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Generalized Additive Games

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Abstract A Transferable Utility (TU) game with n players specifies a vector of $2^n - 1$ real numbers, i.e. a number for each non-empty coalition, and this can be difficult to handle for large n. Therefore, several models from the literature focus on interaction situations which are characterized by a compact representation of a TU-game, and such that the worth of each coalition can be easily computed. Sometimes, the worth of each coalition is computed from the values of single players by means of a mechanism describing how the individual abilities interact within groups of players. In this paper we introduce the class of *Generalized Additive Games* (*GAG*s), where the worth of a coalition $S \subseteq N$ is evaluated by means of an interaction filter, that is a map $\mathcal{M}$ which returns the valuable players involved in the cooperation among players in S. Moreover, we investigate the subclass of *basic GAGs*, where the filter $\mathcal{M}$ selects, for each coalition S, those players that have *friends* but not *enemies* in S. We show that well-known classes of TU-games can be represented in terms of such basic GAGs, and we investigate the problem of computing the core and the semivalues for specific families of GAGs.

Keywords TU-games · core · semivalues · airport games · peer games · argumentation games.

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1 Introduction

Since the number of coalitions grows exponentially with respect to the number of players, it is computationally very interesting to single out classes of games that can be described in a concise way.
In the literature on coalitional games there exist several approaches for defining classes of games whose concise representation is derived by an additive pattern among coalitions. In some contexts, due to an underlying structure among the players, such as a network, an order, or a permission structure, the value of a coalition $S \subseteq N$ can be derived additively from a collection of subcoalitions $\{T_1, \cdots T_k\}$, $T_i \subseteq S \ \forall i \in \{1, \cdots, k\}$. Such situations are modeled, for example, by the graph-restricted games, introduced by Myerson [26] and further studied by Owen [27], the component additive games [11] and the restricted component additive games [12].

Sometimes, the worth of each coalition is computed from the values that single players can guarantee themselves by means of a mechanism describing the interactions of individuals within groups of players. As an example, consider a cost game where n players want to buy online n different objects and the value of a single player in the game is defined as the price of the object he buys. However, such a model may fail to reflect the importance of a subset of players in contributing to the value of the coalition they belong to. In the previous example, it is often the case that, by making a collective purchase, when a certain threshold price is reached, some of the objects will be sold for free and therefore the price that a coalition S should pay will depend only on the price of a subset of purchased objects.

In fact, in several cases the procedure used to assess the worth of a coalition $S \subseteq N$ is strongly related to the sum of the individual values over another subset $T \subseteq N$, not necessarily included in S.
Many examples from the literature fall into this category, among them the well-known *glove game*, the *airport games* ([21],[22]), the *connectivity game* and its extensions ([2],[20]), the *argumentation games* [6] and classes of operation research games, such as the *peer games* [8] and the *mountain situations* [24]: some of them will be described in Section 4.
In all the aforementioned models, the value of a coalition S of players is calculated as the sum of the single values of players in a subset of S. On the other hand, in some cases the worth of a coalition might be affected by external influences and players outside the coalition might contribute, either in a positive or negative way, to the worth of the coalition itself. This is the case, for example, of the *bankruptcy games* [3] and the *maintenance problems* ([19], [7]).

In this paper we introduce a general class of additive TU-games where the worth of a coalition $S \subseteq N$ is evaluated by means of an interaction filter, that is a map $\mathcal{M}$ which returns the valuable players involved in the cooperation among players in S.

Our objective is to provide a general framework for describing several classes of games studied in the literature on coalitional games and to give a kind of taxonomy of coalitional games that are ascribable to this notion of additivity over individual values.
The general definition of the map $\mathcal{M}$ allows various and wide classes of games to be embraced. Moreover, by making further hypothesis on $\mathcal{M}$, our approach enables to classify existing games based on the properties of $\mathcal{M}$. In particular, we introduce the class of *basic GAGs*, which is characterized by the fact that the valuable players in a coalition S are selected on the basis of the presence, among the players in S, of their *friends* and *enemies*, see Definition 4.
Several of the aforementioned classes of games can be described as basic GAGs, as well as games deriving from real-world situations. As an example, this model turns out to be suitable for representing an online social network, where friends and enemies of the web users are determined by their social profiles, as we shall see in Section 4.
The interest of this classification is not only taxonomical, since it also allows to study the properties of solutions for classes of games known from the literature. We indeed provide results on classical solution concepts for basic GAGs and we address the problem of how to guarantee that a basic GAG has a non-empty core.

The outline of the paper is as follows. Section 2 provides the basic definitions and notation regarding coalitional games. We introduce the model in Section 3 and the class of basic GAGs is discussed and characterized in Section 4. Next, Sections 5 and 6 present some results on the core and the semivalues of the GAGs. Finally, Section 7 concludes the paper.

2 Preliminaries

In this section we introduce some preliminary notation and definitions on coalitional games.

A TU-game, also referred to as *coalitional game*, is a pair (N, v), where N denotes the set of players and $v : 2^N \to \mathbb{R}$ is the *characteristic function*, with $v(\emptyset) = 0$. A group of players $S \subseteq N$ is called *coalition* and $v(S)$ is called the *value* or *worth* of the coalition S. If the set N of players is fixed, we identify a coalitional game (N, v) with its characteristic function v.
We shall assume that $N = \{1, \cdots, n\}$ and for a coalition S, we shall denote by s its cardinality $|S|$.

A particular class of games is that of *simple games*, where the characteristic function v can only assume values in $\{0, 1\}$.
A game (N, v) is said to be *monotonic* if it holds that $v(S) \leq v(T)$ for all $S, T \subseteq N$ such that $S \subseteq T$ and it is said to be *superadditive* if it holds that

$$v(S \cup T) \geq v(S) + v(T)$$

for all $S, T \subseteq N$ such that $S \cap T = \emptyset$.
Moreover, a game (N, v) is said to be *convex* or *supermodular* if it holds that

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$$

for all $S, T \subseteq N$.

Given a game v, an *imputation* is a vector $x \in \mathbb{R}^n$ such that $\sum_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N$. An important subset of the set of the imputations is the *core*, which represents a classical solution concept for TU-games. The *core* of v is defined as $C(v) = \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \quad \forall S \subset N\}$ [1].

Another class of solution concepts for coalitional games is the class of semivalues. A *semivalue* $\pi^{\mathbf{p}}$ is defined for all $i \in N$ as:

$$\pi_i^{\mathbf{p}}(v) = \sum_{S \subseteq N \setminus \{i\}} p_s \big[v(S \cup i) - v(S)\big], \tag{1}$$

where $\mathbf{p} = \{p_0, \cdots, p_{n-1}\}$ is such that $p_s \geq 0$ and $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1$: the non-negative number p_s represents the probability that a coalition of size $s+1$ will form. A semivalue is said to be *regular* if $p_s > 0$ for all $s \in \{0, \cdots, n-1\}$. If $p_s = \left[n\binom{n-1}{s}\right]^{-1}$, the corresponding semivalue defined via relation (1) is the *Shapley value* of v and is shortly denoted by $\sigma(v)$, while if $p_s = \frac{1}{2^{n-1}}$, the corresponding semivalue is the *Banzhaf value* of v, shortly denoted by $\beta(v)$. For a general introduction on cooperative games, see Maschler et al. (2013).

3 Generalized Additive Games (GAGs)

In this section we define the class of games that is the object of the paper, and we provide some examples and basic properties.

The basic ingredients of our definition are the set $N = \{1, \ldots, , n\}$, representing the set of players, a map $v : N \to \mathbb{R}$, assigning a real value to each player and a map $\mathcal{M} : 2^N \to 2^N$, called the *coalitional map*, which assigns a coalition $\mathcal{M}(S)$ to each coalition $S \subseteq N$ of players.

Definition 1 We shall call *Generalized Additive Situation* (GAS) any triple $\langle N, v, \mathcal{M} \rangle$, where N is the set of the players, $v : N \to \mathbb{R}$ is a map that assigns to each player a real value and $\mathcal{M} : 2^N \to 2^N$ is a *coalitional map*, which assigns a (possibly empty) coalition $\mathcal{M}(S)$ to each coalition $S \subseteq N$ of players and such that $\mathcal{M}(\emptyset) = \emptyset$.

