Fuzzy Preference Structures

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1. Introduction

- “Alternative a is at least as good as alternative $b \ldots$"
  - Boolean: classical YES/NO
  - Discrete: finite totally ordered set of (linguistic) values $\mathcal{L}$

$$\text{None} \preceq \text{Very Low} \preceq \text{Low} \preceq \text{Medium}$$

$$\preceq \text{High} \preceq \text{Very High} \preceq \text{Perfect}$$

  - Fuzzy: evaluation scale is a compact real interval

2. Boolean preference structures

- Preference structure: result of
  - the pairwise comparison
  - of a set of alternatives A
  - by a decision maker

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  - the pairwise comparison
  - of a set of alternatives A
  - by a decision maker

- Consists of three binary relations on A:
  - strict preference relation P
  - indifference relation I
  - incomparability relation J

2. Boolean preference structures

- A preference structure on a set of alternatives A is a triplet (P, I, J) of relations in A that satisfy:

  (B1) P is irreflexive, I is reflexive and J is irreflexive

  (B2) P is asymmetric, I is symmetric and J is symmetric

  (B3) $P \cap I = \emptyset$, $P \cap J = \emptyset$ and $I \cap J = \emptyset$

  (B4) $P \cup P^t \cup I \cup J = A^2$

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  (B4) $P \cup P^t \cup I \cup J = A^2$

- (P, I, J) is a preference structure on A iff

  (i) I is reflexive and I is symmetric

  (ii) $P(a,b) + P(b,a) + I(a,b) + J(a,b) = 1$

2. Completeness condition (B4)

(C1) $\text{co}(P \cup I) = P^t \cup J$

(C2) $\text{co}(P \cup P^t) = I \cup J$

(C3) $\text{co}(P \cup P^t \cup I) = J$

(C4) $\text{co}(P \cup P^t \cup J) = I$

(C5) $\text{co}(P^t \cup I \cup J) = P$

(C6) $P \cup P^t \cup I \cup J = A^2$

2. Construction and characterization

- Given a reflexive relation R in A, the triplet (P, I, J) defined by

$$\begin{aligned} P &= R \cap \mathrm{co}(R^t) \\ I &= R \cap R^t \\ J &= \mathrm{co}R \cap \mathrm{co}(R^t) \end{aligned}$$

is a preference structure on A such that

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is a preference structure on A such that

$$R = P \cup I \quad \text{and} \quad R^c = P^t \cup J$$

- Consider a preference structure (P, I, J) on A. Define its large preference relation R as

$$R = P \cup I$$

then (P, I, J) can be reconstructed from R.

3. Continuous de Morgan triplets

- A t-norm T is an increasing, commutative and associative binary operation on $[0, 1]$ with neutral element 1
  - minimum operator $T_{\mathbf{M}}(x, y) = \min(x, y)$
  - algebraic product $T_{\mathbf{P}}(x, y) = xy$
  - Lukasiewicz t-norm $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$

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  - Lukasiewicz t-norm $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$
- A t-conorm S is an increasing, commutative and associative binary operation on $[0, 1]$ with neutral element 0
  - maximum operator $S_{\mathbf{M}}(x, y) = \max(x, y)$
  - probabilistic sum $S_{\mathbf{P}}(x, y) = x + y - xy$
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  - probabilistic sum $S_{\mathbf{P}}(x,y) = x+y-xy$
  - bounded sum $S_{\mathbf{L}}(x,y) = \min(x+y,1)$
- An involutive negator N is an involutive decreasing permutation of $[0,1]$
  - standard negator $N_s(x) = 1-x$

3. Continuous de Morgan triplets

- N-dual t-conorm of a t-norm T is the t-conorm T^N:

$$T^N(x, y) = N(T(N(x), N(y)))$$

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- A de Morgan triplet M is a triplet of the type

$$(T, T^N, N)$$

(is called continuous if T is continuous)

- The Lukasiewicz triplet: $(T_{\mathbf{L}}, S_{\mathbf{L}}, N_s)$

3. The Frank t-norm family

- $s \in]0,1[\cup]1,\infty[$:

$$T_s^{\mathbf{F}}(x,y) = \log_s \left(1 + \frac{(s^x-1)(s^y-1)}{s-1}\right)$$

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- limits:

$$\lim_{s\to 0} T_s^{\mathbf{F}}(x,y) = \min(x,y)$$

$$\lim_{s\to 1} T_s^{\mathbf{F}}(x,y) = xy$$

$$\lim_{s\to \infty} T_s^{\mathbf{F}}(x,y) = \max(x+y-1,0)$$

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- $T_0^{\mathbf{F}} = T_{\mathbf{M}}$, $T_1^{\mathbf{F}} = T_{\mathbf{P}}$, $T_\infty^{\mathbf{F}} = T_{\mathbf{L}}$

