Complexity Results for the Empire Problem in Collection of Stars

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Abstract. In this paper, we study the Empire Problem, a generalization of the coloring problem to maps on two-dimensional compact surface whose genus is positive. Given a planar graph with a certain partition of the vertices into blocks of size r, for a given integer r, the problem consists of deciding if s colors are sufficient to color the vertices of the graph such that vertices of the same block have the same color and vertices of two adjacent blocks have different colors. In this paper, we prove that given a 5-regular graph, deciding if there exists a 4-coloration is **NP**-complete. Also, we propose conditional **NP**-completeness results for the Empire Problem when the graph is a collection of stars. A star is a graph isomorphic to $K_{1,q}$ for some $q \geq 1$. More exactly, we prove that for $r \geq 2$, if the $(2r-1)$-coloring problem in $2r$-regular connected graphs is **NP**-complete, then the Empire Problem for blocks of size $r+1$ and $s = 2r-1$ is **NP**-complete for forests of $K_{1,r}$. Moreover, we prove that this result holds for $r = 2$. Also for $r \geq 3$, if the r-coloring problem in $(r+1)$-regular graphs is **NP**-complete, then the Empire Problem for blocks of size $r+1$ and $s = r$ is **NP**-complete for forests of $K_{1,1} = K_2$, i.e., forest of edges. Additionally, we prove that this result is valid for $r = 2$ and $r = 3$. Finally, we prove that these results are the best possible, that is for smallest value of s or r, the Empire Problem in these classes of graphs becomes polynomial.

Keywords: Empire Problem, Coloring in regular graphs, **NP**-completeness, Forests of stars.

1 Introduction

Graph coloring problem is an important optimization problem because scheduling problems appearing in real-life situations may often be modeled as graph coloring problems (see [1,7,13]). For instance, scheduling problems involving only incompatibility constraints correspond to the classical vertex coloring problem. A k-coloration of a graph

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$G = (V, E)$ is a mapping $c : V \rightarrow \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for all $[u, v] \in E$. It is well known that, given an integer k, deciding if a graph admits a k-coloration is a **NP**-complete problem if $k \geq 3$ and polynomial otherwise [6]. The coloring problem consists in finding the minimum k such that G is k-colorable, this number is called the chromatic number of G denoted $\chi(G)$.

Using the Brooks' theorem, it is well known that a connected graph with maximum degree Δ, for any $\Delta \geq 3$, is Δ-colorable, except for $K_{\Delta+1}$, and such a coloration can be found within polynomial time. So, an open question concerning the coloration is to know the complexity of the k-coloring problem in $(k+1)$-regular graphs and such a result will help narrowing down the gap between **P** and **NP**-complete classes for the coloration with respect to the maximum degree of the graph. Dailey [3] proved that the 3-coloring problem in 4-regular graph is **NP**-complete, even if the graph is planar. To our best knowledge, no result exist for $k > 3$. We prove in this paper that the 4-coloring problem in 5-regular graph is **NP**-complete.

The Empire Problem is a generalization of the coloring problem to maps on two-dimensional compact surfaces whose genus is positive. In graph theory terminology, the Empire Problem can be described as follows: given an integer r and a planar graph $G = (V, E)$, what is the minimum number s of colors needed to color a map in which each country has r colonies? A country and its colonies should be colored the same, and no pair of distinct colonies (belonging to two different countries) are adjacent. Each country is called block (sometimes a block is also called empire). We denote the decision version of the Empire Problem by s-$\textsc{Col}_r$.

In 1890, Heawood [8] conjectured that at most $6r$ colors are necessary to color every instance of the Empire Problem in the plane, where blocks are of size at most r. He could only proved the conjecture for $r = 2$. In 1981, Taylor [5] proved Heawood's conjecture for $r = 3$ and $r = 4$. Later, in 1984 this conjecture has completely been proved by Jackson and Ringel [9] for every $r \geq 2$. For random graphs, better bounds exist. For instance, McGrae and Zito [10] proved that for every fixed integer $r > 1$ there exists a positive integer $s_r = O(\frac{r}{\log r})$ such that $s_r < s \leq 2r$ asymptotically almost surely for a random n-vertex tree. Later, Coper, McGrae and Zito [2] improved this result and showed that for every fixed integer $r > 1$, $s_r \leq s \leq s_r + 6$. Furthermore, if $r > \frac{2(s_r-1)^2}{2s_r-3} \log s_r$ then $s_r + 1 \leq s \leq s_r + 6$.

