Super-polynomial time approximability of inapproximable problems

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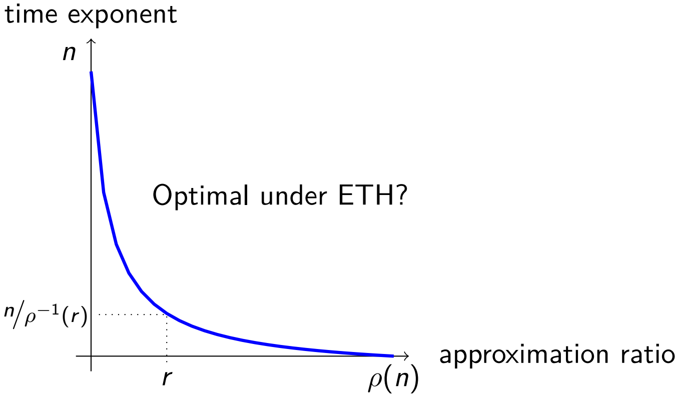
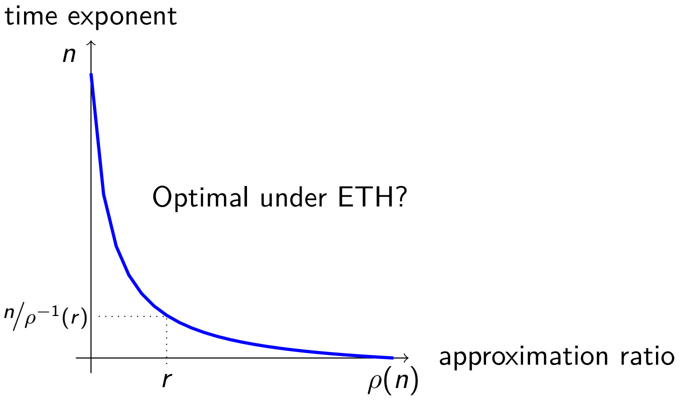
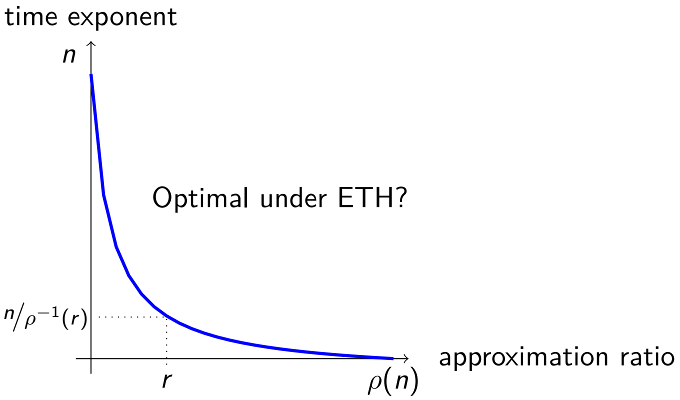


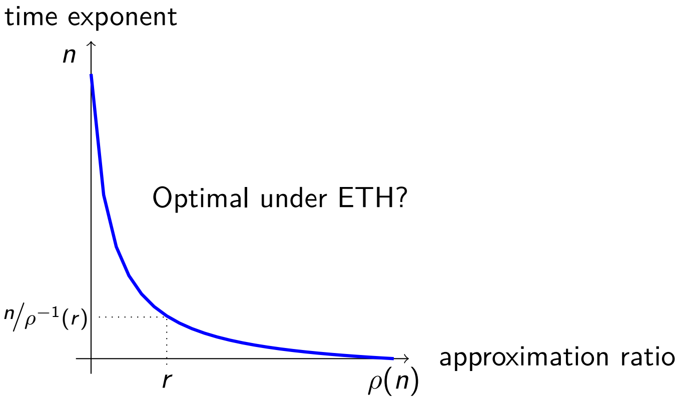
Consider Time-Approximation Trade-offs for Clique.



Clique is $\tilde{\Theta}(n)$-approximable in P and optimally solvable in λ^n.



Clique is r-approximable in time $2^{n/r}$.



Is this the correct algorithm? For every r?

Minimization subset problems

$\mathcal{I}, n$

Minimization subset problems

$\mathcal{I}, n$

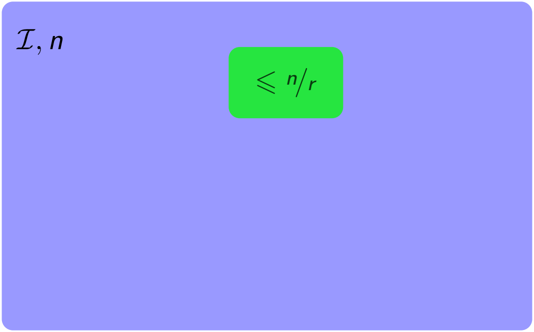
$\leqslant n/r$

Minimization subset problems

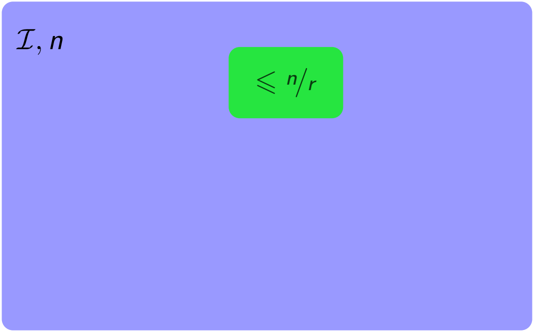
$\mathcal{I}, n$

$\leqslant {}^{n}/_{r}$

Minimization subset problems

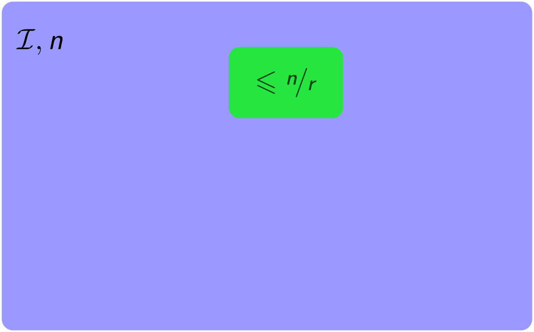


Minimization subset problems



- If a solution is found, it is an optimal solution.

Minimization subset problems



- If a solution is found, it is an optimal solution.
- If not, any feasible solution is an r-approximation.

Weakly monotone maximization subset problems

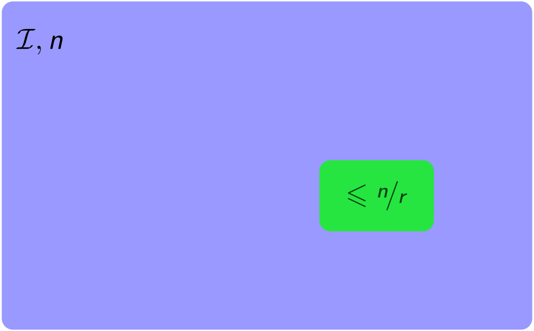
$\mathcal{I}, n$

Weakly monotone maximization subset problems

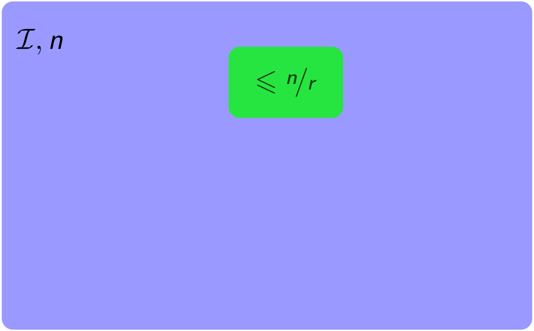
$\mathcal{I}, n$

$\leqslant n/r$

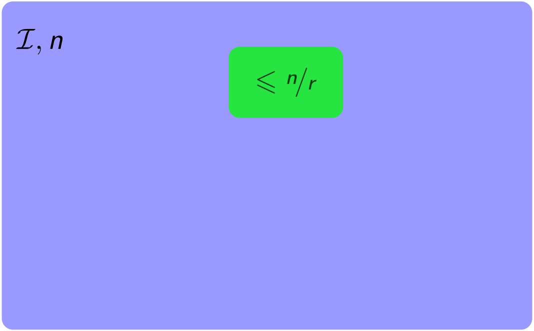
Weakly monotone maximization subset problems



