LIKELIHOOD CONSISTENCY

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A new method is presented for measuring beliefs/likelihoods under uncertainty. It will simplify:

- preference axiomatizations (SEU, PS, CEU);
- quantitative “belief” measurements;
- testing and characterizing qualitative properties.

1. Introduction and History

Savage (1954): First full-blown decision model for uncertainty (restricted to SEU).

Savage'54 is uncertainty-oriented:

- measurement of uncertainty/beliefs was central;
- richness (continuity) imposed on state space;
- measurement of utility is by-product.

Following Debreu (1959) and Arrow (1963), most modern analyses of uncertainty are outcome-oriented:

- measurement of utility is first (cf. micro-economics);
- richness (continuity) imposed on outcome space;
- measurement of uncertainty is indirect.

Uncertainty-oriented references:

1. Expected utility:
   Savage ‘54, von Neumann-Morgenstern ‘44
   (+ Herstein & Milnor ‘53 + Jensen ‘67).

2. Nonexpected utility:
   - Uncertainty: Gilboa ‘87,
     Machina & Schmeidler ’92,
     Grant ’95,
     Epstein & Zhang ’01,
     Kopylov ‘04
   - Risk: Abdellaoui ‘02,
     Nakamura ‘95.

For uncertainty (risk and "ambiguity"):
uncertainty-oriented is most natural.

We do it systematically.
Most general, and simplest, axioms you ever saw!

Extra, mathematical, reason for
simplicity + generality:
It naturally exploits set-theoretic structure on the state space.

The idea of our “revealed-likelihood” method can be recognized in Gilboa’s (’87) axiom P2*; our work builds on it, and aims to give intuition to it.

2. Notation

event A event A^c

A **A^c**

γ β Act, yielding γ under A,
β under A^c.

outcome γ outcome β

Convention: γ is a good outcome, β a bad one,
$\gamma \succ \beta$.

Using one matrix to denote several acts:

A	A^c	
γ	β	} Act, yielding γ under A, β under A^c.
γ'	β'	} Act, yielding γ' under A, β' under A^c.

3. Basic Likelihood Measurements

Point of departure is outcome level β.

Choose between two improvements. (remember: $\gamma \succ \beta$)

	A	A^c		B	B^c	
	β	β	$=$	β	β	
Gamble on A	γ	β	$\prec$	γ	β	Gamble on B

Def. If strict preference $\succ$ for left gamble then: $A \succ_b B$;

Def. If indifference $\sim$ then: $A \sim_b B$;

Def. If strict preference $\prec$ for right gamble then: $A \prec_b B$.

Under subjective expected utility (SEU):

$$\begin{array}{cc} A & A^c \\ \beta & \beta \\ \gamma & \beta \end{array} \quad = \quad \begin{array}{cc} B & B^c \\ \beta & \beta \\ \gamma & \beta \end{array}$$

SEU-gain is $P(A)\big(U(\gamma) - U(\beta)\big)$ SEU-gain is $P(B)\big(U(\gamma) - U(\beta)\big)$

SEU:

$$A \succ_b B \Leftrightarrow P(A) > P(B)$$
$$A \sim_b B \Leftrightarrow P(A) = P(B)$$
$$A \prec_b B \Leftrightarrow P(A) < P(B)$$

Conclusion: Basic revealed likelihood relations elicit probability orderings.

Avoid contradictions in your measurements:

Savage's P4 excludes them for $\succcurlyeq_b$. We rename (and weaken) P4 as basic likelihood consistency:

$$\text{NOT } [A \sim_b B \text{ and } A \succ_b B]$$

In preferences, for all $\gamma \succ \beta$ and $\gamma' \succ \beta'$:

$$
\text{NOT} \left(\begin{array}{lcccccc}
 & \mathbf{A} & \mathbf{A^c} & & \mathbf{B} & \mathbf{B^c} & \\
 & \beta & \beta & = & \beta & \beta & \\
 & \gamma & \beta & \sim & \gamma & \beta & \\
 & \beta' & \beta' & = & \beta' & \beta' & \\
\text{and} & \gamma' & \beta' & \succ & \gamma' & \beta' &
\end{array} \right.
$$

Obviously:

Lemma. SEU $\Rightarrow$ basic likelihood consistency. □

Many other models imply
basic likelihood consistency also.