From a deterministic point of view, the complexity of the Empire Problem in several classes of planar graphs, has been mainly studied by McGrae and Zito in two recent papers [12,11]. The case $r = 1$ corresponds to the coloring problem in planar graphs. Conversely, for any s, r, the complexity of s-$\textsc{Col}_r$ in G is equivalent to the complexity of the s-coloring problem in the reduced graph $R_r(G)$, where $R_r(G)$ is obtained from G by contracting each country into a distinct node. In particular, s-$\textsc{Col}_r$ is polynomial as soon as $s \leq 2$. Hence, the complexity of the Empire Problem seems interesting only when we deal with very particular classes of planar graphs. For instance in [12], the authors give a full dichotomy theorem for trees. More exactly, McGrae and Zito proved that for fixed positive integers r and s, with $r \geq 2$, s-$\textsc{Col}_r$ in trees is **NP**-complete, if $3 \leq s \leq 2r - 1$ and polynomial otherwise. For general planar graphs, McGrae and Zito [12] showed that for fixed positive integers r and s, with $r \geq 2$, s-$\textsc{Col}_r$ is **NP**-complete if $3 \leq s \leq 6(r-1)$. They proved also in [11] that s-$\textsc{Col}_r$ is **NP**-complete if $s < 7$ for

$r = 2$ and $s < 6r - 3$ for $r \geq 3$. Finally, for linear forests, i.e., collection of induced paths, McGrae and Zito [12] proved on the one hand that $(2r-1)$-COL_r is polynomial for paths of length at most $2r-1$, for fixed positive integer $r \geq 2$, and on the other hand, they showed that s-COL_r is **NP**-complete for any fixed positive integers r and s, with $3 \leq s < r$, if the paths have an arbitrary length. In [11], the values of r, s have been improved to any $r \geq 2$ and $3 \leq s \leq 2r - \sqrt{2r + \frac{1}{4}} + \frac{3}{2}$, but the length of the paths remain arbitrary large. Here, as corollary of Theorem 2, we strengthen this result by proving that 3-COL_3 is **NP**-complete, even for linear forests of length exactly 2, i.e., collection of disjoint $K_{1,2}$.

A related problem is the selective graph coloring problem. It consists of selecting one vertex per empire (not all the empire) in such a way that the chromatic number of the resulting induced selection is the smallest possible. In [4], some complexity results for various classes of graphs are given. Hence, the decision version of the selective graph coloring problem can be viewed as the restriction of the Empire Problem to select one vertex per empire.

This paper is organized as follows. In Section 2 we introduce some definition and notation. We show in Section 3 that the 4-coloring problem in 5-regular graph is **NP**-complete. We present in Section 4 some polynomial results and conditional **NP**-completeness results for s-COL_r for special classes of graphs and for given values of s and r. Indeed, we prove in this section that for $r \geq 2$, if the $(2r-1)$-coloring problem in $2r$-regular connected graphs is **NP**-complete, then $(2r-1)$-COL_{r+1} is **NP**-complete for forests of $K_{1,r}$. For $r = 2$, the result is valid without conditions. We prove that for $r \geq 2$ and $s \geq 2r$, s-COL_{r+1} is polynomial for forests of $K_{1,r}$, showing that the previous conditional **NP**-complete result will be the best possible for forests of $K_{1,r}$. Also, for any $r \geq 3$, if the r-coloring problem in $(r+1)$-regular graphs is **NP**-complete, then r-COL_{r+1} is **NP**-complete for graphs of disjoint edges. For $r = 3, 4$, the result is valid without conditions. Moreover, we have that r-COL_r is polynomial in forest of edges. Finally, we conclude by some discussions and perspectives in Section 5.

