Introduction to Auctions (and information aggregation)

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> > October 29, 2008

Outline of the Topics

- Scoring rules
- Prediction markets
- Auctions: clean and oversimplifying view
- Auctions: messy but closer to reality
- Information aggregation results

Information Aggregation

- Surveys
- Opinion Polls
- Aggregating votes / ratings / review scores / ...
- Eliciting and aggregating expert opinions

Issues:

. . .

- info quantity
- info quality
- representativeness
- incentives

Single Data Point Info Quality

- stats
- machine learning

Add incentives:

Example: Binary event E (thumbs up/down, yes/no, 0/1) Elicit probability estimate of E=1: p. **Brier score:** $B(p) = 1 - (E - p)^2$ If *r* reported instead of *p*: $E[B(r)|p] = p(1-(1-r)^2)+(1-p)(1-r^2)$ maximized at r=p (d/dr: 2p(1-r)-2(1-p)r)

Truthful reporting maximizes Brier score

Scoring Rules

- Consider a probability forecast for a discrete event with *n* possible outcomes ("states of the world").
- Let e_i = (0, ..., 1, ..., 0) denote the indicator vector for the ith state (where 1 appears in the ith position).
- Let $\mathbf{p} = (p_1, ..., p_n)$ denote the forecaster's *true* subjective probability distribution over states.
- Let $\mathbf{r} = (r_1, ..., r_n)$ denote the forecaster's *reported* distribution (if different from **p**).

(Also, let $\mathbf{q} = (q_1, ..., q_n)$ denote a *baseline* distribution upon which the forecaster seeks to improve.)

Proper Scoring Rules

- The scoring rule S is [*strictly*] *proper* if
 S(p) ≥[>] S(r, p) for all r [≠p], i.e., if the forecaster's expected score is [uniquely] maximized when she reports her true probabilities.
- S is [strictly] proper iff S(p) is a [strictly] convex function of p.
- If S is strictly proper, then it is uniquely determined from S(p) by McCarthy's (1956) formula:

 $S(\mathbf{r}, \mathbf{p}) = S(\mathbf{r}) + \nabla S(\mathbf{r}) \cdot (\mathbf{p} - \mathbf{r})$

Slide thx to R.Nau

Common scoring rules

The three most commonly used scoring rules are:

• The *quadratic* scoring rule:

 $S(\mathbf{p}, \mathbf{e}_i) = -(\|\mathbf{e}_i - \mathbf{p}\|_2)^2$

- The *spherical* scoring rule: $S(\mathbf{p}, \mathbf{e}_i) = p_i / ||\mathbf{p}||_2$
- The *logarithmic* scoring rule:

 $S(\mathbf{p}, \mathbf{e}_i) = \ln(p_i)$

Score is squared Euclidean distance between \mathbf{p} and \mathbf{e}_i

Score vector lies on the surface of a pseudosphere centered at the origin

Slide thx to R.Nau

Scoring, Cross-entropy, Utility...

- Scoring rules are reward functions for defining subjective probabilities and eliciting them in forecasting applications and experimental economics (de Finetti, Brier, Savage, Selten...)
- **Cross-entropy,** or **divergence**, is a physical measure of information gain in communication theory and machine learning (Shannon, Kullback-Leibler...)
- Utility maximization is the decision maker's objective in Bayesian decision theory and game theory (von Neumann & Morgenstern, Savage...)

Connections (Jose et al. 08)

- Any decision problem under uncertainty may be used to define a scoring rule or measure of divergence between probability distributions.
- The expected score or divergence is merely the expectedutility gain that results from solving the problem using the decision maker's "true" (or posterior) probability distribution *p* rather than some other "baseline" (or prior) distribution *q*.
- These connections have been of interest in the recent literature of robust Bayesian inference and mathematical finance.

Specific results (Jose et al. 08)

- Connections among the best-known parametric families of generalized scoring rules, divergence measures, and utility functions.
- The expected scores obtained by truthful probability assessors turn out to correspond exactly to well-known generalized divergences.
- They also correspond exactly to expected-utility gains in financial investment problems with utility functions from the linear-risk-tolerance (a.k.a. HARA) family.
- These results generalize to incomplete markets via a primal-dual pair of convex programs.



p precise probability of a risk averse decision maker with utility function u

Q = set of imprecise probabilities of risk neutral opponent/market

Finding the payoff vector \mathbf{x} to maximize $E_p[u(\mathbf{x})]$ s.t. $E_q[\mathbf{x}] \le 0$ is equivalent (dual) to finding \mathbf{q} in \mathbf{Q} to minimize the divergence $S(\mathbf{p}||\mathbf{q})$

Slide thx to R.Nau

Betting

Standard truth-telling incentive:

Put your money where your mouth is

Example revisited: Binary event *E* (thumbs up/down, yes/no, 0/1) Let *p* be the probability estimate of E=1.

