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PARTITION CONSTRAINED COVERING OF A SYMMETRIC CROSSING SUPERMODULAR FUNCTION BY A GRAPH*

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Abstract. We are given a symmetric crossing supermodular set function p on V and a partition $\mathcal{P}$ of V. We solve the problem of finding a graph with vertex set V having edges only between the classes of $\mathcal{P}$ such that for every subset X of V the cut of the graph defined by X contains at least $p(X)$ edges. The objective is to minimize the number of edges of the graph. This problem is a common generalization of the global edge-connectivity augmentation of a graph with partition constraints, which was solved by Bang-Jensen et al. [*SIAM J. Discrete Math.*, 12 (1999), pp. 160–207] and the problem of covering a symmetric crossing supermodular set function solved by Benczúr and Frank [*Math. Program.*, 84 (1999), pp. 483–503]. Our problem can be considered as an abstract form of the problem of global edge-connectivity augmentation of a hypergraph with partition constraints, which was earlier solved by the authors [*J. Graph Theory*, 72 (2013), pp. 291–312].

Key words. edge-connectivity augmentation, splitting off, connectivity function

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1. Introduction.

This paper is concerned with edge-connectivity augmentation problems in graphs, hypergraphs, and abstract forms of the problems for "connectivity" set functions. For a survey, we refer the reader to [10].

Our starting point is the problem of *global edge-connectivity augmentation of a graph*, where we have to add a minimum number of new edges to a given graph $G = (V, E)$ in order to obtain a k-edge-connected graph for a given $k \geq 2$. A natural lower bound can be obtained as follows: for a set X of degree $d(X)$ less than k, the deficiency of X is $k - d(X)$; that is, we must add at least $k - d(X)$ edges between X and $V \setminus X$. The deficiency of a subpartition of V is the sum of the deficiencies of its sets. By adding a new edge we may decrease the deficiency of at most two sets of this subpartition, so we may decrease the deficiency of the subpartition by at most two, and hence we obtain the so-called *subpartition lower bound*: $\alpha_G := \lceil$ half of the maximum deficiency of a subpartition of $V \rceil$. The *minimax theorem* due to Watanabe and Nakamura [11] says that this lower bound α_G can always be achieved.

The next step is a generalization of the above problem, namely the problem of *global edge-connectivity augmentation of a hypergraph*, where we have to add a minimum number of new graph edges to a given hypergraph $\mathcal{G} = (V, \mathcal{E})$ in order to obtain a k-edge-connected hypergraph for a given k. Of course, the subpartition lower bound holds also for hypergraphs. However, a new lower bound arises: after deleting $k - 1$ hyperedges, the connected components must be connected by the new

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graph edges, and hence we obtain the *component lower bound*: $\omega_{\mathcal{G}} - 1$, where $\omega_{\mathcal{G}} :=$ maximum number of connected components after deleting $k-1$ hyperedges. The *minimax theorem* due to Bang-Jensen and Jackson [3] says that the lower bound $\max\{\alpha_{\mathcal{G}}, \omega_{\mathcal{G}} - 1\}$ can always be achieved.

Benczúr and Frank [4] considered the abstract form of the previous problem, namely *covering of a symmetric crossing supermodular function by a graph*: given a symmetric, crossing supermodular set function p on V, what is the minimum number of edges of a graph on vertex set V that covers p, that is, for all subsets X of V, the cut defined by X contains at least $p(X)$ edges? The *subpartition* and the *component lower bounds* can be extended for this problem: $\alpha_p := \lceil$ half of the maximum of the sum of the values of the sets in a subpartition of $V\rceil$ and $\dim(p) - 1 :=$ maximum size of a p-full partition -1, where a partition is p-full if each union of some of its sets has value at least one. The *minimax theorem* due to Benczúr and Frank [4] says that the lower bound $\max\{\alpha_p, \dim(p) - 1\}$ can always be achieved.

Now we consider the partition constrained versions of the above problems.