2 Definitions

All graphs in this paper are finite, simple and loopless. Let $G = (V, E)$ be a graph. An edge between u and v will be denoted $[u, v]$. For a vertex $v \in V$, let $N_G(v)$ denote the set of vertices in G that are adjacent to v, i.e., the neighbors of v and the degree of v is $d_G(v) = |N_G(v)|$. A graph $G = (V, E)$ is r-regular if $\forall v \in V$, $d_G(v) = r$. For any $V' \subseteq V$, $G[V']$ is the graph induced by the set of vertices V', i.e., the graph obtained from G by deleting the vertices of $V - V'$ and all edges incident to at least one vertex of $V - V'$. If the graph is directed we denote the set of oriented edges $\overrightarrow{E}$ and (u, v) is the arc from u to v. For a directed graph, let $N^+_{\overrightarrow{G}}(v) = \{u \in V : (v, u) \in \overrightarrow{E}\}$ be the outgoing neighbors of v and $N^-_{\overrightarrow{G}}(v) = \{u \in V : (u, v) \in \overrightarrow{E}\}$ be the incoming neighbors of v. Finally, $d^+_{\overrightarrow{G}}(v) = |N^+_{\overrightarrow{G}}(v)|$ and $d^-_{\overrightarrow{G}}(v) = |N^-_{\overrightarrow{G}}(v)|$.

A simple graph $G = (V, E)$ is called Eulerian if it has a cycle (called Eulerian cycle) which visits each edge exactly once. The famous Euler's theorem asserts that a graph $G = (V, E)$ is Eulerian iff it is connected and the degree of all its vertices is even. An independent set in a graph $G = (V, E)$ is a set $S \subseteq V$ of pairwise nonadjacent vertices. Alternatively, a k-coloration of $G = (V, E)$ can be viewed as a partition of V into k independent sets.

We denote by nG the disjoint union of n copies of a graph G. As usual P_n denotes the path induced by n vertices. The length of a path is the number of its edges. The complete bipartite graph $K_{1,q}$ is also called a q-star, ie., a root adjacent to q leafs. For more graph definitions and notations see [14].

In this paper we will be interested in the following two problems. Let $r \geq 3$ be a fixed integer.

r-COLORING in $(r + 1)$-regular graph
Input: A $(r + 1)$-regular connected graph $G = (V, E)$
Output: Deciding if there is an assignment of vertices to r colors such that no edge is monochromatic (no two adjacent vertices share the same color).

THE EMPIRE PROBLEM (denoted s-COL$_r$)
Input: A planar graph (not necessary connected) $G = (V, E)$ on pr vertices; a partition $\mathcal{V} = (V_1, \ldots, V_p)$ of V where $\forall i = 1, \ldots, p$, $|V_i| = r$.
Output: Deciding if there is an assignment of vertices to r colors such that no edge between V_i and V_j with $1 \leq i < j \leq p$ is monochromatic and all vertices in each V_i have the same colors.

Without loss of generality, we can assume that the partition $\mathcal{V}$ of the Empire Problem is a coloring. For an instance of s-COL$_r$ formed by a graph G, we denote $R_r(G)$ its reduced graph. $R_r(G)$ is obtained from G by contracting each empire to a distinct pseudo-vertex and by adding an edge between a pair of pseudo-vertices if there exists in G at least one edge between one vertex of the first empire of this pair and one vertex of the second empire of this pair.

3 The 4-Coloring Problem in 5-Regular Graphs

We study in this section the complexity of the k-coloring problem in $(k + 1)$-regular graphs. We prove in the following theorem that the problem is **NP**-complete for $k = 4$.

Theorem 1. 4-COLORING *in 5-regular connected graphs is* ***NP****-complete.*

Proof: We propose a polynomial reduction from 3-COLORING in 4-regular connected graphs, proved to be **NP**-complete in [3]. Let $G = (V, E)$ be a 4-regular connected graph on n vertices instance of the 3-coloring problem. We construct a 5-regular connected graph H in the following way: it contains two copies of G where copies of the $i - th$ vertex are v_i, v'_i and a gadget F depicted in Figure 1. This gadget contains $n - 3$ copies $H_1, \ldots, H_{n-3}$ of a same graph, n special vertices $u_1, \ldots, u_n$ where for $i = 1, \ldots, n - 4$, u_i is in copy H_i while the last four vertices $u_{n-3}, \ldots, u_n$ are in

