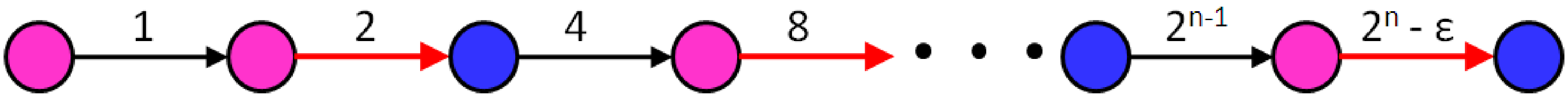
Worst case example



Worst case example



Worst case example



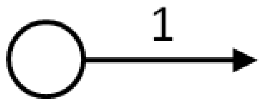
A bird in the hand is worth λ in the bush

- Previous algorithm is $\frac{8}{3}$-competitive.
- Natural to optimize the weighing of risk versus payoff.
- Optimal value is $\lambda = \sqrt{3}$ which gives a $\frac{3\sqrt{3}}{2} \approx 2.6$-competitive algorithm.

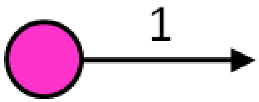
Lower Bound

- Main result: no deterministic algorithm can do better.
- Now we play as the adversary and try to force any algorithm to be λ-competitive.
- Strategy: construct a directed path. Weights are calculated in such a way that if the algorithm places a 1, we immediately win ($\text{OPT} \geq \lambda \text{SOL}$).
- If we can maintain this invariant and the weight decreases at some point, the algorithm must assign 1 and we win!

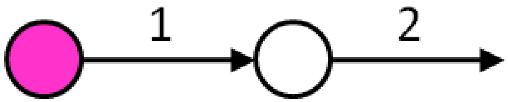
Example: Lower bound of 2



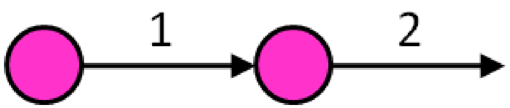
Example: Lower bound of 2



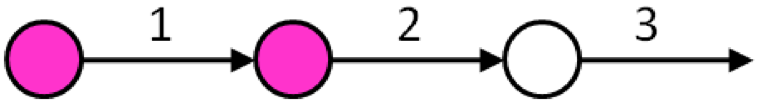
Example: Lower bound of 2



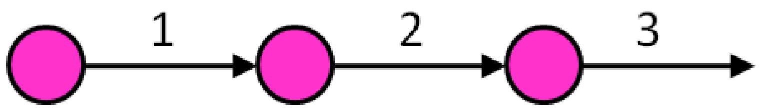
Example: Lower bound of 2



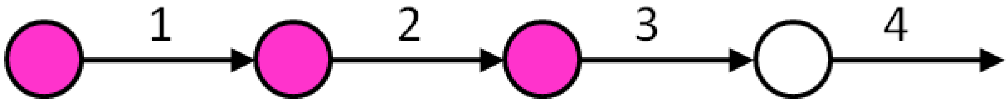
Example: Lower bound of 2



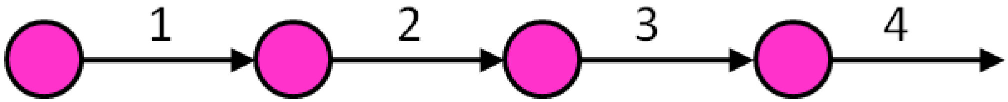
Example: Lower bound of 2



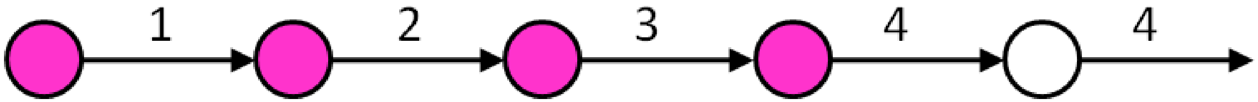
Example: Lower bound of 2



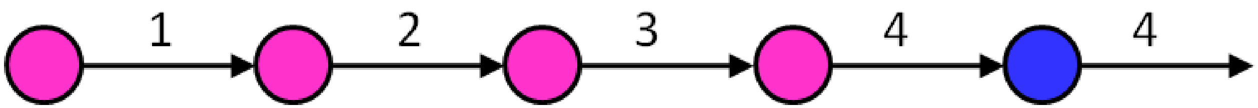
Example: Lower bound of 2



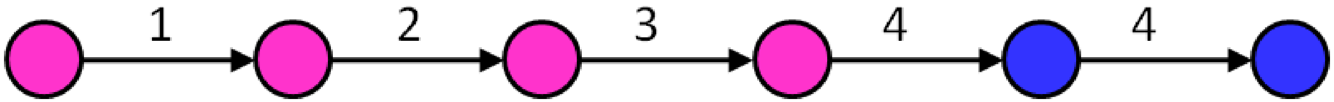
Example: Lower bound of 2



Example: Lower bound of 2



Example: Lower bound of 2



Example: Lower bound of 2

Tight lower bound

- For which λ can the adversary succeed with this strategy?
- We must have $w_k + w_{k-2} + \ldots \geq \lambda w_{k-1}$ for all k, and w_k is not always increasing.
- Cleaner form $w_k = \lambda(w_{k-1} - w_{k-3})$. This is not always increasing for $\lambda < \frac{3\sqrt{3}}{2}$.
- Upper and lower bounds match!

General graphs

- Again, weigh certain payoffs twice as much as potential ones.
- This algorithm is a greedy derandomization of the trivial randomized algorithm.
- $\Rightarrow$ 4-competitive
- Is this the best we can say?

3-competitive algorithm for general graphs

- Actually, we know that SOL $\geq |E|/4 \geq$ OPT$/4$.
- These inequalities cannot be tight at the same time.
- Using first inequality and modified arguments from the case of DAGs we have SOL $\geq$ OPT$/3$.
- There exists a tight example for this algorithm, but best lower bound is the one for DAGs.

Open problems

- How to optimize λ for general graphs?
- How to close upper-lower bound for general graphs?
- Randomized algorithms?
  - Vertices considered in random order?
  - Decisions involving randomization based on vertex degree?

THANK YOU!