Decision under complete uncertainty

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Outline

1. From Decision under Risk or Uncertainty to Decision under Complete Uncertainty
2. Some models of Decision under Complete Uncertainty
3. Axiomatic analysis
4. Hurwicz, Milnor and Chernoff's Complete Uncertainty
5. Open questions

1. From Decision under Risk or Uncertainty to Decision under Complete Uncertainty

Decision under risk

Von Neumann - Morgenstern

Formalism :

- $X = \{a, b, c, ...\}$: set of consequences
- $(a, 0.3 ; b, 0.45 ; c, 0.25)$ is a lottery
- $(c, 0.05 ; a, 0.65 ; d, 0.3)$ is another one
- The DM has preferences over all conceivable lotteries, given X.

Decision under risk

Von Neumann - Morgenstern

Weak points

- The probabilities are not always known.
  Ex. Probability that groundwater be contaminated by nuclear waste in year 2500 ?

- Uncertainty is not always probabilistic.
  Ex. 10 € on "NP=P" or 20 € on Riemann hypothesis.

Decision under uncertainty

Savage

Formalism :

- $X = \{a, b, c, ...\}$: set of consequences
- $S = \{s_1, s_2, s_3, ...\}$: set of states of nature
- An act f is a mapping from S to X

	s_1	s_2	s_3	s_4	s_5
f	a	b	b	a	c
g	a	c	a	b	a

- The DM has preferences over all conceivable acts, given X and S.

Decision under uncertainty

Savage

Weak points :

- The set of states of nature may be very large.
  Ex. Travel time from Brussels to Han sur Lesse by train (and bus) or by car.

Relevant events: rain, fog, snow, flat tyre (my car), trafic jam, out of gas , railway strike, bus strike, flat tyre (bus), lost (10 events)

States of nature: any combination of the 10 above events. (1024 states)

	Fog Snow Flat tyre (car) Trafic jam Railway strike Bus strike	Rain Flat tyre (car) Bus strike Out of gas	Rain Fog Snow Flat tyre (car) Trafic jam Out of gas Railway strike Bus strike	No problem	...
Car	4’00’’	3’30’’	3’00’’	2’00’	...
Train	5’00’’	4’00’’	3’00’’	2’30’’	...

Decision under uncertainty

Savage

Weak points :

- The set of states of nature may be very large.
  Ex. Travel time from Brussels to Han sur Lesse by train or by car.
- The set of states of nature may be unknown.
  Ex. If I take the exam today, I may have B or C.
  Next week, I may have A or B.

Decision under *complete* uncertainty

Kannai - Peleg (Decision under ignorance):

Formalism :

- $X = \{a, b, c, \ldots\}$: set of consequences
- An act A is a finite subset of X
- Ex. $A = \{a, b, c\}$ and $B = \{a, b\}$.

	s_1	s_2	s_3	s_4	s_5	
f	a	b	b	a	c	(A)
g	a	b	a	a	b	(B)

- The DM has preferences over all conceivable acts, given X.

Decision under *complete* uncertainty

Kannai - Peleg (Decision under ignorance):

Weak points :

- We cannot have two different acts with the same consequences.
  Ex. If I take the exam today, I may have A or B.
  Next week, I may have A or B.
  But A is more likely next week than today.

2. Some models of Decision under Complete Uncertainty

Notation and definitions

- $X = \{a, b, c, ...\}$: set of consequences
- $A = \{a, c\}$ is an act.
- Any finite non-empty subset of X is an act.
- $\gtrsim$: a weak order representing the preferences of the DM. Defined over all acts.
- $\{a\}$ is a certain act.

Some models

Maximin (pessimistic model)

- $A \gtrsim B$ iff $\min_{a \in A}\{a\} \gtrsim \min_{b \in B}\{b\}$

Maximax (optimistic model)

- $A \gtrsim B$ iff $\max_{a \in A}\{a\} \gtrsim \max_{b \in B}\{b\}$

Max-min

- $A \gtrsim B$ iff $\max_{a \in A}\{a\} \succ \max_{b \in B}\{b\}$

or $\begin{cases} \max_{a \in A}\{a\} \sim \max_{b \in B}\{b\} \\ \text{and} \\ \min_{a \in A}\{a\} \gtrsim \min_{b \in B}\{b\} \end{cases}$

Min-max

Some models for decision under *complete* uncertainty

Let $a_{(1)}$ denote the largest element in A, $a_{(2)}$ denote the second largest element in A, etc.

