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*Mémoire Intitulé*

# Game theory and practice: models, axioms and algorithms

*présenté par*

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*pour l'obtention du diplôme*

**Habilitation à Diriger des Recherches**

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# Preface

In this monograph I provide a summary of my recent research activity, with a particular emphasis on the results published after my arrival at Paris Dauphine University, where I work since 2010 as an associate scientist of the French National Centre for Scientific Research (French: *Centre national de la recherche scientifique*, CNRS).

The HDR (French: *Habilitation à Diriger des Recherches*) is a French degree which accredits to supervise PhD students, but it is also the opportunity to look at the past research and to plan the future. To make projects for the future is a good practice in every human activity, so it is the same in research. But we live in an uncertain world, and research is not spared from this reality. When we try to formulate models, axioms and algorithms to solve practical problems, we end up in theories. And theories can be generalized or particularized: this process may lead us to consider other practical problems, not even imagined before. This is the the strong interplay between theory and practice, a process that can be hardly predicted and that generates new models, axioms and algorithms, which are the basic ingredients of my research activity. This kind of "constructive" uncertainty, that I experienced several times during my past research work, led me to study problems that are quite different in their nature, and yet are characterized by a common pattern in their conceptualisation.

But what models, axioms and algorithms are? Avoiding an epistemic debate about these abstract objects (well away from the goals of this monograph and from my competences!) I would like to give here some intuitions based on common sense and real life examples.

Models are abstract and hyper-simplified representations of reality. For instance, we use models to teach to our children how to behave well in a society, or to predict at what time we need to wake up in order to catch the train that tomorrow will bring us to an important meeting. On the other hand, our computational resources are limited, we cannot process too much information in our models and, most important, we need to select the relevant parameters. Axioms may drive us to the selection of the "right" model and help us to characterize appropriate "solutions". For example, the problem of purchase a good car or a nice apartment, or the choice of a loyal mate, can be "solved" specifying the axioms that a car, an apartment or a mate should satisfy to be considered good, nice and loyal, respectively. Finally, algorithms are efficient procedures that allow to compute solutions and to compare the results of our models with the observed data, in order to understand whether our predictions are right and, if needed, to update and correct our models (e.g., like improving a receipt to prepare a good dish of pasta through repeated trials over the ingredients' proportions).

In these few lines all terms of the title have been shortly introduced except the first one: *game*. Games are in fact the main subjects of this monograph, and I invite the interested reader to go further in the text to discover the role played by games in my work. After all, once again, in real life as well as in doing research, we need to have fun...

Stefano Moretti
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# Acknowledgements

Throughout my career I had the honour to interact with many exceptionally intelligent and talented people who deeply influenced my approach to research and, often, also to life. My gratitude to these persons remains intact over the time.

Among these persons, here I wish to thank all the members of my committee, for the precious time they spent to evaluate my work, and for being inspiring examples of scientists.

I am grateful to all my co-authors, for their fundamental contributions to the papers described in this dissertation, but also for the nice time spent enjoying their company. I have learned precious lessons from all of them.

I am also indebted with all my colleagues at LAMSADE, for creating a stimulating intellectual environment and generating the conditions for a pleasant everyday professional activity.

Many of these brilliant people are also my friends, and I am grateful to them also for this.

Last but not least, I express all my gratitude to the members of my family for their unceasing support: my gratitude for their love cannot be constrained by the size of any text. This monograph is warmly dedicated to them.

# Contents

# Chapter 1

# Introduction

In Game Theory it is usual to divide interaction situations into two main groups: *cooperative games*, where all kinds of agreements among the interacting agents (that are called *players*) are possible, and *non-cooperative games*, dealing with conflict situations where players cannot make binding agreements.

More precisely, in cooperative games, players may form *coalitions* with the objective to coordinate their actions and to obtain joint payoffs which exceed the sum of the individual payoffs. For example, cooperative games can be used to analyse cost allocation problems, where the players are willing to form coalitions in order to have extra monetary savings as an effect of cooperation. For instance, consider three nearby municipalities (namely, 1, 2 and 3) that must take the decision on whether to cooperate in order to implement a joint project, for instance, a Waste-water Treatment System (WTS). Suppose that each municipality 1, 2 and 3 could implement its own independent WTS at a cost of 5, 3 and 2 million euros, respectively. However, if they cooperate, they can reduce the cost of implementation thanks to a more efficient use of common facilities and resources. Suppose that the cost to implement a WTS is for each group of municipalities as follows (in million euros): $c(\{1,2\}) = 7$, $c(\{1,3\}) = 6$, $c(\{2,3\}) = 4$ and $c(\{1,2,3\}) = 8$ (the cost of a coalition of municipalities $S \subseteq \{1,2,3\}$ is intended as the cost to implement a WTS serving the total area of municiplities in $S$). Together with the individual costs $c(\{1\}) = 5$, $c(\{2\}) = 3$, $c(\{3\}) = 2$, the map $c$, assigning to each coalition $S \subseteq \{1,2,3\}$ the cost of implementing a WTS on $S$, is called *characteristic function* (see Section 1.2.2 for a formal definition of games in *characteristic function form*, also known as *Transferable Utility (TU) games*). Looking at the characteristic function $c$, it seems quite clear that cooperating and forming larger coalitions yields extra savings (in particular, we notice that the map $c$ is sub-additive, i.e., the cost $c(S \cup T)$ of the union of each pair of disjoint coalitions $S$ and $T$ is lower than the sum $c(S) + c(T)$). On the other hand, the opportunity to have extra savings is only a necessary condition to guarantee the cooperation! In fact, all the municipalities will coordinate their action only if they find an agreement on how to share the total cost of 8 million euros, corresponding to a WTS serving all the three municipalities. How to guarantee that coalition $\{1,2,3\}$ forms? and which cost allocation will likely be adopted? These questions are the main issues in the analysis of TU games, and will be further discussed in Chapter 2.

The previous example illustrates an applications of cooperative games to a cost sharing problem. Sometimes, the outcome of a coalition of players does not represent an amount of money. Consider, for instance, three political parties, (again, 1, 2 and 3) that have 47%, 38%, and 15% of the seats of a parliament, respectively. A coalition of parties $S \subseteq \{1,2,3\}$ is said to be winning if $S$ is able to force the adoption of a decision in the parliament. For instance, suppose that decisions are taken using a simple majority rule (a voting requirement of more than half of all seats). So, only coalitions with more than one party will be a winning coalition. We can represent the parliament situation again as a cooperative game with characteristic function $w$ such that $w(\{1,2,3\}) = w(\{1,2\}) = w(\{1,3\}) = w(\{2,3\}) = 1$ and $w(\{1\}) = w(\{2\}) = w(\{3\}) = w(\emptyset) = 0$, where the value $w(S) = 1$ means that coalition $S$ is winning, and $w(S) = 0$ means that $S$ is losing. An important question, here, concerns the distribution of "power" among the players. Looking at the simple structure of winning and losing coalitions in the parliament example, we notice that, despite the differences in terms of number of seats, all the parties may form the same number of winning coalitions in a "symmetric" way (see Section 1.2.2 for more details). Consequently, we argue that they have the same opportunities to force the adoption of a decision. For more complex situations, classical game theoretical tools to measure the power of players are *power indices* [171, 86], that will be further discussed in Chapter 5.

As we already said, in non-cooperative games, players have not the possibility to sign an agreement and each player chooses to act in his own interest, keeping into account that the outcome of the game depends on the actions of all the players involved. Actions can be made simultaneously by players, as in the 'stone, paper, scissors' game or in 'matching pennies', or sequentially at several time moments, as in chess. A classical and illustrative example of non-cooperative *(strategic) game* with simultaneous actions is the "battle of sexes". The interaction situation involves a man and a woman who agreed to go out in the evening, either to a football match or to the opera. Unfortunately, they forgot which of the two special events they had agreed on. The man prefers the football match, the women prefers the opera, but both prefer being together to spending the night alone. They must decide simultaneously and they have no means to communicate [1]. We can represent this situation using the following table, where the available actions correspond to the choice of a row for the man, and to the choice of a column for the women, and the payoff at each combination of row and column must be read as follows: the first number is the payoff of the man, and the second one is the payoff of the woman [2]

| | Football | Opera |
|---|---|---|
| Football | $2,1$ | $0,0$ |
| Opera | $0,0$ | $1,2$ |

One of the main goals in the analysis of non-cooperative games is trying to predict the choice of the players. In the game of the battle of sexes, there is no clear indication of what is the best choice for the man and for the women. However the combination (Football, Football) and (Opera, Opera) are special, in the sense that the man and the woman's actions are "best replies" to each other: if the man chooses Football (Opera) , then it is optimal for the woman to choose Opera (Football). A combination of actions of this type is called *Nash equilibrium* (for a formal definition see 1.2.2; see also Chapter 4 for some results on the existence of Nash equilibria for particular classes of games).

For a general introduction on cooperative situations we suggest the books [158, 151, 160, 148] and on non-cooperative situations we suggest to look at [155, 152, 144, 82] (anyway, most of these readings deal with both cooperative and non-cooperative games).

## 1.1 Overview

The structure of this monograph reflects the organization of my research activity around five main axes: (1) TU-games (Chapters 2), ordinal coalitional situations (Chapters 3), algorithmic game theory (Chapter 4), application of power indices (Chapters 5) and bioinformatics and statistical analysis of biological data (Chapter 6). The first three chapters mainly deal with game theoretical models, whereas the last two are more oriented to game practice and other applications.

An important part of my research activity focuses on the analysis of TU-games and their solutions, and is described in Chapter 2. To be more specific, in Section 2.2 we introduce cost sharing problems arising from connection situations, and, in particular, we discuss new cost allocation protocols for minimum cost spanning tree games [94] based on our contributions published in [10, 16, 22, 22, 23, 51, 26]. In Section 2.3, we describe the family of *generalized additive games*, a class of TU-games games recently introduced in [70] and [71] and where the worth of a coalition is evaluated as a sum over the "valuable" contributions of players involved in the cooperation. Section 2.4 deals with argumentation games, recently published in [43], where we proposed a method to evaluate the impact of arguments in a debate by merging the classical argumentation framework proposed in [111] into a game theoretic coalitional setting.

Chapter 3 focuses on models that we recently introduced to analyse coalitional situations where the strength of agents' interaction is characterized by a "qualitative" information. Section 3.2 focuses on the problem of how to generalize the notion of power index within an ordinal framework and is mainly based on the papers [5] and [72]. Section 3.3 is devoted to the problem of how to extend a ranking over single objects to another ranking over all possible collections of objects, taking into account the fact that objects grouped together can have mutual interaction. This part is mainly based on the articles [46, 3, 59].

Chapter 4 is devoted to algorithmic issues related to non-cooperative games, and in particular to problems arising from considering the dynamics of interaction among the players (e.g., better response dynamic). Section 4.2 is based on the article [47], where we analysed non-cooperative games based on connection situations, which are the counter-part of the cooperative framework considered in Section 2.2. In Section 4.3, we provide

[1] Most of the classical introductory examples of games have been suggested at a time characterized by much less efficient communication facilities. Nowadays, we will assume that the mobile phones of the two players have run out of battery...

[2] Notice that the choice of the payoffs is purely ordinal: for instance, the payoff $(2,1)$ at position (Football, Football) compared to the payoff $(1,2)$ at combination (Opera, Opera) simply means that the men prefers the outcome (Football, Football) over (Opera, Opera), and the opposite for the woman; and both of them prefer being together to staying alone (payoff $(0,0)$)...

an overview of the main results recently published in [6], where we introduced a class of *congestion games* characterized by the fact that each resource is associated both with a capacity level, representing the maximum number of users that such a resource may simultaneously accommodate, and with an ordering on the users, prescribing the priority of accommodation of the users. Finally, Section 4.4, which is based on the paper [44], deals with games on social networks where the players interact only with their neighbours and the relationship between them can be modelled as simple two-player strategic games.

Chapter 5 deals with some recent contributions to the application of power indices to real-life situations. Section 5.2 is devoted to the problem of reducing the energy consumption over computer networks and is based on publications [49] and [48]. Section 5.3, is devoted to the discussion of a recent application of power indices presented in [54] to design a weighted majority voting system for the *Paris Science & Lettre* (PSL) Federal University. Finally, in Section 5.4 the problem of the measurement of "social capital", intended as the ability of individuals to gain benefits by utilizing their position in the society, is presented as an overview of the papers [4] and [8].

Chapter 6 is the last chapter of this monograph and is aimed at introducing and discussing our recent contributions on the application of coalitional games and their solutions to measure the importance of biological factors/variables in producing certain biological or epidemiological effects. We start in Section 6.2 with some results from publications [15, 37, 21, 17, 19, 14, 13, 50, 12, 9], where we introduced and applied alternative coalitional games to the analysis of large data-sets on gene expression. Then, Section 6.3, is focused on a recent approach described in our publication [11] to evaluate the "centrality" of genes in co-expression networks. Section 6.4 concludes with a very short presentation of recent results in the domain of the statistical analysis of biological data of human RNA (Ribonucleic acid) from publications [36, 38, 39, 40, 35], and, more recently, from [33, 31, 29, 32, 34, 30, 28, 27].

All chapters are provided with an introductory overview and a concluding section with future directions of research on the topics discussed in each chapter.

Finally, a list of bibliographic references is given at the end. The first part of the list collects the papers I contributed to (for most of them, the electronic version of this monograph is supplied with a link to the publication or to the journal website where it has been published). The second part (namely, section 'Other papers') lists the other publications cited in this monograph.

### 1.1.1 Issues not discussed in this monograph

For space reasons (this monograph must not exceed fifty pages), I omit from this monograph the issues related to the following papers.

In [62], we developed a mathematical model for the analysis of a cost allocation problem in a consortium of municipalities for garbage collection. In [25], we studied a more general model of cost allocation (also inspired by a waste management problem) and taking into account the fact that collecting information on costs can be itself costly. In [20] we have analysed the role that social rules play in selecting an equilibrium in a contest of strategic interaction arising from (unexpected) states of the world. The results of this paper, which makes use of fuzzy logic and default reasoning, are applied to the context of Corporate Social Responsibility for the theoretical foundations of ethical codes. In [45] we considered Internet as composed of Autonomous Systems exchanging routes via the inter-domain routing protocol and we introduced an auction framework adapted to determine the price at which these routes can be sold. In [61], we have studied a cost allocation problem arising from water resource management in the framework of an irrigation project for the West Delta region, in Egypt. Within the same project framework, in [2] we introduced an application of cooperative game theory to a cost allocation problem taking into account the differences in the regional landscape of land sectors of the project area. In [50] we proposed an approach based on the framework of connection situations to represent the interactions between all possible pairs of genes from gene expression data. In [12] we have presented some contributions from the literature on the analysis of the behaviour of non-rational agents in the domain of computational biology, in particular using evolutionary games and coalitonal games.

For lack of space, I will neither discuss any further the research studies concerning the development of educational tools and learning models introduced in the papers [41, 42, 52, 53].

## 1.2 Preliminaries and notations

In this section we introduce some basic definitions and notations that will be used in the following chapters.

### 1.2.1 Basic definitions on binary relations and graphs

A *binary relation* $R$ on a finite set $N = \{1, \dots, n\}$ is a collection of ordered pairs of elements of $N$, i.e. $R \subseteq N \times N$. For all elements $x, y \in N$, the more familiar notation $xRy$ will be often used instead of the more formal one $(x, y) \in R$. We provide some standard properties for $R$. *Reflexivity*: for each $x \in N$, $xRx$; *transitivity*: for each $x, y, z \in N$, $xRy$ and $yRz \Rightarrow xRz$; *totality*: for each $x, y \in N$, $x \neq y \Rightarrow xRy$ or $yRx$; *antisymmetry*: for each $x, y \in N$, $xRy$ and $yRx \Rightarrow x = y$. A reflexive and transitive binary relation is called *preorder*. A preorder that is also total is called *total preorder*. A total preorder that also satisfies antisymmetry is called *linear order*. The notation $\neg(xRy)$ means that $xRy$ is not true. We denote by $2^N$ the power set of $N$ and we use the notations $\mathcal{T}^N$ and $\mathcal{T}^{2^N}$ to denote the set of all total preorders on $N$ and on $2^N$, respectively. Moreover, the cardinality of a set $S \in 2^N$ is denoted by $|S|$.

Consider a total preorder $\succcurlyeq \subseteq 2^N \times 2^N$ over the subsets of $N$. Often we will use the notation $S \succ T$ to denote the fact that $S \succcurlyeq T$ and $\neg(T \succcurlyeq S)$ (in this case, we also say that the relation between $S$ and $T$ is 'strict'), and the notation $S \sim T$ to denote the fact that $S \succcurlyeq T$ and $T \succcurlyeq S$. For each $i, j \in N$, $i \neq j$, and all $k = 1, \dots, n-2$, we denote by $\Sigma_{ij}^k = \{S \in 2^{N \setminus \{ij\}} : |S| = k\}$ the set of all subsets of $N$ not containing neither $i$ nor $j$ with $k$ elements. Moreover, for each $i, j \in N$, we define the set $D_{ij}^k(\succcurlyeq) = \{S \in \Sigma_{ij}^k : S \cup \{i\} \succcurlyeq S \cup \{j\}\}$ as the set of coalitions $S \in 2^{N \setminus \{ij\}}$ of cardinality $k$ such that $S \cup \{i\}$ is in relation with $S \cup \{j\}$.

We provide now some basic definitions on graphs. An (undirected) *graph* is a pair $< V, E >$, where $V$ is a set of vertices or nodes and $E$ is a set of edges $e$ of the form $\{i, j\}$ with $i, j \in V$, $i \neq j$. The *complete graph* on a set $V$ of vertices is the graph $< V, E_V >$, where $E_V = \{\{i, j\} | i, j \in V \text{ and } i \neq j\}$.

A *path* between $i$ and $j$ in a graph $< V, E >$ is a sequence of nodes $(i_0, i_1, \dots, i_k)$, where $i = i_0$ and $j = i_k$, $k \geq 1$, such that $\{i_s, i_{s+1}\} \in E$ for each $s \in \{0, ..., k-1\}$ and such that all these edges are distinct. A *cycle* in $< V, E >$ is a path from $i$ to $i$ for some $i \in V$. A path $(i_0, i_1, \dots, i_k)$ is *without cycles* if there do not exist $a, b \in \{0, 1, \dots, k\}$, $a \neq b$, such that $i_a = i_b$. Two nodes $i, j \in V$ are connected in $< V, E >$ if $i = j$ or if there exists a path between $i$ and $j$ in $E$. A *connected component* of $V$ in $< V, E >$ is a maximal subset of $V$ with the property that any two nodes in this subset are connected in $< V, E >$.

For any set of edges $E \subseteq E_V$, let $V(E) = \bigcup_{\{i,j\} \subseteq E} \{i, j\}$ be the set of vertices of edges in $E$ and, for each $T \subseteq V$, let $E(T) = \{\{i, j\} \in E : i, j \in T\}$ be the set of edges contained in $T$. Moreover, let $G[T] = < T, E(T) >$ be the *subgraphs* of $< V, E >$ induced by $T$.

A *directed graph* or *digraph* is a pair $\langle V, A \rangle$, where $V$ is a set of vertices or nodes and $A$ is a set of arcs $a$ of the form $(i, j)$ with $i, j \in V$ and $i \neq j$, where arc $(i, j)$ denotes the connection between $i$ and $j$ in the direction from $i$ to $j$. The *complete digraph* on a set $V$ of vertices is the graph $\langle V, A_V \rangle$, where $A_V = V \times V$. Given a digraph $\langle V, A \rangle$, we denote by $A(S) \subseteq A$ the set of arcs in $A$ whose vertices are in $S$, i.e. $A(S) = \{(i, j) \in A | i, j \in S\}$, for each $S \subseteq V$.

### 1.2.2 Basic definitions on games

A *Transferable Utility (TU) game* (also referred to as *coalitional game* or game in *characteristic function form*) is a pair $(N, v)$, where $N = \{1, \dots, n\}$ denotes the set of players and $v : 2^N \to \mathbb{R}$ is the *characteristic function*, (by convention, $v(\emptyset) = 0$). A group of players $S \subseteq N$ is called *coalition* and $v(S)$ is called the *value* or *worth* of the coalition $S$. If the set $N$ of players is fixed, we identify a coalitional game $(N, v)$ with its characteristic function $v$ and we denote as $\mathcal{C}^N$ the class of all coalitional games with $N$ as the set of players.

A game $(N, v)$ is said to be *monotonic* if it holds that $v(S) \leq v(T)$ for all $S, T \subseteq N$ such that $S \subseteq T$ and it is said to be *superadditive* if it holds that

$$v(S \cup T) \geq v(S) + v(T)$$

for all $S, T \subseteq N$ such that $S \cap T = \emptyset$.
Moreover, a game $(N, v)$ is said to be *convex* or *supermodular* if it holds that

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$$

for all $S, T \subseteq N$. Equivalently, a game $(N, v)$ is said to be convex if the *marginal contribution* of any player to any coalition is not strictly larger than his marginal contribution to a larger coalition, *i.e.* if it holds that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \tag{1.1}$$

for all $i \in N$ and all $S \subseteq T \subseteq N \setminus \{i\}$.

The *unanimity game* $(N, u_S)$ on $S \subseteq N$ is the game described by $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$, otherwise, for each $T \in 2^N \setminus \{\emptyset\}$. Every coalitional game $(N, v)$ can be written as a linear combination of unanimity games in a unique way, *i.e.* $v = \sum_{S \subseteq N, S \neq \emptyset} \lambda_S(v) u_S$. The coefficients $\lambda_S(v)$, for each $S \in 2^N \setminus \{\emptyset\}$, are called *unanimity coefficients* of the game $(N, v)$. Given a coalitional game $(N, v)$ and a non-empty coalition $S \subseteq N$, the *subgame* of $v$ on $S$ is defined as the game $(S, v_{|S})$ such that $v_{|S}(T) = v(T)$ for all $T \subseteq S$.

An *imputation* of a game $(N, v)$ is a vector $x \in \mathbb{R}^n$ such that $\sum_{i \in N} x_i = v(N)$ (efficiency) and $x_i \geq v(\{i\})$ for all $i \in N$ (individual rationality). An important subset of the set of the imputations is the *core*, which represents a classical solution concept for TU-games. The *core* $C(v)$ of $v$ is defined as the set of efficient payoff vectors for which no coalition has an incentive to leave the grand coalition $N$, precisely,

$$C(v) = \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \quad \forall S \subset N\}.$$

A *population monotonic allocation scheme* or *pmas* [173] of the game $(N, v)$ is a scheme $x = \{x_{S,i}\}_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ with the properties

i) $\sum_{i \in S} x_{S,i} = v(S)$ for all $S \in 2^N \setminus \{\emptyset\}$;

ii) $x_{S,i} \leq x_{T,i}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ and $i \in N$ with $i \in S \subset T$[3].

A pmas provides an allocation vector for every coalition in a monotonic way, *i.e.* the value allocated to some player increase if the coalition to which he belongs becomes larger. It is easy to check that a pmas provides a core element for the game and all its subgames.

A *one-point solution* (or simply a *solution*) for a class $\mathcal{C}^N$ of coalitional games with $n$ players is a function $\psi$ that assigns a payoff vector $\psi(v)$ to every coalitional game in the class, that is $\psi : \mathcal{C}^N \to \mathbb{R}^N$.

An important family of solutions for TU-games are *probabilistic values* [179]. A probabilistic value (or *probabilistic power index* ) for the game $v$ is an $n$-vector $\pi^p(v) = (\pi_1^p(v), \ldots, \pi_n^p(v))$, such that

$$\pi_i^p(v) = \sum_{S \in 2^{N \setminus \{i\}}} p^i(S) m_i(S) \tag{1.2}$$

where $m_i(S) = v(S \cup \{i\}) - v(S)$ is the *marginal contribution* of $i$ to $S \cup \{i\}$, for each $i \in N$ and $S \in 2^{N \setminus \{i\}}$, and $p = (p^i : 2^{N \setminus \{i\}} \to \mathbb{R}^+)_{i \in N}$, is a collection of non negative real-valued functions fulfilling the condition $\sum_{S \in 2^{N \setminus \{i\}}} p^i(S) = 1$. A probabilistic value $\pi_i^p$ is called *regular* when all the coordinates of $p$ are strictly positive functions. A particular interesting case is when the probabilistic value $\pi^p$ is a *semivalue* [110], which means that non negative weights $p_0, \ldots, p_{n-1}$ are given such that $p^i(S) = p_s$, whenever the cardinality of coalition $S$ is equal to $s$ and $i \in N$[4]; furthermore, it is required that $\sum_{k=0}^{n-1} p_k \binom{n-1}{k} = 1$, in order to fulfil the condition $\sum_{S \in 2^{N \setminus \{i\}}} p_s = 1$; thus $\mathbf{p} = (p_0, \ldots, p_{n-1})$ represents a probability distribution on the family of the subsets of $N$ not containing $i$, and it is the same for all $i \in N$. We shall denote by $\mathbf{p}$ a vector $(p_0, \ldots, p_{n-1})$ as above, and, by a slight abuse of notation, $\pi^{\mathbf{p}}$ is the semivalue engendered by the vector $\mathbf{p}$. Hence, for each $i \in N$

$$\pi_i^{\mathbf{p}}(v) = \sum_{S \in 2^{N \setminus \{i\}}} p_s m_i(S). \tag{1.3}$$

We shall denote by $\mathcal{S}$ the set of all semivalues for the given fixed set $N$. The two most famous regular semivalues (i.e., with $p_s > 0$, for each $s = 0, \ldots, n-1$) are the *Shapley value* [170] $\pi^{\mathbf{p}^\phi}$, with $p_s^\phi = \frac{1}{n\binom{n-1}{s}}$, and the *Banzhaf value* [86] $\pi^{\mathbf{p}^\beta}$, with $p_s^\beta = \frac{1}{2^{n-1}}$, for each $s = 0, \ldots, n-1$. Other regular semivalues are present in the literature [104, 102, 103, 120, 142]. In order to simplify notations, in the following we will often use the notation $\phi(v)$ and $\beta(v)$ to denote, respectively, the Shapley value and the Banzhaf value of game $v$.