2008 Presidential Election Winner (Individual) bartick Obama to win 2008 US Presidential Election

2008.PRES.OBAMA	Lifetime	Last We	ek		Last Da	ау
Trade Last Trade : 87.5 V 0.3 Suggest An Image	Clos 2007	sing Price: 87.8 Advanced gr dd this chart to you	2008 raphing ur website	y r		
Links	Contra	ct Info	BI	D	A	sĸ
Related Intrade Markets	Currency	USD	Qty	Price	Price	Qty
Contract Specific Rules	Session lo/hi	87.5 - 87.5	31	87.5	87.6	107
Contract Specific Rules	Life lo/hi	5.8 - 89.2	13	87.4	87.7	10
Search Yahoo News	Previous Close	97.9	10	87.3	87.8	9940
Search Google	Previous close	07.0	103	87.2	87.9	13
	Open Price	87.5	29	87.1	88.0	1331
	Last Price	87.5	715	87.0	88.1	10
	Last Trade Time	08:42 AM, GMT	20	86.9	88.2	10
	Today's Volume	1	34	86.8	88.3	10
	Total Volume	1071633	15	86.7	88.4	10
	Market Lifetime:	Nov 9 - Nov 9	15	86.6	88.5	20
			160	86.5	88.6	10
			10	86.4	88.7	35
			10	86.3	88.8	57
			103	86.2	88.9	100
			10	86.1	89.0	2

Image from intrade.com



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Betting

Standard truth-telling incentive:

Put your money where your mouth is

Example revisited: Binary event *E* (thumbs up/down, yes/no, 0/1) Let *p* be the probability estimate of E=1.

Contract: \$1 paid if E=1, \$0 paid if E=0. E[Contract]=p\$.

► Buy Contract if Contract price < *p*

► Sell Contract if Contract price >*p*.

IF willing to bet moneyIF no discountingIF risk-neutralIF trading counterparty exists

Prediction Markets

Binary event *E* (thumbs up/down, yes/no, 0/1)

Contract: \$1 paid if E=1, \$0 paid if E=0.

IF willing to bet money, no discounting, risk-neutral: *E[Contract]=p\$*.

▶ Buy Contract if Contract price < *p* (bid, do you want to bid *p*?)

► Sell Contract if Contract price >*p*. (ask, do you want ask *p*?)

"Little details"

- How the market clears?
- How the price forms?

Does the information aggregate? What information?

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BID		ASK		
Qty	Price	Price Qty		
31	87.5	87.6	107	
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Market Clearing

Bid (willing to buy up to the bid price):

 $b_1 \ge b_2 \ge b_3 \ge \dots \ge b_k \ge \dots \ge b_n$

Ask (willing to sell for at least the asking price):

$$a_1 \le a_2 \le a_3 \le \dots \le a_k \le \dots \le a_m$$

- Market clears at the largest k such that $a_k \leq b_{k}$.
- At what price?

Match: $[a_1, b_1], [a_2, b_2], ..., [a_k, b_k]$

Any price possible for any pair.

(e.g., pay your bid and/or get your ask) If uniform price, must be in $[a_k, b_k]$

BID		ASK		
Qty	Price	Price Qty		
31	87.5	87.6	107	
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Double Auction

Bid (willing to buy up to the bid price):

 $\boldsymbol{b}_1 \geq \boldsymbol{b}_2 \geq \boldsymbol{b}_3 \geq \dots \geq \boldsymbol{b}_k \geq \dots \geq \boldsymbol{b}_n$

Ask (willing to sell for at least the asking price):

$$a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_k \leq \ldots \leq a_m$$

Uniform price must be in $[a_k, b_k]$.