Definition 2 Given the GAS $\langle N, v, \mathcal{M} \rangle$, the associated *Generalized Additive Game* (GAG) is defined as the TU-game $(N, v^{\mathcal{M}})$ assigning to each coalition the value

[1] Note that all the previous definitions hold for TU-games where v represents a gain, while the inequalities should be replaced with $\leq$ when v is a cost function.

$$v^{\mathcal{M}}(S) = \begin{cases} \sum_{i \in \mathcal{M}(S)} v(i) \text{ if } \mathcal{M}(S) \neq \emptyset \\ \\ \quad 0 \qquad\qquad \text{otherwise.} \end{cases} \tag{2}$$

Example 1 (simple games) Let w be a simple game. Then w can be described by the GAG associated to $\langle N, v, \mathcal{M} \rangle$ with $v(i) = 1$ for all i and

$$\mathcal{M}(S) = \begin{cases} \{i\} \subseteq S \text{ if } & S \in \mathcal{W} \\ \quad \emptyset & \text{otherwise} \end{cases}$$

where $\mathcal{W}$ is the set of the winning coalitions in w.
In case there is a *veto player*, i.e. a player i such that $S \in \mathcal{W}$ only if $i \in S$, then the game can also be described by $v(i) = 1$, $v(j) = 0 \; \forall j \neq i$ and

$$\mathcal{M}(S) = \begin{cases} T \text{ if } & S \in \mathcal{W} \\ R & \text{otherwise} \end{cases}$$

with $T, R \subseteq N$ such that $i \in T$ and $i \notin R$.

From Example 1 it is clear that the description of a game as GAG need not be unique.

Example 2 (glove game) Let w be the *glove game* defined in the following way. A partition $\{L, R\}$ of N is assigned. Define $w(S) = \min\{|S \cap L|, |S \cap R|\}$. Then w can be described as the GAG associated to $\langle N, v, \mathcal{M} \rangle$ with $v(i) = 1$ for all i and

$$\mathcal{M}(S) = \begin{cases} S \cap L \text{ if } |S \cap L| \leq |S \cap R| \\ S \cap R \qquad \text{otherwise.} \end{cases}$$

Example 3 (bankruptcy games) Consider the bankruptcy game (N, w) introduced by Aumann and Maschler [3], where the value of a coalition $S \subseteq N$ is given by

$$w(S) = \max\{E - \sum_{i \in N \setminus S} d_i, 0\}.$$

Here $E \geq 0$ represents the estate to be divided and $d \in \mathbb{R}^N_+$ is a vector of claims satisfying the condition $\sum_{i \in N} d_i > E$. It is easy to show that a bankruptcy game is the difference $w = v_1^{\mathcal{M}} - v_2^{\mathcal{M}}$ of two GAGs $v_1^{\mathcal{M}}, v_2^{\mathcal{M}}$ arising, respectively, from $\langle N, v^1, \mathcal{M}^1 \rangle$ and $\langle N, v^2, \mathcal{M}^2 \rangle$ with $v^1(i) = E$ and $v^2(i) = d_i$ for all i,

$$\mathcal{M}^1(S) = \begin{cases} \{i\} \subseteq S \text{ if } & S \in B \\ \quad \emptyset & \text{otherwise} \end{cases}$$

and

$$\mathcal{M}^2(S) = \begin{cases} N \setminus S \text{ if } & S \in B \\ \quad \emptyset & \text{otherwise} \end{cases}$$

for each $S \in 2^N \setminus \{\emptyset\}$, and where $B = \{S \subseteq N : \sum_{i \in N \setminus S} d_i \leq E\}$.

Example 4 (connectivity games) ([2], [20]) Let $\Gamma = (N, E)$ be a graph, where N is a finite set of vertices and E is a set of non-ordered pairs of vertices, i.e. the edges of the graph. Consider the (extended) *connectivity game* (N, v_Γ), where each node i of the underlying graph is assigned a weight w_i. The *weighted connectivity game* is defined as the game (N, w), where

$$w(S) = \begin{cases} \sum_{i \in S} w_i \text{ if } S \subseteq N \text{ is connected in } \Gamma \text{ and } |S| > 1 \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

Then w can be described as the GAG associated to $\langle N, v, \mathcal{M} \rangle$ with $v(i) = w_i$ for all i and

$$\mathcal{M}(S) = \begin{cases} S \text{ if } S \subseteq N \text{ is connected in } \Gamma \\ \emptyset \qquad \text{otherwise.} \end{cases}$$

Some natural properties of the map $\mathcal{M}$ can be translated into classical properties for the associated GAG.

Definition 3 The map $\mathcal{M}$ is said to be *proper* if $\mathcal{M}(S) \subseteq S$ for each $S \subseteq N$; it is said to be *monotonic* if $\mathcal{M}(S) \subseteq \mathcal{M}(T)$ for each S, T such that $S \subseteq T \subseteq N$.

Note that a map $\mathcal{M}$ can be monotonic but not proper, or proper but not monotonic. An example of map $\mathcal{M}$ which is not monotonic is the one relative to the glove game. Maps that are not proper will be seen later.

The following results are straightforward.

Proposition 1 *Let $\langle N, v, \mathcal{M} \rangle$ be a GAS with $v \in \mathbb{R}^N_+$ and $\mathcal{M}$ monotonic. Then the associated GAG $(N, v^{\mathcal{M}})$ is monotonic.*

Proposition 2 *Let $\langle N, v, \mathcal{M} \rangle$ be a GAS with $v \in \mathbb{R}^N_+$ and $\mathcal{M}$ proper and monotonic. Then the associated GAG $(N, v^{\mathcal{M}})$ is superadditive.*

Proof Let S and T be two coalitions such that $S \cap T = \emptyset$. By properness it is $\mathcal{M}(S) \cap \mathcal{M}(T) = \emptyset$. By monotonicity it is

$$\mathcal{M}(S) \cup \mathcal{M}(T) \subseteq \mathcal{M}(S \cup T).$$

Thus, since $v \in \mathbb{R}^N_+$,

$$v^{\mathcal{M}}(S \cup T) = \sum_{i \in \mathcal{M}(S \cup T)} v(i) \geq \sum_{i \in \mathcal{M}(S) \cup \mathcal{M}(T)} v(i) = v^{\mathcal{M}}(S) + v^{\mathcal{M}}(T).$$

□

Observe that Propositions 1 and 2 provide only sufficient conditions, for instance the glove game is monotonic and superadditive but the associated map $\mathcal{M}$ is not monotonic.

The following example shows that, if the map $\mathcal{M}$ is proper and monotonic, the corresponding GAG does not need be convex.

Example 5 Let $N = \{1,2,3,4\}$, $v(i) > 0\ \forall i \in N$, and let $\mathcal{M}$ be such that $\mathcal{M}(\{2\}) = \emptyset$, $\mathcal{M}(\{2,3\}) = \{3\}$ and $\mathcal{M}(S) = S$ for all $S \neq \{2,3\}$. Then $\mathcal{M}$ is proper and monotonic but the corresponding GAG is not convex, since it holds that $v^{\mathcal{M}}(S \cup T) + v^{\mathcal{M}}(S \cap T) < v^{\mathcal{M}}(S) + v^{\mathcal{M}}(T)$ for $S = \{1,2,3\}$, $T = \{2,3,4\}$.

The next proposition shows that it is possible to provide sufficient conditions for a monotonic map $\mathcal{M}$ to generate a convex GAG.

Proposition 3 *Let $\langle N, v, \mathcal{M}\rangle$ be a GAS with $v \in \mathbb{R}^N_+$ and $\mathcal{M}$ such that*

$$\mathcal{M}(S) \cap \mathcal{M}(T) = \mathcal{M}(S \cap T), \tag{3}$$

for each $S, T \in 2^N$. Then the associated GAG $(N, v^{\mathcal{M}})$ is convex.