4. The Likelihood Method

- Not identity, but equivalence, as point of departure.
- Under A^c not β, but any outcomes of any act f result.
- Under B^c not β, but any outcomes of any act g result.

We require ~ (instead of =) as point of departure.

$$
\begin{array}{ccccc}
A & A^c & & B & B^c \\
\beta & f & \sim & \beta & g \\
\hline
\gamma & f & \prec & \gamma & g
\end{array}
$$

A is revealed more likely than **B**

Def. If strict preference $\succ$ for left gamble then: A $\succ$ B;

Def. If indifference ~ then: A ~ B;

Def. If strict preference $\prec$ for right gamble then: A $\prec$ B.

Will definitions always reveal a sensible likelihood ordering?

Or will they reveal contradictions (and then signal problems)?

Let us try, work on them, and see where they take us.

Comparison of new with preceding revelation method:

Basic elicitation:	New elicitation:
$\begin{array}{ccccc} A & A^c & & B & B^c \\ \beta & \beta & \sim & \beta & \beta \\ \gamma & \beta & ? & \gamma & \beta \end{array}$	$\begin{array}{ccccc} A & A^c & & B & B^c \\ \beta & f & \sim & \beta & g \\ \gamma & f & ? & \gamma & g \end{array}$
SEU-gain is P(A)(U(γ)–U(β)) ; SEU-gain is P(B)(U(γ)–U(β))	SEU-gain is P(A)(U(γ)–U(β)) ; SEU-gain is P(B)(U(γ)–U(β))
Strict preference $\succ$ for left gamble then: A $\succ_b$ B; etc.	Strict preference $\succ$ for left gamble then: A $\succ$ B; etc.

SEU:

$$A \succ_b B \iff P(A) > P(B)$$
$$A \sim_b B \iff P(A) = P(B)$$
$$A \prec_b B \iff P(A) < P(B)$$

Also if we drop the subscripts b.

Conclusion: Revealed likelihood orderings $\succcurlyeq$ elicit probability orderings, as do the basic orderings $\succcurlyeq_b$.

Under SEU, revealed likelihood orderings give desirable results.

Let us now investigate a general criterion, the analog of Savage's P4, that revealed likelihood orderings should not run into contradictions.

Likelihood consistency:

NOT [A $\sim$ B and A $\succ$ B]

Same as basic likelihood consistency ($\approx$ P4), but with subscript b dropped.

Directly in terms of preferences:

For all $\gamma \succ \beta$, $\gamma' \succ \beta'$, f, f', g, g',

$$\text{NOT}\left\{\begin{array}{lcccccc} & \mathbf{A} & \mathbf{A^c} & & \mathbf{B} & \mathbf{B^c} \\ & \beta & f & \sim & \beta & g \\ & \gamma & f & \sim & \gamma & g \\ & & & & & \\ & \beta' & f' & \sim & \beta' & g' \\ \text{and} & \gamma' & f' & \succ & \gamma' & g' \end{array}\right.$$

Lemma. SEU $\Rightarrow$ likelihood consistency.

General question: When, besides SEU, is revealed likelihood free of contradictions; i.e., when is our measurement instrument OK?

Questions :
We assume “usual things,”

- weak ordering,
- monotonicity on outcomes/events as much as you want.
- continuity on outcomes/events as much as you want.

Then: how strong is likelihood consistency?
Does it imply more than Savage’s P4?
If so, what?

Hypothesis 1. Likelihood consistency $\Leftrightarrow$
P4.
(No addition; implies no more than dominance.)

Hypothesis 2. Likelihood consistency $\Leftrightarrow$
probabilistic sophistication.

Hypothesis 3. Likelihood consistency $\Leftrightarrow$
Choquet expected utility.

Hypothesis 4. Likelihood consistency $\Leftrightarrow$
Subjective expected utility.

Likelihood consistency:

NOT [A $\sim$ B and A $\succ$ B]

Same as basic likelihood consistency (≈ P4), but with subscript b dropped.