Weakly monotone maximization subset problems



Weakly monotone maximization subset problems



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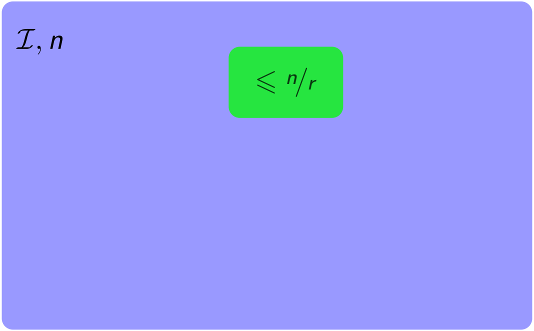
Weakly monotone maximization subset problems



- If a solution is found, it is an r-approximation.
- If not, there is no feasible solution.

The r-approximation takes time
$O^*(\binom{n}{n/r}) = O^*((\frac{en}{n/r})^{n/r}) = O^*((er)^{n/r}) = O^*(2^{n \log(er)/r})$.

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Can we improve this time to $O^*(2^{n/r})$?

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$O^*(\binom{n}{n/r}) = O^*((\frac{en}{n/r})^{n/r}) = O^*((er)^{n/r}) = O^*(2^{n\log(er)/r})$.

Can we improve this time to $O^*(2^{n/r})$?

- In this talk we don't care! (?? sort of)
- Bottom line: $r^{n/r}$ is a **Base-line Trade-off**.
- When can we do better?
- When is it optimal?

Min Asymmetric Traveling Salesman Problem

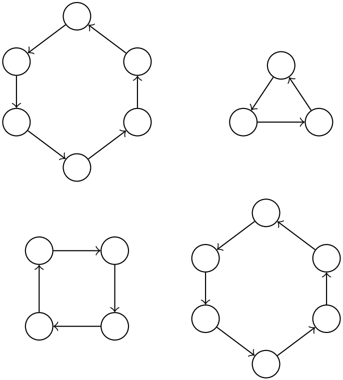
Min ATSP in polytime

- $O(\log n)$-approximation [FGM '82].
- $O(\frac{\log n}{\log \log n})$-approximation [AGMOS '10].

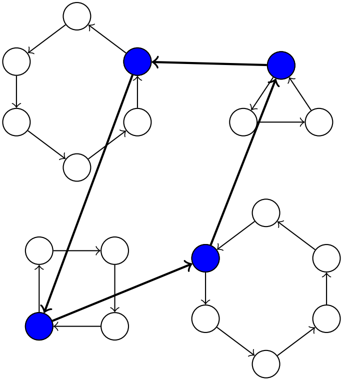
Our goal:

Theorem

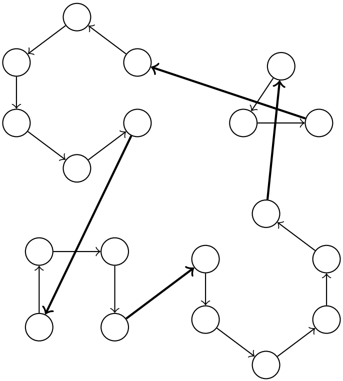
$\forall r \leqslant n$, $\boxed{\text{Min ATSP is } \log r\text{-approximable in time } O^*(2^{n/r}).}$



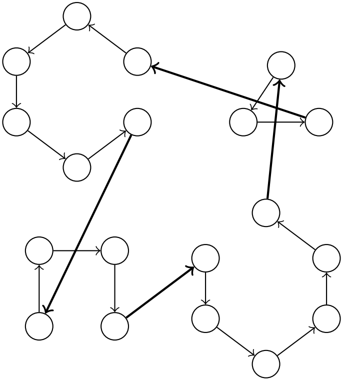
A circuit cover of minimum length can be found in polytime.



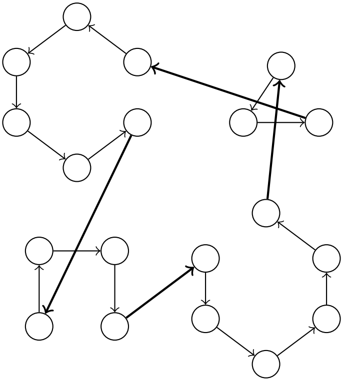
Pick any vertex in each cycle and recurse.



This can only decrease the total length (triangle inequality).



ratio = recursion depth: $\log n$ for polytime; $\log r$ for time $2^{n/r}$.



Is this optimal? NO!
Is this close to optimal? No idea!

Inapproximability in super-polynomial time

(Randomized) Exponential Time Hypothesis:
There is no (randomized) $2^{o(n)}$-time algorithm solving 3-SAT.

Theorem (CLN '13)

Under randomized ETH, $\forall \varepsilon > 0$, for all sufficiently big $r < n^{1/2-\varepsilon}$,

Max Independent Set is not r-approximable in time $2^{n^{1-\varepsilon}/r^{1+\varepsilon}}$.

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SAT formula ϕ with N variables $\rightsquigarrow$ graph G with $r^{1+\varepsilon}N^{1+\varepsilon}$ vertices

- ϕ satisfiable $\Rightarrow \alpha(G) \approx rN^{1+\varepsilon}$.
- ϕ unsatisfiable $\Rightarrow \alpha(G) \approx r^{\varepsilon}N^{1+\varepsilon}$.

Inapproximability in super-polynomial time

Goal: Assuming ETH, Π is not r-approximable in time $2^{o(n/f(r))}$

Inapproximability in super-polynomial time

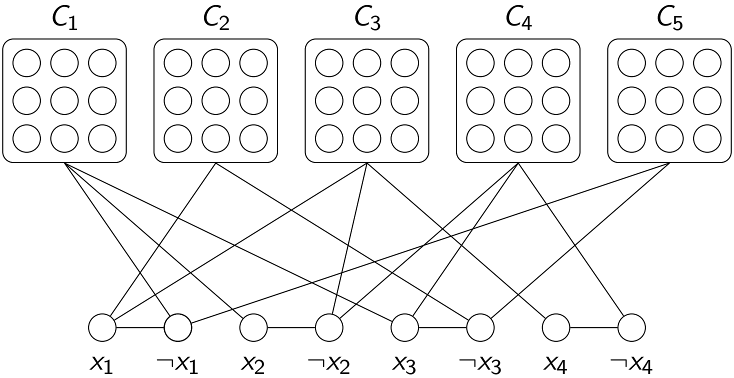
Goal: Assuming ETH, Π is not r-approximable in time $2^{o(n/f(r))}$

SAT formula ϕ with N variables $\rightsquigarrow \mathcal{I}$ instance of Π s.t.