3. The Frank t-norm family

- Frank t-norm family: $(T_s^{\mathbf{F}})_{s\in[0,\infty]}$
- Frank t-conorm family: $(S_s^{\mathbf{F}})_{s\in[0,\infty]}$, $S_s^{\mathbf{F}} = (T_s^{\mathbf{F}})^*$
- Continuous irreducible solutions of the Frank equation:

$$T(x,y) + S(x,y) = x + y$$

4. Additive fuzzy preference structures

- Consider a continuous de Morgan triplet $M = (T, S, N)$.

  An M-FPS on A w.r.t. completeness condition (Ci), $i \in \{1, \ldots, 6\}$, is a triplet (P, I, J) of binary fuzzy relations in A that satisfy:

  (i) P is irreflexive, I is reflexive and J is irreflexive

  (ii) P is T-asymmetric, I is symmetric and J is symmetric

  (iii) $P \cap_T I = \emptyset$, $P \cap_T J = \emptyset$ and $I \cap_T J = \emptyset$

  (iv) (P, I, J) satisfies completeness condition (Ci)

4. Completeness condition

(C1) $\text{co}_N(P \cup_S I) = P^t \cup_S J$

(C2) $\text{co}_N(P \cup_S P^t) = I \cup_S J$

(C3) $\text{co}_N(P \cup_S P^t \cup_S I) = J$

(C4) $\text{co}_N(P \cup_S P^t \cup_S J) = I$

(C5) $\text{co}_N(P^t \cup_S I \cup_S J) = P$

(C6) $P \cup_S P^t \cup_S I \cup_S J = A^2$

4. Completeness condition

- p_i = (P, I, J) is an M-FPS on A w.r.t. (Ci)
  - in general: no relationships
  - in the case of the Lukasiewicz triplet: $i \in \{3, 4, 5\}$

$$\{p_1, p_2\} \quad \Rightarrow \quad p_i \quad \Rightarrow \quad p_6$$

4. Why use the Lukasiewicz triplet?

- Assignment Principle: the decision maker should be able to assign one of the degrees $P(a,b)$, $P(b,a)$, $I(a,b)$ and $J(a,b)$ freely in the unit interval $(a \neq b)$

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- Assignment Principle: the decision maker should be able to assign one of the degrees $P(a,b)$, $P(b,a)$, $I(a,b)$ and $J(a,b)$ freely in the unit interval $(a \neq b)$
- The only suitable continuous de Morgan triplet is the Lukasiewicz triplet. (up to automorphism(s))
- Which completeness condition to use?
  We suggest (C1):
  - strongest condition
  - axiomatic constructions

4. (Minimal) Definition

- An additive fuzzy preference structure on A is a triplet (P, I, J) of fuzzy relations in A that satisfy:

  (F1) P is irreflexive, I is reflexive and J is irreflexive

  (F2) P is $T_{\mathbf{L}}$-asymmetric, I is symmetric and J is symmetric

  (F3) $P \cap_{\mathbf{L}} I = \emptyset$, $P \cap_{\mathbf{L}} J = \emptyset$ and $I \cap_{\mathbf{L}} J = \emptyset$

  (F4) $\text{co}(P \cup_{\mathbf{L}} I) = P^t \cup_{\mathbf{L}} J$

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- (P, I, J) is an additive fuzzy preference structure on A iff

  (i) I is reflexive and I is symmetric

  (ii) $P(a, b) + P(b, a) + I(a, b) + J(a, b) = 1$

5. Axiomatic constructions

- Orlovski (78):

$$P(a,b) = \max(R(a,b) - R(b,a), 0)$$

$$I(a,b) = \min(R(a,b), R(b,a))$$

- Ovchinnikov (81):

$$P(a,b) = \begin{cases} R(a,b) & , \text{ if } R(a,b) > R(b,a) \\ 0 & , \text{ otherwise} \end{cases}$$

$$I(a,b) = \min(R(a,b), R(b,a))$$

5. Axiomatic considerations

- Roubens & Vincke (87):

$$P(a,b) = \min(R(a,b), 1 - R(b,a))$$

$$I(a,b) = \min(R(a,b), R(b,a))$$

$$J(a,b) = \min(1 - R(a,b), 1 - R(b,a))$$

- Roubens (89), Ovchinnikov & Roubens (91), Fodor (91)

5. Axiomatic considerations

- Consider a continuous de Morgan triplet $M = (T, S, N)$ and a reflexive binary fuzzy relation R in A. Construct

$$P = R \cap_T \text{co}_N R^t$$

$$I = R \cap_T R^t$$

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5. Axiomatic considerations

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\end{aligned}
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- When does it hold that $R = P \cup_S I$, i.e.

$$R = (R \cap_T \text{co}_N R^t) \cup_S (R \cap_T R^t)?$$

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- When does it hold that $R = P \cup_S I$, i.e.