Proof It is easy to show that condition (3) implies monotonicity of $\mathcal{M}$. Indeed, given $S \subseteq T \subseteq N$ it holds that $S \cap T = S$ and therefore $\mathcal{M}(S) \cap \mathcal{M}(T) = \mathcal{M}(S)$, which implies that $\mathcal{M}(S) \subseteq \mathcal{M}(T)$. In order to prove the convexity of the associated GAG $(N, v^{\mathcal{M}})$, we observe that the following expression holds for every $S, T \in 2^N$:

$$\begin{aligned} v^{\mathcal{M}}(S) + v^{\mathcal{M}}(T) &= \sum_{i \in \mathcal{M}(S)} v(i) + \sum_{i \in M(T)} v(i) \\ &= \sum_{i \in \mathcal{M}(S) \cup \mathcal{M}(T)} v(i) + \sum_{i \in \mathcal{M}(S) \cap \mathcal{M}(T)} v(i). \end{aligned} \tag{4}$$

By monotonicity, we have that $\mathcal{M}(S) \cup \mathcal{M}(T) \subseteq \mathcal{M}(S \cup T)$. Thus, if $\mathcal{M}(S \cap T) = \mathcal{M}(S) \cap \mathcal{M}(T)$ for each $S, T \in 2^N$, then from relation (4) it follows that

$$\begin{aligned} v^{\mathcal{M}}(S) + v^{\mathcal{M}}(T) &\leq \sum_{i \in \mathcal{M}(S \cup T)} v(i) + \sum_{i \in \mathcal{M}(S \cap T)} v(i) \\ &= v^{\mathcal{M}}(S \cup T) + v^{\mathcal{M}}(S \cap T), \end{aligned}$$

which concludes the proof. □

The condition provided by relation (3) can be useful to construct a monotonic map $\mathcal{M}$ such that the corresponding GAG is convex when $v \in \mathbb{R}^N_+$. The most trivial example is the identity map $\mathcal{M}(S) = S$ for each $S \in 2^N$. Another example is a map $\mathcal{M}$ of a GAS $\langle N, v, \mathcal{M}\rangle$ with $N = \{1,2,3\}$ and $v \in \mathbb{R}^N_+$ such that $\mathcal{M}(\{1,2,3\}) = \{1,2,3\}$, $\mathcal{M}(\{1,2\}) = \{1,2\}$, $\mathcal{M}(\{2,3\}) = \{2\}$, $\mathcal{M}(\{2\}) = \{2\}$ and $\mathcal{M}(\{1\}) = \mathcal{M}(\{3\}) = \mathcal{M}(\{1,3\}) = \emptyset$.

4 Basic GAGs

We now define an interesting subclass of GASs. Consider a *collection* $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$, where $\mathcal{C}_i = \{F_i^1, \dots, F_i^{m_i}, E_i\}$ is a collection of subsets of N such that $F_i^j \cap E_i = \emptyset$ for all $i \in N$ and for all $j = 1, \cdots, m_i$.

Definition 4 We denote by $\langle N, v, \mathcal{C} \rangle$ the *basic* GAS associated with the coalitional map $\mathcal{M}$ defined, for all $S \subseteq N$, as:

$$\mathcal{M}(S) = \{i \in N : S \cap F_i^1 \neq \emptyset, \dots, S \cap F_i^{m_i} \neq \emptyset, S \cap E_i = \emptyset\} \tag{5}$$

and by $\langle N, v^{\mathcal{C}} \rangle$ the associated GAG, that we shall call *basic* GAG.

For simplicity of exposition, we assume w.l.o.g. that $m_1 = m_2 = \cdots = m_n := m$. We shall call each F_i^k, for all $i \in N$ and all $k = 1, \dots, m$, the k-th set of *friends* of i, while E_i is the set of *enemies* of i.

The basic GAG $v^{\mathcal{C}}$ associated with a basic GAS can be decomposed in the following sense: define the collection of n games $v^{\mathcal{C}_i}$, $i = 1, \dots, n$, as

$$v^{\mathcal{C}_i}(S) = \begin{cases} v(i) & \text{if} \quad S \cap E_i = \emptyset, S \cap F_i^k \neq \emptyset, k = 1, \dots, m \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Proposition 4 *The basic GAG $v^{\mathcal{C}}$ associated with the map defined in (5) verifies:*

$$v^{\mathcal{C}} = \sum_{i=1}^{n} v^{\mathcal{C}_i}. \tag{7}$$

A particularly simple case is when every player has a unique set of friends, that we shall denote by F_i.

Example 6 (airport games) ([21], [22]): Let N be the set of players. We partition N into groups $N_1, N_2, \dots,$ N_k such that to each N_j, $j = 1, \dots, k$, is associated a positive real number c_j with $c_1 \leq c_2 \leq \cdots \leq c_k$ (representing costs). Consider the game $w(S) = \max\{c_i : N_i \cap S \neq \emptyset\}$. This type of game (and variants) can be described by a basic GAS $\langle N, (\mathcal{C}_i = \{F_i, E_i\})_{i \in N}, v \rangle$ by setting for each $i \in N_j$ and each $j = 1, \dots, k$:

- the value $v(i) = \frac{c_j}{|N_j|}$,
- the set of friends $F_i = N_j$,

and the set of enemies $E_i = N_{j+1} \cup \ldots \cup N_k$ for each $i \in N_j$ and each $j = 1, \dots, k-1$ and $E_l = \emptyset$ for each $l \in N_k$.

By using similar arguments, it is possible to show that also the maintenance games ([19], [7]), which generalize the airport games, can be represented as basic GAGs.

Example 7 (top-k nodes problem) [32,1] Let (N, E) be a co-autorship network, where nodes represent researchers and there exists an edge between two nodes if the corresponding researchers have co-authored in a paper. Given a value

k, the top-k nodes problem consists in the search for a set of k researchers who have co-authored with the maximum number of other researchers. The problem, introduced in [32], is formalized as follows. For any $S \subseteq N$, we define the function $g(S)$ as the number of nodes that are adjacent to nodes in the set S. Given a value k, the problem of finding a set S of cardinality k such that $g(S)$ attains maximum value is NP-hard [32]. Therefore, in [32] and later in [1] a slightly different problem is studied through a game-theoretical approach, by using the Shapley value of a properly defined cooperative game as a measure of the importance of nodes in the network. In the corresponding game (N, w), the worth of a coalition S, for each $S \subseteq N$, $S \neq \emptyset$, is equal to the number of nodes that are connected to nodes in S via, at most, one edge. Formally, $w(S) = |S \cup \bigcup_{i \in S} N_i(E)|$, where $N_i(E)$ is the set of neighbours of $i \in N$ in the network (N, E). It is easy to check that game w can be described as a basic GAS $\langle N, v, (\mathcal{C}_i = \{F_i, E_i\})_{i \in N} \rangle$, where $v(i) = 1$, $F_i = \{i\} \cup N_i(E)$ and $E_i = \emptyset$ $\forall i \in N$.

We now provide further examples of classes of coalitional games which can be represented as basic GAGs, where in general each player can have several sets of friends (see the related literature for a more detailed description and analysis of such classes of games).

Example 8 (argumentation games) ([6]): Consider a directed graph $\langle N, \mathcal{R} \rangle$, where the set of nodes N is a finite set of *arguments* and the set of arcs $\mathcal{R} \subseteq N \times N$ is a *binary defeat* (or *attack*) *relation* (see Dung 1995). For each argument i we define the set of *attackers* of i in $\langle N, \mathcal{R} \rangle$ as the set $P(i) = \{j \in N : (j, i) \in \mathcal{R}\}$. The meaning is the following: N is a set of arguments, if $j \in P(i)$ this means that argument j attacks argument i. The value of a coalition S is the number of arguments in the opinion S which are not attacked by another argument of S. This type of game (and variants) can be described as a basic GAS $\langle N, v, \{F_i, E_i\} \rangle$ by setting $v(i) = 1$, the set of friends $F_i = \{i\}$ and the set of enemies $E_i = P(i)$. This example still falls in the setting of basic GAGs where each player has only one set of friends. However, there are also different, and natural as well, types of characteristic functions that can be considered. For instance, it is interesting to consider the game $(N, v^{\mathcal{M}})$ such that for each $S \subseteq N$, $v^{\mathcal{M}}(S)$ is the sum of $v(i)$ over the elements of the set $D(S) = \{i \in N : P(i) \cap S = \emptyset \text{ and } \forall j \in P(i), P(j) \cap S \neq \emptyset\}$ of arguments that are not internally attacked by S and at the same time are defended by S from external attacks:

$$v^{\mathcal{M}}(S) = \sum_{i \in D(S)} v(i). \tag{8}$$

It is clear that such a situation cannot be described by a basic GAG where each player has a unique set of friends. The game in (8) can however be described as a basic GAG $\langle N, v^{\mathcal{C}} \rangle$, where, given a bijection $k : P(i) \to \{1, \cdots, |P(i)|\}$, $\mathcal{C}_i = \{F_i^1, \cdots, F_i^{|P(i)|}, E_i\}$ is such that $F_i^{k(j)} = P(j) \setminus P(i)$ for all $j \in P(i)$, and $E_i = P(i)$ for all $i \in N$.