- $|\mathcal{I}| \approx f(r)N$
- ϕ satisfiable $\Rightarrow$ val(Π) $\approx a$
- ϕ unsatisfiable $\Rightarrow$ val(Π) $\approx ra$

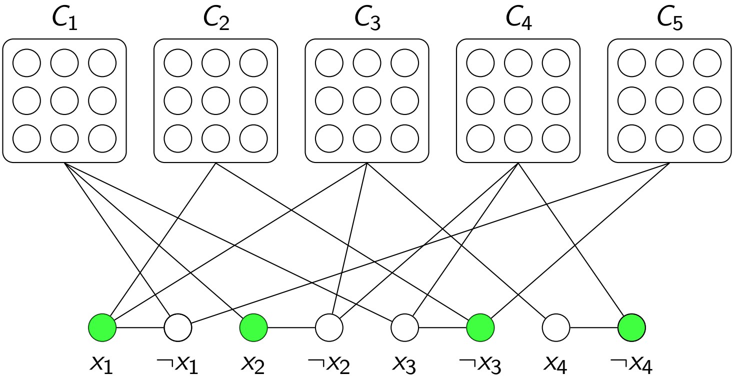
Min Independent Dominating Set

Inapproximability in polytime [I '91, H '93]



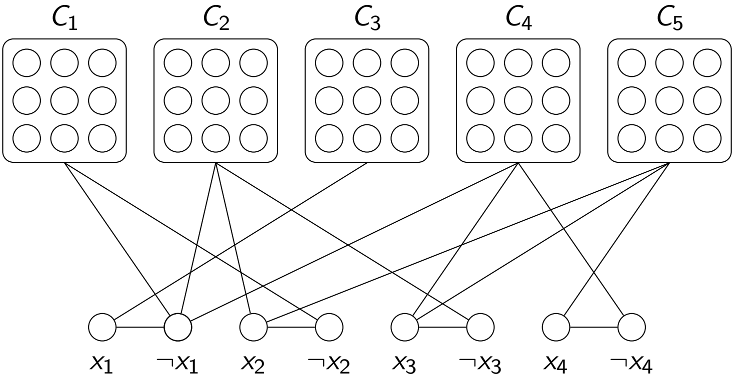
Satifiable CNF formula with N variables and CN clauses

Inapproximability in polytime [I '91, H '93]



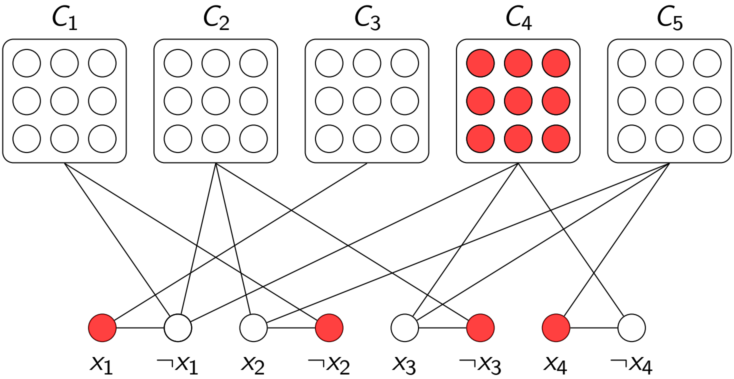
MIDS of size N

Inapproximability in polytime [I '91, H '93]



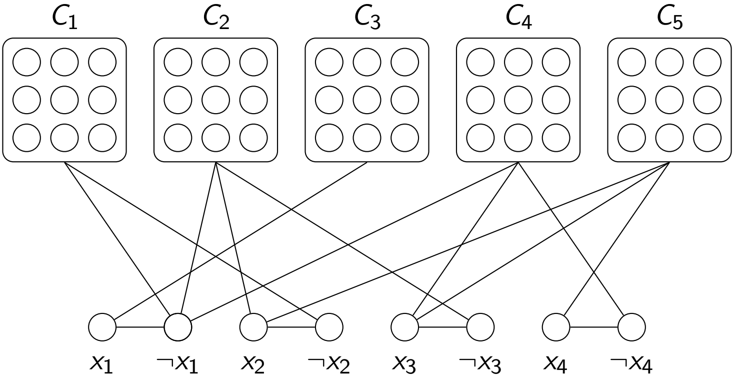
Unsatifiable CNF formula with N variables and CN clauses

Inapproximability in polytime [I '91, H '93]



MIDS of size greater than rN

Inapproximability in polytime [I '91, H '93]



Set $r = N^{9998} \approx n^{\frac{9998}{10000}} \geqslant n^{0.999}$

As $n = 2N + CrN^2 \approx N^{1000}$

(In)approximability in subexponential time

Our goal:

Theorem

Under ETH, $\forall \varepsilon > 0$, $\forall r \leqslant n$,

MIDS is not r-approximable in time $O^*(2^{n^{1-\varepsilon}/r^{1+\varepsilon}})$.

almost matching the r-approximation in time $O^*(2^{n \log(er)/r})$.

(In)approximability in subexponential time

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Under ETH, $\forall \varepsilon > 0$, $\forall r \leqslant n$,

MIDS is not r-approximable in time $O^*(2^{n^{1-\varepsilon}/r^{1+\varepsilon}})$.

- 

In the previous reduction, $\frac{n^{1-\varepsilon}}{r^{1+\varepsilon}} \approx N^{2-\varepsilon'}$.
We need to build a graph with $n \approx rN$ vertices.

(In)approximability in subexponential time

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In the previous reduction, $\frac{n^{1-\varepsilon}}{r^{1+\varepsilon}} \approx N^{2-\varepsilon'}$.
We need to build a graph with $n \approx rN$ vertices.
-  Can we use only r vertices per independent set C_i and use the inapproximability of a CSP to boost the gap?

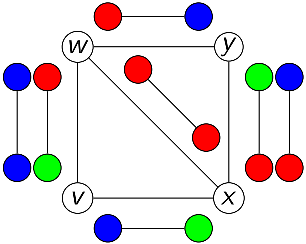
Almost linear PCP with perfect completeness?

Lemma (D '05, BS '04)

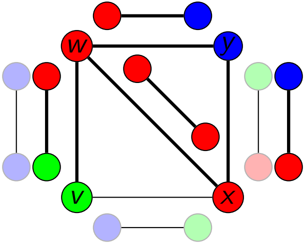
$\exists c_1, c_2 > 0$, *we can transform* ϕ *a SAT instance of size* N *into a constraint graph* $G = \langle (V, E), \Sigma, E \to 2^{\Sigma^2} \rangle$ *such that:*

- $|V| + |E| \leqslant N(\log N)^{c_1}$ *and* $|\Sigma| = O(1)$.
- ϕ *satisfiable* $\Rightarrow$ $UNSAT(G) = 0$.
- ϕ *unsatisfiable* $\Rightarrow$ $UNSAT(G) \geqslant 1/(\log N)^{c_2}$.