We recall some nice properties of the Shapley value $\pi^{\mathbf{p}^\phi}$ of a coalitional game $(N, v)$: *efficiency* (EFF), i.e. $\sum_{i \in N} \phi_i(v) = v(N)$; *symmetry* (SYM), i.e. if $i, j \in N$ are such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$, then $\phi_i(v) = \phi_j(v)$; *dummy player property* (DPP), i.e. if $i \in N$ is such that $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \subseteq N$, then $\phi_i(v) = v(\{i\})$; *additivity* (ADD), i.e. $\phi(v) + \phi(w) = \phi(v + w)$ for each $v, w \in \mathcal{C}^N$. It is well known that the Shapley value is the only solution that satisfies these four properties on the class $\mathcal{C}^N$ of all TU-games with $N$ as the set of players [170].

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[3] Note that all the previous definitions hold for TU-games where $v$ represents a gain, while the inequalities should be replaced with $\leq$ when $v$ is a cost function.

[4] Observe: $p_0 = p^i(\emptyset)$ for all $i \in N$.

The Shapley value $\phi(v)$ of a game $(N,v)$ is often introduced as the average of marginal vectors over all $n!$ possible orders of players (we denote by $S_N$ the set of all bijections $\sigma : N \to N$, where $\sigma(i) = j$ means that with respect to $\sigma$, player $j$ is in the $i$-th position). In formula

$$\phi_i(v) = \frac{1}{n!} \sum_{\sigma \in S_N} m_i^\sigma(v) \text{ for all } i \in N, \tag{1.4}$$

where for each $\sigma \in S_N$, the *marginal vector* $m^\sigma(v)$ is defined by

$$m_i^\sigma(v) = v([i,\sigma]) - v((i,\sigma)) \text{ for all } i \in N,$$

where $[i,\sigma] = \{j \in N : \sigma^{-1}(j) \leq \sigma^{-1}(i)\}$ is the set of predecessors of $i$ with respect to $\sigma$ including $i$, and $(i,\sigma) = \{j \in N : \sigma^{-1}(j) < \sigma^{-1}(i)\}$ is the set of predecessors of $i$ with respect to $\sigma$ excluding $i$.

Basic combinatorial considerations on relation (1.4) lead to the equivalent definition (1.3) with $p_s^\phi = \frac{1}{n\binom{n-1}{s}}$ for each $s = 0, \ldots, n-1$. A still alternative formulation of the Shapley value (that will be used in the following) is provided in terms of the unanimity coefficients $(\lambda_S(v))_{S \in 2^N \setminus \{\emptyset\}}$ of a game $(N,v)$, that is:

$$\phi_i(v) = \sum_{S \subseteq N : i \in S} \frac{\lambda_S(v)}{|S|} \tag{1.5}$$

for each $i \in N$.

A coalitional game $(N,v)$ such that $v(S) \in \{0,1\}$ (i.e., the worth of every coalition is either 0 or 1) for each $S \in 2^N$ and $v(N) = 1$ is said a *simple game*. The standard interpretation for these games is to consider coalitions as "winning" ($v(S) = 1$) or "losing" ($v(S) = 0$). A particular class of simple games is the one of *weighted majority games*, where the players in $N$ are associated to a vector of $n = |N|$ *weights* $(w_1, \ldots, w_n)$ and a *majority quota* $q$ is given. A weighted majority game $(N, v^w)$ on the weight $w$ and the quota $q$ is such that for each $S \in 2^N \setminus \{\emptyset\}$:

$$v^w(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i > q, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

For a monotonic simple game $(N,v)$ and a player $i \in N$, a coalition $S \in 2^N \setminus \{\emptyset\}$ with $i \in S$ and such that $v(S) = 1$ and $v(S \setminus \{i\}) = 0$ is said a *swing* for $i$ (and player $i$ is said *critical* for $S$) and we denote by $s_i(v)$ the number of swings for player $i$ in game $v$. It is easy to check that the *normalized* Banzhaf value $\bar{\beta}(v)$ of $v$ can be defined for monotonic simple games as follows:

$$\bar{\beta}_i(v) = \frac{\beta_i(v)}{\sum_{i \in N} \beta_i(v)} = \frac{s_i(v)}{\sum_{i \in N} s_i(v)}, \tag{1.7}$$

for each $i \in N$.

Given a (communication) network $\langle N, E \rangle$, following the approach in [150], we define a new game $(N, w_E^v)$, where the value $w_E^v(S)$ of a coalition $S \subseteq N$ equals the sum of the values assigned by $v$ to the connected components of the network restricted to this coalition $S$. The game $w_E^v$ is called the *graph-restricted game*. Formally,

$$w_E^v(S) = \sum_{T \in C_{E(S)}} v(T) \tag{1.8}$$

for each $S \in 2^N \setminus \{\emptyset\}$, where, according to the notations of Section 1.2.1, $E(S) = \{e \in E | e \subseteq S\}$ is the set of edges with vertices in $S$ and $C_{E(S)}$ is the set of all the connected components in the subgraph $\langle S, E(S) \rangle$, and with the convention $w_E^v(\emptyset) = 0$. The Shapley value of game $w_E^v$ is known as the *Myerson value* [150] of the *communication situation* $\langle N, v, E \rangle$ and denoted by $\mu(v, E)$.

We give now some basic definitions on strategic games. A *strategic game* is a tuple $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where $N = \{1, \ldots, n\}$ is the set of players, $S_i$ is a finite set of *pure strategies* or *actions* for player $i$ and $u_i : \prod_{j \in N} S_i \to \mathbb{R}$ is a *payoff function* specifying for each *strategy profile* or *state* $s = (s_i)_{i \in N} \in \prod_{j \in N} S_i$ player $i$' s payoff $u_i(s) \in \mathbb{R}$, for each $i \in N$. Given a strategy profile $s = (s_i)_{i \in N} \in \prod_{j \in N} S_i$, in the following, $s_{-i}$ will denote $s$ from which the strategy of player $i$ is removed, i.e. $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, s_n)$ and $(x, s_{-i})$ will denote the strategy profile $s$ from which $s_i$ is replaced by $x \in S_i$, i.e. $(x, s_{-i}) = (s_1, \ldots, s_{i-1}, x, s_{i+1}, s_n)$.

Given a strategic game $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$, a pure strategy $x \in S_i$ is a *better response* of player $i$ with respect to the strategy profile $s = (s_i)_{i\in N} \in \prod_{j\in N} S_i$ if $u_i(x, s_{-i}) \geq u_i(s)$; we say that $x$ is a *best response* to $s_{-i}$ when $u_i(x, s_{-i}) = \max_{y\in S_i} u_i(y, s_{-i})$.

A state $s = (s_i)_{i\in N} \in \prod_{j\in N} S_i$ is a (*pure*) *Nash equilibrium* of the strategic game $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$, if for every player $i \in N$, it holds that $s_i$ is a *best response* to $s_{-i}$ for each $i \in N$.

A *Better Response Dynamic* (*BRD*, also called Nash dynamics) (associated with a strategic game $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$) is a sequence of states $s^0, s^1, s^2, \ldots \in \prod_{j\in N} S_i$ such that each state $s^k \in \prod_{j\in N} S_i$ (except $s^0$) is resulted by a better response of some player from the state $s^{k-1}$. Note that if a better response dynamic reaches a Nash equilibrium after a finite number of states, then no further changes of strategies are expected (if we assume that a player changes his strategy only if he strictly prefers a different strategy).

Sometimes the payoff function $u_i$, $i \in N$, will be replaced by a cots function $c_i$, for each player $i \in N$: in this case we will call the tuple $\langle N, (S_i)_{i\in N}, (c_i)_{i\in N}\rangle$ a *strategic cost game* and the inequality in the definition of better response is replaced with $\leq$, and the max in the definition of best response is replaced with min.

# Chapter 2

# Transferable Utility (TU-)games

## 2.1 Overview of the chapter

This chapter focuses on the property-driven design of solutions for TU-games that are robust against coalitional deviations in a dynamic framework, and on the generation of efficient algorithms for their computation.

In Section 2.2, we introduce and discuss cost sharing problems arising from connection situations [64]. A connection situation takes place in the presence of a group of agents, each of whom needs to be connected directly or via other agents to a source. Since links are costly, agents evaluate the opportunity of cooperating in order to reduce costs. If a group of agents decides to cooperate, a minimum cost spanning tree (mcst) (which minimizes the total cost of connection) is constructed and the total cost of the tree must be shared among the agents of the group. Examples of connection situation are the problem of building a telecommunication network connecting some users with a service provider: players are the users, the source is the service provider and the costs over the links may represent the communication costs of each pair of users, or of a user and the service provider. The problem of finding an mcst can be easily solved by means of alternative algorithms proposed in the literature (e.g. Kruskal's algorithm [136], Prim's algorithm [163], etc.). However, finding an mcst does not guarantee that it is going to be really implemented: players must still support the cost of the mcst and then a cost allocation problem must be addressed. This cost allocation problem was introduced in [106] and has been studied with the aid of cooperative game theory since the basic paper [94]. After this seminal paper, many cost allocation methods have been proposed in the literature on *mcst games* (see, for instance, [97, 115, 128, 92]). More recently, we introduced some alternative approaches aimed at considering connection situations in a dynamic framework [64]. In fact, in many applications the number of agents can vary in time, and also increasing or decreasing of connection costs may occur. In [16], we introduced a new family of cost allocation protocols for connection situations. As discussed in more details in Section 2.2, these allocation protocols charge the agents with 'fractions' of the cost of each edge constructed at each step of the Kruskal's algorithm. It turns out that a subclass of these cost allocation protocols coincides with the class of *Obligations rules* [22], which are *cost monotonic* and induce a population monotonic allocation scheme (pmas) (see Section 1.2.2 for a formal definition of pmas). Connection networks and coalitional games are also the main subjects of the joint-papers [24, 26, 23, 51, 25]. Connection situations where the costs on the edges are interval numbers have been introduced in [10] and are briefly discussed at the end of Section 2.2. On these situations, we generalized the approach introduced in [22, 16] to interval costs and we analyzed the monotonicity and other properties of mcst solutions.

Section 2.3 is devoted to another class of cooperative games, namely, the class of *Generalized Additive Games* (*GAG*s), recently introduced in [70][1]. In a GAG, the worth of a coalition $S \subseteq N$ is evaluated by means of an *interaction filter*, that is a map $\mathcal{M}$ which returns the valuable players involved in the cooperation among players in $S$. The paper starts from the consideration that in some cases the procedure used to assess the worth of a coalition $S \subseteq N$ is strongly related to the sum of the individual values over another subset $S \subseteq N$, not necessarily included in $S$. Several examples from the literature fall into this category (for instance, the class of *argumentation games* introduced in [43] and discussed in Section 2.4, and many other *operation research games*[97]). Moreover, by making further hypothesis on the filter $\mathcal{M}$, our approach in [70] enables to classify existing games based on the properties of the map $\mathcal{M}$. In particular, in [70] we introduced and studied the class

[1]The results presented in the paper [70] originate from a collaboration with Giulia Cesari, student of a joint PhD program at *Paris Dauphine University*, under my supervision, and at *Politecnico di Milano*, in Italy, under the supervision of Roberto Lucchetti.

of *basic GAGs*, which is characterized by the fact that the valuable players in a coalition $S$ are selected on the basis of the presence, among the players in $S$, of their *friends* and *enemies*: precisely, a player contributes to the value of $S$ if and only if $S$ contains at least one of his friends and none of his enemies is present. These situations seem to well represent an online social network where the players (e.g., the members of a social network) are provided with a utility value that may represent their individual activity in a social networking web site (for instance, measured in terms of the productive time spent in uploading content files), and the participation of each player to the global activity of the social network is based on a coalitional structure of friends and enemies that is determined by their social profiles. In [71], we provided formulas for the easy computation, under certain conditions, of several classical solutions from cooperative game theory on basic GAG (e.g., the semivalues and the core).

Finally, Section 2.4 deals with the family of argumentation games introduced in [43], where we proposed a method to compute the relative relevance of arguments by merging the classical argumentation framework proposed in [111] into a game theoretic coalitional setting, where the *worth* of a collection of arguments can be seen as the combination of the information concerning the defeat relation and the preferences over arguments of a "user". In fact, a central problem faced by agents in a multi-agent debate is that they have to put forward arguments taking into account their own goals, but also how the audience may react to their arguments and the rules of the debate. Consider, for instance, the attitude of politicians participating to public debates: their choice to embrace arguments often depends on factors like the popularity of the arguments, a degree of personnel satisfaction, the consensus generated by those arguments in an assembly or in a forum, the contiguity with a political position, etc. This results in a complex decision-making problem, where most of the parameters are likely to be uncertain: what are the arguments known by other agents? what are their own goals? The final goal of the study presented in [43], and shortly resumed in Section 2.4, is to measure the relative importance of arguments for an agent taking part in a debate, and keeping into account both her/his own preferences - as represented by a utility function defined over the set of arguments - and the information provided by the attack relations among arguments. Following [43], we also provide an axiomatic characterization of the Shapley value for coalitional games defined over an argumentation framework, and we show that, for a large family of (coalitional) argumentation frameworks, the Shapley value can be easily computed.

## 2.2 Minimum cost spanning tree games

A *connection situation* or *minimum cost spanning tree* (mcst) situation is represented by a set $N = \{1, \dots, n\}$ of agents that are willing to be connected as cheap as possible to a *source* (*i.e.* a supplier of a service) denoted by 0, based on a given weight (or cost) system of connection. In the sequel we use the notation $S' = S \cup \{0\}$, for each set $S \subseteq N$, and $w$ for the *weight function*, *i.e.* a map which assigns to each *edge* or *link* of the form $\{i, j\}$ (with $i, j \in N'$, $i \neq j$) a non-negative number $w(\{i, j\})$ representing the weight or cost of edge $\{i, j\}$. We denote an *mcst situation* with set of users $N$, source 0, and weight function $w$ by $< N', w >$ (or simply $w$). Further, we denote by $\mathcal{W}^{N'}$ the set of all mcst situations $w$ with node set $N'$.

Let $E_{S'} = \{\{i, j\} | i, j \in S' \text{ and } i \neq j\}$ be the set of all possible edges between elements in $S'$, $S \subseteq N$, $S \neq \emptyset$. The cost of a *network* $< S', \Gamma >$ with $\Gamma \subseteq E_{N'}$ is $w(\Gamma) = \sum_{e \in \Gamma} w(e)$. An indirected graph $< S', \Gamma >$ is a *spanning network* on $S' \subseteq N'$ if for every $e \in \Gamma$ we have $e \in E_{S'}$ and for every $i \in S$ there is a path in $< S', \Gamma >$ from $i$ to the source. For any mcst situation $w \in \mathcal{W}^{N'}$ it is possible to determine at least one *spanning tree* on $N'$, i.e. a spanning network without circuits on $N'$, of minimum cost; each spanning tree of minimum cost is called an *mcst* for $N'$ in $w$ or, shorter, an mcst for $w$. In order to find a mcst, one can use the Kruskal algorithm [136] that works in the following way: in the first step an edge between two nodes in $N \cup \{0\}$ of minimal cost is formed. In every subsequent step, a new edge of minimal cost is formed, under the constraint that no cycles are formed with the edges constructed at the previous steps. In summary, a sequence of edges is produced and after $n$ steps an mcst appears. Since some edges may have the same cost, different mcsts may be selected by the Kruskal algorithm, depending on the ordering of the edges with respect to their increasing costs which has been considered in the Kruskal algorithm.

**Example 1.** *In this example (inspired by [24]) we consider a minimum cost spanning tree situation arising from the problem of car pooling. Suppose that three employees of a firm consider the possibility of car pooling in order to reduce their daily travel cost. The cost of driving a car from one employee to another or from one employee to the firm are given in Figure 2.1. Here the employees are denoted by* 1, 2, *and* 3 *and the firm by* 0. *To each edge* $e \in E_{\{0,1,2,3\}}$ *is assigned a non-negative number* $w(e)$ *representing the cost of edge* $e$. *A minimum cost spanning tree in this mcst situation* $< \{0, 1, 2, 3\}, w >$ *is the network* $\Gamma = \{\{0, 1\}, \{1, 2\}, \{1, 3\}\}$ *with cost*


Figure 2.1: An mcst situation $< \{0,1,2,3\}, w >$ (left side) and a related mcst (right side).

$w(\Gamma) = 48$. *This network* $\Gamma$ *corresponds to the plan of car pooling in which employees* 2 *and* 3 *drive their car in solitude to employee* 1 *where all employees take one car in order to drive together to the firm.*

Let $< N', w >$ be an mcst situation. The *minimum cost spanning tree game* $(N, c_w)$ (or simply $c_w$), corresponding to $< N', w >$, is defined by

$$c_w(S) = \min\{w(\Gamma)| < S', \Gamma > \text{ is a spanning network on } S'\}$$

for every $S \in 2^N \backslash \{\emptyset\}$, with the convention that $c_w(\emptyset) = 0$. We denote by $\mathcal{MCST}^N$ the class of all mcst games corresponding to mcst situations in $\mathcal{W}^{N'}$.

**Example 2.** *Consider the mcst situation* $< N', w >$ *with* $N' = \{0,1,2,3\}$ *and* $w$ *as depicted in Figure 2.1. If* $S = \{1,2\}$ *then a minimum cost spanning network for* $S$ *is* $\Gamma = \{\{1,2\}, \{0,1\}\}$ *with cost* 36*, whereas the minimum cost spanning network for* $S = \{3\}$ *is* $\Gamma = \{\{0,3\}\}$ *with cost* 26*. Proceeding in this way we find that the mcst game* $(N, c_w)$*, corresponding to* $< N', w >$*, is given by*

$$\begin{array}{lll} c_w(123) = 48, & & \\ c_w(12) = 36, & c_w(13) = 36, & c_w(23) = 44, \\ c_w(1) = 24, & c_w(2) = 24, & c_w(3) = 26. \end{array}$$

*Note that the allocation* $(x_1, x_2, x_3) = (24, 12, 12)$*, assigning to each player* $i \in N$ *the cost of the link from* $i$ *to its predecessor on the unique path from the source to* $i$ *in the mcst of Figure 2.1, is in the core of game* $(N, c_w)$*.*

Given a mcst situation $< N', w >$, allocations provided by the procedure assigning to each player $i \in N$ the cost of the link from $i$ to its predecessor on the unique path from the source to $i$ in an optimal tree is called *Bird allocation* [94], and is always in the core of the associated mcst $c_w \in \mathcal{MCST}^N$ [94, 128]. Note that the allocation $(x_1, x_2, x_3) = (24, 12, 12)$ is the Bird allocation associated to the mcst of Figure 2.1.

In [16, 22, 24] we have studied several solutions for mcst situations. Precisely, a *solution* is a map $F : \mathcal{W}^{N'} \to \mathbb{R}^N$ assigning to every *mcst* situation $w \in \mathcal{W}^{N'}$ a unique allocation in $\mathbb{R}^N$. An interesting family of solutions is the class of *Obligation rules*, introduced in [22].

In order to provide a formal definition of Obligation rules we need some further notations. We define the set $\Sigma_{E_{N'}}$ of *linear orders* on $E_{N'}$ as the set of all bijections $\sigma : \{1, \ldots, |E_{N'}|\} \to E_{N'}$, where $|E_{N'}|$ is the cardinality of the set $E_{N'}$. For each mcst situation $< N', w >$ there exists at least one linear order $\sigma \in \Sigma_{E_{N'}}$ such that $w(\sigma(1)) \leq w(\sigma(2)) \leq \ldots \leq w(\sigma(|E_{N'}|))$. We denote by $w^\sigma$ the column vector $\big(w(\sigma(1)), w(\sigma(2)), \ldots, w(\sigma(|E_{N'}|))\big)^t$.

For any $\sigma \in \Sigma_{E_{N'}}$ we define the set

$$K^\sigma = \{w \in \mathbb{R}_+^{E_{N'}} \mid w(\sigma(1)) \leq w(\sigma(2)) \leq \ldots \leq w(\sigma(|E_{N'}|))\}.$$

The set $K^\sigma$ is a cone in $\mathbb{R}_+^{E_{N'}}$, which we call the *Kruskal cone with respect to* $\sigma$ (corresponding to the ordering of the edges considered according to the Kruskal algorithm). One can easily see that $\bigcup_{\sigma \in \Sigma_{E_{N'}}} K^\sigma = \mathbb{R}_+^{E_{N'}}$.

Let $w \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^\sigma$. We can consider a sequence of precisely $|E_{N'}| + 1$ graphs $< N', F^{\sigma,0} >, < N', F^{\sigma,1} >, \ldots, < N', F^{\sigma,|E_{N'}|} >$ such that $F^{\sigma,0} = \emptyset$, $F^{\sigma,k} = F^{\sigma,k-1} \cup \{\sigma(k)\}$ for each

$k \in \{1, \ldots, |E_{N'}|\}$. For each graph $< N', F^{\sigma,k} >$, with $k \in \{0, 1, \ldots, |E_{N'}|\}$, let $\pi^{\sigma,k}$ be the partition of $N'$ consisting of the connected components of $N'$ in $< N', F^{\sigma,k} >$.

**Example 3.** *Consider the mcst situation $< N', w >$ with $N' = \{0, 1, 2, 3\}$ and $w$ as depicted in Figure 2.1. Note that $w \in K^\sigma$, with $\sigma(1) = \{1, 3\}$, $\sigma(2) = \{1, 2\}$, $\sigma(3) = \{2, 3\}$, $\sigma(4) = \{0, 1\}$, $\sigma(5) = \{0, 2\}$, $\sigma(6) = \{0, 3\}$.*

*The sequence of seven graphs $< N', F^{\sigma,k} >$ and the corresponding sequence of partitions $\pi^{\sigma,k}$ are shown in the following table*

| $k$ | $F^{\sigma,k}$ | $\pi^{\sigma,k}$ |
|---|---|---|
| 0 | $\{\emptyset\}$ | $\{\{0\}, \{1\}, \{2\}, \{3\}\}$ |
| 1 | $\{\{1,3\}\}$ | $\{\{0\}, \{1,3\}, \{2\}\}$ |
| 2 | $\{\{1,3\}, \{1,2\}\}$ | $\{\{0\}, \{1,2,3\}\}$ |
| 3 | $\{\{1,3\}, \{1,2\}, \{2,3\}\}$ | $\{\{0\}, \{1,2,3\}\}$ |
| 4 | $\{\{1,3\}, \{1,2\}, \{2,3\}, \{0,1\}\}$ | $\{N'\}$ |
| 5 | $\{\{1,3\}, \{1,2\}, \{2,3\}, \{0,1\}, \{0,2\}\}$ | $\{N'\}$ |
| 6 | $\{\{1,3\}, \{1,2\}, \{2,3\}, \{0,1\}, \{0,2\}, \{0,3\}\}$ | $\{N'\}$ |

Now, let $\Delta(N) = \{x \in \mathbb{R}^N_+ | \sum_{i \in N} x_i = 1\}$. The sub-simplex $\Delta(S)$ of $\Delta(N)$ given by $\Delta(S) = \{x \in \Delta(N) | \sum_{i \in S} x_i = 1\}$ is called, for reasons to be clarified later, the set of *obligation vectors* of $S$.
An *obligation function* is a map $o : 2^N \setminus \{\emptyset\} \to \Delta(N)$ assigning to each $S \in 2^N \setminus \{\emptyset\}$ an obligation vector

$$o(S) \in \Delta(S) \tag{2.1}$$

in such a way that for each $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and for each $i \in S$

$$o_i(S) \geq o_i(T). \tag{2.2}$$

Let $\Theta(N')$ be the family of partitions of $N'$. Such an obligation function $o$ on $2^N \setminus \{\emptyset\}$ induces an *obligation map* $\hat{o} : \Theta(N') \to \mathbb{R}^N$, where $\Theta(N')$ is the family of partitions of $N'$, and

$$\hat{o}(\theta) = \sum_{S \in \theta, 0 \notin S} o(S) \tag{2.3}$$

for each $\theta \in \Theta(N')$.

Note that if $\theta = \{N'\}$, then the resulting empty sum is assumed, by definition, to be the $n$-vector of zeroes: $\hat{o}(\theta) = 0 \in \mathbb{R}^N$.

**Example 4.** *Let $o^* : 2^N \setminus \{\emptyset\} \to \Delta(N)$ be defined by $o^*(S) = \frac{e^S}{|S|}$ for each $S \in 2^N \setminus \{\emptyset\}$, where $e^S$ is the $n$-vector such that $e^S_i = 1$ if $i \in S$ and $e^S_i = 0$ if $i \in N \setminus S$. Then, $o^*$ is an obligation function and the corresponding obligation map is*

$$\hat{o}^*_i(\theta) = \begin{cases} |S(\theta, \{i\})|^{-1} & \text{if } 0 \notin S(\theta, \{i\}) \\ 0 & \text{otherwise,} \end{cases} \tag{2.4}$$

*for each $\theta \in \Theta(N')$ and each $i \in N$. Here $S(\theta, \{i\}) \in \theta$ is the partition element to which $i$ belongs.*

*Note that $o^*(S)$ is the barycenter of $\Delta(S)$ and for $N = \{1, 2, 3, 4\}$, $\theta = \{\{1, 2\}, \{0, 3\}, \{4\}\}$ we have $\hat{o}^*(\theta) = (\frac{1}{2}, \frac{1}{2}, 0, 1)$.*

A characteristic of Obligation rules is that they assign to an mcst situation a vector of cost contributions which can be obtained as a product of a contribution matrix with the cost vector of edges ordered according to $\sigma$.

**Definition 1.** *Let $\hat{o}$ be an obligation map on $\Theta(N')$. Let $\sigma \in \Sigma_{E_{N'}}$. The* contribution matrix *w.r.t $\hat{o}$ and $\sigma$ is the matrix $D^{\sigma,\hat{o}} \in \mathbb{R}^{N \times |E_{N'}|}$ where*

$$D^{\sigma,\hat{o}}_{ik} = \hat{o}_i(\pi^{\sigma,k-1}) - \hat{o}_i(\pi^{\sigma,k})$$

*for each $i \in N$ and each $k \in \{1, \ldots, |E_{N'}|\}$.*

**Definition 2.** *Let $\hat{o}$ be an obligation map on $\Theta(N')$. Let $\sigma \in \Sigma_{E_{N'}}$. We define the map $\phi^{\sigma,\hat{o}} : K^\sigma \to \mathbb{R}^N$ by*

$$\phi^{\sigma,\hat{o}}(w) = D^{\sigma,\hat{o}} w^\sigma, \tag{2.5}$$

*for each mcst situation $w$ in the cone $K^\sigma$.*

Next proposition states that $\phi^{\sigma,\hat{o}}$ does not depend on the choice of $\sigma$ and makes it possible to define an Obligation rule with respect to an obligation map on $\Theta(N')$ as a map on $\mathcal{W}^{N'}$ (differently from the bird allocations, which depend on the selected optimal tree).