Buyers perspective:

 $\boldsymbol{b}_1 \geq \boldsymbol{b}_2 \geq \boldsymbol{b}_3 \geq \dots \geq \boldsymbol{b}_k \geq \dots \mathbf{a}_k \dots$

If price in $[a_k, b_k)$, "winners" bid values don't affect the price. (Analogous for the seller)

Auctions?

Use of Auctions

- Procedures for allocation and pricing Incomplete information Self-interested parties
- Theoretical models for price formation
- Theoretical models for market interactions

Literature

- Historically, microeconomic theory
- INFORMS Journals: 100+ papers from 2003 (Mgmt.Sci. 45).
- Also thousands of recent CS papers (combinatorial auctions, ad auctions, optimization)

Auctions Everywhere

- **Procurement.** GE, CombineNet, FreeMarkets(Ariba), etc.
- **Transportation.** truck/bus routes, containers, airport takeoff/landing rights, etc.
- **Government controlled resources/rights.** oil-leases, timber, frequencies,... (also: privatization sales)
- Commodities. electricity, pollution rights/credits,...
- Finance. government bonds (T-bills), IPOs, acquisitions, ...
- Advertising. sponsored search (Google, Yahoo, etc.), internet ads (e.g., AdSense) traditional media ads.
- **Retail.** eBay, uBid,..., alternative (online) sales channel
- Other ("traditional"). construction contracts, real estate (foreclosures), art, cars, ..., flowers, fish, tobacco, cattle, ...

E-Business revolution redefined what is possible.

Basic Auction Models Standard setup:

single seller *n* bidders (buyers)

- English auction (ascending auction)
- Dutch auction (descending auction)
- Sealed-bid first-price auction
- Sealed-bid second-price auction

Important for procurement: reverse auctions

v = bidder's value (private info) $v^{(j)} = j^{\text{th}}$ highest value (so, $v^{(1)} \ge v^{(2)} \ge \dots \ge v^{(n)}$)

Equilibria:

• 2nd price auction:

• 1st price auction:

v = bidder's value (private info) $v^{(j)} = j^{\text{th}}$ highest value (so, $v^{(1)} \ge v^{(2)} \ge ... \ge v^{(n)}$)

Equilibria:

• 2^{nd} price auction: b(v) = v

(Bidding truthfully is a dominant strategy)

• 1st price auction:

v = bidder's value (private info) $v^{(j)} = j^{\text{th}}$ highest value (so, $v^{(1)} \ge v^{(2)} \ge ... \ge v^{(n)}$)

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• 1st price auction: $b(v) = E[v^{(2)} / v = v^{(1)}]$

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Strategic equivalence:

English \Leftrightarrow 2nd price **Dutch** \Leftrightarrow 1st price

v = bidder's value (private info) $v^{(j)} = j^{\text{th}}$ highest value (so, $v^{(1)} \ge v^{(2)} \ge \dots \ge v^{(n)}$)

Equilibria:

• 2^{nd} price auction: b(v) = v

(Bidding truthfully is a dominant strategy)

• 1st price auction: $b(v) = E[v^{(2)} / v = v^{(1)}]$

Strategic equivalence:English $\Leftrightarrow 2^{nd}$ priceDutch $\Leftrightarrow 1^{st}$ priceNote: If v's also independent, then $E[price] = v^{(2)}$ for all formats.

Beautiful Theorems

The Revelation Principle [Myerson 79, others]

For any (auction) mechanism, there exists an (auction) mechanism in which truthful reporting (bidding) is an equilibrium and all outcomes are the same as in the equilibrium of the original (auction) mechanism.

The Revenue Equivalence Theorem [Myerson 81]

Suppose bidders' values are independent and i.i.d., and bidders are risk neutral. Then *any* symmetric and increasing equilibrium for *any* auction such that

- bidder with $v = v^{(l)}$ wins the object
- bidder with v = 0 pays zero

yields the same expected revenue to the seller.

Theory Disconnect

Auctions proposed by theory, often not found in practice

Auctions used in practice often not found in theory

- overly simplistic bidder information/valuation structures
- fixed supply and demand assumption

Basic Multi-item Auctions

one-shot auctions n bidders, k items unit demand

Discriminatory auction (1st price auction analogue)

Uniform auction (2nd price auction analogue)

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Discriminatory auction (1st price auction analogue)

- *k* highest bidders win
- Winners pay what they bid

Uniform auction (2nd price auction analogue)

Basic Multi-item Auctions

one-shot auctions n bidders, k items unit demand

Discriminatory auction (1st price auction analogue)

- *k* highest bidders win
- Winners pay what they bid

Uniform auction (2nd price auction analogue)

- *k* highest bidders win
- Every winner pays price equal to the highest losing bid Lowest clearing price: (k+1)st price.