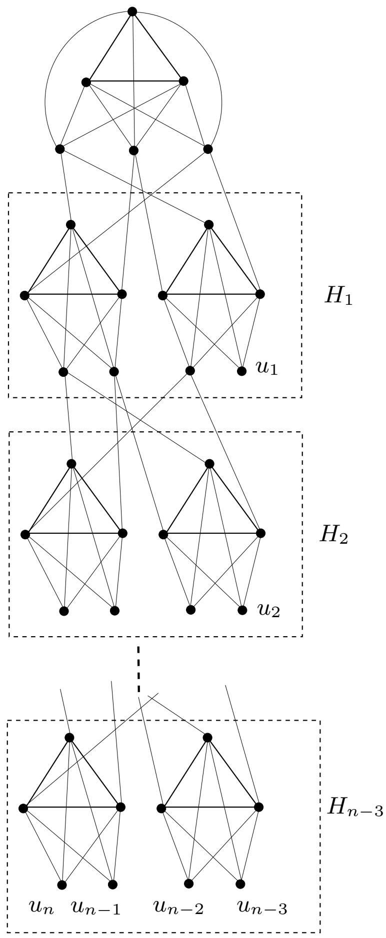

Fig. 1. The gadget F

H_{n-3}. In F, all vertices have a degree 5 except special vertices $u_1, \ldots, u_n$ of degree 3. Finally, each vertex u_i is linked to v_i and v'_i. Hence, H is a 5-regular connected graph and this construction can be done within polynomial time.

We claim that G is 3-colorable iff H is 4-colorable.

Actually, by pointing out that F is 4-colorable and the special vertices $u_1, \ldots, u_n$ must have the same color in any 4-coloration, the result follows. □

4 Complexity Results for the Empire Problem

We propose in this section some conditional **NP**-completeness results as well as polynomial results for the Empire Problem when the graph is a collection of stars.

Theorem 2. *Let $r \geq 2$ be an integer. If $(2r-1)$-*COLORING *in $2r$-regular connected graphs is **NP**-complete, then $(2r-1)$-*COL_{r+1} *is **NP**-complete in $nK_{1,r}$.*

Proof: We propose a polynomial reduction from $(2r-1)$-COLORING in $2r$-regular connected graphs.

Let $r \geq 2$. Consider an instance I of $(2r-1)$-COLORING formed by a $2r$-regular connected graph $G = (V, E)$ with $V = \{1, \ldots, n\}$. We define an orientation on the edges of E according to an Eulerian cycle in G. This cycle exists since G is connected and all vertices of V have an even degree (see Figure 2 for an illustration when $r = 2$).

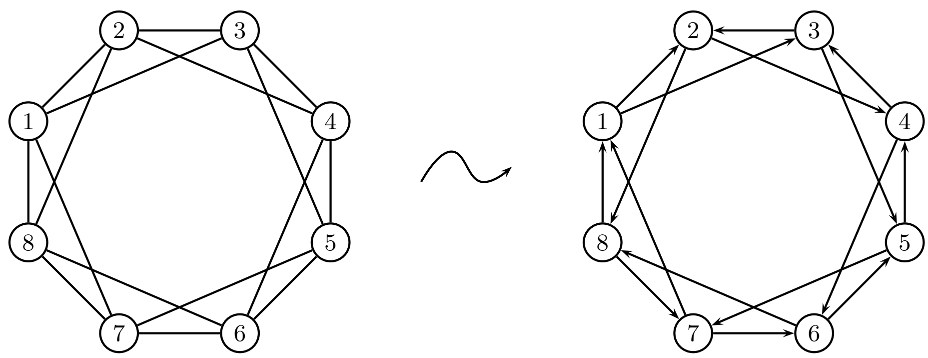

Fig. 2. Case $r = 2$. Orienting the edges of a 3-regular connected graph G according to an Eulerian cycle.

Denote by $\overrightarrow{G} = (V, \overrightarrow{E})$ the corresponding digraph. For $i \leq n$, let $e_\ell(i)$, for $\ell = 1, \ldots, r$, be the arcs incoming in vertex i in $\overrightarrow{E}$. We have $\overrightarrow{E} = \{e_\ell(i), \ell = 1, \ldots, r : i \leq n\}$ because G is $2r$-regular. Let $\overrightarrow{G}_i = (V, \overrightarrow{E_i})$, where $\overrightarrow{E_i} = \{e_\ell(j), \ell = 1, \ldots, r : j \leq i\}$. By construction $\overrightarrow{E_0} = \emptyset$.