**Proposition 1** (from [22]). *Let $\hat{o}$ be an obligation map on $\Theta(N')$. If $w \in K^\sigma \cap K^{\sigma'}$ with $\sigma, \sigma' \in \Sigma_{E_{N'}}$, then $\phi^{\sigma,\hat{o}}(w) = \phi^{\sigma',\hat{o}}(w)$.*

A basic property of fairness for solutions is cost monotonicity, imposing that if some connection costs go down, then no agents will pay more. Formally, a solution $F$ is a *cost monotonic* solution if for all *mcst* situations $w, w' \in \mathcal{W}^{N'}$ such that $w(e) \leq w'(e)$ for each $e \in E_{N'}$ it holds that $F_i(w) \leq F_i(w')$ for each $i \in N$. Unfortunately, the Bird rule and many other solutions for mcst games are not cost monotonic. In [22] we proved the following result.

**Theorem 1** (from [22]). *Obligation rules are cost monotonic.*

As we formally defined in Section 1.2.2, a pmas provides a cost allocation vector for every coalition in a monotonic way. The following result has been proved in [22] too.

**Theorem 2** (from [22]). *Let $\hat{o}$ be an obligation map on $\Theta(N')$, let $\phi^{\hat{o}}$ be the Obligation rule w.r.t $\hat{o}$, and let $w \in \mathcal{W}^{N'}$. Then, the table $[\phi^{\hat{o}_S}(w_{|S'})]_{S \in 2^N \setminus \{\emptyset\}}$ is a pmas for the mcst game $(N, c_w)$.*

The obligation map of Example 4 defines a well known solution for mcst situations, that is the *Equal Remaining Obligation rule*, also called $P$-value [23, 92, 115].

**Definition 3.** *The $P$-value is the map $P : \mathcal{W}^{N'} \to \mathbb{R}^N$, defined by*

$$P(w) = \phi^{\hat{o}^*}(w) \tag{2.6}$$

*for each $w \in \mathcal{W}^{N'}$ and where $\phi^{\hat{o}^*}$ is the Obligation rule w.r.t. the obligation map $\hat{o}^*$ of Example 4.*

Example 5 provides an illustration of the $P$-value.

**Example 5.** *Consider the mcst situation $< N', w >$ with $N' = \{0, 1, 2, 3\}$ and $w$ of Example 1. The contribution matrix $D^{\sigma,\hat{o}^*}$ is*

$$D^{\sigma,\hat{o}^*} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 \end{pmatrix}$$

*and $w^\sigma = (12, 12, 20, 24, 24, 26)^t$.*
*Then, $P(w) = \phi^{\hat{o}^*}(w) = D^{\sigma,\hat{o}^*} w^\sigma = (16, 16, 16)^t$.*

In [23] we also provided an axiomatic characterization of the $P$-value is provided, that is shortly introduced in the following. A *solution for mcst situations* is a map $F : \mathcal{W}^{N'} \to \mathbb{R}^N$ assigning to every mcst situation $w$ a unique cost allocation in $\mathbb{R}^N$. Some interesting properties for solutions for mcst situations are the following.

**Property 1.** *The solution $F$ is* efficient *(EFF) if for each $w \in \mathcal{W}^{N'}$*

$$\sum_{i \in N} F_i(w) = w(\Gamma),$$

*where $\Gamma$ is a minimum cost spanning network on $N'$.*

**Property 2.** *The solution $F$ has the* Equal Treatment *(ET) property if for each $w \in \mathcal{W}^{N'}$ and for each $i, j \in N$ with $C_i(w) = C_j(w)$,*

$$F_i(w) = F_j(w).$$

**Property 3.** *The solution $F$ has the* upper bounded contribution *(UBC) property if for each $w \in \mathcal{W}^{N'}$ and every $(w, N')$-component $C \neq \{0\}$*

$$\sum_{i \in C \setminus \{0\}} F_i(w) \leq \min_{i \in C \setminus \{0\}} w(\{i, 0\}).$$

**Property 4.** *The solution $F$ has the* Cone-wise Positive Linearity *(CPL) property if for each $\sigma \in \Sigma_{E_{N'}}$, for each pair of mcst situations $w, \widehat{w} \in K^{\sigma}$ and for each pair $\alpha, \widehat{\alpha} \geq 0$, we have*

$$F(\alpha w + \widehat{\alpha}\widehat{w}) = \alpha F(w) + \widehat{\alpha} F(\widehat{w}).$$

**Theorem 3** (from [23])**.** *The $P$-value is the unique solution which satisfies the properties EFF, ET, UBC and CPL on the class $\mathcal{W}^{N'}$ of mcst situations.*

We give now a short summary of other interesting results on connection situations provided in our papers [26, 24, 51, 16, 10]. In [26] we showed that, for variants of classical mcst games on directed graphs, a pmas does not necessarily exist. In particular, in [26] we provide a simple algorithm to obtain an mcst and to extend each core elementto a pmas and also to a bi-monotonic allocation scheme [101, 177]. In [24], we introduced the *Subtraction Algorithm* computing, for every mcst situation and each permutation on the set of players, a pmas. This algorithm is based on a decomposition theorem which guarantees that every mcst game can be written as a nonnegative combination of mcst games corresponding to $0-1$ cost functions. In [51], we presented a new way to define the *irreducible core* [94], based on a non-Archimedean semimetric. The Bird core correspondence turns out to have interesting monotonicity and additivity properties, and each stable cost monotonic allocation rule for mcst situations is a selection of the Bird core correspondence. In [16], we studied the class of *Construct and Charge* (*CC*-) rules for mcst situations, that are defined starting from *charge systems*, and specify particular allocation protocols that are also rooted on the Kruskal algorithm for computing an mcst.

In [10] we introduced the model of *minimum interval cost spanning tree* (micst) situations, *i.e.* situations where agents of the set $N = \{1, \ldots, n\}$ are willing to be connected as cheaply as possible to a source based on an interval-valued cost system of connection: to each edge $e \in E_{N'}$ is assigned a closed interval in $\mathbb{R}_+$ representing the uncertain cost of edge $e$ (no probability distribution is assumed for edge costs). In [10] we focused on particular allocation protocols for micst situations that we called *extended obligation rules* generalizing the notion of obligation rule. In this setting, it turns out that cost monotonicity provides extra incentives in favour of a social agreement, where the unique condition of core membership may not be sufficient. We also presented an application to a randomly generated ad-hoc wireless network with many nodes, together with a computer programme in the R language [167] for the computation of the (extended) $P$-value.

## 2.3 Generalized additive games

One of the main issues in the analysis of coalitional games is that the number of coalitions grows exponentially with respect to the number of players. Consequently, it is computationally very interesting to single out classes of games that can be described in a concise way. In the literature on coalitional games there exist several approaches for defining classes of games whose concise representation is derived by an additive pattern among coalitions. In some contexts, due to an underlying structure among the players, such as a network, an order, or a permission structure, the value of a coalition $S \subseteq N$ can be derived additively from a collection of subcoalitions $\{T_1, \cdots T_k\}$, $T_i \subseteq S$ $\forall i \in \{1, \cdots, k\}$. Such situations are modeled, for example, by the graph-restricted games, introduced by Myerson [150] and further studied by Owen [157], the component additive games [109] and the restricted component additive games [108].

Sometimes, the procedure used to assess the worth of a coalition $S \subseteq N$ is strongly related to the sum of the individual values over another subset $T \subseteq N$, not necessarily included in $S$. Several examples from the literature fall into this category, among them the well-known *glove game*, the *airport games* [139, 140], the *connectivity game* and its extensions [80, 138], the *argumentation games* [43] and some classes of operation research games, such as the *peer games* [100] and the *mountain situations* [26]. In these models, the value of a coalition $S$ of players is calculated as the sum of the single values of players in a subset of $S$. On the other hand, in some cases the worth of a coalition might be affected by external influences and players outside the coalition might contribute, either in a positive or negative way, to the worth of the coalition itself. This is the case, for example, of the *bankruptcy games* [83] and the *maintenance problems* [134, 97].

In [70] we introduced a general class of additive TU-games where the worth of a coalition $S \subseteq N$ is evaluated by means of an interaction filter, that is a map $\mathcal{M}$ which returns the valuable players involved in the cooperation among players in $S$. More precisely, a *Generalized Additive Situation* (*GAS*) is a triple $\langle N, v, \mathcal{M}\rangle$, where $N$ is the set of players, $v : N \to \mathbb{R}$ is a map that assigns a value to each player and $\mathcal{M} : 2^N \to 2^N$ is a *coalitional map*, which assigns a (possibly empty) coalition $\mathcal{M}(S)$ to each coalition $S \subseteq N$ of players.

Given the GAS $\langle N, v, \mathcal{M}\rangle$, the associated *Generalized Additive Game* (GAG) is defined as the TU-game $(N, v^{\mathcal{M}})$ assigning to each coalition $S \in 2^N$ the value

$$v^{\mathcal{M}}(S) = \sum_{i \in \mathcal{M}(S)} v(i) \tag{2.7}$$

for each $S \in 2^N \setminus \{\emptyset\}$ and $v^{\mathcal{M}}(\emptyset) = 0$, as usual.

In particular, in [70] we introduced and studied the class of *basic GAGs*, which is characterized by the fact that the valuable players in a coalition $S$ are selected on the basis of the presence, among the players in $S$, of their *friends* and *enemies*. For each $i \in N$, define $\mathcal{C}_i = \{F_i^1, \dots, F_i^{m_i}, E_i\}$ as a collection of subsets of $N$ such that $F_i^j \cap E_i = \emptyset$ for all $i \in N$ and for all $j = 1, \cdots, m_i$. Let $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$. A *basic* GAS is the triple $\langle N, v, \mathcal{C}\rangle$, associated with the coalitional map $\mathcal{M}$ defined as:

$$\mathcal{M}(S) = \{i \in N : S \cap F_i^1 \neq \emptyset, \dots, S \cap F_i^{m_i} \neq \emptyset, S \cap E_i = \emptyset\}. \tag{2.8}$$

Several of the aforementioned classes of games from the literature can be described as basic GAGs, as well as games deriving from real-world situations (see [70] for more details).

**Example 6.** *(airport games [139, 140]): Let $N$ be the set of players* [2]. *We partition $N$ into groups $N_1, N_2, \dots,$ $N_k$ such that to each $N_j$, $j = 1, \dots, k$, is associated a positive real number $c_j$ with $c_1 \leq c_2 \leq \cdots \leq c_k$ (representing costs). Consider an airport game $w$ such that $w(S) = \max\{c_i : i \in S\}$. This type of game (and variants) can be described by a basic GAS $\langle N, (\mathcal{C}_i = \{F_i, E_i\})_{i \in N}, v\rangle$ by setting for each $i \in N_j$ and each $j = 1, \dots, k$:*

- *the value $v(i) = \frac{c_j}{|N_j|}$,*

- *the set of friends $F_i = N_j$,*

*and the set of enemies $E_i = N_{j+1} \cup \ldots \cup N_k$ for each $i \in N_j$ and each $j = 1, \dots, k-1$ and $E_l = \emptyset$ for each $l \in N_k$.*

*By using similar arguments, it is possible to show that also the maintenance games ([134], [97]), which generalize the airport games, can be represented as basic GAGs.*

Moreover, it is possible to produce, for basic GAGs, results concerning important solution concepts, like the core and the semivalues. It is therefore interesting to study under which conditions a GAS can be described as a basic one. To this purpose, the following theorem provides a necessary and sufficient condition when the set of enemies of each player is empty.

**Theorem 4** ([70])**.** *Let $\langle N, v, \mathcal{M}\rangle$ be a GAS. The map $\mathcal{M}$ can be obtained by relation (2.8) via collections $\mathcal{C}_i = \{F_i^1, \dots, F_i^{m_i}, E_i = \emptyset\}$, for each $i \in N$, if and only if $\mathcal{M}$ is monotonic (i.e., $\mathcal{M}(s) \subseteq \mathcal{M}(T)$ for each $S, T$ such that $S \subseteq T \subseteq N$).*

The model of basic GAG turns out to be suitable for representing an online social network, where friends and enemies of the web users are determined by their social profiles. It is well known that the problem of identifying influential users on a social networking web site plays a key role to find strategies aimed at increasing the site's overall view. The main issue is to target advertisement to the site members of the online social network whose activities' levels have a significant impact on the activity of the other site members. The overall influence of a user can be seen as the combination of two ingredients: 1) the individual ability to get the attention of other site members, and 2) the personal characteristic of the social profile, that can be represented in terms of groups or communities to which users belong. A basic GAS $\langle N, v, \mathcal{C}\rangle$ can represent an online social network as described above. More specifically, each player $i \in N$ of the basic GAS is associated to a value $v(i)$ representing her/his individual activity in a social networking web site (for instance, measured in terms of the productive time spent in uploading content files), and the participation of the individuals to the global activity of the social network is based on a coalitional structure $\mathcal{C}$ of friends and enemies that is determined by players' social profiles. Thus, it

[2] In airport games the players are the *landings* that occur during the lifetime of the landing strip of an airport. Since not all players will need a landing strip of the same length, the cost of a coalition $S$ is computed as the cost associated with a landing strip long enough to accommodate all of the landings in $S$.

is interesting to analyze, for this type of games, the behavior of indices aimed at measuring the influence of the players in the game: in particular we consider the Shapley value [171, 170], the Banzhaf Value [86] and other semivalues [110].

In what follows, in order to simplify the notation, we fix $i \in N$ and denote by $f$ the cardinality of $F_i$ and by $e$ the cardinality of $E_i$ (in order to simplify the notation, if $E_i = \emptyset$ we assume by convention that $e = 0$ and $\frac{1}{e} = 0$). We denote by $\langle N, v^{\mathcal{C}} \rangle$ the GAG associated to a basic GAS $\langle N, v, \mathcal{C} \rangle$, and we call such a game *basic* GAG. Consider a basic GAS with a *single* set of friends for each player $i$. The associated basic GAG $v^{\mathcal{C}_i}$ reduces to:

$$v^{\mathcal{C}_i}(S) = \begin{cases} v(i) & \text{if} \quad S \cap F_i \neq \emptyset, S \cap E_i = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

The following proposition holds.

**Proposition 2** ([70]). *Let us consider a basic GAS on $\langle N, v, \{\mathcal{C}_i = \{F_i, E_i\}\}_{i \in N} \rangle$. Then the Shapley value $\phi(v)$ and Banzhaf value $\beta(v)$ for the game $v^{\mathcal{C}_i}$ are given, respectively, by:*

$$\phi_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if} \quad j \in N \setminus (F_i \cup E_i) \\ \frac{v(i)}{f+e} & \text{if} \quad j \in F_i \\ -v(i)\frac{f}{e(f+e)} & \text{if} \quad j \in E_i \end{cases}$$

*and*

$$\beta_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if} \quad j \in N \setminus (F_i \cup E_i) \\ \frac{v(i)}{2^{f+e-1}} & \text{if} \quad j \in F_i \\ -v(i)\frac{2^f - 1}{2^{f+e-1}} & \text{if} \quad j \in E_i. \end{cases}$$

We comment this result with the help of the following example from [70].

**Example 7.** *As a toy example, consider an online social network with four users $N = \{1, 2, 3, 4\}$ where each user spends the same amount of time $T$ in uploading new content files and, according to her/his social profile, each user $i \in N$ belongs to a single community $F_i \subseteq N$ (e.g., the set of users with whom $i$ intends to share her/his content files) which is in conflict with the complementary one $E_i = N \setminus F_i$ (here, enemies in $E_i$ are interpreted as those members that have no permission to access the content files of player $i$). Suppose, for instance, that $F_1 = \{1, 2, 3\}$, $F_2 = \{2, 3\}$, $F_3 = \{3\}$ and $F_4 = \{1, 2, 3, 4\}$. Following the discussion about social networking web sites earlier introduced, we can represent such a situation as a basic GAS $\langle N, v, \{\mathcal{C}_i = \{F_i, E_i = N \setminus F_i\}\}_{i \in N} \rangle$. How to identify the most influential users? According to Proposition 2, the influence vector provided by the Shapley value is: $\phi(v^{\mathcal{C}}) = (\frac{T}{6}, \frac{2T}{3}, T, \frac{-5T}{6})$. So, user 3 results the most influential one, followed by 2, then 1 and finally 4, who is the only user to get a negative index.*

*Suppose now that user 2 wants to improve her/his influence as measured by the Shapley value. It is worth noting that if user 2 removes 3 from her/his set of friends (and all the other sets of friends and enemies remain the same), then player 2 gets exactly the same Shapley value of user 3. Precisely, if now $F_2 = \{2\}$ and $E_2 = \{1, 3, 4\}$, then $\phi_2(v^{\mathcal{C}}) = \phi_3(v^{\mathcal{C}}) = \frac{2T}{3}$. Notice that the fact that an influential player has been removed from his/her list of friends does not impact directly the influence of player 2, but it determines an important reduction of the influence of player 3.*

## 2.4 Argumentation games

A *Dung Argumentation Framework* (DAF) is a directed graph $\langle \mathcal{A}, \mathcal{R} \rangle$, where the set of nodes $\mathcal{A}$ is a finite set of *arguments* and the set of arcs $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ is a *binary defeat* (or *attack*) *relation* (i.e, $(i, j) \in \mathcal{R}$ means that argument $i \in \mathcal{A}$ attacks argument $j \in \mathcal{A}$). We say that a set of arguments $S \subseteq \mathcal{A}$ (also called a *coalition* $S$) attacks another coalition $T \subseteq \mathcal{A}$ in $\langle \mathcal{A}, \mathcal{R} \rangle$ if there exists $(s, t) \in S \times T$ with $(s, t) \in \mathcal{R}$, that is an attacks which originates from an argument in $S$ and is directed against an argument in $T$. For each argument $a$ we define the set of *predecessors* of $a$ in $\langle \mathcal{A}, \mathcal{R} \rangle$ as the set $Pr(\mathcal{R}, a) = \{j \in \mathcal{A} : (j, a) \in \mathcal{R}\}$, and the set of *successors* of $a$ is denoted by $Su(\mathcal{R}, a) = \{j \in \mathcal{A} : (a, j) \in \mathcal{R}\}$ (if clear from the context, we omit notation $\mathcal{R}$ in $Pr(a)$ and $Su(a)$).

The main goal of argumentation theory is to identify which arguments are rationally "acceptable" according to different notions of acceptability. Some of the most common notions of acceptability are the following ones. An argument $a \in \mathcal{A}$ is said *acceptable w.r.t.* $S \subseteq \mathcal{A}$ iff $\forall b \in \mathcal{A}$: if $(b, a) \in \mathcal{R}$, then $\exists c \in S$ such that $(c, b) \in \mathcal{R}$.

A set $S \subseteq \mathcal{A}$ is said to be: (*conflict-free*) iff $S$ does not attack itself; (*stable*) iff $S$ is conflict-free and attacks every argument in $\mathcal{A} \setminus S$; (*admissible*) iff $S$ is conflict-free and $S$ attacks every argument in $\mathcal{A} \setminus S$ that attacks $S$; (*preferred*) iff it is a maximal (w.r.t. $\subseteq$) admissible extension; (*complete*) iff $\forall a \in \mathcal{A}$, if $a$ is acceptable w.r.t. $S$, then $a \in S$; (*grounded*) iff $S$ is the minimal (w.r.t. $\subseteq$) complete extension.

In [43] we introduced a framework for abstract argumentation, keeping into consideration the preferences over (collections of) arguments of an agent (hereafter, called "the user"), who deals with the problem of assessing the relevance of each argument with respect to her/his own objectives, and facing the uncertainty about which combination of arguments will result from the debate. The final goal of this paper, is then to measure the relative importance of arguments for the user, taking into account both her/his own preferences - as represented by a utility function defined over the set of arguments - and the information provided by the attack relation over the arguments. In this direction, we merged the classical argumentation framework proposed in [111] into a game theoretic coalitional setting, where the "worth" of a *coalition of arguments* can be seen as the combination of the information about the preferences of the user over arguments and the information concerning the conflicts.

Precisely, in [43] we defined a *Coalitional Argumentation Framework* (CAF) as a triple $\langle \mathcal{A}, \mathcal{R}, v \rangle$ where $\langle \mathcal{A}, \mathcal{R} \rangle$ is a DAF and $v$ is a map assigning to each coalition $S \subseteq \mathcal{A}$ a number $v(S) \in \mathbb{R}$. The value $v(S)$ represents the *worth* of the coalition $S$ for the user (for example, it could measure the success provided by coalition $S$ according to a criterion specified by the user, e.g., the overall popularity of the arguments in $S$). We assume that for each $a \in \mathcal{A}$, the worth (or *utility*) of the singleton $\{a\}$ is given by a (cardinal) utility function on $\mathcal{A}$. We also assume that a CAF $\langle \mathcal{A}, \mathcal{R}, v \rangle$ satisfies the following conditions of consistency between the map $v$ and the DAF $\langle \mathcal{A}, \mathcal{R} \rangle$: (c.1) if $a \in \mathcal{A}$ is such that $Pr(a) = Su(a) = \emptyset$ ($a$ is not connected to other arguments in $\langle \mathcal{A}, \mathcal{R} \rangle$), then $v(S \cup \{a\}) = v(S) + v(\{a\})$ for each coalition $S \subseteq \mathcal{A} \setminus \{a\}$; (c.2) if $a, b \in \mathcal{A}$ are such that $Pr(a) = Pr(b)$ and $Su(a) = Su(b)$ (i.e., $a$ and $b$ are symmetric in the DAF) and $v(\{a\}) = v(\{b\})$, then $v(S \cup \{a\}) = v(S \cup \{b\})$ for each coalition $S \subseteq \mathcal{A}$.

Given a CAF $\langle \mathcal{A}, \mathcal{R}, v \rangle$ (satisfying the consistency conditions (c.1) and (c.2) as well), we study the problem of providing a measure representing the relevance of arguments, taking into account both the structure of the DAF and the worth of coalitions as measured by $v$. In this direction, we focus on properties that such a measure of relevance should satisfy.

For instance, the SYM[3] property introduced in Section 1.2.2, states that two symmetric players in the DAF should have the same relevance, provided that their worth as singletons is the same. Analogously, rephrasing the notion of dummy player in a CAF, the DPP says that disconnected arguments in a DAF should receive as value of relevance precisely their worth as singletons. Still, the EFF property imposes un upper bound over the scale for measuring the relevance of arguments (precisely, the sum of the relevance values must be equal to $v(\mathcal{A})$). Finally, an interesting reinterpretation of the ADD property suggests that the sum of the relevance values measured over two distinct CAFs $\langle \mathcal{A}, \mathcal{R}, v_1 \rangle$ and $\langle \mathcal{A}, \mathcal{R}, v_2 \rangle$ sharing the same DAF (for instance, $v_1$ and $v_2$ may represent the preference over coalitions of arguments in two distinct populations, like the population of women and the one of men), should be equal to the relevance of arguments measured on $\langle \mathcal{A}, \mathcal{R}, v_1 + v_2 \rangle$.

**Example 8.** *Consider two CAFs,* $\langle \{1,2,3\}, \{(1,2),(2,1)\}, v \rangle$ *and* $\langle \{1,2,3\}, \{(1,2), (2,1)\}, v' \rangle$ *(satisfying conditions (c.1) and (c.2) as well). Consider the CAF* $\langle \{1,2,3\}, \{(1,2),(2,1)\}, \bar{v} = v + v' \rangle$. *By ADD and DPP,*

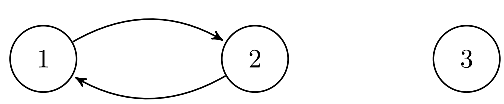


Figure 2.2: The DAF corresponding to the CAFs of Example 8.

*the Shapley value of argument 3 is* $\phi_3(\bar{v}) = v(\{3\}) + v'(\{3\})$, *and by SYM and EFF,* $\phi_1(\bar{v}) = \phi_2(\bar{v}) = \frac{1}{2}(v(\mathcal{A}) + v'(\mathcal{A}) - (v(\{3\}) + v'(\{3\})))$.

In order to *partially* take into account the argumentation system, in [43] we defined a notion of the worth of coalitions combining the preferences of the user over single arguments and a "local" information about the attacks. Precisely, consider a CAF $\langle \mathcal{A}, \mathcal{R}, \hat{v} \rangle$ where the worth of coalitions is additive over non-attacked

[3]Notice that, given a CAF $\langle \mathcal{A}, \mathcal{R}, v \rangle$, there may exist arguments $a$ and $b$ that are symmetric players in $v$ (according to definition provided in Section 1.2.2) that do not necessarily satisfy the condition $Pr(a) = Pr(b)$, $Su(a) = Su(b)$ and $v(\{a\}) = v(\{b\})$ introduced in condition ($c$.2). In a similar way, there may exist arguments $a \in \mathcal{A}$ that are dummy players in $v$ that do not satisfy condition $Pr(a) = Su(a) = \emptyset$.

arguments, i.e. if a coalition $S \subseteq \mathcal{A}$ forms, $\hat{v}(S)$ denotes the sum of the worth of single arguments that are not attacked within coalition $S$. Let $F_S = \{i \in S : \{i\}$ is not attacked by $S \setminus \{i\}\}$ be the set of non-attacked arguments in $S$, then

$$\hat{v}(S) = \sum_{i \in F_S} \hat{v}(\{i\}), \tag{2.9}$$

for each $S \subseteq \mathcal{A}$ (by convention, $\hat{v}(\emptyset) = 0$). We denote by $\hat{\mathcal{V}}^{\mathcal{A}}$ the class of CAFs $\langle \mathcal{A}, \mathcal{R}, \hat{v} \rangle$ on $\mathcal{A}$ introduced above, and by $\mathcal{G}\hat{\mathcal{V}}^{\mathcal{A}}$ the class of corresponding games $\hat{v}$ defined by relation (2.9).

**Example 9.** *Consider the CAF $\langle \{1,2,3\}, \{(1,2),(2,3)\}, \hat{v} \rangle$, such that the preference over each argument $i$ is the same and is equal to* 1*.*

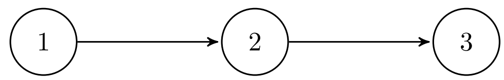

Figure 2.3: The DAF corresponding to the CAF of Example 14.