Directly in terms of preferences:

For all $\gamma \succ \beta$, $\gamma' \succ \beta'$, f, f', g, g',

$$\text{NOT}\left\{\begin{array}{lccccc} & \mathbf{A} & \mathbf{A^c} & & \mathbf{B} & \mathbf{B^c} \\ & \beta & f & \sim & \beta & g \\ & \gamma & f & \sim & \gamma & g \\ & & & & & \\ & \beta' & f' & \sim & \beta' & g' \\ \text{and} & \gamma' & f' & \succ & \gamma' & g' \end{array}\right.$$

Answer:

Hypothesis 4 is correct!

Likelihood consistency $\Leftrightarrow$ subjective expected utility.

(Bayesians: Good news! A new foundation of Bayesianism!
Non-Bayesians: Disappointing? Instrument doesn't bring anything interesting? Please wait. NonEU will result from natural modifications. Comes later.)

For now, we get very simple axiomatization of SEU!

To state it, we turn to technical axioms. Here uncertainty-orientedness also gives improvements.

Weakening Savage's P6:

If both preferences $\prec$ hold,

$$\begin{pmatrix} D & A & L \\ f & \beta & f \end{pmatrix} \prec g \prec \begin{pmatrix} D & A & L \\ f & \gamma & f \end{pmatrix}$$

then for some partition A_γ, A_β of A:

$$g \sim \begin{pmatrix} D & A_\gamma & A_\beta & L \\ f & \gamma & \beta & f \end{pmatrix}$$

"If improving on the whole event is too much to get ~, then you can improve on a subevent."

Archimedean axiom:

The can be no infinite sequences of equally likely disjoint nonnull events.

Monotonicity:

f $\succcurlyeq$ g if f(s) $\succcurlyeq$ g(s) for each state s.

Only weak form needed!

That's all. Outcomes can be general; only richness on state space.

Theorem [generalizing Savage ’54].
Assume “nondegeneracy“ and solvability, with acts measurable w.r.t. a Mosaic of events. Then

Subjective expected utility holds

if and only if:

(i) Weak ordering;
(ii) Monotonicity;
(iii) Archimedeanity;
(iv) Likelihood consistency.

Further,
P is unique;
U is unique up to unit and origin.

More general than Savage in a structural sense:

Every structure satisfying Savage's axioms also satisfies our axioms. Not vice versa.

We can handle:
- general algebras (Kopylov 2004).
- Equally-likely finite state spaces (with atoms);
- No richness in outcomes.

We are not more general in a "logical" sense.
Savage's theorem is not a corollary of ours.
Axioms *per se* are logically independent.

Strange thing:

Where did cardinality get in???

Outcome-oriented approaches always have to do something extra (linearity, mixtures, midpoints, tradeoffs, ...) to get cardinality in. We didn't.

Likelihood consistency seems to concern only ordinal comparisons of likelihood.

How come??

Explanation:

Likelihood consistency is the dual version of tradeoff consistency for outcomes. The latter concerned differences of utility.

How use likelihood revelations for nonEU?

For those of you who like nonEU:
Every violation of SEU can be signaled through a contradictory likelihood revelation.
We can classify the contradictions,
exclude the ones we don’t like
(say the comonotonic violations)
and allow for others and, thus,
get characterizations and measurements of all those models.

So, for generalizing SEU:
we have to restrict the permitted likelihood revelations.

5. Violations of Llc Consistency and Rank-Dependence

Yet another axiomatization of rank-dependence!?
Well-known; so novelty of our general measurement instrument for uncertainty will be clearer.
You can judge it on its didactical merits, i.e. how easy it is.
If you don't know rank-dependence yet:
A very simple explanation is coming up.
We replace comonotonicity restrictions by "ranking position" conditions.

Example. Ellsberg paradox. Urn:

30 R 60 B/Y (unknown proportion)

B	Y	R		R	Y	B
\$	0	0	≺	\$	0	0
\$	\$	0	≻	\$	\$	0

We can get ε > 0 s.t.:

B	Y	R		R	Y	B
\$1+ε	0	0	~	\$	0	0
\$1+ε	\$	0	≻	\$	\$	0

Reveals **Y** ≻ **Y**. Signals that there is a problem!
This and **Y** ~ **Y**: likelihood consistency is violated.
Signals that SEU is violated.