Idem for $b_{(i)}$ in B.

Leximax

$A \succeq B$ iff $A = B$ or

$\#A < \#B$ and $\{a_{(i)}\} \sim \{b_{(i)}\}$, $i = 1 \ldots \#A$ or

$\{a_{(i)}\} \sim \{b_{(i)}\}$ $\forall\, i < j$ and $\{a_{(j)}\} \succ \{b_{(j)}\}$ for some j.

Leximin

A weakness of Maximin, Maximax, Max-min, Min-Max, Leximax, Leximin

Let $A = \{1, 1\,000\,000\}$ and

$B = \{0, 900\,000, 900\,001, 900\,002, \ldots , 900\,999\}$.

All criteria seen so far yield $A \succ B$!

The Uniform Expected Utility model

Uniform Expected Utility , UEU (arithmetic mean)

- $A \succsim B$ iff $\Sigma_{a \in A} u(a)/\#A \geq \Sigma_{b \in B} u(b)/\#B$

Let $A = \{1, 1\,000\,000\}$ and
$B = \{0, 900\,000, 900\,001, 900\,002, \ldots, 900\,099\}$.

The UEU model can yield $B \succ A$

3. Axiomatic analysis

Max en Min based models

Dominance

- $\{a\} \succsim \{b\}$ for all $b \in B$ implies $B \cup \{a\} \succsim B$.
- $\{a\} \precsim \{b\}$ for all $b \in B$ implies $B \cup \{a\} \precsim B$.

Independence

- $\forall\, c \notin A \cup B$, $A \succsim B \Leftrightarrow A \cup \{c\} \succsim B \cup \{c\}$.

Theorem 1 [Kannai and Peleg, JET, 1984]

If $\succsim$ satisfies Dominance and Independence, then

$A \sim \{\min_{a \in A}\{a\}\ , \max_{a \in A}\{a\}\ \}$ for all A.

Proof of Theorem 1.

Dominance : • $\{a\} \succsim \{b\}$ for all $b \in B$ implies $B \cup \{a\} \succsim B$.
• $\{a\} \precsim \{b\}$ for all $b \in B$ implies $B \cup \{a\} \precsim B$.

Independence : $\forall\, c \notin A \cup B$, $A \succsim B \Leftrightarrow A \cup \{c\} \succsim B \cup \{c\}$.

$A = \{a_{(1)}, a_{(2)}, ..., a_{(n)}\}$.

Dom. : $\{a_{(1)}\} \succsim \{a_{(1)}, a_{(2)}\} \succsim \{a_{(1)}, a_{(2)}, a_{(3)}\} \succsim \{a_{(1)}, ..., a_{(n-1)}\}$

WO: $\{a_{(1)}\} \succsim \{a_{(1)}, ..., a_{(n-1)}\}$

Ind. : $\{a_{(1)}, a_{(n)}\} \succsim \{a_{(1)}, ..., a_{(n-1)}, a_{(n)}\} = A$.

Dom. : $\{a_{(n)}\} \precsim \{a_{(n-1)}, a_{(n)}\} \precsim ... \precsim \{a_{(2)}, ..., a_{(n)}\}$

WO: $\{a_{(n)}\} \precsim \{a_{(2)}, ..., a_{(n)}\}$

Ind. : $\{a_{(1)}, a_{(n)}\} \precsim \{a_{(1)}, a_{(2)}, ..., a_{(n)}\} = A$.

$\{a_{(1)}, a_{(n)}\} \sim A$.

Theorem 2 [Kannai and Peleg, JET, 1984]

If $\#X \geq 5$ and there are a_1, a_2, a_3, a_4, a_5 s.t. $a_1 \succeq a_2 \succeq a_3 \succ a_4 \succeq a_5$, then $\succeq$ does not satisfy Dominance and Independence.

Proof.

Suppose $\{a_3\} \succ \{a_2, a_4\}$

Ind. : $\{a_3, a_5\} \succ \{a_2, a_4, a_5\}$

Th.1 : $\{a_3, a_4, a_5\} \succ \{a_2, a_3, a_4, a_5\}$

Contradicts Dominance. So, $\{a_2, a_4\} \succeq \{a_3\}$.

Ind. : $\{a_1, a_2, a_4\} \succeq \{a_1, a_3\}$.

Th.1 : $\{a_1, a_2, a_4\} \succeq \{a_1, a_2, a_3\}$.