*The game $\hat{v}$ is provided in Table 2.1. The Shapley value of such a game is $\phi_1(\hat{v}) = \phi_3(\hat{v}) = \frac{1}{2}$ and $\phi_2(\hat{v}) = 0$. So the greatest relevance is given to arguments* 1 *and* 3 *(note that the coalition $\{1,3\}$ is the only stable one).*

| $S$ : | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
|---|---|---|---|---|---|---|---|
| $\hat{v}(S)$ : | 1 | 1 | 1 | 1 | 2 | 1 | 1 |

Table 2.1: The worth of each coalition $S \subseteq \{1,2,3\}$ in $\hat{v}$.

*Alternatively, suppose that the argument* 2 *is preferred to the other ones, and thus the worth of the singleton coalition $\{2\}$ is, for instance, $\hat{v}(\{2\}) = 1$, and $\hat{v}(\{1\}) = \hat{v}(\{3\}) = 0$. Now the relevance assigned by the Shapley value is $\frac{1}{2}$ to arguments* 2*,* 0 *to argument* 3*, and $-\frac{1}{2}$ to argument* 1*: the user would be worse off by attacking the most beneficial and non-defended argument* 2*, whereas she/he would receive no detriment in adopting argument* 3*.*

In general the Shapley value is hard to calculate, since it requires a number of operations that is exponential in the number of arguments. However, for the specific class $\mathcal{G}\hat{\mathcal{V}}^{\mathcal{A}}$, it is possible to calculate the Shapley value easily, as proved by the following theorem.

**Theorem 5** ([43]). *Consider a CAF $\langle \mathcal{A}, \mathcal{R}, \hat{v} \rangle \in \hat{\mathcal{V}}^{\mathcal{A}}$. Then the Shapley value of game $(\mathcal{A}, \hat{v})$ is*

$$\phi_i(\hat{v}) = \tfrac{\hat{v}(\{i\})}{|Pr(i)|+1} - \textstyle\sum_{j \in Su(i)} \tfrac{\hat{v}(\{j\})}{|Pr(j)|(|Pr(j)|+1)}, \tag{2.10}$$

*for each $i \in \mathcal{A}$.*

Note that the Shapley value of an argument $i$ in game $\hat{v}$ does not depend only on the number of predecessors (attackers) an argument has, but also on the number of successors (arguments attacked by $i$), and on the number of other attackers of the arguments attacked by $i$. In [43] we also introduced an axiomatic characterization of a solution in the specific class of coalitional games arising from CAFs in $\hat{\mathcal{V}}^{\mathcal{A}}$.

For this purpose, we define a *solution* as a map $\psi : \mathcal{G}\hat{\mathcal{V}}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}}$. An interesting property for a solution is the following one.

**Property 5** (Equal Impact of an Attack)**.** *Let $\langle \mathcal{A}, \mathcal{R}, \hat{v} \rangle \in \hat{\mathcal{V}}^{\mathcal{A}}$ and $i, j \in \mathcal{A}$, with $i \neq j$. Consider a CAF $\langle \mathcal{A}, \mathcal{R} \cup \{(i,j)\}, \hat{v}_{ij} \rangle \in \hat{\mathcal{V}}^{\mathcal{A}}$ with $\hat{v}_{ij}(\{k\}) := \hat{v}(\{k\})$ for each $k \in \mathcal{A}$. A solution $\psi : \mathcal{G}\hat{\mathcal{V}}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}}$ satisfies (on $\mathcal{G}\hat{\mathcal{V}}^{\mathcal{A}}$) the property of Equal Impact of an Attack (EIA) iff*

$$\psi_i(\hat{v}) - \psi_i(\hat{v}_{ij}) = \psi_j(\hat{v}) - \psi_j(\hat{v}_{ij}).$$

Property of EIA states that when a new attack between two argument $i$ and $j$ is added to (or removed from) a CAF, then the relevance of the two arguments should be affected in the same way. Differently stated, this property says that a consequence of an attack should be detrimental for both arguments involved in the defeat relation, since an attacks always decreases the worth of coalitions containing the involved nodes.

We also say that a solution $\psi : \mathcal{G}\hat{\mathcal{V}}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}}$ satisfies the property ADD*, if and only if $\psi(v)+\psi(w) = \psi(v+w)$ for each $v, w \in \mathcal{G}\hat{\mathcal{V}}^{\mathcal{A}}$ such that $v + w \in \mathcal{G}\hat{\mathcal{V}}^{\mathcal{A}}$.

An axiomatic characterization of the Shapley value on the class of the CAFs presented in [43] is the following one.

**Theorem 6** ([43]). *The Shapley value is the unique solution that satisfies EFF, SYM, DPP, ADD* and EIA properties on the class* $\mathcal{G}\hat{\mathcal{V}}^{\mathcal{A}}$.

## 2.5 Future directions

In Section 2.2, we have discussed connection situations where the players are often located at some nodes of a network. In many cases, however, the focus of interest of rational agents are the edges of a network. For instance, the authors of [130] recently introduced a model that can be applied to the problem where agents may require the connection between certain nodes of a network, using a single link or via longer paths, and where it is assumed that the set of implemented edges is exogenously fixed and may be "redundant" (see also [149]). A still different class of games has been studied in [84], where the players are the edges of a graph and a coalition of edges gets value one if it is a connected component in the graph, and zero otherwise. All the aforementioned approaches deal with coalitional games where the optimization problem used to compute the cost of a coalition is not based on the problem of finding a network of minimum cost in the corresponding sub-graph. Instead, a different direction is to assume that the optimal network associated to each coalition (of edges) is not fixed and follows a cost optimization procedure.

One of the goal of our future research on the models introduced in Section 2.3 is the application to the analysis of real social network data. As shown by Example 7, the information required to compute classical power indices on basic GAGs representing online social networks (like the users' activity time or the users' social profiles and social affinities) is not very demanding and can be obtained by available records and models from the literature [169]. Moreover, as it has been stressed in the same example, it would be interesting to explore the strategic issues related to the attempt of players to increase their influence (as measured by the Shapley value or by other power indices) on a social network. An interesting direction for future research is indeed that of coalition formation, since for generic basic GAGs associated to GASs with nonnegative $v$, where the sets of enemies are not empty, the grand coalition is not likely to form. In general, we believe that the issue about which coalitions are more likely to form in a basic GAG is not trivial and deserves to be further explored.

In Sections 2.4, a property-driven approach has been used to support the adoption of the Shapley value as a measure of the relevance of arguments. On the other hand, we may consider a multi-agent interaction protocol where arguments are introduced by agents one after the other, respecting the protocol's rule according to which an argument can be introduced at a certain stage of the debate only if it attacks or it is attacked by another argument previously introduced. According to such a protocol, assuming that all coalitions of arguments could be formed is not meaningful, and then it would be interesting to look at other semivalues engendered by probability distributions that are protocol-specific.

# Chapter 3

# Ordinal coalitional situations

## 3.1 Overview of the chapter

This chapter focuses on models that we recently introduced in [5, 72, 46, 3, 59] to analyse coalitional situations where the "intensity" of the agents' interaction is characterized by a "qualitative" information. This kind of problems are studied in [5] (see Section 3.2.1) with the objective to generalize the notions of coalitional game and power index within an ordinal framework. Given a total preorder representing the relative strength of coalitions (namely, a *power relation*), a particular *social ranking* over the player set is provided in [5] according to a notion of *ordinal influence* strongly connected to the classical Banzhaf index of a "canonical" coalitional game. More precisely, in [5] we provided an axiomatic characterization of a *social ranking* (i.e., a map assigning to each power relation over the subsets of $N$ a ranking over the single players in $N$) by means of properties dealing with the ordinal structure of power relations. A first property used in the characterization is a *coalitional dominance* axiom, which states that whatever coalition $S$ is going to form, a player with more opportunities to form coalitions stronger than $S$ should be ranked higher than another one with less. The second property, namely the *additivity* axiom, allows for the composition of power relations with opposite social rankings.

A similar problem has been studied in [72] (see Section 3.2.2) with the goal to characterize social rankings starting from the very basic properties of a power relation over coalitions, and without the use of any particular coalitional game or power index. The properties for social rankings that we analyse in [72] have classical interpretations, such as *anonymity*, saying that the ranking should not depend on the name of the players, or the *dominance*, saying that a player $i$ should be ranked higher than a player $j$ whenever $i$ dominates $j$, i.e. the coalition $S \cup \{i\}$ is stronger than coalition $S \cup \{i\}$ for each $S$ not containing neither $i$ nor $j$. Other two properties, namely, *independence from irrelevant coalitions* and *separability*, have been also discussed in [72], and we obtained some axiomatic characterizations of social rankings satisfying combinations of the previous properties over different restricted families of power relations.

In [46] and in [3] (see Section 3.2), we address the problem of how to extend a ranking over single objects to another ranking over all possible collections of objects, taking into account the fact that objects grouped together can have mutual interaction. This problem has been carried out in the tradition of the literature on extending an order on a set $N$ to its *power set* (denoted by $2^N$) with the objective to axiomatically characterize families of ordinal preferences over subsets (see, for instance, [59] for a survey on this issue). More precisely, the question raised in [46, 3] is related to the extension problem under the interpretation of *sets as final outcomes*. Under this interpretation, if objects are goods, one could guess that to have $\{x, y\}$ is better then to have $x$ or $y$ alone, because the agent will receive both $y$ and $x$. But in reality, this assumption of monotonicity depends on the context. In some decision problems, the judgement depends on the nature of $x$ and $y$ and on possible effects of incompatibility between the two objects. Nevertheless, most of the axiomatic approaches from literature focused on properties suggesting that *interactions* among single objects should not play a relevant role in establishing the ranking among subsets. Another example is the property of *responsiveness*, which requires that a set $S \subseteq N$ is preferred to a set $T \subseteq N$ whenever $S$ is obtained from $T$ by replacing some object $t \in T$ with another $i \in N$ not in $T$ which is preferred to $t$ (according to the primitive ranking over the single elements of the set $N$). In other words, the responsiveness property prevents complementarity or incompatibility effects among objects within sets of the same cardinality. In [3], the idea of *alignment* with a probabilistic value (the Banzhaf value, the Shapley value, or, more in general, the family of *semivalues* [110]) was developed in order to have meaningful extensions of a total preorder on a finite set $N$ to the set of its subsets, and keeping into account the possibility that objects within each subset may interact. Specifically, the fact that an extension must be *aligned* with a semivalue means that the ranking of the objects according to the semivalue must be the same and must preserve the primitive total preorder on the singletons, no matter which utility function is used to describe the preorder over the coalitions. In the most favourable situation it remains the same for the whole simplex of the semivalues (if and only if the total preorder on $2^N$ fulfils a particular condition provided in [3]).

## 3.2 Ordinal power

In practical situations the information concerning the strength of coalitions and their effective possibilities of cooperation is not easily accessible due to hardly quantifiable factors like bargaining abilities, moral and ethical codes and other "psychological" attributes [143]. For instance, in addition to what it can gain by itself, a coalition may obtain some more power by threatening not to cooperate with other players [143].

As an illustrative example of a situation dealing with the ordinal power of coalitions, consider a company where the employer must evaluate the job performance of some employees with the objective to award bonus shares or to decide promotions. In order to rank the employees on the basis of the quality of their past job activities, the evaluation should take into account the ability of each employee to work alone on its own initiative and with others as a team. From the analysis of the job records of three employees 1, 2 and 3, it can be derived that when employees 1 and 3 work independently by each other, the job performance of $\{1\}$ as a singleton coalition is significantly lower than the job performance of $\{3\}$. In addition, because of a strong incompatibility between 2 and 3, the performance of the team $\{2,3\}$ is strictly lower than the performance of any other team; instead, a complementarity effect between 1 and 2 makes $\{1,2\}$ the most successful team. Summing up, the job performance of the teams turns out to be $\{1,2\} \succ \{3\} \succ \{1\} \succ \{2,3\}$, where the notation $S \succ T$ means that a team $S$ performs strictly better than a team $T$. Based on this evidence, who is better between 1 and 3?

We provided a first attempt to solve this kind of problems in [5], where, given a total preorder representing the relative strength of coalitions, a social ranking over the player set is provided according to a notion of *ordinal influence* and using the Banzhaf index [86] of a "canonical" coalitional game (see Section 3.2.1). In [72] we introduced an alternative approach, using general properties that are not based on coalitional games (see Section 3.2.2).

### 3.2.1 A game theoretic approach

Adopting a marginalist approach to solve the problem illustrated in the previous section, in [5] we argued that employee 1 should be ranked higher than employee 3 whenever the probability of joining a team with two members is higher than the probability of working alone (i.e., joining an empty coalition). To be more specific, take a map $v$ assigning to each subset $S \subseteq \{1,2,3\}$ a hypothetical score $v(S)$ measuring the job performance of a team $S$ (assuming $v(\emptyset) = 0$), and compute the expected marginal contribution $\pi_i(v)$ of $i$ in $v$ and over all possible teams according to the definition of a probabilistic power index (more precisely, a semivalue [110], see also Section 1.2.2) for a 3-players coalitional game. In formula we have:

$$\pi_i(v) = p_0(v(i) - v(\emptyset)) + p_1(v(i,j) - v(j) + v(i,k) - v(k)) + p_2(v(i,j,k) - v(j,k)),$$

where $i, j, k$ are distinct elements in $\{1,2,3\}$, and where $p_k$, for each $k = 0,1,2$, represents the probability that a coalition of size $k$ forms (i.e., $p_0, p_1$ and $p_2$ are positive real numbers such that $p_0 + 2p_1 + p_2 = 1$). It is easy to check that the difference $\pi_i(v) - \pi_j(v)$ for each $i, j \in \{1,2,3\}$ can be written as follows:

$$\pi_i(v) - \pi_j(v) = (p_0 + p_1)(v(i) - v(j)) + (p_1 + p_2)(v(i,k) - v(j,k)).$$

Now, taking $i = 1$, $j = 3$ and $k = 2$, and supposing $p_2 > p_0$, from the above formula we have that $\pi_1(v) - \pi_3(v) > 0$ for all possible performance scores $v$ such that $v(1,2) > v(3) > v(1) > v(2,3)$ (which means that $v$ is compatible with the ordinal constraint provided by the evaluation of the employees). Differently stated, according to a marginalist approach, the fact that a performance score $v$ satisfies the constraint $v(1,2) > v(3) > v(1) > v(2,3)$ implies that employee 1 is ranked higher than employee 3 (if the probability $p_2$ of participating to a coalition with two employees is larger than the probability $p_0$ of entering in the empty coalition). So, in this case, the probability of forming coalitions plays a key role. On the other hand, this is not always the case. Consider for instance employee 2, and suppose that, again on the basis of the past job records, the evaluation puts in evidence that the singleton coalition $\{2\}$ works very well, strictly better than each other team, and that the job performance of team $\{1,3\}$ is strictly higher than the job performance of each other team of precisely two employees. Together with the other comparisons involving 1 and 2, the job performance turns out to be such that $\{2\} \succ \{1,3\} \succ \{1\} \succ \{2,3\}$. Of course, in this situation, if a cardinal scale to compare the performances of teams is not given, the marginalist approach cannot help: it is easy to check that the difference $\pi_1(v) - \pi_2(v)$ of expected marginal contributions between 1 and 2, for any strictly positive probability $p_0$ and $p_2$, can be made negative or positive with a suitable choice of $v$ (still compatible with the constraint $v(2) > v(1,3) > v(1) > v(2,3)$).

In [5], our goal is to provide an analytical method to describe how the relative strength of coalitions may influence the ranking of individuals in a society. Therefore, we define a social ranking as a map assigning to each total preorder on the set of all the coalitions, a total preorder on the set of players (actually, in [5], we simply required that $\rho(\succcurlyeq)$ is a total binary relation, i.e., we always want to express the relative comparison of two agents, but we do not exclude *a priori* the possibility of cycles in $\rho(\succcurlyeq)$). In other words, in the model we study in this paper we do not exclude the possibility that coalitions may be threatened by internal agents.

We call the map $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$, assigning to each power relation on $2^N$ a total preorder on $N$, a *social ranking solution* or, simply, a *social ranking*. Then, given a power relation $\succcurlyeq$, we will interpret the relation $\rho(\succcurlyeq)$ associated to $\succcurlyeq$ by the social ranking $\rho$, as the relative power of players in a society under relation $\succcurlyeq$. Precisely, for each $i, j \in N$, $i\rho(\succcurlyeq)j$ stands for '$i$ is considered at least as influential as $j$ according to the social ranking $\rho(\succcurlyeq)$', where the influence of a player is intended as her/his ability to join coalitions in the strongest positions of a power relation.

Given a power relation $\succcurlyeq \in \mathcal{T}^{2^N}$, consider a bijection $\theta : \{1, \ldots, 2^n\} \to 2^N$ such that

$$S \succ T \Rightarrow \theta^{-1}(S) < \theta^{-1}(T), \tag{3.1}$$

for every $S, T \in 2^N$. Now, for each $i \in N$, let $\Gamma^i(\succcurlyeq)$ be a $2^n$-vector of natural numbers such that the $k$-th component represents the number of coalitions containing $i$ that are in relation with $\theta(k)$, i.e.

$$\Gamma^i_k(\succcurlyeq) = |\{S \subseteq N : i \in N \text{ and } S \succcurlyeq \theta(k)\}|$$

for each $k = 1, \ldots, 2^n$. One can see $\theta^{-1}$ as a priority function over the subsets of $N$ according to the power relation $\succcurlyeq$ (that is, from priority 1 given to a strongest subset, to priority $2^n$ assigned to a weakest subset), and $\Gamma^i_k(\succcurlyeq)$ as the number of subsets of $N$ containing $i$ that are at least as strong as the subset with priority $k$. Note that vector $\Gamma(\succcurlyeq)$ does not depend on the choice of the bijection $\theta$, since $\Gamma^i_k(\succcurlyeq) = \Gamma^i_l(\succcurlyeq)$ for every $k$, $l$ such that $\theta(k) \succcurlyeq \theta(l)$ and $\theta(l) \succcurlyeq \theta(k)$.

In the following, for every $i, j \in N$, we will say that $\Gamma^i(\succcurlyeq)$ *dominates* $\Gamma^j(\succcurlyeq)$ (denoted by $\Gamma^i(\succcurlyeq) \geq \Gamma^j(\succcurlyeq)$) iff $\Gamma^i_k(\succcurlyeq) \geq \Gamma^j_k(\succcurlyeq)$ for each $k = 1, \ldots, 2^n$; and we will say that $\Gamma^i(\succcurlyeq)$ *strictly dominates* $\Gamma^j(\succcurlyeq)$ (denoted by $\Gamma^i(\succcurlyeq) > \Gamma^j(\succcurlyeq)$) iff $\Gamma^i(\succcurlyeq)$ dominates $\Gamma^j(\succcurlyeq)$ and there exists $k \in \{1, \ldots, 2^n\}$ such that $\Gamma^i_k(\succcurlyeq) > \Gamma^j_k(\succcurlyeq)$. We can now introduce the first property for social rankings.

The first property is the *coalitional dominance axiom*, which states that a player $i$ is ranked better than $j$ if, for every coalition $S$, the number of coalitions stronger than $S$ that contain $i$ is larger than the number of those that contain $j$. The interpretation of this property is the following: whatever coalition $S$ is going to form, a player with more opportunities to form coalitions stronger than $S$ should be ranked higher than another one with less.

**Definition 4** (CDOM). *A social ranking $\rho$ satisfies the coalitional dominance property iff for all $i, j \in N$ and $\succcurlyeq \in \mathcal{P}^{2^N}$,*

$$\Gamma^i(\succcurlyeq) > \Gamma^j(\succcurlyeq) \Rightarrow i\rho(\succcurlyeq)j \text{ and } \neg(j\rho(\succcurlyeq)i)$$

*and*

$$\Gamma^i(\succcurlyeq) = \Gamma^j(\succcurlyeq) \Rightarrow i\rho(\succcurlyeq)j \text{ and } j\rho(\succcurlyeq)i.$$

In [5] we show that a player $i \in N$ dominates a player $j \in N$ with respect to the power relation $\succcurlyeq$ if and only if the Banzhaf value $\beta_i(v)$ of player $i$ is larger than (or equal to) the Banzhaf value $\beta_j(v)$ of player $j$, for every characteristic function $v \in V(\succcurlyeq)$.

**Theorem 7** ([5]). *Let $\succcurlyeq \in \mathcal{P}^{2^N}$. Then, for each $i, j \in N$*

$$\Gamma^i(\succcurlyeq) \geq \Gamma^j(\succcurlyeq) \Leftrightarrow [\beta_i(v) \geq \beta_j(v) \text{ for every } v \in V(\succcurlyeq)]. \tag{3.2}$$

**Corollary 1.** *Let $\succcurlyeq \in \mathcal{P}^{2^N}$ and let $\rho$ be a social ranking which satisfies CDOM. Then, for each $i, j \in N$*

$$[\beta_i(v) \geq \beta_j(v) \text{ for every } v \in V(\succcurlyeq)] \Rightarrow i\rho(\succcurlyeq)j. \tag{3.3}$$

**Example 10.** *After the Italian political election held on February 24$^{th}$ and 25$^{th}$, 2013 for the determination of the 630 members of the Chamber of Deputies and the 315 members of the Senate of the Italian Republic, no political party had the majority of seats in the Senate. Precisely, the* 315 *seats of the Senate were distributed as follows:* 123 *to the Center-Left (hereafter, L) alliance,* 117 *to the Center-Right (hereafter, R) alliance,* 54 *to the the Five Star Movement (hereafter, M), and the remaining* 21 *seats were distributed among other minor parties. As a consequence, in order to have a simple majority in the Senate (and to proceed with the formation of a new government), a coalition of at least two major parties among L, R and M was required. Holding a clear majority of seats in the Chamber of Deputies (the other Chamber of the Italian Parliament), the L alliance was supposed to explore all the possibilities to form a majority coalition with the other political forces in the Senate. Due to constitutional constraints, formal talks to form a new government started four weeks after the election, when the President of the Italian Republic officially designated the L alliance to lead the formation of a new government. Initially, the L alliance ruled-out any possibility of a coalition with the R one and tried to form a minority government supported by M. In fact, the political program of M was characterized by issues originating from ecological themes (like sustainable mobility and protection of the territory) and other political principles inspired by a participative attitude of citizens and traditionally close to the positions of L. Therefore, immediately after the election, a coalition between L and M appeared more likely than a coalition between L and its historical antagonist R. In that moment, the most realistic representation of the relative strength of coalitions was* $\{L, M\} \succ \{L, R\} \succ \{R, M\}$*. Moreover, each coalition of at least two parties was considered more powerful than any singleton coalition. On the other hand, the comparison among individual parties still reflected the electoral consensus of individual parties in the election of the Italian Parliament, that is* $\{L\} \succ \{R\} \succ \{M\}$*.*

*In the light of the considerations illustrated above, a compatible representation of the relative power between coalitions of major parties in the Italian Senate after the election held on February 2013 is represented by the following power relation:* $\{L, M\} \succ \{L, R\} \succ \{L, R, M\} \succ \{R, M\} \succ \{L\} \succ \{R\} \succ \{M\} \succ \emptyset$[1]*.*

*The vectors* $\Gamma^i(\succ)$*, for each* $i \in \{L, M, R\}$ *are provided in Table 3.1. Note that,* $\Gamma^L(\succ)$ *dominates both* $\Gamma^R(\succ)$ *and* $\Gamma^M(\succ)$*, but no relation of dominance exists between* $\Gamma^R(\succ)$ *and* $\Gamma^M(\succ)$*.*

| $S$ | $L, M$ | $L, R$ | $L, R, M$ | $R, M$ | $L$ | $R$ | $M$ | $\emptyset$ | score |
|---|---|---|---|---|---|---|---|---|---|
| $\Gamma^L(\succ)$ | 1 | 2 | 3 | 3 | 4 | 4 | 4 | 4 | 25 |
| $\Gamma^M(\succ)$ | 1 | 1 | 2 | 3 | 3 | 3 | 4 | 4 | 21 |
| $\Gamma^R(\succ)$ | 0 | 1 | 2 | 3 | 3 | 4 | 4 | 4 | 21 |

Table 3.1: The Γ vectors and their scores for the power relations used to model the Italian election of February 2013 (curly brackets are omitted for coalitions).

In order to characterize a social ranking on the class of all power relations $\mathcal{P}^{2^N}$, in [5] we introduced a second property, namely the *additivity axiom* (for the analogy with the use of the homonym property in the theory of coalitional games [171]; see also Section 1.2.2), that allows for the combination of social rankings with an opposite relative comparison. More precisely, if a power relation $P_0$ can be obtained as the intersection of two power relations $P_1$ and $P_2$, such that player $i$ is ranked better than player $j$ in the social ranking on $P_1$ and player $j$ is ranked better than player $i$ in the social ranking on $P_2$, then the relative social ranking of $i$ and $j$ in $P_0$ is determined by the comparison of the "intensity" of the dominance of $i$ over $j$ and of $j$ over $i$ on $P_1$ and on $P_2$, respectively. As a particular measure of the intensity of the dominance in opposite social rankings, we study a notion of total capacity of players to threaten coalitions, that is computed as the sum (over all possible coalitions $S$) of the number of coalitions containing $i$ which are stronger than $S$ (see the last column of Table 3.1 for an example). Surprisingly, on the class of all power relations, we show that a social ranking that satisfies both the dominance and the additivity axioms coincides with the ranking provided by the Banzhaf value of a particular coalitional game related to the numerical representation of the power relation (see [5] for more details).

### 3.2.2 A social choice approach

In [72], we characterized social rankings starting from the very basic properties of a power relation over coalitions, and without the use of any particular coalitional game, that would necessarily require the conversion of the

[1]Here, the ranking of coalition $\{L, R, M\}$ is motivated by a preliminary attempt of $L$ to obtain a larger consensus in a government of grand coalition.

(purely ordinal) information about the relative strength of coalitions into a quantitative assessment of their power (as, in fact, we did in [5]).