Example 9 (peer games) [8] Let $N = \{1, \ldots, n\}$ be the set of players and $T = (N, A)$ a directed rooted tree describing the hierarchy between the players, with N as the set of nodes, 1 as the root (representing the leader of the entire group) and $A \subset N \times N$ is a set of arcs. Each agent i has an individual potential a_i which represents the gain that player i can generate if all players at an upper level in the hierarchy cooperate with him. For every $i \in N$, we denote by $S(i)$ the set of all agents in the unique directed path connecting 1 to i, i.e. the set of superiors of i. Given a *peer group situation* (N, T, a) as described above, a *peer game* is defined as the game (N, v^P) such that for each non-empty coalition $S \subseteq N$

$$v^P(S) = \sum_{i \in N : S(i) \subseteq S} a_i.$$

A peer game (N, v^P) can be represented as the GAG associated to the basic GAS on N where $v(i) = a_i$ and where $\mathcal{M}$ is defined by relation (5) with collections $\mathcal{C}_i = \{F_i^1, \ldots, F_i^n, E_i\}$ such that:

$$F_i^j = \begin{cases} \{j\} \text{ if } j \in S(i) \\ \{i\} \ \text{ otherwise} \end{cases}$$

and $E_i = \emptyset$ for all $i \in N$.

Besides the aforementioned classes of games from the literature, this model turns out to be suitable for representing social network situations, as further discussed in Section 6.

With the aim of providing a tool for the analysis of a wide range of coalitional games, in Section 5 we address the problem of how to guarantee that a basic GAG has a non-empty core, while in Section 6 we analyse classical solutions from coalitional game theory, like the well known Shapley value [30], the Banzhaf value [4] and other semivalues [15].

4.1 A characterization of basic GASs

As it has been shown in the previous sections, a variety of classes of games that have been widely investigated in the literature can be described using the formalism provided by basic GASs. Moreover, as we shall see in the next sections, it is possible to produce, for basic GAGs, results concerning important solution concepts, like the core and the semivalues. It is therefore interesting to study under which conditions a GAS can be described as a basic one. To this purpose, the following theorem provides a necessary and sufficient condition when the set of enemies of each player is empty.

Theorem 1 *Let $\langle N, v, \mathcal{M} \rangle$ be a GAS. The map $\mathcal{M}$ can be obtained by relation (5) via collections $\mathcal{C}_i = \{F_i^1, \ldots, F_i^{m_i}, E_i = \emptyset\}$, for each $i \in N$, if and only if $\mathcal{M}$ is monotonic.*

Proof It is obvious that every map $\mathcal{M}$ obtained by relation (5) over a collections $\mathcal{C}_i = \{F_i^1, \dots, F_i^{m_i}, E_i = \emptyset\}$, for each $i \in N$, is monotonic.

Now, consider a monotonic map $\mathcal{M}$ and, for each $i \in N$, define the set $\mathcal{M}_i^{-1} = \{S \subseteq N : i \in \mathcal{M}(S)\}$ of all coalitions whose image in $\mathcal{M}$ contains i. Let $\mathcal{S}^{\mathcal{M},i}$ be the collection of minimal (with respect to set inclusion) coalitions in $\mathcal{M}_i^{-1}$, formally:

$$\mathcal{S}^{\mathcal{M},i} = \{S \in \mathcal{M}_i^{-1} : \text{ it does not exist } T \in \mathcal{M}_i^{-1} \text{ with } T \subset S\}.$$

For each $i \in N$, consider the collection $\mathcal{C}_i = \{F_i^1, \dots, F_i^{m_i}, E_i = \emptyset\}$ such that

$$\{F_i^1, \dots, F_i^{m_i}\} = \{T \subseteq N : |T \cap S| = 1 \text{ for each } S \in \mathcal{S}^{\mathcal{M},i} \text{ and } |T| \leq |\mathcal{S}^{\mathcal{M},i}|\}, \tag{9}$$

where each set of friends F_i^k, $k \in \{1, \dots, m_i\}$, contains precisely one element in common with each coalition in $S \in \mathcal{S}^{\mathcal{M},i}$ and no more than $|\mathcal{S}^{\mathcal{M},i}|$ elements.

Denote by $\mathcal{M}^*$ the map obtained by relation (5) over such collections $\mathcal{C}_i$, $i \in N$. We need to prove that $\mathcal{M}(S) = \mathcal{M}^*(S)$ for each $S \in 2^N$, $S \neq \emptyset$.

First note that for each $i \in N$ and for every coalition $S \in \mathcal{M}_i^{-1}$, we have $i \in \mathcal{M}^*(S)$. Let us prove now that $i \notin \mathcal{M}^*(S)$ for each $S \notin \mathcal{M}_i^{-1}$. Suppose, by contradiction, that there exists $T \subseteq N$ with $T \notin \mathcal{M}_i^{-1}$ such that $F_i^k \cap T \neq \emptyset$, for each $k \in \{1, \dots, m_i\}$. Consequently, by the definition of $\mathcal{S}^{\mathcal{M},i}$, we have that for every $S \in \mathcal{S}^{\mathcal{M},i}$, $S \setminus T \neq \emptyset$. Define a coalition $U \subseteq N$ of no more than $|\mathcal{S}^{\mathcal{M},i}|$ elements and such that U contains precisely one element of $S \setminus T$ for each $S \in \mathcal{S}^{\mathcal{M},i}$, i.e. $|U \cap (S \setminus T)| = 1$ for each $S \in \mathcal{S}^{\mathcal{M},i}$ and $|U| \leq |\mathcal{S}^{\mathcal{M},i}|$.

By relation (9), U must be a set of friends in the collection $\{F_i^1, \dots, F_i^{m_i}\}$, which yields a contradiction with the fact that $U \cap T = \emptyset$. It follows that for each $i \in N$, $i \in \mathcal{M}^*(S)$ if and only if $i \in \mathcal{M}(S)$ for each $S \subseteq N$, which concludes the proof.

□

Based on the arguments provided in the proof of Theorem 1, the following example shows a procedure to represent a GAS with a monotonic map $\mathcal{M}$ as a basic GAS.

Example 10 Consider a GAS $\langle N, v, \mathcal{M}\rangle$ with $N = \{1, 2, 3, 4\}$ and $\mathcal{M}$ such that $\mathcal{M}(\{1, 2, 3\}) = \{3\}$, $\mathcal{M}(\{3, 4\}) = \{2, 3\}$, $\mathcal{M}(\{2, 3, 4\}) = \{2, 3\}$, $\mathcal{M}(\{1, 3, 4\}) = \{2, 3, 4\}$, $\mathcal{M}(N) = \{2, 3, 4\}$, and $\mathcal{M}(S) = \emptyset$ for all other coalitions. The sets of minimal coalitions are as follows: $\mathcal{S}^{\mathcal{M},1} = \emptyset$, $\mathcal{S}^{\mathcal{M},2} = \{\{3, 4\}\}$, $\mathcal{S}^{\mathcal{M},3} = \{\{1, 2, 3\}, \{3, 4\}\}$, $\mathcal{S}^{\mathcal{M},4} = \{\{1, 3, 4\}\}$. Such a map can be represented via relation (5) with the collections: $F_1^1 = \emptyset$, $\{F_2^1, F_2^2\} = \{\{3\}, \{4\}\}$, $\{F_3^1, \dots, F_3^4\} = \{\{1, 4\}, \{2, 4\}, \{3\}, \{3, 4\}\}$, $\{F_4^1, \dots, F_4^3\} = \{\{1\}, \{3\}, \{4\}\}$, where such collections of friends are obtained via relation (9).