Ind. : $\{a_4\} \succeq \{a_3\}$. Contradiction.

Theorem 2 [Kannai and Peleg, JET, 1984]

If $\#X \geq 5$ and then there are a_1, a_2, a_3, a_4, a_5 s.t. $a_1 \succeq a_2 \succeq a_3 \succ a_4 \succeq a_5$, then $\succeq$ does not satisfy Dominance and Independence.

Conclusion

Independence or Dominance are too strong.

Dominance is unescapable (satisfied by all models seen so far).

So, Independence is too strong.

Some axioms for the leximax

Top Independence

- If $\{d\} > \{c\}$ $\forall\, c \in A \cup B$ then $[A > B \Leftrightarrow A \cup \{d\} > B \cup \{d\}]$.

Neutrality

- $\forall\, A, B \subseteq X\ (\neq \emptyset)$, $\forall$ one-to-one mapping $f : A \cup B \to X$,
  $[\ \{a\} \succeq \{b\}$ iff $f(a) \succeq f(b)$ and
  $\{b\} \succeq \{a\}$ iff $f(b) \succeq f(a)\ ]$ $\forall\, a \in A, \forall\, b \in B$
  implies
  $A \succeq B$ iff $f(A) \succeq f(B)$ and $B \succeq A$ iff $f(B) \succeq f(A)$

Disjoint Independence

- If $A \cap B = \emptyset$ and $c \notin A \cup B$ then $[A > B \Leftrightarrow A \cup \{c\} > B \cup \{c\}\ ]$.

A characterization

Theorem 3. [Pattanaik & Peleg (SCW,1984)]

- The relation $\succeq$ is the leximax weak order if and only if $\succeq$ satisfies Dominance, Top Independence, Disjoint Independence and Neutrality.

There are similar characterizations of maximax, maximin, max-min, min-max, leximin.

Axioms for the UEU model

Restricted Independence

- $A \gtrsim B$ iff $A \cup C \gtrsim B \cup C$ $\quad (A \cap C = \varnothing = B \cap C,\ \#A = \#B)$

Averaging

- $A \gtrsim B$ iff $A \gtrsim A \cup B$ iff $A \cup B \gtrsim B$ $\quad (A \cap B = \varnothing)$

Attenuation

- $A \sim B$, $\#A > \#B$, $A \gtrsim C$ implies $A \cup C \gtrsim B \cup C$
- $A \lesssim C$ implies $A \cup C \lesssim B \cup C$

  $(A \cap C = \varnothing = B \cap C)$

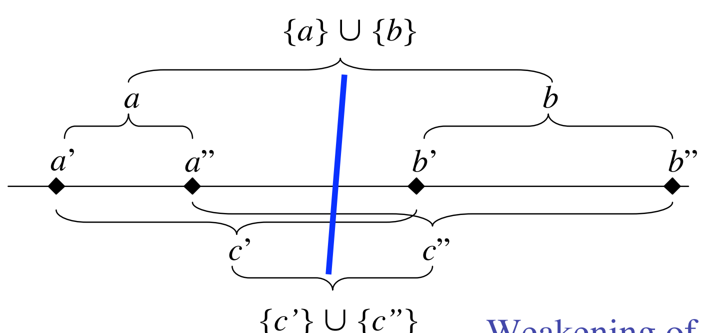
Bisymmetry

- $\{a'\} \cup \{a''\} \sim \{a\}$, $\{b'\} \cup \{b''\} \sim \{b\}$
  $\{a'\} \cup \{b'\} \sim \{c'\}$, $\{a''\} \cup \{b''\} \sim \{c''\}$
  implies $\{a\} \cup \{b\} \sim \{c'\} \cup \{c''\}$

Axioms for the UEU model

Bisymmetry

- $\{a'\} \cup \{a''\} \sim \{a\}$, $\{b'\} \cup \{b''\} \sim \{b\}$
  $\{a'\} \cup \{b'\} \sim \{c'\}$, $\{a''\} \cup \{b''\} \sim \{c''\}$
  implies $\{a\} \cup \{b\} \sim \{c'\} \cup \{c''\}$



Weakening of Associativity

Structural axioms for the UEU model

Certainty Equivalence

- For all non-empty $A \subseteq X$,there is a in X such that $A \sim \{a\}$.

Restricted Solvability

- For all non-empty $A, B \subseteq X$ and c^*, c_* in X,

  $A \cup \{c^*\} > B > A \cup \{c_*\}$ implies

  there is c in X such that $A \cup \{c\} \sim B$.