We construct an instance I' of $(2r-1)$-COL_{r+1} problem formed by a graph $G' = (V', E')$ as follows (see Figure 3 for the case $r = 2$ and the 3-regular graph G given in Figure 2). We associate to each vertex $i \in V$, $r+1$ copies in V', for $p = 0, \ldots, r$, denoted i^p, which form an empire of size $r+1$ in G'. Let $N^-_{\overrightarrow{G}}(i) = \{h_1, \ldots, h_r\}$ be the incoming neighbors vertices of i. For each vertex i, for $i = 1, \ldots, n$, noting $e_\ell(i) = (h_\ell, i)$, for $\ell = 1, \ldots, r$, we add to G' the edges $[h_\ell^{d^+_{\overrightarrow{G}_{i-1}}(h_\ell)+1}, i^0]$, for $\ell = 1, \ldots, r$. Since $d^+_{\overrightarrow{G}}(i) = r, \forall i \in V$, we have $d_{G'}(i^p) \leq r, \forall i \leq n$ and $\forall p = 0, \ldots, r$. Indeed, by construction :

- $d_{G'}(i^0) = r$, because $d_{G'}(i^0)$ corresponds to the r arcs $e_\ell(i)$, for $\ell = 1, \ldots, r$, the only r arcs incoming in vertex i in $\overrightarrow{G}$.
- $d_{G'}(i^p) = 1$, for $p = 1, \ldots, r$, because by construction i^p has an edge incident if and only if $d^+_{\overrightarrow{G}_{j_p - 1}}(i) = p - 1$ at the iteration j_p. Since $d^+_{\overrightarrow{G}}(i) = r$, the iteration j_p exists.

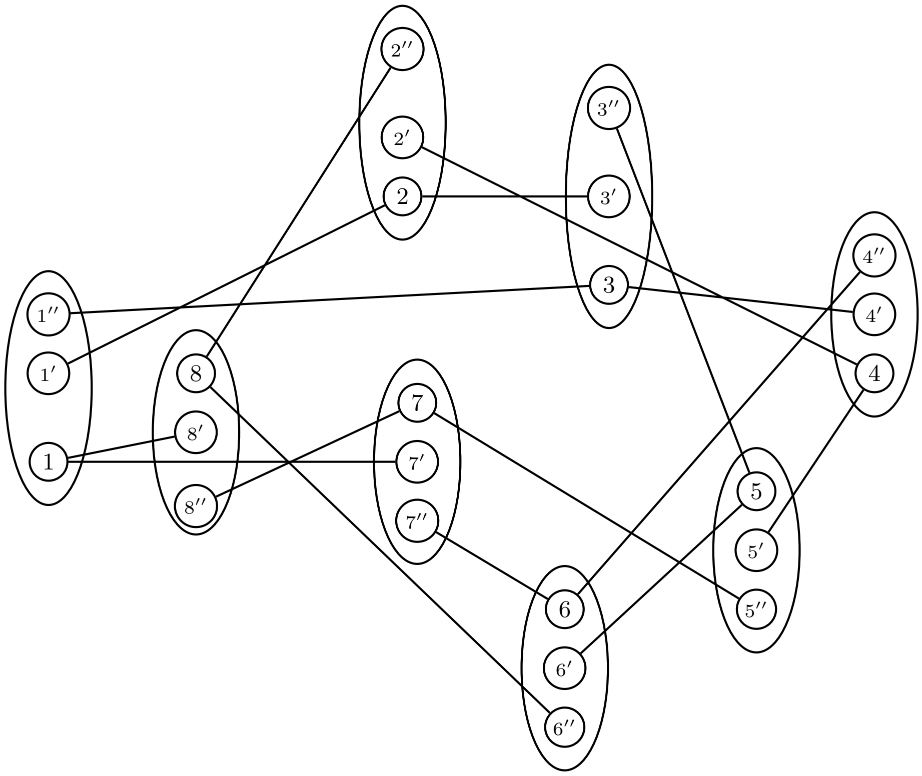

Fig. 3. Construction of G' from G when $r = 2$

Thus, since by construction $d_{G'}(i^0) = r$ and $d_{G'}(i^p) = 1$, for $p = 1, \ldots, r$ and $i = 1, \ldots, n$, G' is a $nK_{1,r}$ and this construction can be done within polynomial time. We remark that $R_{r+1}(G') = G$.