Motivated by a problem from the theory of rigidity, Bang-Jensen et al. [2] introduced the problem of *partition constrained global edge-connectivity augmentation of a graph*: given a graph $G = (V, E)$, an integer $k \geq 2$, and a partition $\mathcal{P} = \{P_1, \dots, P_r\}$ of V, what is the minimum number of new edges, between different members of $\mathcal{P}$, whose addition results in a k-edge-connected graph? We have a new *partition constrained lower bound* because we cannot add a new edge in $P \in \mathcal{P}$: $\beta_G :=$ maximum deficiency of a subpartition of P over all $P \in \mathcal{P}$. The *minimax theorem* due to Bang-Jensen et al. [2] says that the lower bound $\max\{\alpha_G, \beta_G\}$ can be achieved, except if the graph contains a C_4- or a C_6-configuration, in which case one more edge is needed.

Bernáth, Grappe, and Szigeti [6] considered a generalization of the above problem, the problem of *partition constrained global edge-connectivity augmentation of a hypergraph*: given a hypergraph $\mathcal{G} = (V, \mathcal{E})$, an integer k, and a partition $\mathcal{P} = \{P_1, \dots, P_r\}$ of V, what is the minimum number of new graph edges, between different members of $\mathcal{P}$, whose addition results in a k-edge-connected hypergraph? The *minimax theorem* due to [6] says that the lower bound $\max\{\alpha_{\mathcal{G}}, \beta_{\mathcal{G}}, \omega_{\mathcal{G}} - 1\}$ can be achieved except if the hypergraph contains a $\mathcal{C}_4$- or a $\mathcal{C}_6$-configuration, extension of the above configurations, in which case one more edge is needed.

We emphasize that the above-mentioned papers contain polynomial algorithms solving the corresponding problems.

In this paper we solve the abstract version of the previous problem, a common generalization of all the above-mentioned problems, namely the *partition constrained covering of a symmetric crossing supermodular function by a graph*: given a symmetric crossing supermodular set function p on V and a partition $\mathcal{P}$ of V, what is the minimum number of edges, between different members of $\mathcal{P}$, resulting in a graph that covers p? The *partition constrained lower bound* can be extended for this problem: $\beta_p :=$ maximum of the sum of the values of the sets in a subpartition of P over all $P \in \mathcal{P}$. We show that the lower bound $\max\{\alpha_p, \beta_p, \dim(p) - 1\}$ can be achieved except if a C_4^*-, C_5^*-, or a C_6^*-configuration exists for $(p, \mathcal{P})$, in which case one more edge is needed. This result strictly generalizes the partition constrained problem for hypergraphs. Indeed, a new configuration arises, and it extends an application of Benczúr and Frank [4] that cannot be treated in the framework of hypergraphs; see section 6.

We will follow the classical approach of Frank [7]. First we treat the so-called *degree-specified version* of the above problem in section 4, which is the following: given a symmetric crossing supermodular set function p on V, a partition $\mathcal{P}$ of V, and a function $m : V \to \mathbb{Z}_+$ (called *degree specification*), the task is to decide whether a graph G covering p exists that has only edges connecting different members of $\mathcal{P}$ and that satisfies $d_G(v) = m(v)$ for every $v \in V$. We show the natural necessary conditions of the existence of such a graph, and we characterize the exceptional structures (called *obstacles*). Obstacles are the only cases that satisfy these conditions, yet there does not exist a solution. Then in section 5 we turn to the above *minimization version* of our problem, and we solve it the following way. First, in section 5.2 we try to find a degree specification m with $m(V)$ as small as possible, avoiding the obstacles and satisfying the necessary conditions given earlier: these necessary conditions correspond to natural lower bounds for $m(V)$. Second, in section 5.3 we exhibit the structures (called *configurations*) where we can avoid creating obstacles only if we augment $m(V)$ by 2. Third, we derive our main result in section 5.4. Finally, we provide applications of our theorem in section 6. We show in section 7 that this approach provides a polynomial algorithm.

2. Definitions.

Graphs and set functions. Let us be given a finite ground set V. By $X \subset V$ we mean a proper subset X of V, and $\overline{X} = V \setminus X$. Two subsets X and Y of V are *crossing* if none of $X \setminus Y$, $Y \setminus X$, $X \cap Y$, and $V \setminus (X \cup Y)$ is empty. A family $\mathcal{F}$ of subsets of V is *laminar* if for all $X, Y \in \mathcal{F}$ either X and Y are disjoint or one of them contains the other one. For a family $\mathcal{M} = \{M_1, \dots, M_\ell\}$ of subsets of V, let $M_0^\star = \bigcap_{i=1}^{\ell} M_i$ and $M_i^\star = M_i \setminus \bigcup_{j \neq i} M_j$.