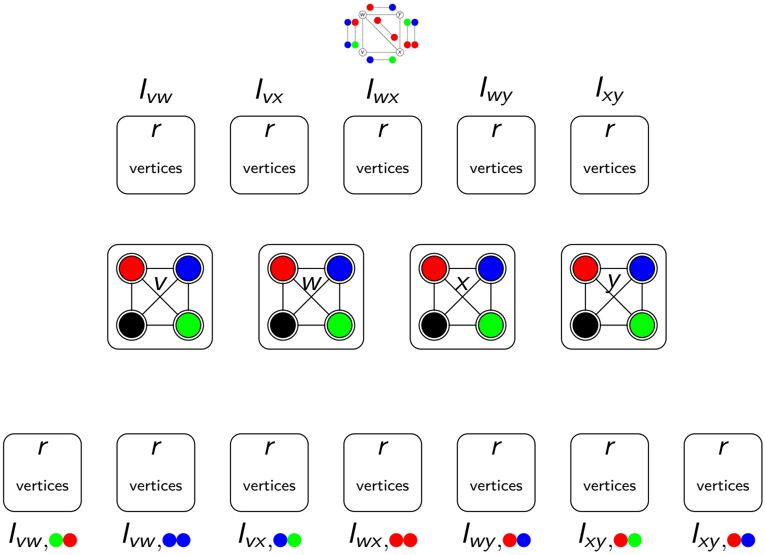
Constraint graph



Constraint graph



v[●] w[●] x[●] y[●]


w
y
v
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I_{vw}
I_{vx}
I_{wx}
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r vertices
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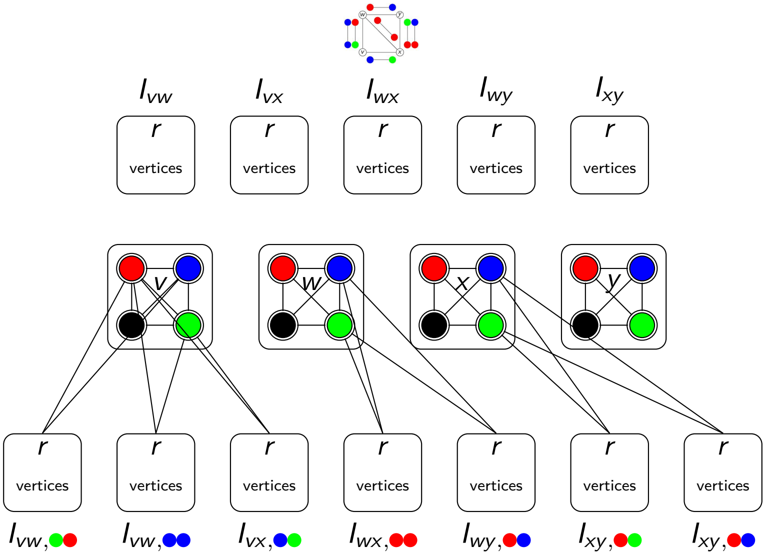
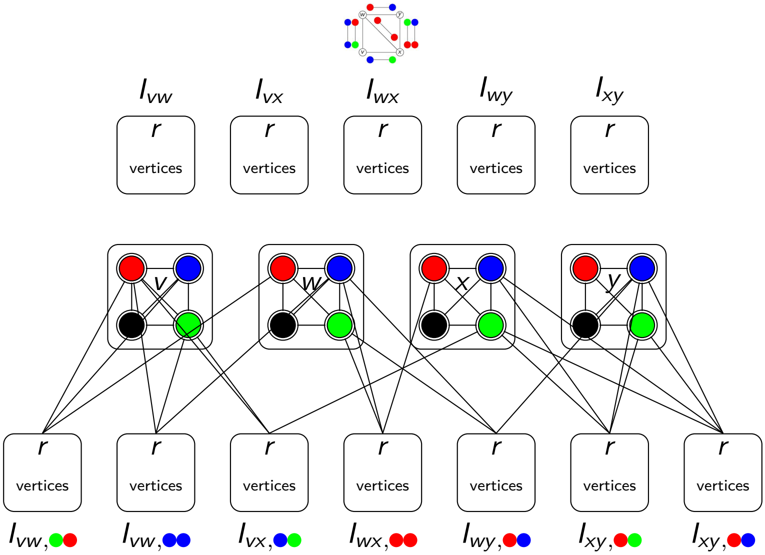
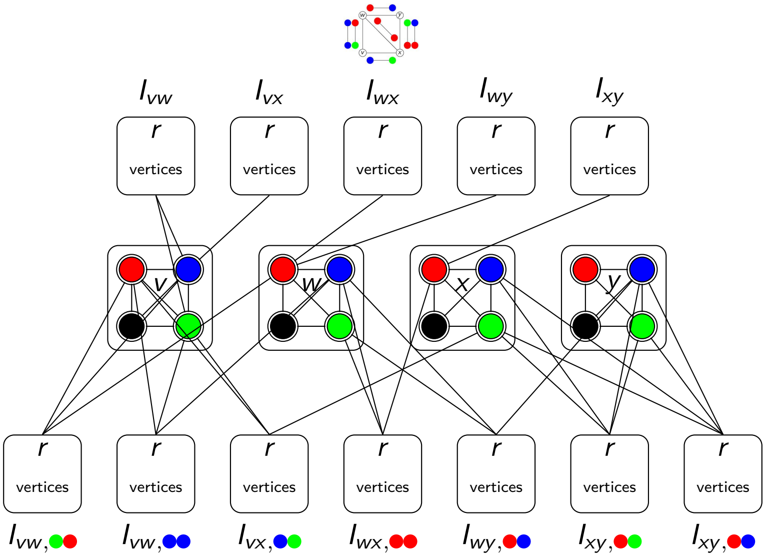


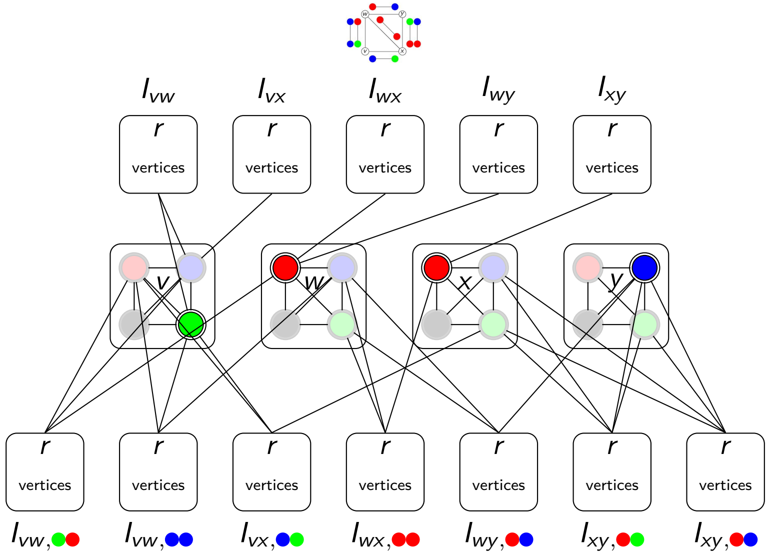
$$I_{st,ab} \leftrightarrow s[\neq a]$$



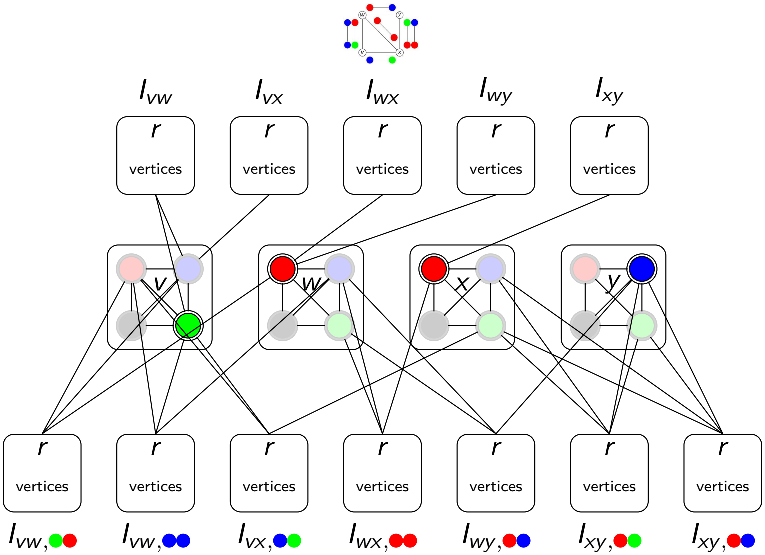
$I_{st,ab} \leftrightarrow t[b']$ if ab' satisfies st