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- Answer: in general, never (Alsina, 1985).

5. Axioms of Fodor and Roubens

Consider a continuous de Morgan triplet (T, S, N).

(IA) Independence of Irrelevant Alternatives:

$$\begin{aligned} P(a,b) &= p(R(a,b), R(b,a)) \\ I(a,b) &= i(R(a,b), R(b,a)) \\ J(a,b) &= j(R(a,b), R(b,a)) \end{aligned}$$

(PA) Positive Association Principle: The mappings $p(x, N(y))$, $i(x, y)$ and $j(N(x), N(y))$ are increasing.

(S) Symmetry: The mappings i and j are symmetric.

5. Axioms of Fodor and Roubens

(LP) Preserving Large Preference:

$$P \cup_S I = R$$

$$P \cup_S J = \text{co}_N R^t$$

Underlying functional equations:

$$S(p(x,y), i(x,y)) = x$$

$$S(p(x,y), j(x,y)) = N(y)$$

5. Axioms of Fodor and Roubens

- If (T, S, N, p, i, j) satisfies the above axioms then

$$(T, S, N) = (T_{\mathbf{L}}, S_{\mathbf{L}}, N_s)$$

(up to automorphism) and, for any $(x, y) \in [0, 1]^2$:

$$\begin{aligned} T_{\mathbf{L}}(x, 1-y) \leq \quad & p(x, y) \quad && \leq \min(x, 1-y) \\ T_{\mathbf{L}}(x, y) \leq \quad & i(x, y) \quad && \leq \min(x, y) \\ T_{\mathbf{L}}(1-x, 1-y) \leq \quad & j(x, y) \quad && \leq \min(1-x, 1-y). \end{aligned}$$

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- For any reflexive binary fuzzy relation R in A, the triplet (P, I, J) defined by means of (p, i, j) is an AFPS on A such that

$$R = P \cup_{\mathbf{L}} I \quad \text{and} \quad R^c = P^t \cup_{\mathbf{L}} J$$

5. Axioms of Fodor and Roubens

- Consider two continuous t-norms T_1 and T_2. Define p and i by

$$p(x,y) = T_1(x, 1-y)$$

$$i(x,y) = T_2(x,y)$$

then $(T_{\mathbf{L}}, S_{\mathbf{L}}, N_s, p, i, j)$ satisfies the above axioms iff $\exists s \in [0, \infty]$ such that

$$T_1 = T^{\mathbf{F}}_{1/s}$$

$$T_2 = T^{\mathbf{F}}_{s}$$

In this case, we have that $j(x,y) = i(1-x, 1-y)$.

6. Characteristic behaviour

- Given a reflexive binary fuzzy relation R in A and $s \in [0, \infty]$, the triplet (P, I, J) defined by

$$(P, I, J) = (R \cap_{1/s} R^d, R \cap_s R^t, R^c \cap_s R^d)$$

is an AFPS on A such that $R = P \cup_{\mathbf{L}} I$ and $R^c = P^t \cup_{\mathbf{L}} J$. Note that

$$R(a, b) = P(a, b) + I(a, b)$$

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$$R(a, b) = P(a, b) + I(a, b)$$

- Characteristic behaviour: Consider an AFPS (P, I, J) on A. Define its fuzzy large preference relation as

$$R = P \cup_{\mathbf{L}} I.$$

How can (P, I, J) be reconstructed from R?

6. T-norm-based constructions

- An s-AFPS on A is an AFPS (P, I, J) on A that satisfies:

  (D1) for $s \in \{0, 1, \infty\}$, the condition

  $$P \cap_s P^t = I \cap_{1/s} J$$

  (D2) for $s \in]0, 1[\cup]1, \infty[$, the condition

  $$s^{P \cap_s P^t} + s^{-(I \cap_{1/s} J)} = 2$$

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  (D2) for $s \in]0, 1[\cup]1, \infty[$, the condition

$$s^{P \cap_s P^t} + s^{-(I \cap_{1/s} J)} = 2$$

- Condition (D1) is equivalent to:

  (i) for $s = 0$: $\min(P(a, b), P(b, a)) = 0$

  (ii) for $s = 1$: $P(a, b)\, P(b, a) = I(a, b)\, J(a, b)$

  (iii) for $s = \infty$: $\min(I(a, b), J(a, b)) = 0$

- Construction and characterization work!