Now we introduce the properties for social rankings studied in [72]. The first axiom is the *dominance* one: if each coalition $S$ containing agent $i$ but not $j$ is stronger than coalition $S$ with $j$ in the place of $i$, then agent $i$ should be ranked higher than agent $j$ in the society, for any $i, j \in N$. Precisely, given a power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ and $i, j \in N$ we say that $i$ *dominates* $j$ in $\succcurlyeq$ if $S \cup \{i\} \succcurlyeq S \cup \{j\}$ for each $S \in 2^{N \setminus \{i,j\}}$ (we also say that $i$ *strictly dominates* $j$ in $\succcurlyeq$ if $i$ dominates $j$ and in addition there exists $S \in 2^{N \setminus \{i,j\}}$ such that $S \cup \{i\} \succ S \cup \{j\}$).

**Definition 5** (DOM). *A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ satisfies the dominance (DOM) property on $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ if and only if for all $\succcurlyeq \in \mathcal{C}^{2^N}$ and $i, j \in N$, if $i$ dominates $j$ in $\succcurlyeq$ then $i\rho(\succcurlyeq)j$ [and $\neg(j\rho(\succcurlyeq)i)$ if $i$ strictly dominates $j$ in $\succcurlyeq$].*

The following axiom states that the relative strength of two agents $i, j \in N$ in the social ranking should only depend on their effect when they are added to each possible coalition $S$ not containing neither $i$ nor $j$, and the relative ranking of the other coalitions is irrelevant. Formally:

**Definition 6** (IIC). *A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ satisfies the Independence of Irrelevant Coalitions (IIC) property on $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ iff*

$$i\rho(\succcurlyeq)j \Leftrightarrow i\rho(\sqsupseteq)j$$

*for all $i, j \in N$ and all power relations $\succcurlyeq, \sqsupseteq \in \mathcal{C}^{2^N}$ such that for each $S \in 2^{N \setminus \{i,j\}}$*

$$S \cup \{i\} \succcurlyeq S \cup \{j\} \Leftrightarrow S \cup \{i\} \sqsupseteq S \cup \{j\}.$$

Let $\succcurlyeq \in \mathcal{T}^{2^N}$ and $i, j, p, q \in N$ be such that, for each $k = 0, \ldots, n-2$, and for all coalitions $S$ of cardinality $k$, the number of times that $S \cup \{i\}$ is stronger than $S \cup \{j\}$ equals the number of times that $S \cup \{p\}$ is stronger than $S \cup \{q\}$ (and the number of times that $S \cup \{j\}$ is stronger than $S \cup \{i\}$ equals the number of times that $S \cup \{q\}$ is stronger than $S \cup \{p\}$). In this symmetric situation, the following axiom states a principle of equivalence between the pairs $i, j$ and $p, q$.

**Definition 7** (SYM). *A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ satisfies the symmetry (SYM) property on $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ iff*

$$i\rho(\succcurlyeq)j \Leftrightarrow p\rho(\succcurlyeq)q$$

*for all $i, j, p, q \in N$ and $\succcurlyeq \in \mathcal{C}^{2^N}$ such that $|D^k_{ij}| = |D^k_{pq}|$ and $|D^k_{ji}| = |D^k_{qp}|$ for each $k = 0, \ldots, n-2$.*

The effect of the combination of these axioms is illustrated by the following theorems from [72].

**Theorem 8** ([72]). *Let $|N| > 3$. There is no social ranking rule $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ which satisfies DOM and SYM on $\mathcal{T}^{2^N}$.*

On the other hand, the properties of IIC and SYM in combination determine a flattening of the social ranking on power relations where the relevant information is represented by coalitions of a given cardinality.

**Theorem 9** ([72]). *Let $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ be a social ranking satisfying IIC and SYM. Let $\succcurlyeq \in \mathcal{T}^{2^N}$ and $k \in \{0, \ldots, |N|-2\}$ be such that $S \cup \{i\} \succcurlyeq S \cup \{j\}$ and $S \cup \{j\} \succcurlyeq S \cup \{i\}$, for all $S \in 2^{N \setminus \{i,j\}}$ with $|S| \neq k$, $D^k_{ij}(\succcurlyeq) \setminus D^k_{ji}(\succcurlyeq) \neq \emptyset$ and $D^k_{ji}(\succcurlyeq) \setminus D^k_{ij}(\succcurlyeq) \neq \emptyset$ for all $i, j \in N$. Then $i\rho(\succcurlyeq)j$ and $j\rho(\succcurlyeq)i$ for each $i, j \in N$.*

Theorem 9 suggests how to deal with situations where coalitions are of a fixed size (such situations are not so eccentric in real life). For instance, let us imagine that we have committees with a given number ($k$) of persons and that we have a ranking on them (for instance $N = \{1, 2, 3, 4\}$ and $k = 2$, with $12 \succcurlyeq 13 \succcurlyeq 14 \succcurlyeq 34 \succcurlyeq 24 \succcurlyeq 23$). Since committees are always formed by two persons, no information is available on subsets of $N$ with $l \neq k$ elements (or such information is irrelevant). How to define a social ranking in this case? One solution could be to consider all the other comparisons indifferent. Then, by Theorem 9, we know that SYM and IIC properties can be used in order to support a unanimous social ranking.

**Example 11.** *Consider a power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ with $N = \{1, 2, 3, 4, 5\}$ and*

$$13 \succ 23 \succ 12 \succ 24 \succ 14 \succ 34 \succ 15 \sim 25 \succ 35 \succ 45,$$

*all the other coalitions of the same size being indifferent (i.e., $S \cup \{i\} \succcurlyeq S \cup \{j\}$ and $S \cup \{j\} \succcurlyeq S \cup \{i\}$, for all $S \in 2^{N \setminus \{i,j\}}$ with $|S| \neq 1$ and $i, j \in \{1, 2, 3\}$). We rewrite the relevant informations about $\succcurlyeq$ and elements $1, 2$ and $3$ by means of Table 11. If a social ranking $\rho$ satisfies both SYM and DOM, then by Theorem 9, all the elements in $\{1, 2, 3\}$ are in relation with each other in $\rho(\succcurlyeq)$ (i.e. they are all indifferent).*

| 1 vs. 2 | 2 vs. 3 | 1 vs. 3 |
|---|---|---|
| $1 \sim 2$ | $2 \sim 3$ | $1 \sim 3$ |
| $13 \succ 23$ | $12 \prec 13$ | $12 \prec 23$ |
| $14 \prec 24$ | $24 \succ 34$ | $14 \succ 34$ |
| $15 \sim 25$ | $25 \succ 35$ | $15 \succ 35$ |
| $134 \sim 234$ | $124 \sim 134$ | $124 \sim 234$ |
| $135 \sim 235$ | $125 \sim 135$ | $125 \sim 235$ |
| $1345 \sim 2345$ | $1245 \sim 1345$ | $1245 \sim 2345$ |

Table 3.2: The relevant informations about $\succcurlyeq$ of Example 11 and the elements $1, 2$ and $3$.

A special class of power relations (namely, the *per size-strong dominant* relations) are also considered in [72]: theyr are characterized by the fact that a relation of dominance always exists with respect to coalitions of the same size, but the dominance may change with the cardinality (for instance, an element $i$ could dominate another element $j$ when coalitions of size $s$ are considered, but $j$ could dominate $i$ over coalitions of size $t \neq s$). We first need to introduce the notion of $s$-strong dominance.

**Definition 8.** *Let $\succcurlyeq \in \mathcal{T}^{2^N}$, $i, j \in N$ and $s \in \{0, \dots, n-2\}$. We say that $i$ $s$-strong dominates $j$ in $\succcurlyeq$, iff*

$$S \cup \{i\} \succ S \cup \{j\} \text{ for each } S \in 2^{N \setminus \{i,j\}} \text{ with } |S| = s. \tag{3.4}$$

**Definition 9.** *We say that $\succcurlyeq \in \mathcal{T}^{2^N}$ is per size-strong dominant (shortly, ps-sdom) iff for each $s \in \{0, \dots, n-2\}$ and all $i, j \in N$, we have either*

$$[i \text{ } s\text{-strong dominates } j \text{ in } \succcurlyeq] \text{ or } [j \text{ } s\text{-strong dominates } i \text{ in } \succcurlyeq].$$

*The set of all ps-sdom power relations is denoted by $\mathcal{S}^{2^N} \subseteq \mathcal{T}^{2^N}$.*

In [72] we argue that if a social ranking satisfies both DOM and IIC on the set of ps-sdom power realtions $\mathcal{S}^{2^N}$, then it must exist a cardinality $t^* \in \{0, \dots, n-2\}$ whose relation of $t^*$-strong dominance (dictatorially) determines the social ranking over all power relations in $\mathcal{S}^{2^N}$.

**Theorem 10** ([72]). *Let $\rho : \mathcal{S}^{2^N} \longrightarrow \mathcal{T}^N$ be a social ranking satisfying IIC and DOM on $\mathcal{S}^{2^N}$. There exists $t^* \in \{0, \dots, n-2\}$ such that*

$$i\rho(\succcurlyeq)j \Leftrightarrow i \text{ } t^*\text{-strong dominates } j \text{ in } \succcurlyeq,$$

*for all $i, j \in N$ and $\succcurlyeq \in \mathcal{S}^{2^N}$.*

**Example 12.** *Consider a power relation $\succcurlyeq \in \mathcal{S}^{2^N}$ with $N = \{1, 2, 3, 4\}$ and such that*

$$1 \succ 2 \succ 3 \succ 4$$
$$34 \succ 24 \succ 14 \succ 23 \succ 13 \succ 12$$
$$123 \succ 134 \succ 124 \succ 234.$$

*We rewrite the relevant informations about $\succcurlyeq$ by means of Table 12. Theorem 10 says that if a social ranking*

| 1 vs. 2 | 2 vs. 3 | 1 vs. 3 | 1 vs. 4 | 2 vs. 4 | 3 vs. 4 |
|---|---|---|---|---|---|
| $1 \succ 2$ | $2 \succ 3$ | $1 \succ 3$ | $1 \succ 4$ | $2 \succ 4$ | $3 \succ 4$ |
| $13 \prec 23$ | $12 \prec 13$ | $12 \prec 23$ | $12 \prec 24$ | $12 \prec 14$ | $13 \prec 14$ |
| $14 \prec 24$ | $24 \prec 34$ | $14 \prec 34$ | $13 \prec 34$ | $23 \prec 34$ | $23 \prec 24$ |
| $134 \succ 234$ | $124 \prec 134$ | $124 \succ 234$ | $123 \succ 234$ | $123 \succ 134$ | $123 \succ 124$ |

Table 3.3: The relevant informations about $\succcurlyeq$.

*satisfies DOM and IIC on $\mathcal{S}^{2^N}$, then it must yield on $\succcurlyeq$ one of the following three possible linear orders: $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3\rho(\succcurlyeq)4$ (corresponding to the relation of 0-strong dominance); $4\rho(\succcurlyeq)3\rho(\succcurlyeq)2\rho(\succcurlyeq)1$ (corresponding to the relation of 1-strong dominance); $4\rho(\succcurlyeq)1\rho(\succcurlyeq)3\rho(\succcurlyeq)2$ (corresponding to the relation of 2-strong dominance).*

*For instance, suppose that the social ranking is $4\rho(\succcurlyeq)2\rho(\succcurlyeq)3\rho(\succcurlyeq)1$. Define a new power relation $\sqsupseteq \in \mathcal{S}^{2^N}$ such that (again, the main changes with respect to $\succcurlyeq$ are shown in bold):*

$1 \sqsupset 2 \sqsupset 3 \sqsupset 4$
$34 \sqsupset \mathbf{23} \sqsupset \mathbf{24} \sqsupset \mathbf{13} \sqsupset \mathbf{14} \sqsupset 12$
$123 \sqsupset 134 \sqsupset \mathbf{234} \sqsupset \mathbf{124}$

*We rewrite the relevant informations about $\sqsupseteq$ by means of Table 12. By DOM we have that $3\rho(\sqsupseteq)4$ and*

| 1 vs. 2 | 2 vs. 3 | 1 vs. 3 | 1 vs. 4 | 2 vs. 4 | 3 vs. 4 |
|---|---|---|---|---|---|
| $1 \sqsupset 2$ | $2 \sqsupset 3$ | $1 \sqsupset 3$ | $1 \sqsupset 4$ | $2 \sqsupset 4$ | $3 \sqsupset 4$ |
| $13 \sqsubset 23$ | $12 \sqsubset 13$ | $12 \sqsubset 23$ | $12 \sqsubset 24$ | $12 \sqsubset 14$ | $\mathbf{13} \sqsupset \mathbf{14}$ |
| $14 \sqsubset 24$ | $24 \sqsubset 34$ | $14 \sqsubset 34$ | $13 \sqsubset 34$ | $23 \sqsubset 34$ | $\mathbf{23} \sqsupset \mathbf{24}$ |
| $134 \sqsupset 234$ | $124 \sqsubset 134$ | $\mathbf{124} \sqsubset \mathbf{234}$ | $123 \sqsupset 234$ | $123 \sqsupset 134$ | $123 \sqsupset 124$ |

Table 3.4: The relevant informations about $\sqsupseteq$.

*$\neg(4\rho(\sqsupseteq)3)$. By IIC we have $4\rho(\sqsupseteq)2$ and $2\rho(\sqsupseteq)3$ (the columns '2 vs. 4' and '2 vs. 3' are the same in the Tables 12 and 12, respectively), which yields a contradiction with the transitivity of $\rho(\sqsupseteq)$.*

## 3.3 Preference extensions

The "inverse problem" of the one introduced in Section 3.2 is summarized by the following question: how to derive a ranking over the set of all subsets of $N$ in a way that is "compatible" with a primitive ranking over the single elements of $N$? This question has been carried out in the tradition of the literature on extending an order on a set $N$ to its *power set* (the set of all possible subsets of $N$) with the objective to axiomatically characterize families of ordinal preferences over subsets (see, for instance, [87, 88, 98, 99, 122, 118, 132, 135]). In this context, an order $\succcurlyeq$ on the power set of $N$ is required to be an *extension* of a primitive order $P$ on $N$. This means that the relative ranking of any two singleton sets according to $\succcurlyeq$ must be the same as the relative ranking of the corresponding alternatives according to $P$.

The different axiomatic approaches in the literature are related to the interpretation of the properties used to characterize extensions, which is deeply related to the meaning that is attributed to sets. According to the survey [88], the main contributions from the literature on ranking sets of objects may be grouped in three main classes of problems: 1) *complete uncertainty*, where a decision-maker (DM) is asked to rank sets which are considered as formed by mutually exclusive objects (i.e., only one object from a set will materialize), and taking into account that the DM cannot influence the selection of an object from a set (see, for instance, [87, 132, 154]); 2) *opportunity sets*, where sets contain again mutually exclusive objects but, in this case, a DM compares sets taking into account that he will be able to select a single element from a set (see, for example, [99, 135, 165]); 3) *sets as final outcomes*, where each set contains objects that are assumed to materialize simultaneously (if that set is selected; for instance, see [98, 118, 168]).

In [46] and [3] we focus on the problem of the third class, where sets of elements materialize simultaneously. A standard application of this kind of problems is the *college admissions problem* [168, 121], where colleges need to rank sets of students based on their ranking of individual applicants.

The model introduced in [46] (and further studied in [3]) relies on the fact that an utility function attached to a total preorder on $2^N$ represents a coalitional game (provided we set the utility of the empty set to be equal to zero). Since probabilistic values [179, 102, 103, 142, 158] do provide a natural ranking among the elements of the set $N$ (the players in the game theoretical context, the objects in this approach), according to [46] and [3] we can use them in the following sense: given a probabilistic value $\pi^p$ (recall the definition provided in Section 1.2.2), a total preorder $\succcurlyeq$ on $2^N$ will be *$\pi^p$-aligned* , provided that from $\{i\} \succcurlyeq \{j\}$ it follows that $\pi^p$ ranks better $i$ than $j$ for *every* possible choice of the utility function representing $\succcurlyeq$. In other terms, an extension $\succcurlyeq$ on $2^N$ is such that it never changes the mutual positions in the rankings of the objects in $N$ according to $\pi^p(v)$, whatever coalitional game $v$ is used to numerically represent the preference relation $\succcurlyeq$ on $2^N$.

To be more specific, In [46], we introduce the class of *Shapley extensions*, for their attitude to preserve the ranking provided by the *Shapley value* [170, 171], whereas in [3] we are interested in extensions that are $\pi$-aligned to all probabilistic values, or alternatively, to subfamilies of these values.

In [46], a central property for total preorders on $2^N$ is the *responsiveness* (RESP) property, introduced by [168] for the analysis of the college admission problem:

**Definition 10** (responsiveness, RESP). *A total preorder $\succcurlyeq$ on $2^N$ satisfies the RESP property on $2^N$ if for all $i, j \in N$ and all $S \in 2^N$, $i, j \notin S$ we have that*

$$\{i\} \succcurlyeq \{j\} \Leftrightarrow S \cup \{i\} \succcurlyeq S \cup \{j\}. \tag{3.5}$$

An extension that satisfies the RESP property does not take into account the fact that some objects together can present some form of incompatibility or, on the contrary, of mutual enforcement.

Restricted to sets $A$ of fixed cardinality $q \in \mathbb{N}$, representing the maximal number of students the college can admit, the RESP property was used by [98] to characterize (together with another property called *fixed-Cardinality neutrality*, saying that the labelling of the alternatives is irrelevant in establishing the ranking among sets of fixed cardinality $q$) the family of *lexicographic rank-ordered extensions*.

We now introduce another property for extensions, namely, the *monotonicity* property [135, 165].

**Definition 11** (Monotoncity, MON). *A total preorder $\succcurlyeq$ on $2^N$ satisfies the* monotonicity *property iff for each $S, T \in 2^N$ we have that*

$$S \subseteq T \Rightarrow T \succcurlyeq S.$$

The MON property states that each set of objects is weakly preferred to each of its subsets. In other words, the MON property excludes the possibility that some objects in a set $S \in 2^N$ may be incompatible with some others not in $S$.

Let $\succcurlyeq$ be a total preorder on $2^N$. For each $S \in 2^N \setminus \{\emptyset\}$, a *sub-extension* $\succcurlyeq_S$ is a relation on $2^S$ such that for each $U, V \in 2^S$,

$$U \succcurlyeq V \Leftrightarrow U \succcurlyeq_S V.$$

We may now introduce the last property of this section, namely the *sub-extendibility* property for Shapley extensions.

**Definition 12** (Sub-Extendibility, SE). *A Shapley extension $\succcurlyeq$ on $2^N$ satisfies the* sub-extendibility *property iff for each $S \in 2^N \setminus \{\emptyset\}$ we have that $\succcurlyeq_S$ is a Shapley extension on $2^S$.*

The SE property states that the effects of interaction among objects must be "compatible" not only with the information provided by the original preference on single elements of $N$, but also with the information provided by all restrictions of such a preference to each non-empty subset $S$ of $N$. This means that the personal attribution of importance assigned to objects, and taking into account the effects of interaction, must be consistent with the primitive ranking, independently from the size of the universal set considered.

The following definition formally introduces the notion of Shapley extension.

**Definition 13.** *A total prorder $\succcurlyeq$ on $2^N$ is a Shapley extension iff for each numerical representation $v \in \mathcal{G}^N_{\succcurlyeq}$ of $\succcurlyeq$ we have that*

$$\{i\} \succcurlyeq \{j\} \Leftrightarrow \phi_i(v) \geq \phi_j(v)$$

*for all $i, j \in N$.*

Next theorem is important in establishing the connection between MON, SE and RESP properties.

**Theorem 11** (from [46]). *Let $\succcurlyeq$ be a total preorder on $2^N$ which satisfies the MON property. The following two statements are equivalent:*

- *(i) $\succcurlyeq$ satisfies the RESP property.*
- *(ii) $\succcurlyeq$ is a Shapley extension and satisfies the SE property.*

By Theorem 11 we have characterized a class of Shapley extensions aimed to rank subsets of objects in absence of complementarity effects. Another goal in [46] and, in particular, in [3], is to analyze properties of Shapley extensions that, due to the effects of interaction among objects, may "invert" (with respect to the conditions imposed by the RESP property) the relative ranking of a limited number of subsets. To this purpose, in [3] we required that Definition 13 holds for every semivalue where the probability distribution over the subsets $S$ of $N$ not containing $i$ is a function of the cardinality of $S$, and it is the same for all $i$ in $N$.

**Definition 14** ($\pi^{\mathbf{P}}$-alignement). *Given a set $N$, a total preorder $\succcurlyeq$ on $2^N$ and a semivalue $\pi^{\mathbf{P}} \in \mathcal{S}$, we shall say that $\succcurlyeq$ is $\pi^{\mathbf{P}}$-aligned if*

$$\{i\} \succcurlyeq \{j\} \Leftrightarrow \pi^{\mathbf{P}}_i(v) \geq \pi^{\mathbf{P}}_j(v)$$

*for each $v \in V(\succcurlyeq)$.*

Another condition considered in [46] and in [3] is the *permutational responsiveness* (PR) property, which generalizes the notion of RESP property and also guarantees the alignment of a total preorder with any semivalue. For a formal definition of PR we need some further notations. For each subset $\mathcal{F} \in 2^N$ and $i \in N$, we define the vector $\theta(\mathcal{F}, i)$ of ordered coalitions in $\mathcal{F}$, precisely $\theta(\mathcal{F}, i) = (S_1 \cup \{i\}, S_2 \cup \{i\}, \dots)$ such that $S_1 \cup \{i\} \succcurlyeq S_2 \cup \{i\} \succcurlyeq \dots \succcurlyeq S_{|\mathcal{F}|} \cup \{i\}$ and with $\{S_1, S_2, \dots, S_{|\mathcal{F}|}\} \equiv \mathcal{F}$.

**Definition 15** (PR). *We say that a total preorder $\succcurlyeq$ on $2^N$ satisfies the permutational responsiveness (PR) property if for each $i, j \in N$ we have that*

$$\{i\} \succcurlyeq \{j\} \Leftrightarrow \theta(\Sigma_{ij}^s, i)_k \succcurlyeq \theta(\Sigma_{ij}^s, j)_k \tag{3.6}$$

*for every $k = 1, \dots, |\Sigma_{ij}^s|$ and every $s = 0, \dots, n-2$ (see Section 1.2.1 for the definition of $\Sigma_{ij}^s$).*

In other terms, for each $i, j \in N$ such that $\{i\} \succcurlyeq \{j\}$ and for each $s = 0, \dots, n-2$, the PR property admits the possibility of relative rankings which violate the conditions imposed by the RESP property (i.e., $S \cup \{j\}$ is preferred to $S \cup \{i\}$) due to the effect of mutual interaction within the objects in $S$. Nevertheless, such an interaction should be compatible with the requirement that, between sets of the same cardinality, the original relative ranking between $\{i\}$ and $\{j\}$ should be preserved with respect to the position of subsets in $\Sigma_{ij}^s$ and $\Sigma_{ji}^s$, when they are arranged in descending order of preference (i.e., the most preferred subsets in $\Sigma_{ij}^s$ should be preferred to the most preferred subsets in $\Sigma_{ji}^s$, the second most preferred subsets in $\Sigma_{ij}^k$ should be preferred to the second most preferred subsets in $\Sigma_{ji}^k$, etc.). So, differently from RESP, the PR property can keep into account possible interaction effects among the elements of $N$.

In [46] it is shown that the PR condition is sufficient to guarantee that a total preorder is aligned with all semivalues. The following example, instead, displays a total preorder on $2^{\{1,2,3,4\}}$ which is $\pi^{\mathbf{p}}$-aligned for all $\pi^{\mathbf{p}} \in \mathcal{S}$ and that does not satisfy the PR property.

**Example 13.** *Let $X = \{1,2,3,4\}$ and let $\succcurlyeq$ be a total preorder such that $\{1,2,3,4\} \succ \{2,3,4\} \succ \{3,4\} \succ \{4\} \succ \{3\} \succ \{2\} \succ \{2,4\} \succ \{1,4\} \succ \{1,3\} \succ \{2,3\} \succ \{1,3,4\} \succ \{1,2,4\} \succ \{1,2,3\} \succ \{1,2\} \succ \{1\} \succ \emptyset$.*

*Note that $\succcurlyeq$ does not satisfy PR because $\{2\} \succ \{1\}$, $\{2,4\}$ is strictly preferred to $\{1,4\}$ and $\{1,3\}$ is strictly preferred to $\{2,3\}$. However, $\succcurlyeq$ is $\pi^{\mathbf{p}}$-aligned for all semivalues $\pi^{\mathbf{p}} \in \mathcal{S}$ (see the Appendix in [3] for the details of the proof).*

In [3] we also introduce the property that is necessary and sufficient for the alignment of the preorder to all semivalues. For every $i, j \in N$, we set $\mathcal{D}_{ij}^s$ to be the set $\mathcal{D}_{ij}^s = \Sigma_{ij}^s \cup \Sigma_{ij}^{s+1}$ for $s = 0, \dots, n-3$. With a little abuse of notation, set $\mathcal{D}_{ij}^{n-2} = \Sigma_{ij}^{n-1}$.

**Definition 16** (double permutational responsiveness, DPR). *We say that a total preorder on $2^N$ satisfies the double permutational responsiveness (DPR) property if for each $i, j \in N$ we have that*

$$\{i\} \succcurlyeq \{j\} \Leftrightarrow \theta(\mathcal{D}_{ij}^s, i)_k \succcurlyeq \theta(\mathcal{D}_{ij}^s, j)_k \tag{3.7}$$

*for every $k = 1, \dots, |\mathcal{D}_{ij}^s|$ and every $s = 0, \dots, n-2$.*

Intuitive examples of the meaning of the DPR property could involve the comparison of committees whose size is not fixed *a priori* but, according to different external contingencies, may vary and differ of at most one member.

**Theorem 12** (from [3]). *Let $\succcurlyeq$ be a total preorder on $2^N$. The following statements are equivalent:*

*1) $\succcurlyeq$ fulfils the DPR property;*

*2) $\succcurlyeq$ is $\pi^{\mathbf{p}}$-aligned for all semivalues.*

## 3.4 Future directions

An interesting direction for problems discussed in Section 3.2, is the analysis of more realistic classes of power relations, where, for instance, certain relative comparisons of strength are not possible (e.g., for lack of information, or the impossibility to compare on the same scale some social and political attributes of the coalitions, etc.). Note that assuming that two coalitions $S$ and $T$ are not comparable does not imply that $S$ and $T$ cannot form or cannot be compared to other coalitions (with respect to this aspect, the model introduced in Section

3.2 is more general than the classical models of coalitional games with restrictions in cooperation [150], where certain coalitions cannot form at all without the help of external players).