The following proposition characterizes monotonic basic GAGs.

Proposition 5 *Let $\langle N, v, \mathcal{C}\rangle$ be a basic GAS with $v \in \mathbb{R}_+^N$ and $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$. Then the associated GAG $(N, v^{\mathcal{C}})$ is monotonic if and only if $E_i = \emptyset$ $\forall i \in N$.*

Proof The sufficient condition is obvious. Moreover, suppose $E_i \neq \emptyset$ for some i and let $j \in E_i$. Consider $S = F_i^1 \cup \cdots \cup F_i^m$. Then $i \in \mathcal{M}(S)$, while $i \notin \mathcal{M}(S \cup j)$. □

By Proposition 1, Theorem 1 and Proposition 5 we have the following corollary.

Corollary 1 *Let $\langle N, v, \mathcal{M} \rangle$ be a basic GAS with $v \in \mathbb{R}_+^N$. Then the associated basic GAG (N, v^C) is monotonic if and only if $\mathcal{M}$ is monotonic.*

5 The Core of GAGs

In this section we present some results concerning the core of a GAG. The first one is quite simple, and relates to the core of general GAGs. Moreover, we provide some sufficient conditions for the nonemptyness of the core of basic GAGs and we show by an example how they might be useful to derive several allocations in the core for particular subclasses of games.

Proposition 6 *Let $\langle N, v, \mathcal{M} \rangle$ be a GAS such that $v \in \mathbb{R}_+^N$ and $\mathcal{M}$ is proper and such that $\mathcal{M}(N) = N$. Then, the core of the associated (reward) GAG $(N, v^{\mathcal{M}})$ is non-empty.*

Proof Let $x \in \mathbb{R}^N$ be the allocation with $x_i = v(i)$ for each $i \in N$. Consider the game w defined as: $w(S) = \sum_{i \in S} v(i)$. Notice that $x \in C(w)$. Moreover, it holds that $v^{\mathcal{M}}(S) \leq w(S)\ \forall S \subseteq N$, by properness of $\mathcal{M}$, and $v^{\mathcal{M}}(N) = w(N)$. Thus $x \in C(v^{\mathcal{M}})$. □

We now turn our attention to basic GAGs. To start with, we observe that in general $\mathcal{M}$ is not proper. However sufficient conditions in order to apply Proposition 6 are easily seen:

1. for every i there is j such that $F_i^j = \{i\}$
2. $E_i = \emptyset$ for all i.

Condition 1 implies that the map $\mathcal{M}$ is proper, while condition 2 is equivalent to saying that $\mathcal{M}(N) = N$. Next, let us observe that $C(v^{\mathcal{C}}) \neq \emptyset$ if $C(v^{\mathcal{C}_i}) \neq \emptyset$ for all i. Thus let us see now conditions under which $C(v^{\mathcal{C}_i}) \neq \emptyset$. Denote by $I_i = \{j \in N : \exists\ F_i^k \in \mathcal{C}_i \text{ s.t. } F_i^k = \{j\}\}$ the set of players that appear in collection $\mathcal{C}_i$ as singletons. Note that I_i may be empty. Otherwise, players in I_i are veto players in the associated game $v^{\mathcal{C}_i}$. From the above considerations the following Proposition holds, which characterizes the core of the game $v^{\mathcal{C}_i}$.

Proposition 7 *Consider the game $v^{\mathcal{C}_i}$, where $\mathcal{M}$ is defined by relation (5) with collections $\mathcal{C}_i = \{F_i^1, \ldots, F_i^m, E_i = \emptyset\}$. Then $C(v^{\mathcal{C}_i}) \neq \emptyset$ if and only if $I_i \neq \emptyset$. Moreover, if $I_i \neq \emptyset$, then it holds that:*

$$C(v^{\mathcal{C}_i}) = \{x \in \mathbb{R}_+^N : \sum_{j \in I_i} x_j = v(i)\} \tag{10}$$

Proof Note that I_i is the set of veto players in $v^{\mathcal{C}_i}$. Therefore, $C(v^{\mathcal{C}_i}) \neq \emptyset$ if and only if $I_i \neq \emptyset$. Moreover, if $I_i \neq \emptyset$, relation (10) simply follows from the fact that $v^{\mathcal{C}_i}(N) = v(i)$.

□

It follows that if $I_i \neq \emptyset \ \forall i \in N$, then $C(v^{\mathcal{C}}) \neq \emptyset$. However, when games $v^{\mathcal{C}_i}$ such that $I_i = \emptyset$ are combined with games $v^{\mathcal{C}_j}$ such that $I_j \neq \emptyset$, the resulting GAG can have a nonempty core, as shown in the following example.

Example 11 Consider a two-person basic GAS $\langle N, v, \mathcal{C}\rangle$ with $N = \{1, 2\}$ such that $v(1) = \alpha$, $v(2) = 2$ and $\mathcal{C}_1 = \{\{1\}, \{2\}\}, E_1 = \emptyset\}$, $\mathcal{C}_2 = \{\{1, 2\}, E_2 = \emptyset\}$. By Proposition 7, we have that $C(v^{\mathcal{C}_1}) \neq \emptyset$, and $C(v^{\mathcal{C}_2}) = \emptyset$. The core of the resulting GAG $v^{\mathcal{C}} = v^{\mathcal{C}_1} + v^{\mathcal{C}_2}$ is non-empty when $\alpha \geq 2$.

The situation described in the previous example can be generalized as follows. We first define the set $\mathcal{I} = \{i \in N : I_i \neq \emptyset\}$ as the set of players that have at least one singleton among their sets of friends. The following proposition provides a necessary and sufficient condition for the non-emptiness of the core of a special class of basic GAGs of the type introduced in Example 11.

Proposition 8 *Consider the GAG $v^{\mathcal{C}}$ corresponding to a basic GAS $\langle N, v, \mathcal{C}\rangle$ with $v(i) \geq 0$ and $\mathcal{C}_i = \{F_i^1, \ldots, F_i^m, E_i = \emptyset\}$ for each $i \in N$. Suppose there exists a coalition $S \subseteq N$, $S \neq \emptyset$, satisfying the following two conditions:*
i) $S \subseteq I_i$ for each $i \in \mathcal{I}$;
ii) for each $i \in N \setminus \mathcal{I}$, there exists $k \in \{1, \ldots, m\}$ such that $F_i^k = S$.
Define the equal split allocation among players in S as the vector y such that

$$y = e^S \frac{v^{\mathcal{C}}(N)}{s},$$

where s is the cardinality of S and where $e^S \in \{0, 1\}^N$ is such that $e_k^S = 1$, if $k \in S$ and $e_k^S = 0$, otherwise. The allocation y is in the core of the game $v^{\mathcal{C}}$ iff

$$v^{\mathcal{C}}(N) \geq s \sum_{i \in N \setminus \mathcal{I}} v(i). \tag{11}$$

Proof Condition (i) means that the players in S appear as singletons in the set of friends of every player that has at least one singleton among his sets of friends, while condition (ii) means that, for those players that do not have singletons among their sets of friends, there exists a set of friends coinciding with S. First, we prove that condition (11) is necessary. Suppose that (11) does not hold, i.e. $v^{\mathcal{C}}(N) < s \sum_{i \in N \setminus \mathcal{I}} v(i)$. Consider a coalition $T \subseteq N$, such that $|T| \geq 2$, $T \cap S = \{t\}$ and $T \cap F_i^k \neq \emptyset \ \forall i \in N \setminus \mathcal{I}$, $\forall k = 1, \cdots, m$. It holds that:

$$\sum_{j \in T} y_j = y_t = \frac{v^{\mathcal{C}}(N)}{s} < \sum_{i \in N \setminus \mathcal{I}} v(i) = v^{\mathcal{C}}(T)$$

and therefore $y \notin C(v^{\mathcal{C}})$.
Moreover, we prove that condition (11) is also sufficient. Clearly,

$$\sum_{i \in N} y_i = \sum_{i \in S} y_i = v^{\mathcal{C}}(N) = \sum_{i \in N} v(i). \tag{12}$$