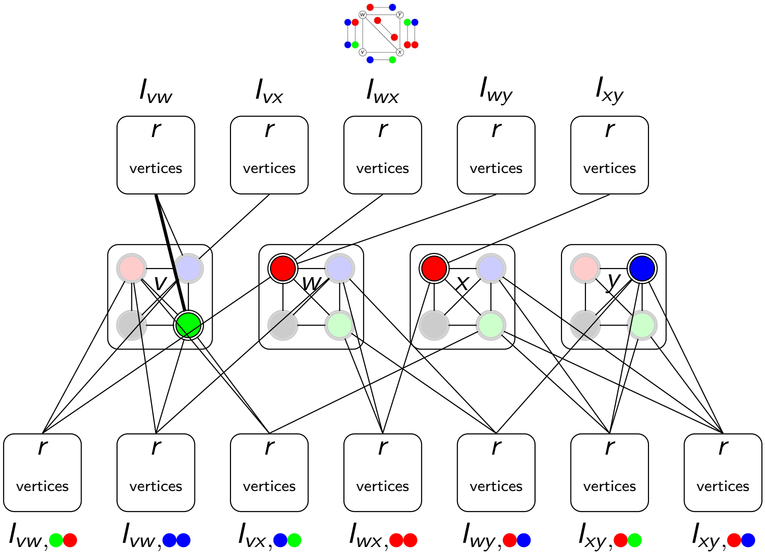
$I_{st} \leftrightarrow s[a]$ if $\exists b$, ab satisfies st



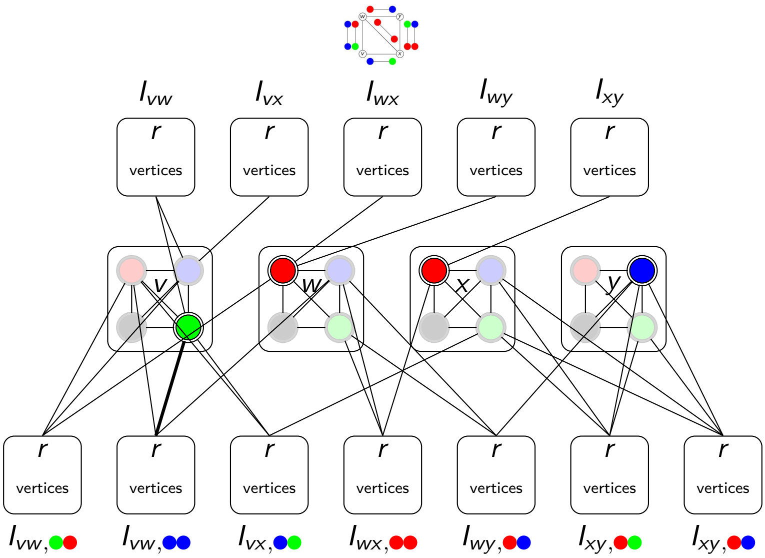
st is satisfied by the *coloration* iff I_{st} and $\bigcup_{a,b} I_{st,ab}$ are dominated.



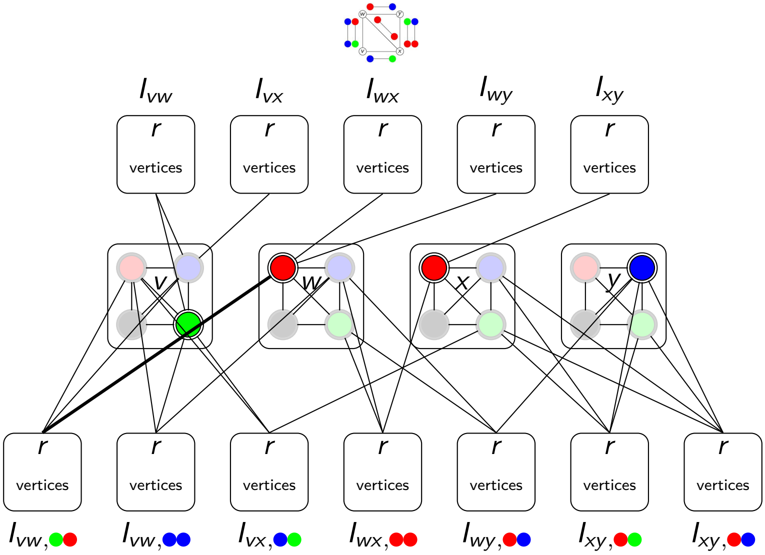
Take for instance *vw* satisfied by ●●.



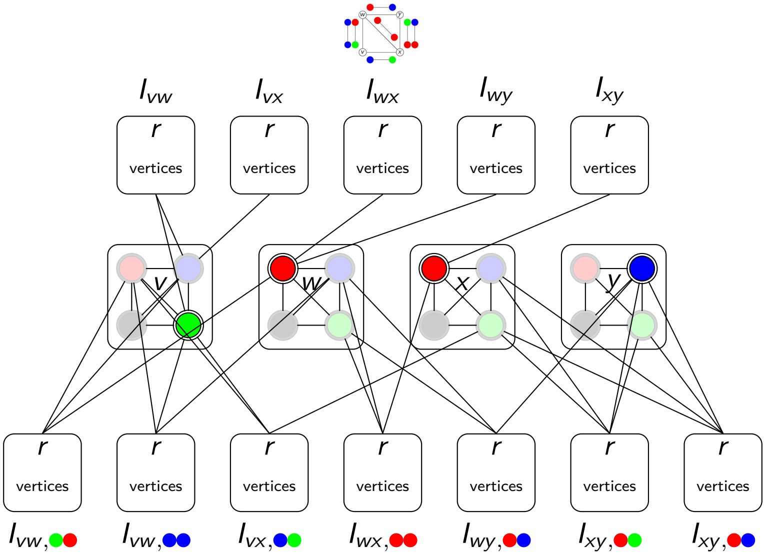
v[●] dominates I_{vw} ($\exists$●, ●● satisfies vw).