For the game theoretic approach introduced in Section 3.2.1, it would be interesting to investigate alternative notions of "intensity" of coalitional dominance, other than the one of relative score considered in [5]. For instance, in the power relation $\{1,2,3\} \succ \{2\} \succ \{1,3\} \succ \{1,2\} \succ \{3\} \succ \{1\} \succ \emptyset \succ \{2,3\}$ (where the reader may check that both players 1 and 2 coalitionally dominate player 3), one could argue that the dominance of player 2 over 3 is stronger than the one of player 1 over 3 because among the singletons coalitions it holds that $\{2\} \succ \{3\}$ and $\{3\} \succ \{1\}$.

Another possible direction for future research on the topic discussed in Section 3.2.2 is the open question about which axioms could be used to characterize a social ranking over the domain of all possible power relations. In view of our results, each combination of the axioms we propose in this paper is not satisfactory. In this respect, it is worth noting that all the properties that we analysed are based on the comparison of subsets having the same number of elements. Therefore, it would be interesting to study properties based on the comparison among subsets with different cardinalities.

# Chapter 4

# Algorithmic game theory

## 4.1 Overview of the chapter

This chapter is devoted to the discussion of algorithms for the computation and the analysis of equilibria in non-cooperative games. Classical non-cooperative situations that are considered in the domain of algorithmic game theory are those where a group of rational players share a common resource, and where the outcome of their interaction is influenced by the action taken by each single player. Typical examples are *strategic connection situations*, that are the non-cooperative counterpart of situations considered in Section 2.2. As in the cooperative framework, each player corresponds to a node of the graph and wants to be connected, directly or via other agents, with a source. Again, links are costly (e.g., due to the energy consumption needed to send a message to a remote agent), but now we suppose that each player-node is allowed to construct a single link which connects himself to another node in the network (i.e., another player or a source) and the decision on which link to construct is taken individually (i.e., it is not allowed to sign binding agreements with other agents). The cost incurred by each player depends both on the cost of the network constructed under a certain strategy profile and on the protocol used to allocate among the players the total cost of such a network. In this framework it is interesting to look at the dynamic process of improvement of each agent's payoff (i.e, the reduction of individual costs) in response to actions made by other agents, that can be modelled via a better response dynamics (BRD) (see Section 1.2.2). Do the players converge to an equilibrium via a BRD? Do we have convergence after a small number of deviations, starting fro any state? Is the state or strategy profile corresponding to an equilibrium also efficient? This kind of questions are in general (and not only on connection situations) of major importance in algorithmic game theory. Answers to questions of these type and on different strategic situations are provided in this chapter.

We start in Section 4.2, with a summary of the main results published in the article [47], where we analysed strategic games based on (non-cooperative) connection situations with the aim of coordinating rational agents (placed on the vertices of a graph) and whose objective is to construct an efficient network. In [47], we first analysed the effects of monotonicity and other basic properties on the optimality of a cost allocation protocol. Successively, we studied the problem of designing cost allocation protocols that can guarantee the convergence of a BRD in these games.

Section 4.3 deals with the model studied in [6] for *congestion situations*. Significant interest has been addressed over the last years to the analysis of practical congestion problems on Internet. Data delays and losses due to data congestions, or the network collapse as a consequence of exceeding the data flow capacity of some links or nodes, is an important issue on Internet. Several policies have been proposed to control congestion, in order to regulate and improve the availability of broadband access to the Internet. Priority rules, for instance, have been adopted to regulate the users who enter into the network, with the objective to prevent congestion and to obtain a Quality of Service (QoS) that otherwise would not be available to users. To this aim, in [6] we introduced the class of congestion games with capacitated resources, where each resource is associated both with a capacity level, representing the maximum number of users that such a resource may simultaneously accommodate, and with an ordering on the users, prescribing the priority of accommodation of the users.

Finally, Section 4.4 is based on paper [44], where we considered strategic interaction situations on *social networks*. On those social networks, the players interact only with their neighbours and the relationship between them can be modelled in terms of two-players games. More precisely, on a particular subclass of problems, in [44] we suggested how to design mechanisms that make possible to influence the players' behaviour towards a desired direction. In other words, we studied some indirect mechanisms in which agents repeatedly play a base game and, at each time step, they are prescribed to choose the best-response to the strategies currently selected by other agents. Roughly speaking, this class of mechanisms takes advantage of the dynamic nature of many systems to induce the desired outcome. In [44], we also provided several definitions of fairness based on different aspects of the problem.

## 4.2 Strategic games on connection situations

In Section 2.2, we have discussed a model of cost sharing on connection situations using cooperative games. Recently, several authors have studied connection situations in a non-cooperative setting [116, 93, 125], by means of games in strategic form where each player-node is allowed to construct a single edge which connects himself to another node in the network (that may be another player or the source). Like in the cooperative framework, each edge has a cost, and the cost of being disconnected from the source is supposed to be larger than any finite cost that could guarantee the connection with the source. Consequently, players want to be connected to the source at any (finite) cost, but of course they are self-interested to save their own money. Cost incurred by players depend both on the cost of the network that is constructed under a certain strategy profile, and also on the protocol used to allocate among the players the total cost of such a network.

For instance, in [125] the authors focused on a strategic game corresponding to the cost allocation protocol provided by the Bird rule [94]. As shown in [125], the Bird rule satisfies a variety of desirable properties. In particular, the associated BRD converges to a pure Nash equilibrium. However such a Nash equilibrium may not correspond to a graph of minimum cost. The question about the existence of a cost allocation protocol that guarantees the convergence of each BRD to a network of minimum cost remained open until the paper [47], where we introduced specific protocols that guarantee the convergence of the BRD.

To be more precise, using the same notations introduced in Section 1.2.1 and also in Section 2.2, given a set of nodes (and a source) $N' = N \cup \{0\}$, consider a graph $\langle N', E\rangle$, where $E \subseteq E_{N'}$ is the set of edges. Suppose also that it is given a *weight function* $w : E \to \mathbb{R}^+$ assigning to each edge $\{i, j\} \in E$ a non-negative number $w(\{i, j\})$ representing the weight or cost of edge $\{i, j\}$.

In the following, we describe the ingredients of a *connection game*, that is a strategic cost game $\mathcal{CG}^E = \langle N, (S_i)_{i\in N},\ (c_i)_{i\in N}\rangle$ on the player-set $N$, where each $i \in N$ needs to be connected to the source 0, either directly or via other nodes which are connected to the source. The strategy space of every player $i \in N$ coincides with its set of neighbours in the graph, i.e. $S_i = \{j \in V : \{i, j\} \in E\}$. When a player $i$ plays a neighbour $j$, then edge $\{i, j\}$ is built. Therefore, a state $s$ is a vector $(s_1, s_2, \cdots, s_n) \in \mathcal{S} = S_1 \times S_2 \times \ldots \times S_n$. With a small abuse of notation, the set of edges built by the players and associated with state $s \in \mathcal{S}$ is denoted by $E(s) = \{\{i, s_i\} : i \in N\}$. For a state $s \in \mathcal{S}$, $\mathtt{con}(s)$ and $\mathtt{dis}(s)$ denote the players who are connected and disconnected from the source, respectively. Finally, the last ingredient of game $\mathcal{CG}^E$ is the *cost allocation protocol* (or, simply, *protocol*) $c_i : \mathcal{S} \to \mathbb{R}$: for each strategy profile $s \in \mathcal{S}$, $c_i(s)$ indicates the cost incurred by player $i \in N$. It is assumed that a player not connected to 0 has an infinite cost. Of course, it is also assumed that players want to minimize their costs.

Consider the graph $T_s =< \mathtt{con}(s) \cup \{0\}, E(\mathtt{con}(s)) \cap E(s) >$ (recall from Section 1.2.1, that $E(\mathtt{con}(s)) = \{\{i, j\} \in E : i, j \in \mathtt{con}(s)\}$). Note that $T_s$ is a tree, since $T_s$ is connected by construction and contains exactly $|\mathtt{con}(s)| + 1$. The social cost of a state $s$ is defined as $\sum_{i\in N} c_i(s)$.

A Nash equilibrium $s \in \mathcal{S}$ in $\mathcal{CG}^E$ is said to be *efficient* in game $\mathcal{CG}^E$ iff the corresponding graph $< N', E(s) >$ is a mcst for $< N', E >$ with respect to the map $w$ (i.e. $w(E(s)) = \sum_{e\in E(s)} w(e)$ equals the minimum cost over all networks connecting all nodes in $N'$). In the following, we provide some properties studied in [47] for cost allocation protocols. A cost allocation protocol $c$ is said to be :

i) *Budget Balanced* (BB) iff $\sum_{i\in \mathtt{con}(S)} c_i(s) = w(E(s))$ for every state $s$;

ii) *Consistent* (Cons) iff every associated BRD reaches a NE;

iii) *Optimal* (Opt) iff every associated BRD reaches an efficient NE;

iv) *Individual Monotonic* (IM) iff for every $s \in \mathcal{S}$, $i \in \mathtt{con}(s)$ and $\hat{s}_i \in S_i$, $w(i, \hat{s}_i) \geq w(i, s_i) \Rightarrow c_i(\hat{s}_i, s_{-i}) \geq c_i(s)$;

v) *Independent from Disconnected Components* (IDC) iff for every state $s$, for every weight vector $w, w' \in (\mathbb{R}^+)^m$ with $w(e) = w'(e)$ for every $e \in E(\mathtt{con}(s)) \cap E(s)$, then $c_i(w, s) = c_i(w', s)$ for every $i \in \mathtt{con}(s)$.

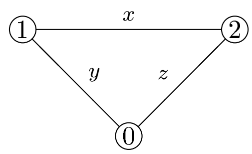


Figure 4.1: A 2-player instance of the connection game.

Cons and Opt properties deal with the notion of convergence of the associated BRD. Property BB implies that the cost of the edges in the network connected to the source is fully supported by its users. Property IM states that if a player $i$, who is connected to the source in a state $s$, decides to construct a more expensive edge to another neighbour $\hat{s}_i$, then the protocol will charge $i$ in state $(\hat{s}_i, s_{-i})$ more than in $s$. Property IDC says that the allocation of the cost of the network $T_s$ connected to the source in a state $s$ should depend only on the edges in $T_s$ (note that property IDC is called "State Independence" in [47]).

In [47], we proved that there is no BB protocol satisfying both IM and Opt properties, as illustrated by Figure 4.1, where $w(1,2) = x$, $w(1,0) = y$, $w(2,0) = z$ and with $z > y > x > 0$. To see this, just take the suboptimal strategy profile where player 1 plays 2 while player 2 plays 0. If player 2 deviates, playing strategy 1, the network is disconnected from the source, and the cost of 2 (and of 1) would be infinite. If player 1 deviates, playing strategy 0, then the cost allocated to 1 by a protocol $c$ that satisfies property IM in state $(0,0)$ should be larger than in state $(2,0)$. This means that the strategy profile $(2,0)$ is a Nash equilibrium, but it is not efficient (by the BB property); so, $c$ is not optimal. Similarly, there is no BB protocol which is both IDC and Opt [47]. These impossibility results imply that a large family of solutions from the literature on (cooperative) mcst games like *obligation rules* [22], which are cost monotonic, and *CC-rules* [16], which are IDC, are not optimal in the strategic framework. Nevertheless, in [47] we show that optimal and budget balanced protocols exist, as it is shortly discussed in the following.

For each state $s \in \mathcal{S}$, let $T^*_s = \langle \mathtt{con}(s)', E^* \rangle$ be one specific minimum cost spanning tree on the sub-graph $G[\mathtt{con}(s)]$. The state which corresponds to $T^*_s$ is denoted by $s^*$ and we say that a player $i \in \mathtt{con}(s)$ follows $T^*_s$ in $s$ iff $s_i = s^*_i$: this means that the strategy of player $i$ is his first neighbour in the unique path from $i$ to the source 0 in $E^*$. We define $\hat{V}(s)$ as the players of $\mathtt{con}(s)$ such that $s_i \neq s^*_i$ and $\mathtt{con}(s) = \mathtt{con}(s^*_i, s_{-i})$. Players in $\hat{V}(s)$ do not follow $T^*_s$ and are such that if they unilaterally change their strategy to follow it, then the set of connected players remains unchanged. It is possible to prove that if there exists $i \in \mathtt{con}(s)$ such that $s_i \neq s^*_i$, then $\hat{V}(s) \neq \emptyset$ (see [47] for more details).

Let $M(s)$ be the total weight of the optimal tree $T^*_s$, i.e. $M(s) = \sum_{e \in E^*} w(e)$. The difference between the weight of the edges built by the players connected to the source and the minimal weight for connecting these players is denoted by $\Delta(s) = \sum_{e \in E(\mathtt{con}(s)) \cap E(s)} w(e) - M(s)$. Two protocols satisfying Opt exist.

In the *first protocol*, denoted by $\bar{c}$, all connected players equally share the cost of an optimal network (namely $\frac{M(s)}{|\mathtt{con}(s)|}$) except one player, denoted by $f(s)$, who is charged $\frac{M(s)}{|\mathtt{con}(s)|}$ plus the extra cost of the current state $\Delta(s)$, for each state $s \in \mathcal{S}$. More precisely, let $f(s) = \min \hat{V}(s)$ be the node of $\hat{V}(s)$ with minimum index if $\hat{V}(s) \neq \emptyset$ and $f(s) = \emptyset$ otherwise. Formally, for each state $s \in \mathcal{S}$, the players' costs according to $\bar{c}$ are the following:

- if $i \in \mathtt{con}(s) \setminus \{f(s)\}$, then $\bar{c}_i(s) = \frac{M(s)}{|\mathtt{con}(s)|}$,
- if $f(s) \neq \emptyset$, then $\bar{c}_{f(s)}(s) = \frac{M(s)}{|\mathtt{con}(s)|} + \Delta(s)$,
- if $i \in \mathtt{dis}(s)$, then $\bar{c}_i(s) = +\infty$.

In the *second protocol*, denoted by $\hat{c}$, all connected players who follow the optimal strategy profile $T^*_s$ pay according to the Bird rule while the other connected players (who do not follow $T^*_s$) pay what they should pay in $T^*_s$ with the Bird rule plus an extra cost, for each state $s \in \mathcal{S}$. We assume that this extra cost is shared equally. More precisely, for each state $s \in \mathcal{S}$, according to protocol $\hat{c}$, the players' costs are the following:

- if $i \in \mathtt{dis}(s)$, then $\hat{c}_i(s) = +\infty$,
- if $i \in \mathtt{con}(s) \setminus \hat{V}(s)$, then $\hat{c}_i(s) = w(i, s_i)$,
- if $i \in \hat{V}(s)$, then $\hat{c}_i(s) = w(i, s^*_i) + \frac{\Delta(s)}{|\hat{V}(s)|}$ (actually here, it is possible to define alternative protocols using any cost function $w(i, s^*_i) + g_i(s)$ such that $g_i(s) > 0$ and $\sum_{i \in \mathtt{con}(s)} g_i(s) = \Delta(s)$).

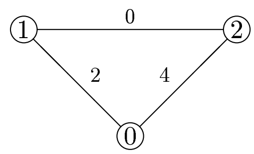


Figure 4.2: A connection situation with only one mcst.

One can observe that the total weight of $E(s)$, for each state $s \in \mathcal{S}$, is always covered by the connected players according to both protocols $\bar{c}$ and $\hat{c}$. Therefore, both protocols are BB.

**Example 14.** *Consider the connection situation depicted in Figure 4.2. Note that there is a unique mcst of cost* 2*, that is the tree* $\{\{1,2\}, \{0,1\}\}$*. The connection games associated with protocol* $\bar{c}$ *and* $\hat{c}$*, in the connection situation depicted in Figure 4.2 are shown in Table 4.1 (player* 1 *is the row player and player* 2 *is the column player).*

| | 0 | 1 |
|---|---|---|
| 0 | $1,5$ | $1,1$ |
| 2 | $3,1$ | $\infty,\infty$ |

| | 0 | 1 |
|---|---|---|
| 0 | $2,4$ | $2,0$ |
| 2 | $4,0$ | $\infty,\infty$ |

Table 4.1: Connection games corresponding to the graph of Figure 4.2 using protocol $\bar{c}$ (left side) and protocol $\hat{c}$ (right side) respectively.

In [47], we proved that the cost protocols $\bar{c}$ and $\hat{c}$ satisfy the properties of cons and Opt, and the BRD converges after at most $n^3$ if all players play their *best* response under protocol $\bar{c}$, whereas the BRD converges after at most $|E|n^2$ if the players play their *best* response under protocol $\hat{c}$.

## 4.3 Congestion games

Congestion games [166] deal with interaction situations where the cost associated with the use of a resource or facility depends on the number of players that use it. In order to model more realistic situations, congestion games have been generalized in several directions including models where players have different weights, or when the cost of using a resource depends on the identity of the players that are using it, etc. (see, for instance, [146, 78]).

More precisely, a congestion situation is composed by a finite set of players $N = \{1, \ldots, n\}$, a finite set of facilities $\mathcal{R}$ and a map $d_r : \{0, \cdots, n\} \to \mathbb{N}$ which associates to each facility $r \in \mathcal{R}$ the cost of facility $r$ as a function of the number of its users. Let $S_i \subseteq 2^{\mathcal{R}}$ be the strategy space of player $i \in N$. We denote by $n_r(s)$ the number of players using the resource $r$ according to a strategy profile $s = (s_i)_{i \in N} \in \prod_{i \in N} S_i$. The cost for a player $i$ in state $s$ is denoted by $c_i(s)$, and is defined as the sum of the costs of the resources used by $i$ in $s$, that is $c_i(s) = \sum_{r \in s_i} d_r(n_r(s))$. The associated congestion game is then defined as the strategic game $\langle N, (S_i)_{i \in N}, (c_i)_{i \in N} \rangle$.

More precisely, in [6], we enriched the original congestion situations introduced in [166] assuming that all resources are provided with a capacity and an ordering over the players: if the number of users of a resource does not exceed its capacity then it behaves like the classical model of congestion; otherwise, if the capacity is exceeded, only a given number among the first users in the ordering are retained.

In [6] we associated with each facility $r \in \mathcal{R}$ a capacity $\kappa_r \in \mathbb{R}$ and a bijection $\succ_r: N \to N$ representing the priority of the players for the facility $r$. According to the model we introduced in [6], the first $\kappa_r$ users in the ordering $\succ_r$ face a cost for using facility $r$ equal to $d_r(n_r(s))$ (like in the original congestion game introduced in [166]), whereas the remaining players, if any, face an infinite cost. The final goal in [6] is to provide necessary and sufficient conditions for the existence of (pure) Nash equilibria of these games and to propose efficient algorithms to find them.

First, in [6] we showed that congestion games with capacities and priorities may not have a Nash equilibrium. In the game corresponding to Figure 4.3, for instance, there are two players and three facilities $x$, $y$ and $z$. The strategy space is $\{\{x\}, \{y, z\}\}$. The facility $x$ has a capacity of 1 and $d_x(1) = 2$. The facility $y$ has a capacity of 2 and $d_y(1) = 3$, $d_y(2) = 0$. The facility $z$ has a capacity of 1 and $d_z(1) = 0$. The priority is always given to player 1 for all facilities.

In [6] we also proved the following results.

| | $\{x\}$ | $\{y,z\}$ |
|---|---|---|
| $\{x\}$ | $2$, $+\infty$ | $2$, $3$ |
| $\{y,z\}$ | $3$, $2$ | $0$, $+\infty$ |

Figure 4.3: A situation with two players and three facilities without Nash equilibria [[6]].

**Theorem 13** ([6]). *All congestion games with capacity and priorities over the facilities and with only two facilities ($|\mathcal{R}| = 2$) have a Nash equilibrium that can be computed in polynomial time.*

Moreover, in [6], we focused on the class of congestion games with capacities and priorities where players can use only one facility (so, $|s_i| = 1$ for each strategy $s_i \in S_i$ and each player $i \in N$), and we provided the following theorem.

**Theorem 14** ([6]). *Every congestion game with singleton strategies (and with capacities and priorities over the facilities) is a potential game.*

Another interesting result shown in [6] is that the BRD of these games converges to the Nash equilibrium in at most $\max\{|N|, |\mathcal{R}|\}$ steps.

## 4.4 Games on social networks

Another interesting class of (non-cooperative) network games are *social coordination games*, that we studied in the paper [44]. In *social coordination games*, each player is in a node of a graph and prefers to coordinate with their *neighbours* in the graph, (i.e., to take the same action or choose the same product), rather than conflicting with them. For example, given a network $\langle N, E\rangle$, each player/node has at most two available strategies, say 0 and 1, and the *local coordination games* $\mathcal{G}_e$ (see also [117]) corresponding to the edge $e = \{u, v\} \in E$ is given by the following cost matrix:

| | 0 | 1 |
|---|---|---|
| 0 | $\alpha_u^e(0), \alpha_v^e(0)$ | $\beta_u^e(0), \beta_v^e(1)$ |
| 1 | $\beta_u^e(1), \beta_v^e(0)$ | $\alpha_u^e(1), \alpha_v^e(1)$ |

where the costs for agreements are smaller or equal to the costs for disagreements (even if these costs may vary depending on which strategy a player adopts), i.e. $\beta_k^e(b) \geq \alpha_k^e(b') \geq 0$ for all $b, b' \in \{0,1\}$ and $k = u, v$.

Social coordination games have been largely adopted for modelling the spread of innovation in social networks. Here, network members have to choose between a new and an old technology, for example, a chat protocol. Clearly, each node prefers to choose the same chat protocol as her neighbours. Thus, once the number of neighbours adopting the new technology is greater than the number of users of the old technology, a node will change for the new chat protocol. It is also assumed that players prefer to use the new technology more than the previous one (that is, the old technology has a strategy cost higher than the new one) or, vice versa, that the price of the new technology is higher than the price of the old one. These strategy costs influence the threshold, making a node eager to adopt the new technology.

**Example 15.** *Consider the following coordination game with two players (the row player $i_0$ and column player $i_1$) both with strategies $\{0,1\}$:*

| | 0 | 1 |
|---|---|---|
| 0 | $0, \epsilon$ | $1-\epsilon, 1-\epsilon$ |
| 1 | $1-\epsilon, 1-\epsilon$ | $\epsilon, 0$ |

*Assume moreover that the row player incurs a preference cost $b$ for playing strategy $b \in \{0,1\}$ while the column player has a preference cost for playing strategy $b$ of $1-b$. The actual costs faced by the players are then as follows:*

| | 0 | 1 |
|---|---|---|
| 0 | $0, 1+\epsilon$ | $1-\epsilon, 1-\epsilon$ |
| 1 | $2-\epsilon, 2-\epsilon$ | $1+\epsilon, 0$ |

*Notice that for* $0 < \epsilon < \frac{1}{3}$*, the two configurations of minimal total cost are* $(0,0)$ *and* $(1,1)$*, but the unique Nash equilibrium of the game is* $(0,1)$*. This example shows that it is often impossible to force the agents to play according to the optimal solution because the optimum may not be an equilibrium.*

In [44] we focused on *incentive-compatible best-response mechanisms* [153], where a desired outcome is induced by means of a repeated playing of a *base game* where agents are prescribed to choose the best-response to the strategies currently selected by other agents. For particular classes of interaction situations, it is in fact possible to modify conveniently the players' cost functions so that players have no incentive to deviate from this prescribed behaviour and the mechanism converges to a desired equilibrium. In particular, in [44] we considered a special way of modifying a cost function in local coordination games. First, we assumed that the mechanism may assign to players playing the desired strategy special *fees* (possible negative) in place of the costs arising from social relationships. Second, we assume that the mechanism is *frugal*, which means that the mechanism can be implemented by a designer without any cost. This means that whenever inducing a player to play the target strategy has a cost for the designer, it should be possible to collect in advance the necessary amount of money from other players.

In [44] we also showed that in social coordination games, made by local coordination games where each player has two strategies as in the example above, the optimal strategy profile can be efficiently computed and it is always possible to efficiently design a frugal best-response mechanism for inducing this optimal profile. Thus, an authority can always find policies that allow to exploit the dynamical nature of a system to induce the desired outcome.

Another aspect on which we focused in [44] is the property of *collusion-resistance* for mechanisms, which means that we are interested in mechanisms such that no coalition has any incentive to leave the strategy profile induced by the mechanism, even if side-payments are allowed. Interestingly, we showed in that a frugal best-response mechanism that is collusion-resistant always exists and we provided a characterization of the property of collusion-resistance in terms of solutions of a suitable cooperative game.

## 4.5 Future directions

The method used to define the cost allocation protocols for the strategic connection situations considered in Section 4.2 might lead to the definition of many other optimal protocols, depending on the rule according to which the cost of a mcst is allocated among the players. As illustrated by numerical examples in [47], the inherent limitations of the proposed optimal protocols is that symmetric players may be treated differently, depending on the choice of an a priori selected mcst. The question about the existence of optimal protocols which treat symmetric players in a more equitable manner remains open.

Considering congestion models discussed in Section 4.3, on a dynamic (and more realistic) perspective, it would be interesting to study an analogous model where the priorities of users depend on their timing of using resources (for routing problems, this could represent the arrival time to the starting node of an edge).

The focus of the model presented in Section 4.4 is the design of mechanisms through which an authority may influence the bargaining among the components of a social network for inducing optimal states as the result of the convergence of natural dynamics. We think that this approach can be promising for the design of mechanisms also in different settings, not necessarily based on social networks.

# Chapter 5

# Power indices and their applications

## 5.1 Overview of the chapter

Solutions of cooperative games have been widely used to evaluate the "power" of the players (agents, political parties, nations, etc.) involved in a collective decision process, i.e. their ability to force a decision in situations like a parliament, a governing council, a management board, etc. For instance, *probabilistic power indices* [110], like the Banzhaf index [86] and the Shapley index [171], are computed using the characteristic function of a coalitional game providing the information about which coalitions of players are winning or not and, if available, taking into account additional information about the interaction possibilities of the players (see also [150, 156, 110, 129]).