We show that G is $(2r-1)$-colorable if and only if G' is $(2r-1)$-colorable.

Suppose first that G is $(2r-1)$-colorable. We assign to vertices i^p in V', for $p = 0, \ldots, r$, the same color as i in V, for $i = 1, \ldots, n$. Hence, the coloring defines a truth assignment of vertices of G' to $2r-1$ colors. Therefore G' is $(2r-1)$-colorable.

Suppose now that G' is $(2r-1)$-colorable. Since $R_{r+1}(G') = G$, we have that G is $(2r-1)$-colorable by assigning to every vertex $i \in V$ the same color as its copies in V'. □

In [12,11], McGrae and Zito have obtained, on the one hand that for any fixed integer $k \geq 1$, $\lceil \frac{2kr}{k+1} \rceil$-$\textsc{Col}_r$ in collection of paths of length at most k is polynomial (p. 182 in [12]. Indeed, it is a corollary of a more general result in sparse graphs). For instance, $\lceil \frac{4r}{3} \rceil$-$\textsc{Col}_r$ in collection of paths of length at most 2 are polynomial. On the other hand, they proved that s-$\textsc{Col}_r$ is **NP**-complete for any $r > s \geq 3$ in linear forests, i.e., collection of paths of arbitrary length (Theorem 3 in [12]) and (in [11], Theorem 6), they have strengthen this result to any $r \geq 2$ and $3 \leq s \leq 2r - \sqrt{2r + 1/4} + 3/2$. Here, we deduce from Theorem 2 that 3-COL$_3$ is **NP**-complete in collection of paths of length exactly 2.

Corollary 1. *3-COL$_3$ is **NP**-complete in $nK_{1,2} = nP_3$.*

Proof: Since 3-COLORING in 4-regular connected graphs has been proved **NP**-complete in [3], the result follows from Theorem 2 with $r = 2$. □

Now, we prove that the result given in Theorem 2 is the best possible according the parameter s.

Proposition 1. *Let r and s be two integers such that $r \geq 2$ and $s \geq 2r$. s-COL$_{r+1}$ is polynomial in $nK_{1,r}$.*

Proof: Let G be a $nK_{1,r}$. G is a planar graph containing no induced subgraph of average degree larger than $\frac{2r}{r+1}$. According to Theorem 1 of [11], s-COL$_r$ is polynomial for planar graphs containing no induced subgraph of average degree larger than $\frac{s}{r}$. Since, $s \geq 2r$ we have that $\frac{s}{r} \geq \frac{2r}{r+1}$. Therefore, s-COL$_{r+1}$ is polynomial in $nK_{1,r}$ for $r \geq 2$ and $s \geq 2r$. □

Theorem 3. *If r-COLORING in $(r+1)$-regular graphs is **NP**-complete, then r-COL$_{r+1}$ is **NP**-complete in $nK_{1,1} = nK_2 = nP_2$.*

Proof: We propose a polynomial reduction from r-COLORING defined on $(r+1)$-regular graph.

Consider an instance I of r-COLORING formed by a $(r+1)$-regular graph $G = (V, E)$ with $V = \{1, \ldots, n\}$ and $E = \{e_1, \ldots, e_m\}$. We construct an instance I' of r-COL$_{r+1}$ problem formed by a graph $G' = (V', E')$ as follows. We associate to each vertex $i \in V$, $r+1$ copies i^{e_h} in V' with $e_h \in E_G(i)$ where $E_G(i)$ are the subset of edges incident to i in G. These $r+1$ copies form an empire of size $r+1$ in G'. For each edge $e_\ell = [i, j] \in E$, we add the edge $[i^{e_\ell}, j^{e_\ell}]$ in G'.

Since $d_G(i) = r+1$ and each edge is exactly incident to two vertices, we have that $d_{G'}(i^{e_\ell}) = 1$, $\forall i^{e_\ell} \in V'$. Thus G' is isomorphic to nK_2 and this construction can be done within polynomial time. We remark that $R_{r+1}(G') = G$.