Valuation Structure

Value of an item to bidder *j*:

 $u(V,X_j)$

with:

- *u* increasing in both variables
- *V* is a r.v. with commonly known pdf.
- X_i has a pdf f(x|v) satisfying MLRP:

 $(x > x' \& v > \overline{v'}) \Longrightarrow f(x|v)/f(x'|v') \ge f(\overline{x'|v)}/f(x|v')$

- **Special cases:**
- **Common value:** u(V,X) = V (have private signal x_i , value is common)
- Affiliated private values: u(V,X)=X
- IPV: no uncertainty in V







Common Value

•Value of an item: *v* (*V* is a r.v. with commonly known pdf.)

•Each bidder gets a private signal x_{i} .

 X_j has a pdf f(x|v) satisfying MLRP: $(x \ge x' \& v \ge v') \Longrightarrow f(x|v)/f(x'|v') \ge f(x'|v)/f(x|v')$

Winner's curse!

Assuming for simplicity f(x|v) symmetric around v: $x^{(l)} \ge x^{(2)} \ge \dots \ge x^{(n)}$ So, $E[x^{(l)}] \ge v$ and cannot just bid $x^{(l)}$.

Symmetric Increasing Equilibrium

For bidder's signal x and another signal y define:

$$v(x,y) = E[v | x, Y^{(k),n-1} = y]$$

where $Y^{(k)}$ is the k^{th} order statistic of n-1 other signals, with conditional pdf $f_{k,n-1}(y|x)$ and conditional cdf $F_{k,n-1}(y|x)$

Uniform Auction SIE Candidate $b^{u}(x) = v(x,x) = E[v | x, Y^{(k)}=x]$

Discriminatory Auction SIE Candidate $(b^{d}(x))' = (v(x,x)-b^{d}(x)) (f_{k,n-1}(x|x)/F_{k,n-1}(x|x))$

 $b^{d}(x) = \int_{t}^{x} v(t,t) (f_{k,n-1}(t|t)/F_{k,n-1}(t|t)) \exp[-\int_{t}^{x} (f_{k,n-1}(s|s)/F_{k,n-1}(s|s)) ds] dt$

Need v(x,y) increasing. Sufficient condition: MLRP

 $(x \ge x' \& v \ge v') \Longrightarrow f(x|v)/f(x'|v') \ge f(x'|v)/f(x|v')$

Uniform Signals

Let *x* be drawn from a uniform distribution on [v-1/2, v+1/2]

Let $u(v,x) = \lambda v + (1 - \lambda)x$ (convex mix of private and common values)

 $b^{u}(x) = x - \lambda(1/2 - k/n)$ $b^{d}(x) = x - \lambda(1/2 - k/n) - k/n$

Uniform Signals

Let *x* be drawn from a uniform distribution on [v-1/2, v+1/2]

Let $u(v,x) = \lambda v + (1 - \lambda)x$ (convex mix of private and common values)

 $b^{u}(x) = x - \lambda(1/2 - k/n)$ $b^{d}(x) = x - \lambda(1/2 - k/n) - k/n$

Private values $(\lambda=0)$: $b^u(x)=x$

 $b^d(x) = x - k/n$

Common value (λ =1):

 $b^{u}(x) = x - 1/2 + k/n,$ $b^{d}(x) = x - 1/2$

Revenue Rankings

The Revenue Ranking Theorem [Milgrom & Weber 82] In the unique symmetric equilibrium and for risk neutral bidders, the expected revenue in the uniform auction is at least as large as the expected revenue in the corresponding discriminatory auction.

But...

- IPV: Revenue Equivalence Theorem
- risk-averse bidders: Discriminatory ≥ Uniform (Holt '80, Maskin& Riley '84)
- multi-demand: Discriminatory ≥ Uniform (Back&Zender '93)
- participation uncertainty: Discriminatory ≥ Uniform (P&Tsetlin '08)

Information Aggregation

Each bidder has a private signal x_i .

If these signals were public, could we get a good estimate of v?