Section 5.2 is devoted to the problem of reducing the energy consumption over communication networks and using power indices. Several models to solve this energy consumption problem have been proposed in the literature: for instance, an interesting "green networking" technology suggests to adapt the network topology to the traffic demand, with the objective to concentrate the traffic on a partial network obtained by allowing some network nodes or links to enter into a low-energy "sleep" mode [105]. In a similar direction, in [49] and [48], we studied the problem of reducing energy consumption in computer networks according to an energy-aware routing approach and keeping into account the contribution of devices in the network in order to provide a good level of Quality of Service (QoS). In Section 5.2 we present some methods aimed to summarize the contribution of devices in a network. These methods are based on the Shapley index of particular coalitional games defined over the set of the elements of a backbone network. Those particular coalitional games incorporate the information about the network structure (e.g., the connectivity of sub-networks), the amount of traffic that the devices are routing and the network robustness (i.e., possible failure scenarios). The ranking provided by the Shapley value of such games has been used to drive a resource consolidation process, i.e., the selection of those devices that should be switched off first in order to reduce the energy consumption.

Section 5.3 is devoted to the discussion of a recent application of power indices presented in [54] to design a weighted majority voting system in practice. More specifically, we solve an *inverse Banzhaf index problem* in order to decide the weight of "great electors" within the electoral college for the election of the members of the Administration Board and the Academic Senate of the *Paris Sciences & Lettres* University. To be more specific, the inverse Banzhaf index problem can be formulated as follows: given a vector $P = (p_1, \ldots, p_n)$ of $n$ real numbers and an appropriate "metric" to evaluate the distance between real-valued vectors, find a voting system with $n$ voters such that the distance (or *error*) between the Banzhaf index computed on such a voting system and the vector $P$ is smaller than a predefined value (see, for instance, [85, 79] for further details).

Section 5.4 shortly introduces the problem of measuring *social capital* in coalitional situations. This part is based on the papers [4] and [8]. In [4] we considered a more advanced approach to the measurement of social capital, which builds upon the recent literature that uses concepts rooted in cooperative game theory for the analysis of social networks. With a similar objective, in [8] we studied two important families of indicators for social-ecological features, specifically, *resilience*, aimed to analyse the ability of ecosystems to absorb changes on both human and ecological variables (and where the structure of interactions between social and ecological components of the systems are taken into account) and social capital, here intended as an assessment of the set of all those human relations which are important for the sustainability issue.

## 5.2 Green networking

Telecommunication infrastructures are responsible of a large part of the carbon offprint of Information and Communications Technology (ICT) systems. As a consequence, the research community is showing an ever increasing interest in studying techniques and algorithms to reduce the energy consumption of ICT systems.

A computer network is represented by a graph $\langle N, E \rangle$, where $N$ is a finite set of computer devices and $E$ is the set of links between the devices, and is associated with a *traffic matrix*, $TM = (t_{ij})_{i,j \in N}$, in which an element $t_{ij}$ represents the volume of traffic entering the network through $i$ and exiting through $j$. Given a computer network $\langle N, E \rangle$ and an associated traffic matrix $TM$, in [49], we modelled the resource consolidation problem as a coalitional game, called the *Green-Game* (or G-Game for short), where $N$ is the set of devices and the value of each coalition $S \subseteq N$ is the amount of traffic that the restriction of the network to $S$ can transport, precisely,

$$v(S) = \sum_{i,j \in S} t_{ij} \mathbf{1}_{\{G[S]\}}(i,j),$$

where $\mathbf{1}_{\{G[S]\}}(i,j) = 1$ whenever $i$ and $j$ are connected in the subgraph $\rangle S, E(S) \rangle$ (i.e. there exists a path in $G[S]$ from $i$ to $j$) and zero otherwise (as usual, by convention $v(\emptyset) = 0$). The Shapley value of a node in a G-Game is interpreted as an indication of how much the node contributes in the traffic delivery process and how its absence would affect the network on "average" (i.e., over all possible network configurations). Differently stated, the Shapley value on the G-Game defines a joint topology-aware and traffic-aware ranking of the network devices, that is considered to establish which nodes can be switched off first.

In order to provide a relevant evaluation of the method based on the Shapley value, in [49] a realistic scenario has been considered as depicted in Figure 5.1 (for the technical details see [49]). The light-shaded nodes (1 to 8 in Figure 5.1) are access nodes (i.e., source and destination of traffic requests, and can not be switched off). The dark nodes (9 to 21) are transit nodes, performing only traffic transport, and can be switched off. Node $T$ is the traffic collection point, providing access to the core network and the big Internet, with whom nodes typically exchange the majority of the traffic.

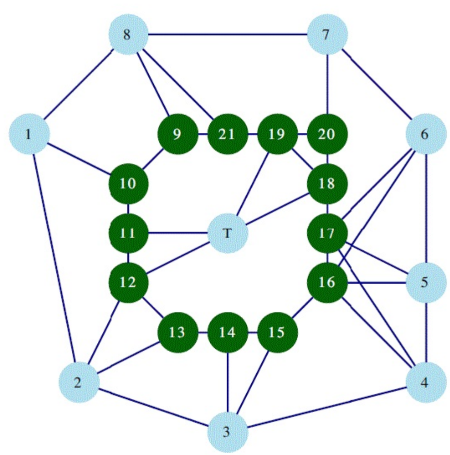


Figure 5.1: The reference topology of a computer network.

In a computer network like the one represented in Figure 5.1, the "criticality" of nodes can be evaluated relatively easily based on the sole topology, or on the sole volume of traffic routed by each node. Taking into account the topology, the most widely used rankings are based either on the connectivity of each node (Degree centrality [89]), or on the number of shortest paths passing through each node (Betweenness centrality [91]), or the average distance between each couple of nodes (Closeness centrality [95]), or even on the relative importance of neighbours nodes (Eigenvector centrality [172]). A completely different criticality criterion is the one proposed by [105] and denoted by "*Load*" hereafter, which merely consists in sorting nodes depending on the amount of traffic load they effectively carry in a standard network configuration. The above indexes either consider the topology (degree, betweenness, closeness, eigenvector centrality) or the traffic (Load criterion), but not both. The Shapley value used in the G-Game instead takes into account (i) the traffic expressed by the traffic matrix and (ii) the importance of the node in the routing process. In fact, the node importance is evaluated in the G-Game by taking into account the number of paths a node lies on, similarly to the betweenness centrality.

However, unlike betweenness centrality, the Shapley value takes into account failure scenarios by considering not only the shortest paths, but also longer ones that can provide alternate routes in degraded scenarios.

All the aforementioned criticality indexes have been evaluated on the reference network scenario of figure Figure 5.1. We also compared two different versions of the Shapley value: (i) a simplified index that reflects only the network topology, considering the G-Game with a uniform traffic matrix, referred to as G-Game U-TM hereafter; (ii) the full G-Game earlier defined, that considers the actual traffic matrix.

In Figure 5.2 we report the link utilization distributions for the different rankings when the first less critical nodes are switched off and the baseline configuration, where no node is switched off. Notice that the Shapley value yields to excellent performances, as the link distribution is roughly equivalent to the baseline configuration.

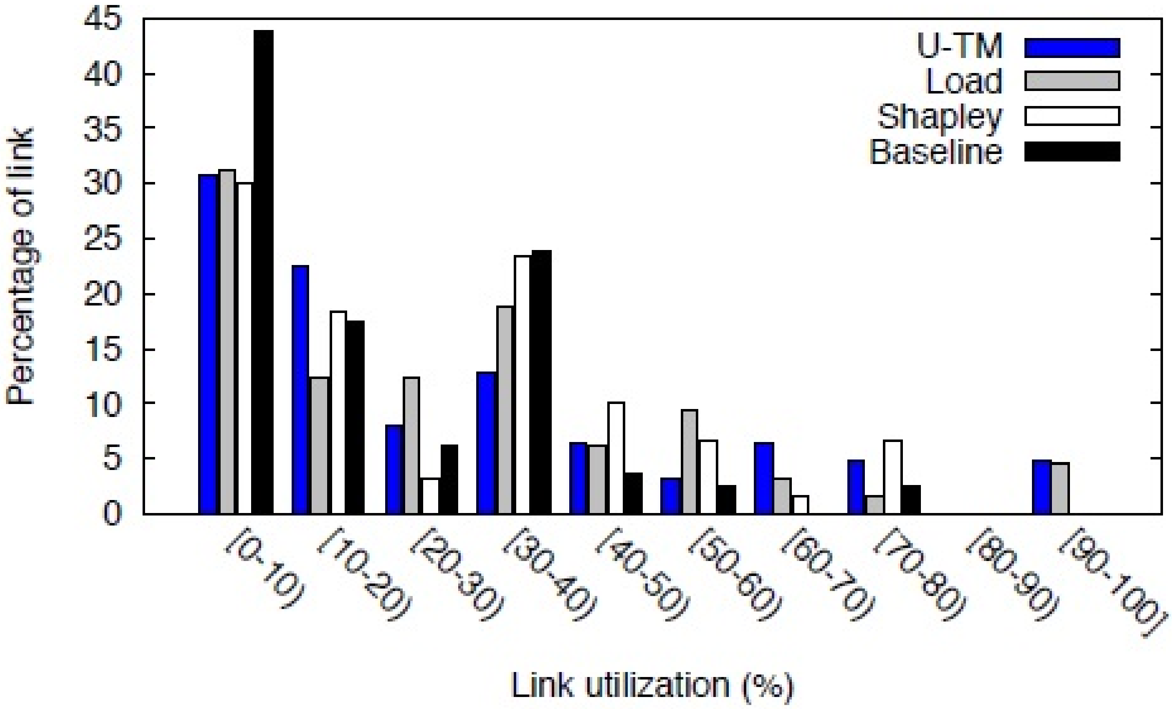


Figure 5.2: Distribution of the link utilization, considering different ranks and in the Baseline configuration.

In particular, the maximum link utilization does not increase under G-Game with respect to the full network configuration: this means that energy saving is obtained without compromising the expected QoS. Conversely, some links reach an utilization higher than 90% for the U-TM and Load strategies. The Load strategy results in worse link distribution since it passes through longer alternate paths (i.e., ignores fault cases), while the worse QoS results of the U-TM strategy are due to its traffic unawareness (i.e., it takes into account only the topology). In contrast, the approach based on G-Games explicitly considers different nodes combinations, which means that it explores configurations where some nodes are excluded (i.e., which is precisely what happens when nodes are switched off in the resource consolidation process).

A similar approach has been studied in [48], this time computing the *importance* of links in a network: the players in the game considered in [48] are the links of a computer network and a coalition corresponds to a network partition (subgraph) formed by the combination of all these links. The surplus of a coalition is defined as the amount of traffic that can be accommodated by the corresponding network partition, and the "importance" of a link is again measured according to the Shapley value of such a game, and thus measures the (average) additional traffic that can be accommodated thanks to the addition of a link.

## 5.3 An application of the inverse Banzhaf index problem

*Paris Sciences & Lettres* (PSL) is a federal university that brings together 25 education and research institutions in Paris (to be hereafter denominated the *PSL Institution Members* or, simply, the *Institutions*). Founded in

2010, the organisation of PSL is based on two main bodies[1]: the PSL Foundation for Scientific Cooperation (*Fondation de Coopération scientifique*), which is mainly responsible for the management of key actions of the PSL project (e.g., the recruitment of chairs of excellence, the development of strategic international partnerships, the implementation of innovative programs in research and training, etc.) and the Community of Universities and Institutions (*Communauté d'Universités et Etablissements*, also denominated *ComUE*) which is responsible of the decisions concerning the training and graduation policy of PSL, and of other decisions over the common actions related with the educational and research community (e.g., the joint coordination of research policies and international projects for knowledge dissemination, the activation of digital actions, the implementation of joint strategies concerning students' life, etc.).

The *ComUE* is governed by an Administration Board (AB) (*Conseil d'Administration*) of 30 members, assisted by an Academic Senate (AS) (*Conseil Académique*) of 120 members with a consultative role. The members of the AB and the AS of the ComUE are representatives of the different Institutions but, for statutory reasons, only 16 Institutions (see Table 5.1) out of the 25 of PSL participate to the electoral process for their designation. Notice that the PSL Foundation and the *ComUE* itself are both represented as independent establishments within the AB and the AS of the *ComUE*.

The members of the AB and the AS are indirectly elected, among the candidates of the different Institutions, by a college of "great electors" designated by the Institutions according to their own statutes (usually, via general elections within their respective Institutions). Moreover, the members of the AB and the AS must be appointed in the respect of their professional categories (e.g., teachers, researchers, administrative and technical staff, etc.) according to the proportion specified in the Internal Regulations [164], which addresses the general recommendations of the relevant national legislation. The same internal rules also specify that the "great electors" must have different amounts of say (weight) and should appoint candidates members for the AB and the AS using a simple majority mechanism.

The *ComUE* is also provided with a Steering Committee (SC) (*Comité des Membres*) formed by the PSL institutions heads, the president and the vice-president of PSL, and the deans of the main departments. The objective of the SC is to ensure the proper functioning of PSL and to address the implementation of the guidelines provided by the Internal Regulations [164].

| | **Institution** | **Short name** |
|---|---|---|
| 1 | *École nationale supérieure de chimie de Paris* | Chimie ParisTech |
| 2 | *Centre national de la recherche scientifique* | CNRS |
| 3 | *Conservatoire national supérieur d'art dramatique* | CNSAD |
| 4 | *Conservatoire national supérieur de musique et de danse de Paris* | CNSMDP |
| 5 | *Communauté d'universités et établissements PSL* | ComUE PSL |
| 6 | *École normale supérieure* | ENS |
| 7 | *École nationale supérieure des arts décoratifs* | ENSAD |
| 8 | *École nationale supérieure des beaux-arts* | ENSBA |
| 9 | *École supérieure de physique et de chimie industrielles de la ville de Paris* | ESPCI |
| 10 | *Fondation de coopération scientifique PSL* | FCS PSL |
| 11 | *Institut national de recherche en informatique et en automatique* | INRIA |
| 12 | *Institut Curie* | Institut Curie |
| 13 | *Fondation européenne des métiers de l'image et du son* | La Fémis |
| 14 | *École Nationale Supérieure des Mines de Paris* | Mines ParisTech |
| 15 | *Observatoire de Paris* | Observatoire |
| 16 | *Université Paris-Dauphine* | Paris-Dauphine |

Table 5.1: The 16 Institutions taking part to the electoral college for the appointment of the AB and AS members.

In [54], we shortly introduced an approach based on coalitional games and on power indices to establish a "fair" distribution of the weights among the "great electors" within the PSL electoral college for the appointment of the AB and the AS members. More precisely, the weight system we propose is the result of an *inverse power*

[1]For a more detailed description of the governance bodies of PSL, see the PSL official website: https://www.univ-psl.fr/fr

*index problem*: if a vector of desired individual powers is given (for instance, as the outcome of a negotiation process), can we determine a voting method where a certain power index yields a good approximation of the desired vector?

According to the recommendation of the SC [174], an "ideal" voting system should take into account the following three criteria: 1) all the Institutions should participate with a non-negligible power to the process of taking decisions in the AB and the AS of the *ComUE*; 2) larger Institutions (in terms of number of staff employed) should play a more relevant role; 3) Institutions with a major academic offer should be fairly represented. Because of an important disproportion of students over the different Institutions (many Institutions have no student at all), the SC of the *ComUE* considers this last principle less relevant than the first two, in order not to destabilise the "global economy" of the decision-making process within PSL. According to this "ideal" triple-majority rule recommended by the SC [174] a subset of the 16 Institutions of the *ComUE* shown in Table 5.1 is a winning coalition if it satisfies the following three criteria:

i) it is formed by a simple majority (> 50%) of the Institutions of the *ComUE*, all in favour;

ii) it represents a qualified majority (> 66%) of the total number of employees working in the Institutions of the *ComUE*;

iii) it represents at least one-fourth (> 25%) of the entire population of students enrolled in the Institutions of the *ComUE*;

A simple game $(N, v^t)$, with $N = \{1, \dots, 16\}$ as the set of players representing the 16 Institutions Members of PSL, was defined according to the above triple majority mechanism. Precisely, let $e_i$ and $s_i$, be, respectively, the number of employees and of students of each Institution $i \in N$, and take a coalition of Institutions $S \subseteq N$, $S \neq \{\emptyset\}$, then we have that:

$$v^t(S) = \begin{cases} 1 & \text{if } |S| > 8 \text{ and } \sum_{i \in S} e_i > \frac{2}{3}\rho \text{ and } \sum_{i \in S} s_i > \frac{1}{4}\sigma, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

where $\rho = \sum_{i \in N} e_i$ is the total size of personnel affiliated to the 16 Institutions of the *ComUE* and $\sigma = \sum_{i \in N} s_i$ is the total number of enrolled students. The normalized Banzhaf index $\bar{\beta}(v^t)$ of game $v^t$ has been computed according to relation (1.7) and by means of the computer program introduced in [137] and the *Mathematica* [180] functions introduced in [90]. The vector $\bar{\beta}(v^t)$ is shown in the second column of Table 5.2.

Our goal in [54] is then to solve an *inverse Banzaf index problem* [79] with the objective to compute the weights of the "great electors" of PSL such that the Banzhaf index of the electoral college (based on a simple majority mechanism) is as close as possible to the Banzhaf index computed on the "ideal" triple majority system recommended by the SC [174] and yielding the game $v^t$. We consider the inverse Banzhaf index problem as introduced in the next definition.

**Definition 17** (Inverse Banzhaf index problem). *Let $N = \{1, \dots, n\}$ be a finite set of players, let $\lambda \in [0, 1]$ be a predefined quota (expressed as a fraction of the total weight) and let $\epsilon \in [0, 1]$ be the maximum tolerated error.*

*Given another vector $P = (p_1, ..., p_n) \in [0, 1]^N$ of $n$ real numbers on the interval $[0, 1]$ and such that $\sum_{i \in N} p_i = 1$ (i.e., $P$ is normalized), find a vector of non-negative integer numbers $w = (w_1, \dots, w_n) \in \mathbb{Z}^N$ with $w_i \geq 0$ for each $i \in N$ and such that*

$$\sum_{i \in N} |\bar{\beta}_i(v^w) - p_i| < \epsilon, \tag{5.2}$$

*where $v^w$ is the weighted majority game on $N$ defined according to relation (1.6) with weights $w$ and a quota $q = \lambda \sum_{i \in N} w_i$, and $|\bar{\beta}_i(v^w) - p_i|$ is the absolute value of the difference between the normalized Banzhaf index $\bar{\beta}_i(v^w)$ of player $i \in N$ in game $v^w$ and the "ideal" value $p_i$.*

Notice that in the above definition, it is asked to find a vector of integer weights and the quota is provided *a priori* (for different formulations of the inverse Banzhaf index problem see [85, 79]).

In order to find a solution of the previously introduced instance of the inverse Banzhaf index problem (specifically, the one introduced in Definition 17 with $P := \bar{\beta}(v^t)$, $\lambda = \frac{1}{2}$ and $\epsilon = 0.05$), we apply a trial-and-error procedure (see [54] for more details). This procedure yielded the vector of integer weights shown in the last column of Table 5.2 together with the Banzhaf value $\bar{\beta}_i(v^w)$ of the corresponding weighted majority game (see also Figure 5.3 for a comparison between $\bar{\beta}_i(v^w)$ and $\bar{\beta}_i(v^t)$ of PSL members ordered according to their Banzhaf index in the AB).

| *Institution* | $\bar{\beta}(v^t)$ | $\bar{\beta}(v^w)$ | $\lvert\bar{\beta}(v^w) - \bar{\beta}(v^t)\rvert$ | *Weight* |
|---|---|---|---|---|
| Chimie ParisTech | 0.039 | 0.037 | 0.002 | 3 |
| CNRS | 0.023 | 0.024 | 0.001 | 2 |
| CNSAD | 0.016 | 0.013 | 0.003 | 1 |
| CNSMDP | 0.073 | 0.076 | 0.003 | 6 |
| ComUE PSL | 0.018 | 0.013 | 0.005 | 1 |
| ENS | 0.213 | 0.208 | 0.005 | 15 |
| ENSAD | 0.041 | 0.037 | 0.004 | 3 |
| ENSBA | 0.042 | 0.037 | 0.005 | 3 |
| ESPCI | 0.049 | 0.050 | 0.001 | 4 |
| FCS PSL | 0.011 | 0.013 | 0.002 | 1 |
| INRIA | 0.013 | 0.013 | 0 | 1 |
| Institut Curie | 0.088 | 0.090 | 0.002 | 7 |
| La Fémis | 0.039 | 0.037 | 0.002 | 3 |
| Mines ParisTech | 0.079 | 0.090 | 0.011 | 7 |
| Observatoire de Paris | 0.071 | 0.076 | 0.005 | 6 |
| Université Paris-Dauphine | 0.186 | 0.186 | 0 | 14 |
| **Total** | *1* | *1* | *0.049* | *77* |

Table 5.2: Normalized Banzhaf index of the "ideal" triple-majority game $v^t$ and of the weighted majority game $v^w$. The absolute value of the differences between the two Banzhaf indices $|\bar{\beta}(v^w) - \bar{\beta}(v^t)|$ is shown in the third column (notice that the total error is less than the maximum tolerated one $\epsilon = 0.05$). The vector of weights is shown in the last column (the quota is then fixed at $q = \frac{77}{2}$).

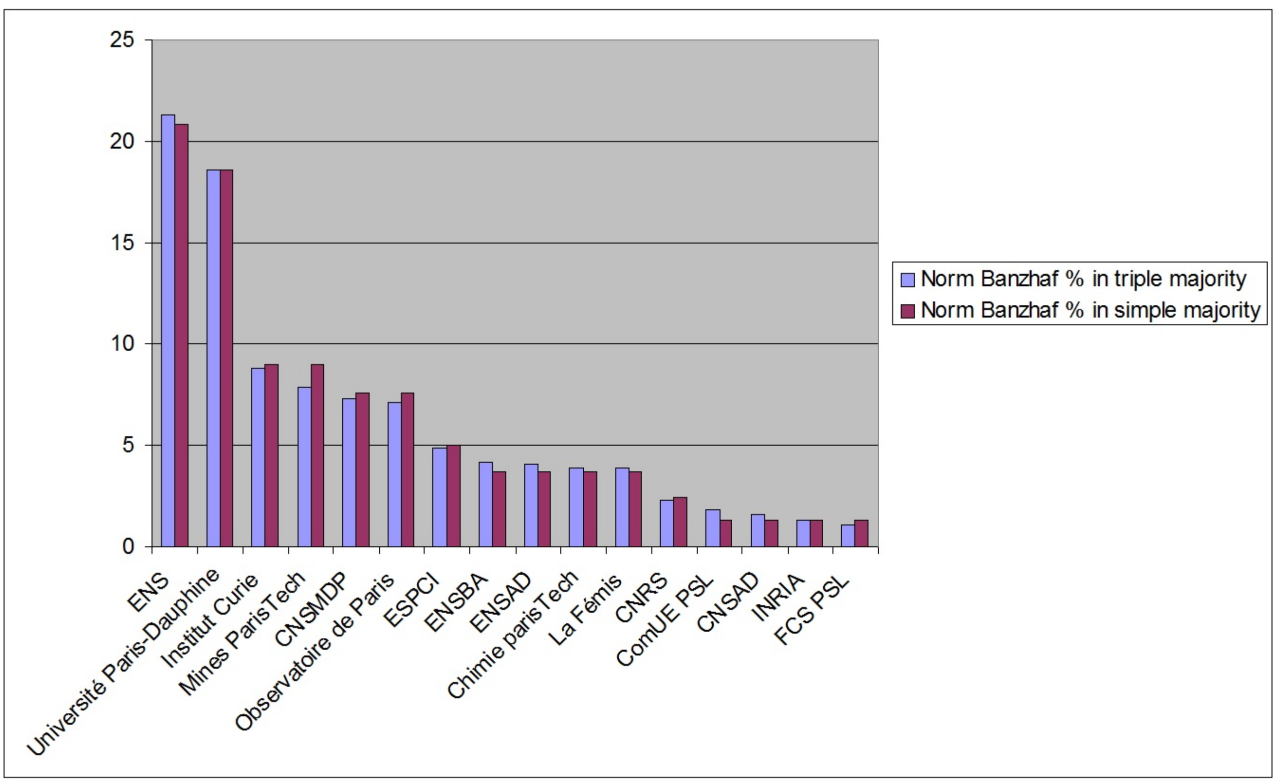


Figure 5.3: Comparison of the normalized Banzhaf index of the "ideal" triple-majority game $v^t$ (left column) and the Banzhaf index of the weighted majority game $v^w$ (right column) for each Institution of the *ComUE*.

The method proposed in [54] for the design of the electoral college of the *ComUE* should be seen as an attempt to provide an objective basis for the negotiation of the weights among the PSL Institutions. This is, at least, how it has been perceived by the members of the SC of the *ComUE* [174], where our proposition has been debated and finally approved under few modifications during the PSL-SC meeting held in Paris on April 21st, 2015.

## 5.4 Social capital

Although power indices are able to capture some aspects related to the ability of players to influence collective decisions, in general they fail to capture other aspects more related to the social interaction. For instance, *social capital* is one of the fundamental concepts in sociology [107]. Intuitively, we can think of social capital as the ability of individuals to gain benefits by utilizing their position in the society or, in other words, their connections in a social network [112]. Although it has been extensively studied in various bodies of the literature, there is no single definition or measure that exactly captures all facets of this concept.

Social capital is understood in the literature in two conceptually different way [96], called *group* and *individual* social capital. The former interprets social capital as the quality or performance of a given group of individuals in a social network and the latter interprets social capital as the value of an individual's social connections, which are seen as potential sources of information, power, or opportunities. While existing measures quantify each of the above types of social capital separately, most of them are limited to the aspects of social capital related solely to network topology and none of them sheds light on the interactions between the two types (group vs. individual) of social capital.

In [4] and [8] we provided an analysis from the literature of some models based on coalitional games that can be used to measure social capital on a micro scale, and that have proved their feasibility and efficiency even on macro situations where the number of nodes of a social network scales up dramatically. Basically, we start from the consideration that in order to evaluate and compare the effects of social capital, it is first necessary to define analytical methods for measuring characteristics related to social capital, like collective efficacy, psychological sense of community, neighbourhood cohesion and community competence [141].