In order to prove that $y \in C(v^{\mathcal{C}})$ we have to show that if relation (11) holds then $\sum_{i \in T} y_i \geq v^{\mathcal{C}}(T)$ for each coalition $T \subseteq N$.
First consider a coalition $T \subseteq N$ such that $S \subseteq T$. Note that $\sum_{i \in N} v(i) \geq v^{\mathcal{C}}(T)$, and then by relation (12), $\sum_{i \in T} y_i \geq \sum_{i \in S} y_i = \sum_{i \in N} v(i) \geq v^{\mathcal{C}}(T)$.
Now consider a coalition $T \subseteq N$ such that $S \cap T = \emptyset$. By condition (i) and (ii), we have that $v^{\mathcal{C}}(T) = 0$, which is not greater than $\sum_{i \in T} y_i$ since $y_i \geq 0$ for each $i \in N$.
Finally, consider a coalition $T \subseteq N$ such that $S \cap T \neq \emptyset$ and $S \nsubseteq T$. Since $S \subseteq I_i$ for each $i \in \mathcal{I}$, then no term $v(i)$ with $i \in \mathcal{I}$ contributes to the worth of T. This means that

$$v^{\mathcal{C}}(T) = \sum_{i \in \mathcal{M}(T)} v(i), \tag{13}$$

with $\mathcal{M}(T) \subseteq N \setminus \mathcal{I}$ and where $\mathcal{M}$ is defined by relation (5). Now consider a player $i \in S \cap T$. If condition (11) holds, then we have that

$$\sum_{j \in T} y_j \geq y_i \geq \sum_{i \in N \setminus \mathcal{I}} v(i) \geq v^{\mathcal{C}}(T),$$

where the last inequality follows by relation (13), which concludes the proof.
□

Observe that, even if the equal split allocation does not belong to the core, the core might be non empty, as the following example shows.

Example 12 Consider a GAS $G = \langle N, v, M \rangle$ with $N = \{1, 2, 3\}$ and such that $\mathcal{C}_1 = \{\{1\}, \{2\}, \{3\}, E_1 = \emptyset\}$, $\mathcal{C}_2 = \{\{2, 3\}, E_2 = \emptyset\}$ and $\mathcal{C}_3 = \{\{2, 3\}, E_2 = \emptyset\}$. The coalition $\{2, 3\}$ satisfies the hypothesis in Proposition 8 but relation (11) is not satisfied. However, the allocation $(v(1), v(2), v(3))$ belongs to the core of $v^{\mathcal{C}}$.

The previous Propositions provide sufficient conditions for the nonemptiness of the core for basic GAGs. If a game can be represented in terms of a basic GAG, these conditions can be directly verified by considering only the collections of friends and enemies of each player, without having to check any further property of the characteristic function (for instance, the balancedness property [5,31]), that in general involves much more complex procedures. Consequently, the results provided in this section can be used to construct non-trivial classes of games with a non-empty core, or to easily derive core allocations. For instance, even if a game is not itself representable in terms of a basic GAG, Proposition 7 may be applied to generate allocations in the core of games that can be described as the sum of proper basic GAGs where no player has enemies, as next example shows.

Example 13 Consider the game introduced in [13], in which the players are nodes of a graph with weights on the edges, and the value of a coalition is determined by the total weight of the edges contained in it. Formally, an undirected graph $G = (N, E)$ is given, with weight $w_{i,j}$ on the edge $\{i, j\}$[2], and the game v is defined, for every $S \subseteq N$, as $v(S) = \sum_{i,j \in S} w_{i,j}$. If all weights are nonnegative, the game is convex and therefore the core is non empty. However, finding allocations in the core is not straightforward and in [13] necessary and sufficient conditions for the Shapley value to belong to it are provided.

Indeed, game v can be described as the sum of n basic GAGs, one for each player $i \in N$, where each other player $j \neq i$ contributes to the worth of a coalition $S \subseteq N$ with half of the weight $w_{i,j}$ if and only if i and j belong to S, while i contributes to any coalition it belongs to with the weight $w_{i,i}$. Formally, $v = \sum_{i \in N} v^{\mathcal{C}^i}$, where $v^{\mathcal{C}^i}$ is a proper basic GAG associated to collections $\mathcal{C}^i_j = \{F^1_j = \{i\}, F^2_j = \{j\}, E_j = \emptyset\}$ and $v(j) = \frac{w_{i,j}}{2}$, for every $j \in N$, $j \neq i$, while $\mathcal{C}^i_i = \{F^1_j = \{i\}, E_j = \emptyset\}$ and $v(i) = w_{i,i}$. As a sum of n proper GAGs such that $\mathcal{M}(N) = N$, Proposition 6 immediately implies the non-emptiness of the core of game v. Moreover, notice that according to Proposition 4 each basic GAG $v^{\mathcal{C}^i}$ can be decomposed as the sum $v^{\mathcal{C}^i} = \sum_{j \in N} v^{\mathcal{C}^i_j}$, for each $i, j \in N$, and then the repeated application of Proposition 7 on each $v^{\mathcal{C}^i_j}$ can be used to efficiently derive allocations in the core of the sum game v.

Following similar intuitions, we argue that the simple structure of basic GAGs could be useful to generalize some of the complexity results about the problem of finding core allocations provided in [13], for instance, considering classes of more sophisticated games that can be generated as a positive linear combination of basic GAGs.

6 Semivalues and GAGs

It is well known that the problem of identifying influential users on a social networking web site plays a key role to find strategies aimed at increasing the site's overall view [29]. The main issue is to target advertisement to the site members of the online social network whose activities' levels have a significant impact on the activity of the other site members. The overall influence of a user can be seen as the combination of two ingredients: 1) the individual ability to get the attention of other site members, and 2) the personal characteristic of the social profile, that can be represented in terms of groups or communities to which users belong.

A basic GAS $\langle N, v, \mathcal{C} \rangle$ can represent an online social network as described above (see also Example 14). More specifically, each player $i \in N$ of the basic GAS is associated to a value $v(i)$ representing her/his individual activity in a social networking web site (for instance, measured in terms of the productive

[2] Since the graph is undirected, we assume by convention that, if an edge between i and j is present, $w_{i,j} \neq 0$ and $w_{j,i} = 0$ for $i < j$.

time spent in uploading content files), and the participation of the individuals to the global activity of the social network is based on a coalitional structure $\mathcal{C}$ of friends and enemies that is determined by players' social profiles. Thus, it is interesting to analyze, for this type of games, the behavior of indices aimed at measuring the influence of the players in the game: in particular we consider the Shapley value [30], the Banzhaf value [4] and other semivalues [15].

Since we are interested in evaluating additive power indices for the players in basic GAGs, thanks to Proposition 4 it becomes possible to evaluate the indices on the games $v^{\mathcal{C}_i}$ defined via relation (6). First of all, we present some results concerning the Shapley and Banzhaf values on interesting subclasses of basic GAGs, where each player has a unique set of friends. Furthermore, we extend our analysis to a generic basic GAG, with multiple sets of friends.

In what follows, in order to simplify the notation, we fix $i \in N$ and denote by f the cardinality of F_i and by e the cardinality of E_i (in order to simplify the notation, if $E_i = \emptyset$ we assume by convention that $e = 0$ and $\frac{1}{e} = 0$).
When we consider a basic GAG with a *single* set of friends for each player i, the game $v^{\mathcal{C}_i}$ reduces to:

$$v^{\mathcal{C}_i}(S) = \begin{cases} v(i) & \text{if} \quad S \cap F_i \neq \emptyset, S \cap E_i = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and the following proposition holds.