We investigate more closely what is going on, from perspective of rank-dependence. Familiar to many of you. Therefore, suited to demonstrate the applicability of revealed likelihood, and uncertainty-orientedness.

30 R   60 B/Y

B	**Y**	**R**		**R**	**Y**	**B**
$	0	0	$\prec$	$	0	0
$	$	0	$\succ$	$	$	0

Left acts: **Y** ranked worse than **B**.
Right acts: **Y** ranked worse than **R**.

Decision weight of **Y** depends on ranking position. **Y** "adds" more to **B** (giving known probability) than to **R**. This may explain why **Y** is weighted more for left acts than for right acts.
Basic idea of rank-dependence: the weight of an event can depend on its "ranking position," i.e. the event yielding better outcomes.
Notation: $\mathbf{Y}^{\mathrm{B}} \succ \mathbf{Y}^{\mathrm{R}}$.

In general:

Because weight of A depends on its ranking position, we should specify ranking position and write A^D instead of A. A^D is ranked event.

Let us reconsider revealed likelihoods from this perspective.

D	A	L		D'	B	L'
f	β	f	~	g	β	g
f	γ	f	$\prec$	g	γ	g

increasingly preferred outcomes

increasingly preferred outcomes

A^D is revealed more likely than $B^{D'}$.

Def. If strict preference $\succ$ for left gamble then: $A^D \succ B^{D'}$;

Def. If indifference ~ then: $A^D \sim B^{D'}$;

Def. If strict preference $\prec$ for right gamble then: $A^D \prec B^{D'}$.

Again, the concept of study should be ranked events A^D, and not just events.

Our relation directly reveals decision-weight orderings of ranked events!

Lemma. Under Choquet expected utility:

$A^D \succ B^{D'} \quad \Rightarrow \quad \pi(A^D) \; > \; \pi(B^{D'})$, i.e.

$$\pi(A^D) = \frac{W(A \cup D)}{W(D)} > \frac{W(B \cup D')}{W(D')} = \pi(B^{D'})$$

$A^D \sim B^{D'} \quad \Rightarrow \quad \pi(A^D) \; = \; \pi(B^{D'})$

$A^D \prec B^{D'} \quad \Rightarrow \quad \pi(A^D) \; < \; \pi(B^{D'})$. $\square$

Ellsberg is fine now.

Contradictory $Y \succ Y$ has been replaced by noncontradictory $Y^B \succ Y^R$; $\pi(Y^B) > \pi(Y^R)$.

Let us reconsider likelihood consistency, now for ranked events.

Rank-dependent likelihood consistency:

$$[A^{D} \sim B^{D'} \text{ and } A^{D} \succ B^{D'}]$$

cannot be.

Necessary for

Choquet expected utility.

Directly in terms of preferences:

For all $\gamma \succ \beta$, $\gamma' \succ \beta'$, f, f', g, g',

$$\text{NOT}\ \left\{\begin{array}{ccccccc} D & A & L & & D' & B & L' \\ f & \beta & f & \sim & g & \beta & g \\ f & \gamma & f & \sim & g & \gamma & g \\ f' & \beta' & f' & \sim & g' & \beta' & g' \\ f' & \gamma' & f' & \succ & g' & \gamma' & g' \end{array}\right.$$

increasingly preferred outcomes

increasingly preferred outcomes

Gilboa '87
Theorem [generalizing ~~Savage '54~~].
Assume "nondegeneracy" and solvability, with acts measurable w.r.t. a general algebra of events. Then

Choquet
~~Subjective~~ expected utility holds

if and only if:

(i) Weak ordering;
(ii) Monotonicity;
(iii) Archimedeanity;
(iv) likelihood consistency.
Rank-dependent

Further,
W ~~P~~ is unique;
U is unique up to unit and origin.

Gilboa '87

More general than ~~Savage~~ in structural sense
and also in logical sense:
His assumptions and axioms directly imply ours.