Archimedeanness

- Let $\{c_i\}$, $i = 1, 2, \ldots$ be a sequence where $c_i \in X$ for all i. Suppose $a, b \in X$, $\{a\} > \{b\}$, $a \neq c_i \neq b$ for all i and $\{c_i , a\} \sim \{c_{i+1} , b\}$ for all i. If there is $d, e \in X$ such that $\{d\} > \{c_i\} > \{e\}$ for all i, then the sequence is finite.

Characterization of UEU

Theorem 2 [Gravel, Marchant & Sen, 2007]

Suppose $\gtrsim$ satisfies Certainty Equivalence and Restricted Solvability. There is then $u : X \to \Re$ such that

$$A \gtrsim B \quad \text{iff} \quad \Sigma_{a \in A}\, u(a)/\#A \;\geq\; \Sigma_{b \in B}\, u(b)/\#B$$

$\forall\, A, B \subseteq X$ and finite

iff $\gtrsim$ satisfies Averaging, Attenuation, Bisymmetry, Restricted Independence and Archimedeanness.

The utility function u is unique up to a positive affine transformation.

Remark

In Theorem 2, X is infinite or $\succsim$ is trivial.
This follows from Averaging and Certainty equivalent.

Proof

Suppose there are a and b such that $\{a\} \succ \{b\}$.

By Averaging, $\{a\} \succ \{a, b\} \succ \{b\}$.

By C.E., there is $c : c \sim \{a, b\}$.

So, $\{a\} \succ \{c\} \succ \{b\}$.

By Averaging, $\{a\} \succ \{a, c\} \succ \{c\}$.

By C.E., there is $d : d \sim \{a, c\}$.

So, $\{a\} \succ \{d\} \succ \{c\} \succ \{b\}$.

And so on.

Weak Point of the UEU criterion

Suppose $X = \Re$ and $u(a) = a$.

Let $A_x = \{1, 100\} \cup \{100 - x\} \cup \{100 + x\}$ with $x \in \Re$ and $B = \{60, 80\}$.

With the UEU, $A_x > B$ for all $x \neq 0$

$A_x < B$ for $x = 0$ (or close to 0)

4. Hurwicz, Milnor and Chernoff's Complete Uncertainty

Hurwicz, Milnor and Chernoff (±1950) use Savage's framework (acts and states) but consider that we do not have any information about the likelihood of the states of nature.

	s_1	s_2	s_3	s_4
f	a	a	b	b
g	a	b	a	a

In SEU, the information about the likelihood is not explicit, it is derived from $\succsim$. For example, if $b \succ a$ and $f \succ g$, then we derive $P(s_3 \cup s_4) > P(s_2)$.

What is then Hurwicz, Milnor and Chernoff's Complete Uncertainty ?

Hurwicz, Milnor and Chernoff impose the following condition.

Anonymity. $\succsim$ does not depend on the labelling of the states.

Some information is discarded.

Models.

Maximin, Maximax, Leximin, Leximax, UEU (Laplace).

$$f \succsim g \quad \text{iff} \quad \Sigma_{s \in S}\, u(f(s))/\#S \geq \Sigma_{s \in S}\, u(g(s))/\#S$$

The uniform distribution is on the states, not on the consequences.

Characterized by Chernof (1954) and Milnor (1954).

	Hillary Clinton	*Not Hillary Clinton*
f	a	c
g	b	a

$f \succsim g$ iff $u(a) + u(c) \geq u(a) + u(b)$ S

	Hillary Clinton	*Other democrat*	*No democrat*
f	a	c	c
g	b	a	a

$f \succsim g$ iff $u(a) + 2\,u(c) \geq 2u(a) + u(b)$

5. Open questions

Open questions

- Characterization of some models
  - Weighted max-min : α max + (1- α) min.
  - Uniform Geometric Expected Utility (UGEU) : $(u_1(x_1) \times u_2(x_2) \times \ldots \times u_{\#A}(x_{\#A}))^{1/\#A}$.
  - Infinite subsets (intervals)
  - ...
- Empirical validation of various models
- Elicitation of u.

Disclaimer

In a large part (but not all) of the literature, ≽ is not necessarily a weak order, but the restriction of ≽ to singletons is assumed to be a linear order (weak order without ties).

Most results and axioms presented in this paper are not the original ones. They have been adapted to fit into the framework where ≽ is a weak order.

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