Proposition 9 *Let us consider a basic GAS on $\langle N, v, \{\mathcal{C}_i = \{F_i, E_i\}\}_{i \in N} \rangle$. Then the Shapley and Banzhaf values for the game $v^{\mathcal{C}_i}$ are given, respectively, by:*

$$\sigma_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if } j \in N \setminus (F_i \cup E_i) \\ \frac{v(i)}{f+e} & \text{if } \quad j \in F_i \\ -v(i)\frac{f}{e(f+e)} & \text{if } \quad j \in E_i \end{cases}$$

and

$$\beta_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if } j \in N \setminus (F_i \cup E_i) \\ \frac{v(i)}{2^{f+e-1}} & \text{if } \quad j \in F_i \\ -v(i)\frac{2^f-1}{2^{f+e-1}} & \text{if } \quad j \in E_i. \end{cases}$$

Proof Clearly, players not in $F_i \cup E_i$ are null players. Now, let us call a player *decisive* for a coalition S if $v^{\mathcal{C}_i}(S \cup \{i\}) - v^{\mathcal{C}_i}(S) \neq 0$. Consider a player $j \in F_i$. He is decisive for S (with worth $v(i)$) if and only if no player i in F_i and no player k in E_i is in S.
Now, let $j \in E_i$. He is decisive for S (with worth $-v(i)$) if and only if at least one player i in F_i is in S and no player k in E_i is in S.
These facts provide the formulas for β, and σ is derived by considering all the possible orderings of the players.

□

We comment the results with the help of the following example.

Example 14 As a toy example, consider an online social network with four users $N = \{1, 2, 3, 4\}$ where each user spends the same amount of time T in uploading new content files and, according to her/his social profile, each user $i \in N$ belongs to a single community $F_i \subseteq N$ (e.g., the set of users with whom i intends to share her/his content files) which is in conflict with the complementary one $E_i = N \setminus F_i$ (here, enemies in E_i are interpreted as those members that have no permission to access the content files of player i). Suppose, for instance, that $F_1 = \{1, 2, 3\}$, $F_2 = \{2, 3\}$, $F_3 = \{3\}$ and $F_4 = \{1, 2, 3, 4\}$. Following the discussion about social networking web sites introduced in Section 1, we can represent such a situation as a basic GAS $\langle N, v, \{\mathcal{C}_i = \{F_i, E_i = N \setminus F_i\}\}_{i \in N}\rangle$. How to identify the most influential users? According to Proposition 9, the influence vector provided by the Shapley value is: $\sigma(v^{\mathcal{C}}) = (\frac{T}{6}, \frac{2T}{3}, T, \frac{-5T}{6})$. So, user 3 results the most influential one, followed by 2, then 1 and finally 4, who is the only user to get a negative index.

Suppose now that user 2 wants to improve her/his influence as measured by the Shapley value. It is worth noting that if user 2 removes 3 from her/his set of friends (and all the other sets of friends and enemies remain the same), then player 2 gets exactly the same Shapley value of user 3. Precisely, if now $F_2 = \{2\}$ and $E_2 = \{1, 3, 4\}$, then $\sigma_2(v^{\mathcal{C}}) = \sigma_3(v^{\mathcal{C}}) = \frac{2T}{3}$. Notice that the fact that an influential player has been removed from his/her list of friends does not impact directly the influence of player 2, but it determines an important reduction of the influence of player 3.

When $E_i = \emptyset$, the formulas in Proposition 9 are further simplified and the following corollary holds.

Corollary 2 *Let us consider a basic GAS on $\langle N, v, \{\mathcal{C}_i = \{F_i, E_i = \emptyset\}\}_{i \in N}\rangle$. Then the Shapley value σ and the Banzhaf value β for the game $v^{\mathcal{C}_i}$ are given, respectively, by:*

$$\sigma_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if } j \in N \setminus F_i \\ \frac{v(i)}{f} & \text{if } \quad j \in F_i \end{cases}$$

and

$$\beta_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if } j \in N \setminus F_i \\ \frac{v(i)}{2^{f-1}} & \text{if } \quad j \in F_i. \end{cases}$$

Proposition 9 and Corollary 2 represent a useful tool for computing the Shapley and Banzhaf values of a subclass of basic GAGs. Their advantage relies on the fact that, once a game is described in terms of basic GAGs, the formulas can be derived in a straightforward way from the individual values of the players and the cardinalities of the sets of friends and enemies. As an example, consider the game introduced in Example 7. The Shapley value of such a game has been proposed as a measure of centrality in networks in [32], where an approximate algorithm for its computation is provided. Moreover, in [1], an exact formula for the Shapley value of such game is provided, but its proof relies on elaborate combinatorial and probabilistic arguments. On the other hand, the description of that game as a basic GAG, which can be easily derived from the definition of the game itself, leads to the same formula in a direct and intuitive way, since the Shapley value (and the Banzhaf value) can be directly derived from Corollary 2 and relation (7).

When generalizing the previous results to the case of a basic GAG with multiple sets of friends, it is natural to extend the analysis to other solutions, beyond the Shapley and Banzhaf values. In what follows, we focus on the class of semivalues. Let $F = \bigcup_{k=1}^{m} F_i^k$ and let $\Gamma_i = \{1, \ldots, f_i^1\} \times \ldots \times \{1, \ldots, f_i^m\}$, where f and f_i^k are the cardinalities of F and F_i^k, for each $k = 1, \ldots, m$. The following Theorem generalizes the results in Proposition 9.

Theorem 2 *Consider a GAS situation* $\langle N, v, \{F_i^1, \ldots, F_i^m, E_i\}_{i \in N}\rangle$ *with* $F_i^j \cap F_i^k = \emptyset$ *for all* $i \in N$ *and* $j, k = 1, \cdots, m$, $j \neq k$. *For all* $j \in N \setminus (F \cup E_i)$, *we have that* $\pi_j^{\mathbf{P}}(v^{\mathcal{C}_i}) = 0$.
Take $j \in F_i^b$, *with* $b \in \{1, \ldots, m\}$. *Then* $\pi_j^{\mathbf{P}}(v^{\mathcal{C}_i})$ *is equal to the following expression:*

$$v(i) \sum_{(k_i^1, \ldots, k_i^{b-1}, 0, k_i^{b+1}, \ldots, k_i^m) \in \Gamma} \sum_{l=0}^{n-e-f} \binom{f_i^1}{k_i^1} \times \ldots \times \binom{f_i^m}{k_i^m} \times \binom{n-e-f}{l} p_{h+l} \tag{14}$$

where $e = |E_i|$ *and* $h = \sum_{j=1}^{m} k^j$. *Now, take* $j \in E_i$. *Then*

$$\pi_j^{\mathbf{P}}(v^{\mathcal{C}_i}) = -v(i) \sum_{(k_i^1, \ldots, k_i^m) \in \Gamma} \sum_{l=0}^{n-t-f} \binom{f_i^1}{k_i^1} \times \ldots \times \binom{f_i^m}{k_i^m} \times \binom{n-e-f}{l} p_{h+l}. \tag{15}$$

Proof Players in $N \setminus (F \cup E_i)$ are dummy players, so they receive nothing. Now, consider the case $j \in F_i^b$ and take a coalition $S \subseteq N \setminus \{j\}$ that does not contain j. The marginal contribution of j to coalition S is $v^{\mathcal{C}_i}(S \cup \{j\}) - v^{\mathcal{C}_i}(S) = v(i)$ if S contains at least one friend from each set of friends F_i^t with $t \neq b$ (i.e, $S \cap F_i^t \neq \emptyset$ for $t \neq b$), and S does not contain neither any element of F_i^b nor any element of E_i; otherwise, $v^{\mathcal{C}_i}(S \cup \{j\}) - v^{\mathcal{C}_i}(S) = 0$.
Given a vector $(k_i^1, \ldots, k_i^{b-1}, 0, k_i^{b+1}, \ldots, k_i^m) \in \Gamma$ (i.e., $k_b = 0$) and $l \in \{0, \ldots, n-e-f\}$, the product $\binom{f_i^1}{k_i^1} \times \ldots \times \binom{f_i^m}{k_i^m} \times \binom{n-e-f}{l}$ represents the number of sets S containing k_i^t elements of F_i^t, for each $t \in \{1, \ldots, m_i\}$ with $t \neq b$, l elements of $N \setminus (F \cup E_i)$ and such that $v^{\mathcal{C}_i}(S \cup \{j\}) - v^{\mathcal{C}_i}(S) = v(i)$. Of course, the probability of such a set S to form is p_{h+l}, and relation (14) follows. Now, consider the case $j \in T$ and take a coalition $S \subseteq N \setminus \{j\}$ that does not contain j. The marginal contribution of j to coalition S $v^{\mathcal{C}_i}(S \cup \{j\}) - v^{\mathcal{C}_i}(S) = -v(i)$ if S contains at least one friend from each set of friends F_i^t for each t (i.e, $S \cap F_i^t \neq \emptyset$ for each $t = 1, \ldots, m_i$), and S does not contain any element of E_i; otherwise, S $v^{\mathcal{C}_i}(S \cup \{j\}) - v^{\mathcal{C}_i}(S) = 0$. Given a vector $(k_i^1, \ldots, k_i^m) \in \Gamma$ and $l \in \{0, \ldots, n-e-f\}$, the product $\binom{f_i^1}{k_i^1} \times \ldots \times \binom{f_i^m}{k_i^m} \times \binom{n-e-f}{l}$ represents the number of sets S containing k_i^t elements of F_i^t, for each $t = 1, \ldots, m_i$, l elements of $N \setminus (F \cup E_i)$ and such that $v^{\mathcal{C}_i}(S \cup \{j\}) - v^{\mathcal{C}_i}(S) = -v(i)$. Of course, the probability of such a set S to form is p_{h+l}, and relation (15) follows.