We show that G is r-colorable if and only if G' is r-colorable.

Suppose first that G is r-colorable. We assign to vertices i^{e_h} in V', for $e_h \in E_G(i)$, the same color as i in V, for $i = 1, \ldots, n$. This coloring constitutes a truth assignment of vertices of G' to r colors. Therefore G' is r-colorable.

Suppose now that G' is r-colorable. Since $R_{r+1}(G') = G$, we have that G is r-colorable by assigning to every vertex $i \in V$ the same color as its copies in V'. □

From [12], we know that s-COL$_r$ in collection of disjoint edges and isolated vertices is polynomial for $s \geq r$. Here, we deduce that this result is the best possible for $r = 3, 4$.

Corollary 2. *r-COL$_{r+1}$ is **NP**-complete in nK_2 for $r = 3, 4$.*

Proof: Since 3-COLORING in 4-regular graphs and 4-COLORING in 5-regular graphs have been proved **NP**-complete in [3] and in Theorem 1 respectively, the result follows from Theorem 3 with $r = 3$ and $r = 4$. □

Proposition 2. *Let $r \geq 1$ be integer. r-COL$_r$ is polynomial in forest of edges.*

Proof: In [12,11], McGrae and Zito showed that $\left\lceil \frac{2kr}{k+1} \right\rceil$-COL$_r$ can be decided in polynomial time for forests of paths of length at most k. Thus, for $k = 1$ we have that r-COL$_r$ is polynomial in forest of edges. □

5 Conclusion

We are interested in this paper to some coloration problems. First, we gave a partial answer to the open question about the complexity of the k-coloring problem in $(k+1)$-regular graphs, or more generally in graphs of maximum degree $k + 1$. To the best of our knowledge, this question has never been raised in the literature, although the case $k = 3$ was solved in 1980 by Dailey [3]. Here, we have continued the investigation of this problem for the particular case of $k = 4$. Indeed, we showed that 4-COLORING in 5-regular connected graphs is **NP**-complete. We are not able to solve the cases $k \geq 5$, but we think that this question is important to better understand the complexity of the coloring problem. Based on this open problem, we have proposed some complexity results on the Empire Problem in sparse planar graphs. More exactly, we studied the Empire Problem a generalization of the k-coloring problem for which we proposed some conditional **NP**-completeness results for collection of stars. Hence, we proved that if $(2r-1)$-COLORING in $2r$-regular connected graphs is **NP**-complete, then $(2r-1)$-COL$_{r+1}$ is **NP**-complete for forests of $K_{1,r}$. This result holds for $r = 2$ and strengthens the results obtained by McGrae and Zito in [12,11] for s-COL$_r$ in forests of paths of arbitrary length. Furthermore, we proved that if r-COLORING in $(r+1)$-regular connected graphs is **NP**-complete, then r-COL$_{r+1}$ is **NP**-complete for forests of edges. This results is valid for $r = 2$ and $r = 3$. Also, we showed that these results are best possible, that is for smallest values of s and r, the Empire Problem in these classes of graphs becomes polynomial.

Looking for the approximation, we show that the optimization version of the Empire Problem with blocks of size r, noted MIN COL$_r$, is $\frac{4}{3}$-approximable for forests of paths of length at most 2. Indeed, if $R(G)$ is bipartite then G is 2-empire colorable. Otherwise, in [12,11], the authors showed that $\left\lceil \frac{2kr}{k+1} \right\rceil$-COL$_r$ can be decided in polynomial time for forests of paths of length at most k. Since G is not bipartite, G is at least 3-colorable which means that the value of an optimal coloration of G is larger than 3. Therefore, MIN COL$_r$ is $\frac{4}{3}$-approximable for forests of paths of length at most 2. It is then interesting to study the approximation of MIN COL$_r$ more generally for forests of stars. Using Proposition 1, we trivially get that MIN COL$_r$ is $\frac{2r}{3}$-approximable in $nK_{1,r}$. Thus, an interesting perspective is to try to improve this approximation ratio.

Another perspective is to study the complexity and approximation of the Empire Problem for other classes of sparse planar graphs with small average degree like nK_3 and nK_4.

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