Can an auction aggregate privately held information and provide a good estimate of *v*?

Specifically, is auction price a good estimate of v?

Assumptions: symmetric equilibrium behavior

Issues: equilibrium existence and uniqueness

IA Definition

Let $A_n(k_n)$ denote an auction with *n* bidders and k_n items and let P_n denote the unique symmetric equilibrium price in $A_n(k_n)$.

Information Aggregation.

An auction (format) aggregates information if for any sequence of auctions, $A_n(k_n)$ with $n \rightarrow \infty$, and for every $\varepsilon > 0$, $Pr(|P_n - V| > \varepsilon) \rightarrow 0$

• Case of uniform auctions and $k_n = k$: aggregation not likely

Theorem [Wilson 77, Milgrom 81] Uniform auction for *k* items aggregates information if and only if for any v,v' such that v < v': $\inf_x f(x|v)/f(x|v') = 0$

Very strong and totally unrealistic condition on signal distribution.

IA and Double Largeness

• If large number of bidders, then there ought to be a large number of items if we even want to hope for information to aggregate

Theorem [Pesendorfer and Swinkels 97] Uniform auction for k_n items aggregates information if and only if $k_n \rightarrow \infty \& (n-k_n) \rightarrow \infty$

What about other auction formats?

Theorem [Kremer 02]

Suppose the amount of information in any one signal is limited.

- •Discriminatory auctions do not aggregate information.
- •English auctions aggregate information whenever corresponding uniform auctions aggregate information.

Participation Uncertainty

N potential risk neutral bidders Stochastic process $\Omega = \{(n_1, p_1), (n_2, p_2), ..., (n_m, p_m)\}$ Ω is assumed to be common knowledge (sorry) (Matthews 87, McAfee & McMillan 87)

• How about uncertainty about *k*?

Supply/Demand Uncertainty

N potential bidders

Stochastic process $\Omega = \{(n_1, k_1, p_1), (n_2, k_2, p_2), ..., (n_m, k_m, p_m)\}$

NOTE:

If n_i bidders participate, the probability of bidder participating is n_i/N . However, if a bidder participates, the probability of n_i bidders is:

 $p_i n_i / \Sigma p_j n_j$

Uniform SIE

Theorem [Harstad, P., Tsetlin] In a *uniform* auction with supply/demand uncertainty a symmetric increasing equilibrium might not exist. When it does, it is a weighted average of equilibrium bidding functions with fixed *n* and *k*.

• Critical:

v(x,y) = E[v | my signal is x and pivotal signal is y]If v(x,y) is increasing in both variables, then $b^*(x)=v(x,x)$

• Weights:

probability of n_i bidders conditional on bidding and on being tied with a pivotal rival.

IA with Supply/Demand Uncertainty

Theorem [Harstad, P. and Tsetlin]

For a sequence of common value uniform auctions with risk neutral bidders, the auction price in the increasing symmetric equilibrium converges to common value if and only if $max(k_i/n_i) - min(k_i/n_i)$ converges to zero.

- Does not hold for discriminatory auctions
- Note: information aggregation can be restored by keeping supply/demand (*k/n*) ratio constant
- Full asymptotic extraction: *E[k(V-price)]* converges to zero

Uniform Common Value Auction *X*~*U*[*v*-1/2,*v*+1/2]

 $n_1 = 100$ $n_2 = 200$ $k_1 = 20$ $k_2 = ?$



Note: Auction Cost per item = *E*[(*v*-*price*)]

IA and Prediction Markets

Does the information aggregate? What information?

- double auction vs. uniform auction
- unit demand
- valuation structure
- SIE?
- k/n ratio constant as n grows?
- convergence rates? (*sqrt(k)* convergence rate, Hong&Shum 04)

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Image from intrade.com

Contract Info

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Open Price	87.5
Last Price	87.5
Last Trade Time	08:42 AM, GMT
Today's Volume	1
Total Volume	1071633
Market Lifetime:	Nov 9 - Nov 9



- Localized computing on a network structure
- Sequential decisions
- Quality of Information (beyond scoring or monetary incentives)
- Voting, Cooperative Games?

Auction Theory: Handle with Care

- valuation structures are rarely IPV
- supply/demand uncertainty is important
- implications for auction format choice
- managing supply: many open issues
- implications for pricing decisions
- need to get closer to practice (e.g., ad auctions)

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