Not vice versa.

We can handle:

general algebras (~~Kopylov earlier & more general here~~).

Equally-spaced finite state spaces (with atoms).

If no atoms, then still no convex-rangedness of ~~P~~.
W

~~We are not more general in a "logical" sense. Savage's theorem is not a corollary of ours. Axioms per se are logically independent.~~

Likelihood consistency aims to popularize Gilboa's P2*.

Applications for Rank-Dependent Models

Rank-dependent revealed likelihood directly observes orderings of decision weights, and is, therefore, a useful tool for analyzing properties of Choquet expected utility.

Better-suited than earlier tools because uncertainty-oriented.

Quantitative measurements of capacity W:

Take "equally likely" partition $A_1,\ldots,A_n$ such that $A_i^{A_1 \cup \ldots \cup A_{i-1}} \sim A_1^b$ for all i.

All have decision weight 1/n.

Such equally likely ("uniform") partitions could not be defined easily heretofore.

Testing qualitative properties of capacity W:

Convexity of W falsified if
$A^{D} \succ A^{D \cup E}$.

Characterizing qualitative properties of W:

W is convex
iff
never $A^{D} \succ A^{D \cup E}$.

W_2 more convex/pessimistic/amb.av. than W_1 if
$(A \cup F)^{D} \sim_1 A^{D \cup E} \Rightarrow (A \cup F)^{D} \preccurlyeq_2 A^{D \cup E}$;

etc.

Applications to Other Studies of Ambiguity

Machina & Schmeidler '92, probabilistic sophistication.
Let $\gamma \succ \beta$.

Not:

$$\underbrace{\begin{array}{cccccccc} \mathbf{A} & \mathbf{B} & \mathbf{R} & & \mathbf{B} & \mathbf{A} & \mathbf{R} \\ \beta & \beta & h & \sim & \beta & \beta & h \\ \gamma & \beta & h & \succcurlyeq & \gamma & \beta & h \end{array}}_{A \succcurlyeq_{ms} B}$$

$$\underbrace{\begin{array}{cccccccc} \mathbf{A} & \mathbf{B} & \mathbf{R} & & \mathbf{B} & \mathbf{A} & \mathbf{R} \\ \beta' & \beta' & h' & \sim & \beta' & \beta' & h' \\ \gamma' & \beta' & h' & \prec & \gamma' & \beta' & h' \end{array}}_{A \prec_{ms} B}$$

Epstein & Zhang 2001:
T is linearly unambiguous if, for all $\gamma \succ \beta$,

NOT:

$$\underbrace{\begin{array}{cc} \mathbf{T} & \mathbf{R} \\ \beta & f \\ \gamma & f \end{array} \begin{array}{c} \\ \preccurlyeq \\ \succ \end{array} \begin{array}{cc} \mathbf{T} & \mathbf{R} \\ \beta & g \\ \gamma & g \end{array}}$$

Same for T^c

I.e., NOT: $T \succ T$

Epstein & Zhang 2001:
T is unambiguous if, for $\gamma \succ \beta$,

NOT:

$$\underbrace{\begin{array}{cccc} T & A & B & R \\ \beta & \lambda & \mu & h \\ \gamma & \lambda & \mu & h \end{array} \begin{array}{c} \\ \preccurlyeq \\ \succ \end{array} \begin{array}{cccc} T & A & B & R \\ \beta & \mu & \lambda & h \\ \gamma & \mu & \lambda & h \end{array}}$$

Same for T^c

I.e., not: $T \succ T$ if measured under … specify the restrictions such and such …

6. Conclusion

For studying uncertainty, uncertainty orientedness seems to be optimal.
The likelihood Method: general tool for measuring uncertainty.

Everything becomes nicer:
- preference axiomatizations;
- quantitative measurements;
- testing and characterizing qualitative properties;

We showed some applications:
- Generalizing and simplifying Savage's '54 SEU.
- Generalizing and simplifying Gilboa's '87 CEU.
- Quantitative measurements of capacities.
- Transparent characterization of convex capacities and more-convex-than.
- New interpretations of probabilistic sophistication of M&S'92, unambiguity of E&Z'01.