□

Formulas for the semivalues on $v^{\mathcal{C}_i}$ when each $i \in N$ has only one set of friends can be derived directly from the previous theorem. Indeed, the following corollaries hold.

Corollary 3 *Let us consider a basic GAS on $\langle N, v, \{\mathcal{C}_i = \{F_i, E_i\}\}_{i\in N}\rangle$. Then a semivalue $\pi^{\mathbf{p}}$ for the game $v^{\mathcal{C}_i}$ is given by:*

$$\pi_j^{\mathbf{p}}(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if } j \in N \setminus (F_i \cup E_i) \\ v(i) \sum_{k=0}^{n-f-e} \binom{n-f-e}{k} p_k & \text{if } \quad j \in F_i \\ -v(i) \sum_{k=1}^{f} \binom{f}{k} \sum_{h=0}^{n-f-e} \binom{n-f-e}{h} p_{k+h} & \text{if } \quad j \in E_i. \end{cases}$$

Corollary 4 *Let us consider a basic GAS on $\langle N, v, \{\mathcal{C}_i = \{F_i, E_i = \emptyset\}\}_{i\in N}\rangle$. A semivalue $\pi^{\mathbf{p}}$ for the game $v^{\mathcal{C}_i}$ is given by:*

$$\pi_j^{\mathbf{p}}(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if } j \in N \setminus F_i \\ v(i) \sum_{k=0}^{n-f} \binom{n-f}{k} p_k & \text{if } \quad j \in F_i. \end{cases}$$

Note that, in the basic GASs $\langle N, v, \{\mathcal{C}_i = \{F_i, E_i\}\}_{i\in N}\rangle$ considered in Corollary 3, a semivalue $\pi^{\mathbf{p}}$ for the game $v^{\mathcal{C}_i}$ assigns to a player $j \in F_i$ a positive share of v_i, proportionally to the probability (according to $\mathbf{p}$) that j enters a coalition not containing any player of the set $F_i \cup E_i$; on the contrary, each player $l \in E_i$ receives a negative share of v_i, proportionally to the probability that l enters a coalition containing at least one player of F_i. In particular, the unique semivalue such that $\sum_{i\in F_i} \pi_i^{\mathbf{p}}(v^{\mathcal{C}_i}) = \sum_{i\in E_i} \pi_i^{\mathbf{p}}(v^{\mathcal{C}_i})$, for every $F_i, E_i \subseteq N$, $F_i \cap E_i = \emptyset$, $F_i \neq \emptyset$ and $E_i \neq \emptyset$, is the Shapley value.
We finally observe that the equivalence of the formula in Corollary 3 with the formulas of Proposition 9 for the Shapley value can be verified through the following combinatorial identities: $\sum_{k=0}^{n} \binom{n}{k} (k)!(n+t-k)! = \frac{(n+t+1)!}{t+1}$, $\sum_{k=0}^{n} \binom{n}{k} (k+1)!(n+t-k-1)! = \frac{(n+t+1)!}{t(t+1)}$, which hold for all $n, t \in \mathbb{N}$ and can be proved by means of products of formal series.

7 Conclusions

In the present paper, the class of generalized additive games is introduced, where the worth of a coalition of players is evaluated by means of a map $\mathcal{M}$ that selects the valuable players in the coalition.
Several examples from the literature of classical coalitional games that can be described within our approach are presented, in particular for the class of basic GAGs, where the worth of each coalition is calculated additively over the individual contributions and keeping into account social relationships among groups of players.
Our approach enables to classify existing games based on the properties of $\mathcal{M}$. The interest of the classification is not only taxonomical, since it also allows to study the properties of solutions for several classes of games known from the literature.

We showed that, in many cases, basic GAGs allow for an easy computation of several classical solutions from cooperative game theory and, at the same time, provide quite simple representations of practical situations (for instance, arising from online social networks).

One of the goal of our future research is to apply these models on real social network data. As shown by Example 14, the information required to compute classical power indices on basic GAGs representing online social networks (like the users' activity time or the users' social profiles and social affinities) is not very demanding and can be obtained by available records and models from the literature [28].
Moreover, as it has been stressed in the same example, it would be interesting to explore the strategic issues related to the attempt of players to increase their influence (as measured by the Shapley value or by other power indices) on a social network.
An interesting direction for future research is indeed that of coalition formation, since for generic basic GAGs associated to GASs with nonnegative v, where the sets of enemies are not empty, the grand coalition is not likely to form. In general, we believe that the issue about which coalitions are more likely to form in a basic GAG is not trivial and deserves to be further explored.

To conclude, observe that further extensions could be introduced, generalizing the idea of coalitional map, in order to embrace a wider range of games represented in a compact way into our framework.
The definition of GAG is based on the coalitional map $\mathcal{M}$, assigning a coalition $\mathcal{M}(S) \subseteq N$ to each coalition $S \subseteq N$ of players. But one can instead consider a more general map $\mathcal{M} : 2^N \to 2^{2^N}$, which assigns to each coalition $S \subseteq N$, a subset $\mathcal{M}(S) \subseteq 2^N$. This can embrace the idea of *graph-restricted game* introduced by Myerson (1977), where the worth of a coalition is evaluated on the connected components induced by an underlying graph.
Analogously, the definition of a basic GAG is based on a collection of sets $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$, one for each player, where $\mathcal{C}_i = \{F_i^1, \ldots, F_i^m, E_i\}$ is a collection of subsets of N that satisfy some particular properties.
If we provide each player i with multiple collections $\{\mathcal{C}_i^1, \cdots, \mathcal{C}_i^k\}$, with $\mathcal{C}_i^k = \{F_i^{k1}, \ldots, F_i^{km}, E_i^k\}$, we are then able to represent those games that are associated to marginal contribution nets (MC-nets), introduced by Ieong and Shoham (2005).
The basic idea behind marginal contribution nets is to represent in a compact way the characteristic function of a game, as a set of rules of the form: *pattern* $\longrightarrow$ *value*, where a *pattern* is a Boolean formula over a set of n variables (one for each player) and a *value* is a real number (see also [10] for a formal definition and a deeper analysis).

Every coalitional game can be represented through MC-nets by defining one rule for each coalition $S \subseteq N$, where the pattern contains all the variables corresponding to the players in S and the negation of all the other variables, and the corresponding value is equal to the value of S in the game.

A game deriving from a MC-nets representation can be described as a generalization of a basic GAG, where each player has multiple collections of set of friends and enemies, one for each rule, and is assigned a vector of values $v(i) = \{v_1(i), \cdots, v_m(i)\}$. See [9] for more details.
In this way, we are indeed able to describe every TU-game, since the representation of MC-nets is complete. The computational complexity of such representation is in general high. However, when a game can be described by a small collection of rules, and therefore the associated extended GAS is described in a relatively compact way, the complexity of its representation and of the computation of solutions is consequently reduced.

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