6. T-norm-based constructions

Consider a reflexive binary fuzzy relation R in A, then we can construct the following fuzzy preference structures on A:

- a 0-AFPS (P_0, I_0, J_0):

$$\begin{aligned} P_0(a,b) &= \max(R(a,b) - R(b,a), 0) \\ I_0(a,b) &= \min(R(a,b), R(b,a)) \\ J_0(a,b) &= \min(1 - R(a,b), 1 - R(b,a)) \end{aligned}$$

- a 1-AFPS (P_1, I_1, J_1):

$$\begin{aligned} P_1(a,b) &= R(a,b)(1 - R(b,a)) \\ I_1(a,b) &= R(a,b)R(b,a) \\ J_1(a,b) &= (1 - R(a,b))(1 - R(b,a)) \end{aligned}$$

6. T-norm-based constructions

- an ∞-AFPS $(P_\infty, I_\infty, J_\infty)$:

$$\begin{aligned} P_\infty(a,b) &= \min(R(a,b), 1 - R(b,a)) \\ I_\infty(a,b) &= \max(R(a,b) + R(b,a) - 1, 0) \\ J_\infty(a,b) &= \max(1 - R(a,b) - R(b,a), 0) \end{aligned}$$

7. Generator triplets (with B. De Baets)

- A triplet (p, i, j) of $[0,1]^2 \to [0,1]$ mappings is called a generator triplet compatible with a continuous t-conorm S if for any reflexive fuzzy relation R on A it holds that the triplet (P, I, J) defined by:

$$\begin{aligned} P(a,b) &= p(R(a,b), R(b,a)) \\ I(a,b) &= i(R(a,b), R(b,a)) \\ J(a,b) &= j(R(a,b), R(b,a)) \end{aligned}$$

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  is an AFPS on A such that

  $P \cup_S I = R$ and $P^t \cup_S J = R^c$

- If (p, i, j) is a generator triplet compatible with a continuous t-conorm S, then S must be nilpotent.

7. Generator triplets

- (p, i, j) is a generator triplet iff
  - (i) $i(1, 1) = 1$
  - (ii) $i(x, y) = i(y, x)$
  - (iii) $p(x, y) + p(y, x) + i(x, y) + j(x, y) = 1$
  - (iv) $p(x, y) + i(x, y) = x$

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  - (iv) $p(x,y) + i(x,y) = x$
- A generator triplet is uniquely determined by, for instance, the generator i:

$$\begin{aligned} p(x,y) &= x - i(x,y) \\ j(x,y) &= i(x,y) - (x + y - 1) \end{aligned}$$

- $T_{\mathbf{L}} \leq i \leq T_{\mathbf{M}}$

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- A generator triplet is uniquely determined by, for instance, the generator i:

$$\begin{aligned} p(x,y) &= x - i(x,y) \\ j(x,y) &= i(x,y) - (x + y - 1) \end{aligned}$$

- $T_{\mathbf{L}} \leq i \leq T_{\mathbf{M}}$
- From any symmetrical i such that $T_{\mathbf{L}} \leq i \leq T_{\mathbf{M}}$ a generator triplet can be built: the generator i

7. The arrival of quasi-copulas

- A generator triplet (p, i, j) is called monotone if:
  (i) p is increasing in the first and decreasing in the second argument
  (ii) i is increasing in both arguments
  (iii) j is decreasing in both arguments

7. The arrival of quasi-copulas

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  - (i) p is increasing in the first and decreasing in the second argument
  - (ii) i is increasing in both arguments
  - (iii) j is decreasing in both arguments

- A generator triplet (p, i, j) is monotone iff

  i is a commutative quasi-copula

  ($i(0, x) = 0$, $i(1, x) = x$, increasing and 1-Lipschitz)

7. Frank again

- Consider a generator triplet (p, i, j) such that i is a t-norm, then the following statements are equivalent:

  (i) the mapping $j(1 - x, 1 - y)$ is a t-norm
  (ii) the mapping $p(x, 1 - y)$ is commutative
  (iii) i is an ordinal sum of Frank t-norms

  and also the following ones:

  (iv) the mapping $p(x, 1 - y)$ is a t-norm
  (v) i is a Frank t-norm

8. Conclusion

- Definition, construction and characterization of AFPS
- Generator triplets: the indifference generator i
- Further work based only on i:
  - Propagation of transitivity-related properties (Ph.D. Susana Díaz)
- Future work: (appropriate classes of) left-continuous de Morgan triplets – how far can we go?