In [8], we presented several applications from the literature that illustrate the fact that coalitional games allow for a richer description of agents relationships in a social network, where it is quite realistic to figure out the mutual influence of individuals in producing a certain social outcome. In network analysis, classical centrality measures [89, 91, 95, 172] are quite appropriate to compute the "importance" of nodes in situations where it is justified to make the assumption that nodes failures occur independently. Another strong assumption that justifies the use of classical centrality measures is that the consequence of the failure of each node in the system is important (for, instance it determines the collapse of the system) and it is the same for all nodes. On the contrary, in real-world social networks, assuming that the actions of the agents on the nodes are independent is not realistic at all. Similarly, the consequences of an action on the social system can be appreciated only if a consistent number of possibly connected agents take the same action. These important aspects of social interaction are taken into account by alternative game theoretic notions of centrality [145, 4].

For example, the Myerson value [150] (i.e. the Shapley value of the graph-restriced game) has been used in [124] to asses the social capital of individuals in a social network. Specifically, given a society of $N$ individuals, in [124], the social influence of each coalition $S \subseteq N$ is represented as a coalitional game $(N, v)$, and all possible relations among individuals by a social network $\langle N, E \rangle$. The authors in [124] compute the restricted game $w_E^v$ and consider the Shapley value of the difference $w_E^v - v$ as an index to reveal the influence of players on the outcome of the game. In other words, they interpret the Shapley value difference $\phi(w_E^v) - \phi(v)$ as a *social capital index* representing the differential power of players between the graph-restricted game $w_E^v$ and the game $v$ where all communication possibilities among players are feasible: the higher the difference $\phi(w_E^v) - \phi(v)$, the higher the social capital of individuals due to the existing possibilities of interaction in the social network.

## 5.5 Future directions

In future work related to the consolidation procedure for computer networks discussed in Section 5.2, it would be interesting to broaden our experimental studies over a wider set of network topologies and traffic matrices. In particular, it would be useful to further study the correlation between the criticality of nodes and different traffic matrices for any given topology. Another open point that deserves more attention, is the evaluation of the robustness, by considering the impact of unexpected faults or changes in the traffic conditions to an already consolidated network.

Dealing with the inverse Banzhaf index problem considered in Section 5.3, an interesting open question is related to the problem of measuring the loss of information in the transformation of a multiple majority weighted game (original game) into a new weighted voting game with the same voting powers. How "far" the new simple game is from the original one? is it possible to guarantee that certain *a priori*-selected winning coalitions in the original game remain winning also in the new one?

With respect to the problem of measuring the social capital in coalitional games as illustrated in Section 5.4, we are currently studying different notions of criticality for players. For instance, in [1] we look at a player $i$ that is not critical for a winning coalition $S \subseteq N$, $i \in S$ (that is, $S$ and $S \setminus \{i\}$ are both winning coalitions), but there exists in $S$ another player $j$, different from $i$, s.t. $S$ becomes losing when both the players leave. This "second order" notion of criticality of players in simple games is useful to better understand blackmail power of players in simple games, which is the possibility of players of threatening coalitions to cause them loss using arguments that are (only apparently) not justified.

# Chapter 6

# Bioinformatics and statistical analysis of biological data

## 6.1 Overview of the chapter

In this chapter, we introduce some of our recent contributions on the application of coalitional games and their solutions to measure the importance of biological factors/variables in producing certain biological or epidemiological effects, and, more in general, to the analysis of gene expression data.

The starting point of this research is the variety of innovative experimental technologies in medical research that in the last few decades has permitted the collection of a large amount of biological data, markers and other relevant factors at once. In the papers [15, 37, 21, 17, 19, 14, 13], we introduced and applied alternative coalitional games to the analysis of large data-sets of gene expression data. Gene expression data may be collected, for instance, by means of *microarray* technology [159, 123]. Within a single experiment of this sophisticated technology, the level of expression of thousands of genes can be estimated in a sample of cells under given conditions (genetic diseases, environmental expositions, pharmacologic treatments, etc.). Many different approaches have been proposed in literature to discover "central" genes in the huge amount of information provided by this technology.

In Section 6.2 we introduce and discuss *microarray games* [21, 15], a class of coalitional games aimed to asses the relevance of genes in regulating or provoking a condition of interest, and taking into account the observed relationships in all subgroups of genes. Via a dichotomization technique applied to gene expression data, it is constructed a game whose characteristic function takes values in the interval $[0, 1]$. The objective of such a game is to stress the relevance ("sufficiency") of groups of genes in relation to a specific condition. The definition of relevance index for genes is provided in terms of the Shapley value [170, 18], that is contextualized and justified as a relevance index by means of an axiomatic approach.

In Section 6.3, we introduce a method based on coalitional games (and published in [11]) to evaluate the "centrality" of genes in *co-expression networks*. The Myerson value for coalitional games is used to express the power of each gene in interaction with the others and to stress the centrality of certain hub-genes in the regulation of biological pathways of interest. The main improvement of this contribution, with respect the model discussed in Section 6.2, consists in a finer resolution of the genes' interaction, which is based on pairwise relationships of genes in a co-expression network. In addition, the new approach allows for the integration of *a priori* knowledge about genes playing a key function on a certain biological process.

Finally, in Section 6.4, we briefly introduce some of our recent papers in the domain of the statistical analysis of biological data of human RNA (Ribonucleic acid), mainly based on the publications [36, 38, 39, 40, 35] and, more recently, on [33, 31, 29, 32, 34, 30, 28, 27].

## 6.2 Microarray games

By means of modern technologies it is possible to consistently generate a matrix of *gene expression* data, in which the rows index the genes (i.e., the variable) and the columns index the study samples/experiments (e.g. several patients with a genetic disease), which are the observations or effects (for example, see [159] for a general introduction to the technology of *microarrays* and related statistical aspects).

In [21] and [65] we introduced a method based on coalitional games where the Shapley value has been proposed for inferring, from a matrix of gene expression data, the relevance of genes keeping into account their individual behaviours and their interactions when the biological system is studied under a condition of interest (e.g. a disease state, the exposure to environmental or therapeutic agents, etc.). According to this method, the frequency of *associations* of each subset of genes with a *condition* of interest has been described by means of a coalitional game (namely, a *microarray game*) where players are genes. The relevance of genes is assessed by means of the Shapley value of a microarray game: the higher the number attributed by the Shapley value to a certain gene in a given microarray game, the higher the relevance of that gene for the mechanisms governing the genomic effects of the condition under study.

We briefly introduce here the model introduced in [21]. Let $N = \{1, \dots, n\}$ be a set of genes. On a single microarray experiment on $N$, a sufficient requirement to realize in a subset (a *coalition of genes*) $S \subseteq N$ the association between a condition and an expression property is that all the genes showing that expression property belong to coalition $S$ (*sufficiency principle*). Different *expression properties* for genes might be considered (*e.g.*, abnormal expression, up- or down-regulation, etc). Moving to $k \geq 1$ microarray experiments on $N$, we refer to a Boolean matrix $\mathbf{B} \in \{0,1\}^{n \times k}$, where the Boolean values $0-1$ represent two complementary expression properties, for example the property of normal expression (coded by 0) and the property of abnormal expression (coded by 1). Let $\mathbf{B}_{\cdot j}$ be the $j$-th column of $\mathbf{B}$. We define the *support* of $\mathbf{B}_{\cdot j}$, denoted by $sp(\mathbf{B}_{\cdot j})$, as the set $sp(\mathbf{B}_{\cdot j}) = \{i \in N \mid \mathbf{B}_{ij} = 1\}$. The microarray game corresponding to $\mathbf{B}$ is the coalitional game $(N, v)$, where $v : 2^N \to I\!\!R_+$ is such that $v(T)$ is the rate of occurrences of the coalition $T$ as a superset of supports in $\mathbf{B}$; in formula, we define $v(T)$, for each $T \in 2^N \setminus \{\emptyset\}$, as the value

$$v(T) = \frac{|\Theta(T)|}{k}, \tag{6.1}$$

where $\Theta(T) = \{j \in K \mid sp(\mathbf{B}_{\cdot j}) \subseteq T,\ sp(\mathbf{B}_{\cdot j}) \neq \emptyset\}$, with $K = \{1, \dots, k\}$ and where $|\Theta(T)|$ is the cardinality of $\Theta(T)$. Finally, we define $v(\emptyset) = 0$.

**Example 16.** *Consider the boolean matrix* $\mathbf{B} \in \{0,1\}^{4\times 3}$ *such that*

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Then* $sp(\mathbf{B}_1) = \{2,3\}$, $sp(\mathbf{B}_2) = \{1,3\}$ *and* $sp(\mathbf{B}_3) = \{1,2,4\}$. *By equation (6.1), the corresponding microarray game* $(\{1,2,3,4\}, v)$ *is such that* $v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = v(\{1,4\}) = v(\{2,4\}) = v(\{1,2\}) = v(\{3,4\}) = 0$*;* $v(\{1,3\}) = v(\{2,3\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = v(\{1,2,4\}) = \frac{1}{3}$*;* $v(\{1,2,3\}) = \frac{2}{3}$, $v(\{1,2,3,4\}) = 1$. *The Shapley value of the microarray game* $(\{1,2,3,4\}, v)$ *is* $(\frac{5}{18}, \frac{5}{18}, \frac{1}{3}, \frac{1}{9})$.

One of the main objectives of the applications of coalitional games to biological problems, is the contextualization of the basic model of coalitional games in the biological domain, and the justification of game theoretic solutions as measures of the importance of the variables governing the biological system. For example, in [21] we provided an axiomatic characterization of the Shapley value on the class of microarray games, using five axioms with a genetic interpretation: 1) the *null gene* property says that a relevance index should attribute null relevance to genes that are never up- or down- regulated under a certain condition; 2) in order to bring smaller gene pathways into prominence, another reasonable property is that if two disjoint sets of genes are up- or down-regulated in a same rate of samples, then genes in the smaller set should receive a higher relevance index than genes in the bigger one (*Partnership Monotonicity* property); 3) The *partnership rationality* property and the 4) *partnership feasibility* property determine, respectively, a lower and an upper bound of the power of "partnerships" of genes; 5) finally, a special version of additivity, namely the *equal splitting* property, is used with a natural interpretation of giving the same reliability to different microarray experiments. In [21], we proved that the Shapley value is the unique relevance index which satisfies the five properties listed above on the class of microarray games.

In [37] we presented the first biological validation for the use of the Shapley value of microarray games as a relevance index for genes. Precisely, a set of genes involved in the pathogenesis of *neuroblastic tumors* has been selected in [37] according to the Shapley value of a microarray game defined on gene expression data from *neuroblastoma cells*. In [19] we presented and discussed an application of the method to a gene expression dataset concerning blood cells of 23 children from the region of Teplice (Czech Republic), a mining district characterized by high levels of airborne pollutants including carcinogens. For other applications on real data-sets see also [113, 176, 114].

The problem to compare the relevance of genes under two different conditions has been also studied in [19] and in [14]. In these papers we considered two groups of microarray experiments on the same set of genes $N = \{1, \ldots, n\}$, collected under two different conditions 1 and 2. A statistical procedure to test the null hypothesis that each gene in $N$ has the same Shapley value between the two conditions 1 and 2 is presented in [19] and is further discussed in [14]. Such a test procedure is based on a non-parametric Bootstrap method of re-sampling with replacement that we called in [19] *Comparative Analysis of Shapley value*.

Other game theoretic relevance indices for genes have been proposed in the literature. For example, in [13] we provided a characterization of the Banzhaf value [86] on the class of microarray games, and compared the results given by the Banzhaf and the Shapley value when they are applied to real data. The ranking of the genes according to the two values are, in general, significantly different and it can happen that the Banzhaf value divides the list of genes in very few classes of equal relevance [13, 142]. This happens because the relevance computed according to the Banzhaf value for a single player decreases exponentially (and not linearly, as in the Shapley's case) with respect to the size of the support of a certain column (see relation (6.1)); consequently, due to round off-errors, the contribution of large supports to the Banzhaf value of single genes is practically null.

Another application of coalitional games deals with the machine learning technique called *feature selection* (also known as *variable selection*), which denotes a family of algorithms aimed to select a subset of relevant variables (or *features*) for building robust learning models. In the computational biology domain, the technique is also called *gene selection*, and it is usually applied to detect genes with high discriminative power (e.g., genes that can successfully be used to generate *classification rules* which are able to predict, as accurately as possible, the true class of biological samples). In [17], we introduced *classification games* to estimate the contribution of a feature for the classification task. Using the classification games in [17], where genes are again in the role of players, we represented the power of groups of genes to classify samples into the right classes (for instance, the class of normal tissues or the class of tumour tissues). Classification games turn out to be closely related to microarray games and, on some numerical examples, the Shapley value and the *Interaction Index* (see, for instance, [126, 127]) have been studied as criteria for gene selection.

## 6.3 Game centrality on biological networks

Another interesting field related to the analysis of genetic data deals with *gene networks*, that are increasingly used to explore the system-level functionality of proteins and genes [131]. In [11] we introduced a new method based on coalitional games to evaluate the centrality of genes in *co-expression networks* [1] keeping into account the interactions among genes.

Following the approach introduced in [11], an *association game* $(N, v)$ is first defined, where $N$ is the set of genes under study (for instance, analysed by means of a gene expression data-set) and the characteristic function $v$ assigns a worth to each coalition of genes $S \subseteq N$ representing the overall magnitude of the "interaction" between the genes in $S$ and a given set of *key-genes* (e.g., a set of genes known *a priori* to be involved in biological pathways related to chromosome damage). More precisely, suppose to have a set $K$ of *key-genes* and let $I \subseteq \{\{i, k\} | i \in N, k \in K\}$ be the set of *interactions* between genes in $N$ and the key-genes in $K$. Given a set of genes $S \subseteq N$, the higher the number of key-genes which interact with genes in $S$, the higher the likelihood that genes in $S$ are also involved in the regulation of the biological process of interest. The map $v : 2^N \to \mathbb{N}$ assigning to each coalition $S \in 2^N \setminus \{\emptyset\}$ the number $v(S)$ of key-genes in $K$ which only interact (in $I$) with genes in $S$ (again, by convention, $v(\emptyset) = 0$) is the *association game* corresponding to $(N, K, I)$.

**Example 17.** *Consider a set of genes $N = \{1, 2, 3, 4\}$, a set of key-genes $K = \{a, b, c\}$ and a set of interactions $I = \{\{1, a\}, \{1, b\}, \{3, b\}, \{3, c\}, \{4, c\}\}$, as depicted in Figure 6.1. This information is sufficient to calculate the corresponding association game: if $S = \{1, 2, 3\}$, for example, we have that key-genes $a$ and $b$ only interact with genes in $S$, whereas the key-gene $c$ interact with gene 3 but also with gene 4, which is not in $S$; so $v(1, 2, 3) = 2$. Similarly, we can define the entire characteristic function of the association game $(N, v)$, which is such that $v(\emptyset) = v(2) = v(3) = v(4) = v(2, 3) = v(2, 4) = 0$, $v(1, 3) = v(1, 2, 3) = 2$, $v(1, 3, 4) = v(1, 2, 3, 4) = 3$ and $v(S) = 1$ for all the remaining coalitions.*

In order to study the cascade of activation/deactivation among genes, in [11] we considered a second game, where gene interaction is restricted to the connections within an associated undirected graph $\langle N, E \rangle$, the nodes

[1] Roughly speaking, a co-expression network is a network where the node correspond to the genes, and a link between two genes is established if they are simultaneously expressed in a dataset (see, for instance, [175] for more details on co-expression networks).

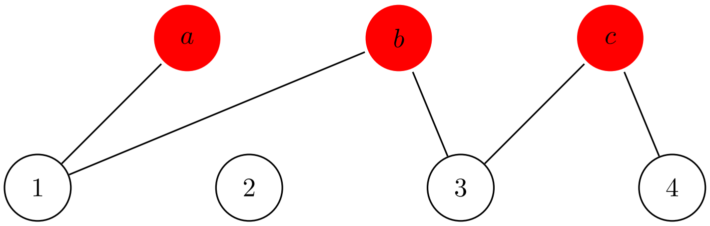


Figure 6.1: A set of genes $N = \{1, 2, 3, 4\}$ and their interactions with a set of key-genes $K = \{a, b, c\}$ (red nodes).

of the graph being the genes. The set of edges $E$ indicates interaction ties between pairs of genes, i.e. a set $\{i, j\} \subseteq N$ is an element of $E$ if and only if $i$ and $j$ have an interaction (for instance, they are significantly co-expressed). Following the approach in [150], in [11] we used the structure of network $\langle N, E\rangle$ to define the graph-restricted game $(N, w_E^v)$ (see Section 1.2.2).

**Example 18.** *Consider the association game $(N, v)$ introduced in Example 17 and the gene network $(N, E)$ where $E = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.*

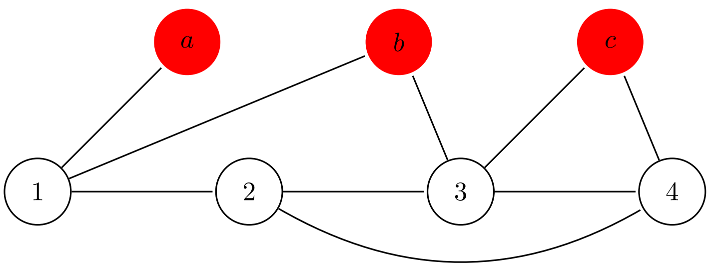


Figure 6.2: The situation of Example 17 with the interaction among the genes 1, 2, 3 and 4.

*If we consider coalition $S = \{1, 3\}$, for example, we have that key-genes $a$ and $b$ only interact with genes in $S$, but node 1 is not connected to 3 in $\langle S, E(S)\rangle$; so, $w_E^v(1, 3) = v(1) + v(3) = 1$.*

*The graph-restricted game $(N, w_E^v)$ is such that $w_E^v(3, 4) = w_E^v(2, 3, 4) = 1$, $w_E^v(1) = w_E^v(1, 2) = w_E^v(1, 4) = w_E^v(1, 2, 4) = w_E^v(1, 3) = 1$, $w_E^v(1, 2, 3) = w_E^v(1, 3, 4) = 2$, $w_E^v(1, 2, 3, 4) = 3$ and $w_E^v(S) = 0$ for all the remaining coalitions.*

The difference of the Shapley values computed on the two coalitional games $(N, v)$ and $(N, w_E^v)$ is considered in [11] as a gene *centrality measure*. Precisely, the *gamma value* $\gamma(v, E)$ is defined by

$$\gamma_i(v, E) = \phi_i(w_E^v) - \phi_i(v), \tag{6.2}$$

for each $i \in N$, where $\phi(v)$ is the Shapley value of the association game $v$ and $\phi(w^v)$ is the Shapley value of the corresponding graph-restricted game $w_E^v$ (i.e., the Myerson value). According to relation (6.2), genes with strictly positive $\gamma$ represent those genes with a positive differential power between the graph-restricted game and the association game.

**Example 19.** *Consider the association game $(N, v)$ introduced in Example 17 and the gene network of Example 18. According to relation (1.3), we can compute the Shapley value in the two games $\phi(v) = (\frac{3}{2}, 0, 1, \frac{1}{2})$ and $\phi(w_E^v) = (\frac{4}{3}, \frac{1}{3}, \frac{5}{6}, \frac{1}{2})$. Thus, by relation (6.2), the gamma value is $\gamma(v, E) = (-\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, 0)$. Note that gene 2 is the unique gene with strictly positive centrality according to $\gamma$.*

An important issue in the analysis of gene data using coalitional games and power indices is the computational one. In some cases, the calculation of the Shapley value can be easy, independently from the number of genes involved in the analysis. This is the case, for instance, of microarray games. In fact, a microarray game $(N, v)$ defined by relation (6.1) can be equivalently formulated via the following relation

$$\bar{v}(T) = \sum_{j=1,\ldots,k} \frac{u_{sp(\mathbf{B}_j)}(T)}{k}, \qquad T \in 2^N \setminus \emptyset, \tag{6.3}$$

where $(N, u_{sp(\mathbf{B}_j)})$ is the unanimity game on the set $sp(\mathbf{B}_j)$, and the Shapley value of game $v$ can be easily calculated in view of relation (1.5) (see [21] for more details). The very simple formulation of the Shapley value for microarray games follows from the particular definition of the worth of a coalition of genes, which is based on the underlying assumption behind the aforementioned sufficiency principle, which says that all the abnormally expressed genes are, as a whole, equally responsible for the "disease" (or another condition of interest) in each experiment. An extension of microarray games, which allows for using weighted majority games in order to differentiate the role of genes within the group of abnormally expressed ones, has been presented in [142].

In other cases, the problem of understanding whether an efficient algorithm to compute the exact Shapley value exists remains open, and the only possibility so far is adopting approximation methods. This is the case, for instance, of the method introduced in [11] and shortly discussed in this section. For gene networks, in fact, it is easy to show that the characteristic function $v$ of an association game can be written as a sum of unanimity games according to the following relation,

$$v = \sum_{k \in K, N_k \neq \emptyset} u_{N_k}, \tag{6.4}$$

where $N_k = \{i \in N | \{i, k\} \in I\}$ denotes the set of genes in $N$ which have a strong interaction with a key-gene $k \in K$. A natural decomposition of a graph-restricted game based on the reformulation of the association game given in (6.4) can be also provided (see [11] for further details), but it requires to consider all minimal components containing $N_k$, for each key-gene $k \in K$, and all of their combinations (see equation (6) in [11]). However, as the number of minimal components in a graph can be very large (especially for graphs generated from realistic data-sets with thousands of variables) this option is computationally too expensive. For practical computational reasons, in [11] we limited the decomposition of the graph-restricted game to the "smallest" minimal components connecting the *most associated genes* (i.e., genes that directly interact with key-genes) on a graph, which are those minimal components where the most associated genes are connected to each other by a shortest path. This procedure was used to calculate an approximate $\gamma$ centrality for a large gene network with 201 nodes and 2083 edges. Of course, the price for using the method based on shortest paths was that genes outside those particular paths received a null value of approximated $\gamma$ centrality, even if their exact $\gamma$ value was not null.

Looking at the axiomatic approach for applications to gene networks, in [11] we noticed that both the gamma value and its approximated version satisfy the intuitive property requiring that smaller pathways of genes are more central, since they provide a less complex explanation of the observed network of interactions (this property has been named in [11] *Total Aggregation Monotonicity*).

## 6.4 Statistical analysis of biological data

A non-negligible part of my research has been devoted to the statistical analysis of biological data in a framework of international collaborations with different cancer research laboratories. The majority of our contributions in the domain of the statistical analysis of biological data deals with the analysis of data of *human RNA* (*Ribonucleic acid*). More precisely, most of the papers we published in this domain, focus on the statistical analysis, the design and the quality control of gene expression data from experiments concerning the study of genetic disorders, and in particular the research on genetic mechanisms regulating the *neuroblastoma*, a rare malignant tumour, and the most common solid extra-cranial cancer in childhood [36, 38, 39, 40, 35].

More recently, our goal in [33] was to identify proteins expressed by the metastatic cells of neuroblastoma, which may be relevant to prognostic and therapeutic purposes. Metastases in the bone marrow, are important prognostic factors in patients with neuroblastoma. Sixty-six children over 18 months, with a diagnosis of "stage 4 neuroblastoma" were included in the study presented in [33]. Among other results, we have shown that the *calprotectin*, a potent inflammatory protein, and another protein called *HLA-G* may represent new biomarkers and/or targets for intervention therapy in patients with high-risk neuroblastoma. In [31], we compared the expression data of thousands of genes of children with localized and metastatic neuroblastoma against the data of healthy children. Among the genes analysed, we found that the expression of *CXCL12* gene is almost completely cancelled in patients with a metastatic disease. To investigate the role of patients' age in tumour aggressiveness in [29] we performed array-CGH (roughly speaking, an experimental technique aimed at finding aberrations in the DNA) and expression profiles of three groups of metastatic neuroblastoma. In [32] we tried to identify new molecular prognostic markers with the objective to better predict relapse risk estimate for children with high-risk metastatic neuroblastoma and we have shown that patients with less than three *segmental copy number aberrations* in their tumors represent a molecular subgroup defined with a high probability of survival observed in the patients group. In the article [34], we studied *multipotent migratory cells* of the *embryonic neural crest*.

We also elaborated biological studies related to other diseases different from the neuroblastoma. In [30], we evaluated the effects of a certain treatment (*leflunomide* combined with a low dose of *prednisone*), on the expression of genes responsible for inflammation in the peripheral blood mononuclear cells of patients with rheumatoid arthritis, a chronic inflammatory degenerative disease. In [28], we analysed the relationship between the the downregulation of a specific gene (namely, the *DKK3* gene) and the *medulloblastoma*, an highly malignant embryonic tumour of the cerebellum which accounts for 20% of all intracranial tumours of childhood. We showed for the first time that $DKK3$ gene is significantly downregulated on different groups of Medulloblastoma patients as compared to normal cerebellum. More recently, in [27], we studied the storage time impact on the transcriptome (i.e., the set of all RNA molecules, including non-coding RNA) transcribed in one cell or a population of cells of slowly frozen cryopreserved human oocytes. Oocyte cryopreservation is a largely-used technique for storage of surplus oocytes in vitro fertilization (IVF) cycles or to allow flexibility if an IVF cycle has to be halted, as well as for fertility preservation (i.e., women at risk of losing fertility because of endometriosis, premature ovarian failure, or gonadotoxic therapies) and oocyte donation programs. For the first time, in [27], we demonstrated that the length of cryostorage has no effect on the gene expression profile of human oocytes.

## 6.5 Future directions

With respect the models discussed in Section 6.2, an interesting question concerns the statistical significance of the Shapley value of microarray games computed on real data-sets. In risk attributions, [133] proposed a measure of the uncertainty of the Shapley value based on its probabilistic interpretation, defining the corresponding uncertainty as the variance of the marginal contributions. Moreover, making the assumption that the marginal contributions are distributed approximately normally (which is likely to happen in large games), then the statistical results for normal distributions are applicable together with the standard test of significance. However, these aspects, with the exception of the examples provided in [133], seem to be neglected from the literature of coalitional games based on real and large data-sets.

The problem of understanding whether an efficient algorithm to compute the exact Shapley value of gene networks presented in Section 6.3 remains open, and the only possibility so far is adopting approximation methods. We are also exploring a different approach to the problem of computing the relevance of genes on networks using the idea of game theoretic centrality generalizing classical notions of centrality as recently introduced in [145].

We want to stress the fact that some of the approaches illustrated in this chapter still need an extensive comparative analysis of their performance with respect to other methods already in use in the literature of reference. This is the case, for example, of classification games [17], where it would be interesting to compare the Shapley value and the Interaction Index with more classical feature selection methods for classification problems.

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