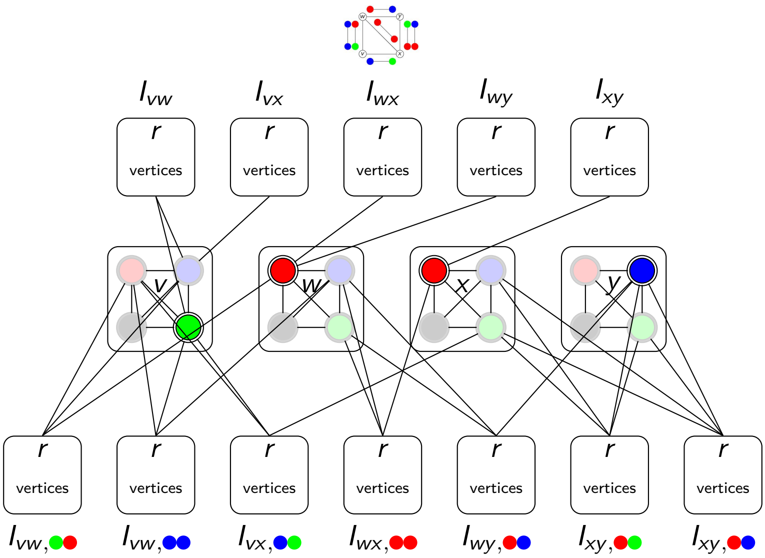
v[●] dominates $I_{vw,●●}$ (and potentially all the $I_{vw,ab}$ with $a \neq$ ●).



w[●] dominates $I_{vw,}$●● (and potentially all the $I_{vw,ab}$ with $a =$ ●).



Reciprocally, I_{st} needs $s[a]$ with ab satisfying st for some b.



Then, $I_{st,ab}$ can only be dominated by $t[b']$ (if ab' satisfies st).

$$\text{SAT } (\phi) \rightsquigarrow \text{CG } (V, E) \rightsquigarrow \text{MIDS } (V', E')$$

Recall $|V| + |E| \leqslant N(\log N)^{c_1}$ and $\Sigma = O(1)$.

- ϕ satisfiable $\Rightarrow$ MIDS of size $|V| \approx N$.
- ϕ unsatisfiable $\Rightarrow$ MIDS of size $|V| + r\frac{|E|}{(\log N)^{c_2}} \approx rN$
- $n := |V'| \leqslant (|\Sigma| + 1)|V| + (1 + |\Sigma|^2)r|E| \approx rN$

Max Induced Path/Forest/Tree

Theorem

Under ETH, $\forall \varepsilon > 0$, $\forall r \leqslant n^{1/2-\varepsilon}$,

Max Induced Forest has no r-approximation in time $2^{n^{1-\varepsilon}/(2r)^{1+\varepsilon}}$.

A max induced forest has size in $[\alpha(G), 2\alpha(G)]$.

Theorem

Under ETH, $\forall \varepsilon > 0$, $\forall r \leqslant n^{1/2-\varepsilon}$,

Max Induced Forest has no r-approximation in time $2^{n^{1-\varepsilon}/(2r)^{1+\varepsilon}}$.

A max induced forest has size in $[\alpha(G), 2\alpha(G)]$.

- An independent set is a special forest.
- A forest has an independent set of size at least the half.

Theorem

Under ETH, $\forall \varepsilon > 0$, $\forall r \leqslant n^{1/2-\varepsilon}$,

Max Induced Tree has no r-approximation in time $2^{n^{1-\varepsilon}/(2r)^{1+\varepsilon}}$.

Add a universal vertex v to the gap instances of MIS: $G \rightsquigarrow G'$.

Theorem

Under ETH, $\forall \varepsilon > 0$, $\forall r \leqslant n^{1/2-\varepsilon}$,

Max Induced Tree has no r-approximation in time $2^{n^{1-\varepsilon}/(2r)^{1+\varepsilon}}$.

Add a universal vertex v to the gap instances of MIS: $G \rightsquigarrow G'$.

- G' has an induced tree of size $\alpha(G) + 1$.
- If T is an induced tree of G', $\alpha(G) \geqslant |T|/2$.

PCP-free inapproximability

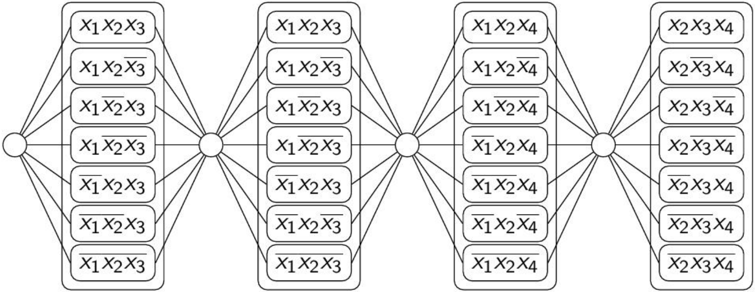
Our goal:

Theorem

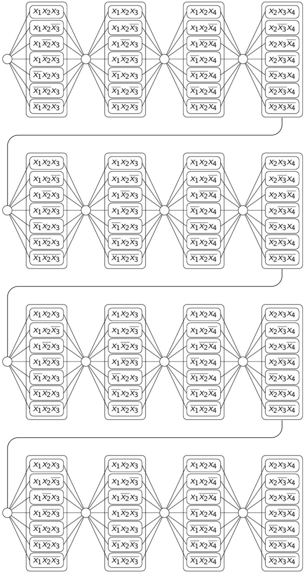
Under ETH, $\forall \varepsilon > 0$ *and* $\forall r \leqslant n^{1-\varepsilon}$,

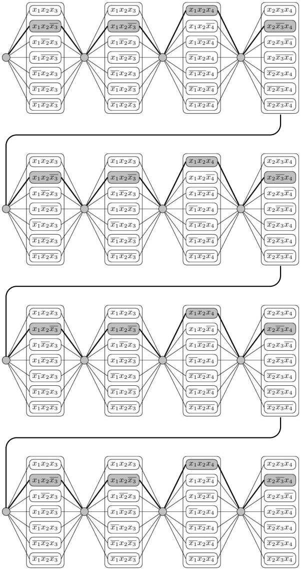
Max Induced Path has no r*-approximation in time* $2^{o(n/r)}$.

Walking through partial satisfying assignments



Contradicting edges are not represented





Max Minimal Vertex Cover

Approximability in polytime [BDP '13]

- MMVC admits a $n^{1/2}$-approximation,
- but no $n^{1/2-\varepsilon}$-approximation for any $\varepsilon > 0$, unless P=NP.

Approximability in polytime [BDP '13]

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- but no $n^{1/2-\varepsilon}$-approximation for any $\varepsilon > 0$, unless P=NP.

Our goal:

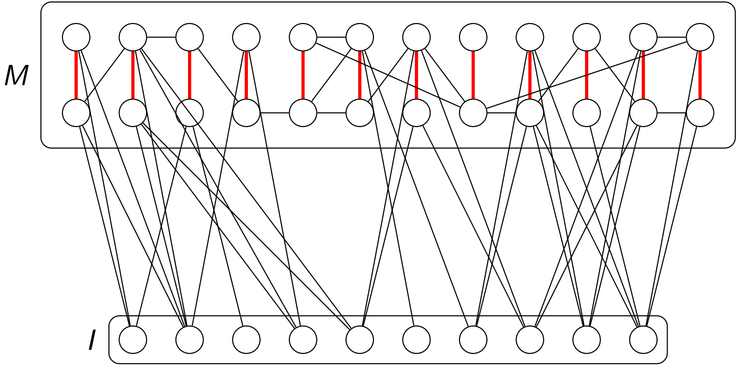
Theorem

For any $r \leqslant n$, *MMVC is r-approximable in time $O^*(3^{n/r^2})$.*

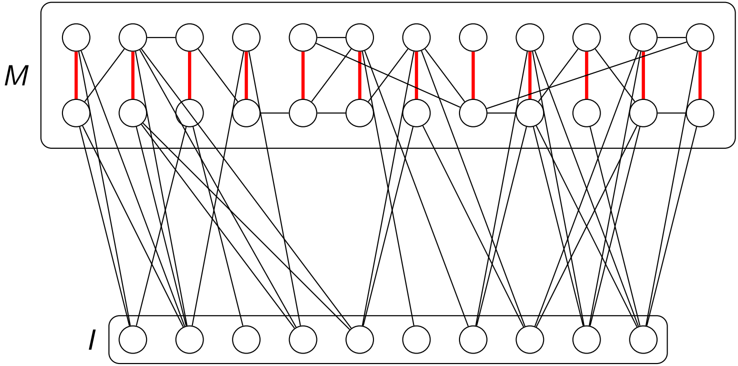
Theorem

Under ETH, $\forall \varepsilon > 0$, $\forall r \leqslant n^{1/2-\varepsilon}$,

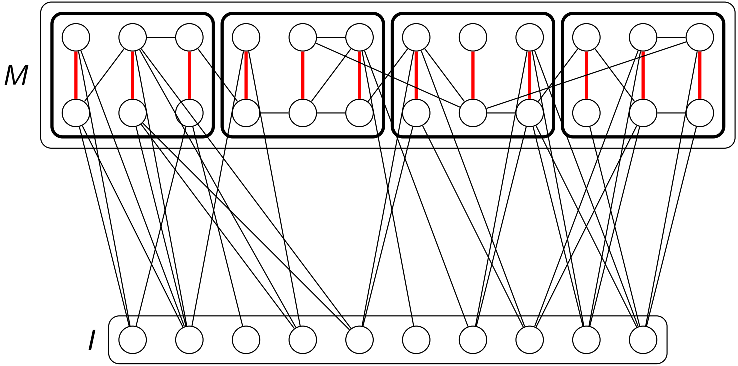
MMVC is not r-approximable in time $O^*(2^{n^{1-\varepsilon}/r^{2+\varepsilon}})$.



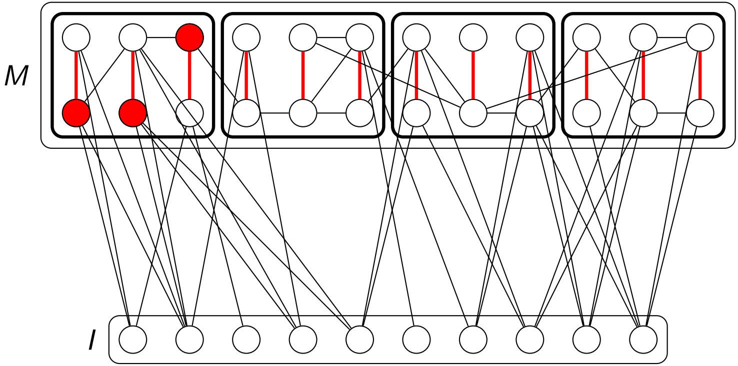
Compute any maximal matching M.



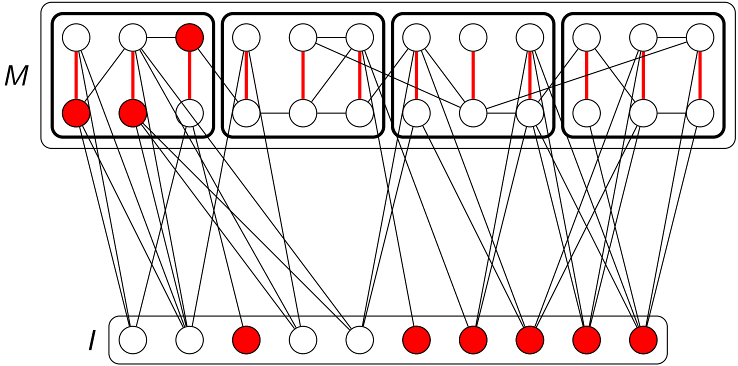
If $|M| \geqslant {}^{n}/_{r}$, then any (minimal) vertex cover contains $\geqslant {}^{n}/_{r}$.



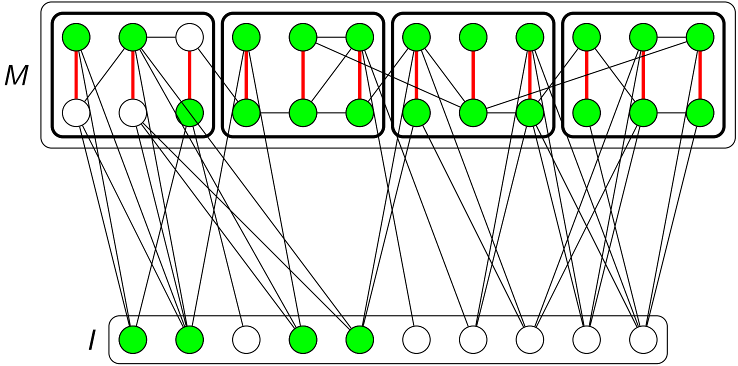
Otherwise split M into r parts $(A_1, A_2, \ldots, A_r)$ of size $\leqslant n/r^2$.



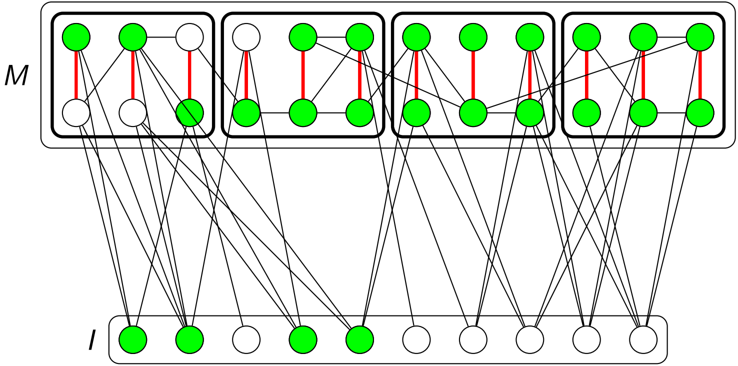
For each of the $\leqslant 3^{n/r^2}$ independent sets of each $G[A_i]$,



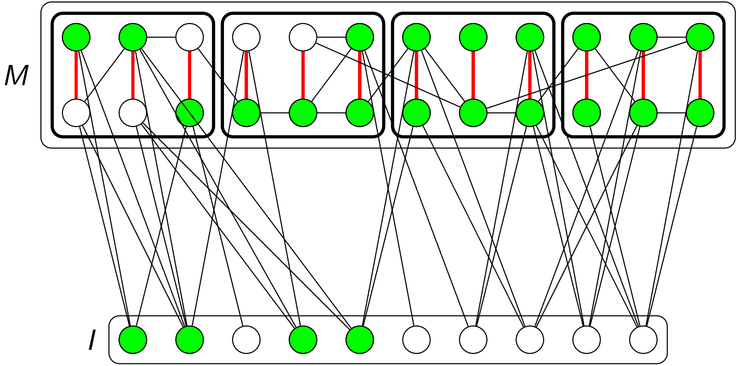
add all the non dominated vertices of I,



and compute a minimal vertex cover from the complement.



and compute a minimal vertex cover from the complement.



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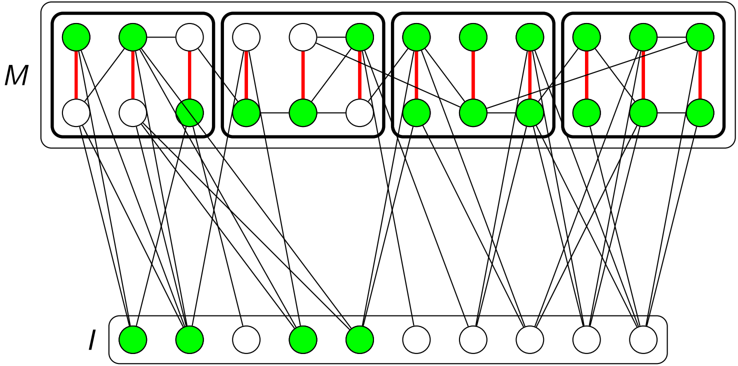

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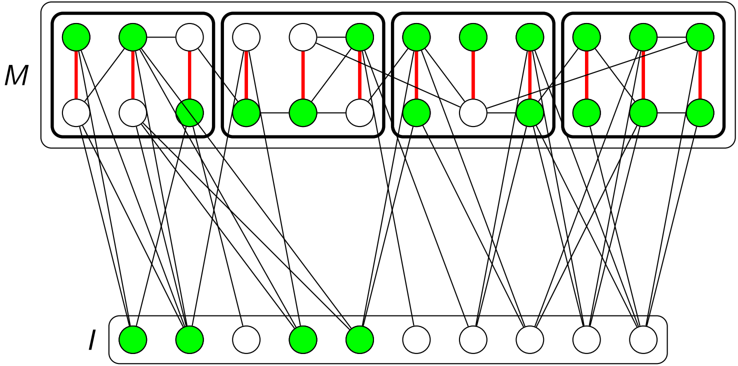

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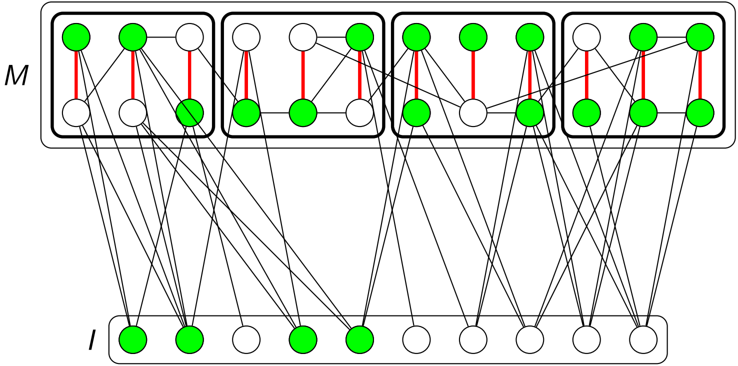

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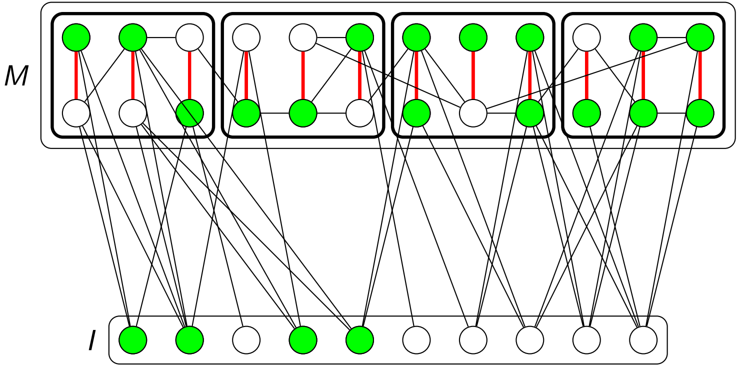
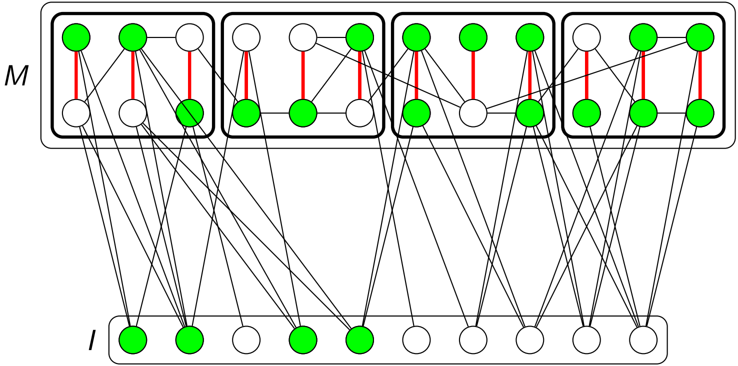
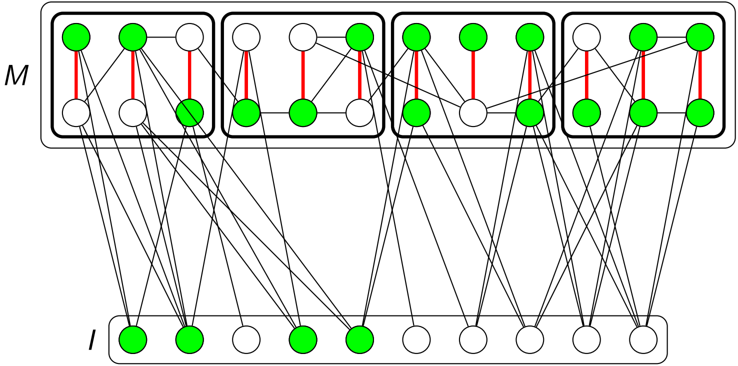


An optimal solution $R = N(\overline{R}) = N(\overline{R} \cap I) \cup \bigcup_i N(\overline{R} \cap A_i)$.

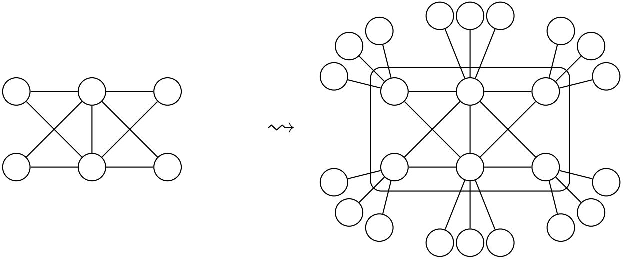


$$\exists i,\ |N(\overline{R} \cap I) \cup N(\overline{R} \cap A_i)| \geqslant \frac{|N(\overline{R})|}{r}.$$



$\overline{R} \cap A_i$ will be tried, and completed with a superset of $\overline{R} \cap I$.

MIS ($\approx rN$ vertices) $\rightsquigarrow$ MMVC ($\approx r^2N$ vertices)



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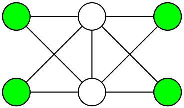

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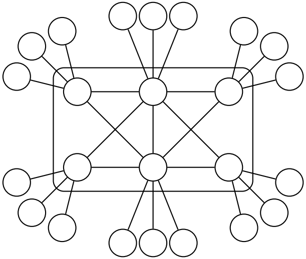

ϕ satisfiable $\Rightarrow |\text{IS}| \approx rN$; ϕ unsatisfiable $\Rightarrow |\text{IS}| \approx N$.

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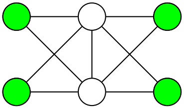

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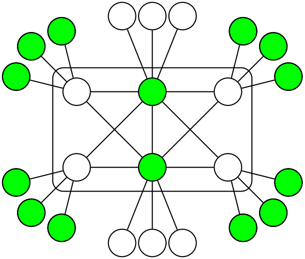

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Thank you for your attention!