Let $G = (V, E(G))$ be a graph. We will often denote $E(G)$ by E. For $X, Y \subseteq V$, $d_G(X, Y)$ denotes the number of edges between $X \setminus Y$ and $Y \setminus X$, $\overline{d}_G(X, Y) = d_G(X, V \setminus Y)$, and $d_G(X) = d_G(X, V \setminus X)$. Given a partition $\mathcal{X}$ of V, $E_{\delta(\mathcal{X})}$ will denote the set of edges connecting two members of $\mathcal{X}$, and $E_{\mathcal{X}}$ the set of edges contained in members of $\mathcal{X}$. It is well known that the following equalities hold for all $X, Y \subseteq V$:

$$d_G(X) + d_G(Y) = d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X, Y), \tag{1}$$

$$d_G(X) + d_G(Y) = d_G(X \setminus Y) + d_G(Y \setminus X) + 2\overline{d}_G(X, Y). \tag{2}$$

All the functions in this paper are integer valued but not necessarily nonnegative, and they have value 0 on the empty set. A set function $p : 2^V \to \mathbb{Z}$ is *symmetric* if $p(X) = p(V \setminus X)$ for all $X \subseteq V$, and is called *crossing supermodular* if it satisfies (3) for all crossing sets $X, Y \subseteq V$ with $p(X), p(Y) > 0$. Note that if $X \setminus Y = \emptyset$ or $Y \setminus X = \emptyset$, then (3) is trivially satisfied with equality. A set $X \subseteq V$ with $p(X) > 0$ is called *p-positive*. A symmetric crossing supermodular set function p also satisfies (4) for crossing p-positive set pairs X, Y. Note that if $X \cap Y = \emptyset$ or $X \cup Y = V$, then (4) is trivially satisfied with equality.

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y), \tag{3}$$

$$p(X) + p(Y) \leq p(X \setminus Y) + p(Y \setminus X). \tag{4}$$

A nonnegative function $m : V \to \mathbb{Z}_+$ is called a *degree specification*. The graph G is said to *cover* the function p if (5) holds, and it is said to *satisfy* the degree specification m if (6) holds. Moreover, m is called *p-admissible* if (7) holds, where $m(X) = \sum_{x \in X} m(x)$:

$$d_G(X) \geq p(X) \quad \text{for all } \emptyset \neq X \subset V, \tag{5}$$

$$d_G(v) = m(v) \quad \text{for all } v \in V, \tag{6}$$

$$m(X) \geq p(X) \quad \text{for all } \emptyset \neq X \subset V. \tag{7}$$

The following theorem due to Frank is fundamental in every result on edge-connectivity augmentation: it gives the connection between the minimization version and the degree-specified version of these problems. The theorem is true under more general circumstances; we only state what we need in this paper.

THEOREM 1 (see [7], [1]). *If* $p : 2^V \to \mathbb{Z}$ *is crossing supermodular, then*

$$\begin{aligned} \min\{m(V) : m \textit{ is a } p\textit{-admissible degree specification}\} \\ = \max\left\{\sum_{i=1}^{t} p(V_i) : \{V_1, \dots, V_t\} \textit{ is a subpartition of } V\right\}. \end{aligned} \tag{8}$$

The maximum value in (8) is denoted by σ_p. A degree specification m that achieves the minimum value in (8) will be called *minimal.* Note that, for the parameter α_p defined in the introduction, we have

$$\alpha_p = \left\lceil \frac{1}{2}\sigma_p \right\rceil. \tag{9}$$

Given a partition $\mathcal{X} = \{X_1, \dots, X_t\}$ of V, the index j of X_j is considered modulo t; that is, for example, $X_{t+1} = X_1$. Let $J \subset \{1, \dots, t\}$ be an index set. Let $\overline{J} := \{1, \dots, t\} \setminus J$. We call J *consecutive* if $J = \{j, j+1, \dots, k\}$ modulo t for some j and k. Let us say that J is *p-positive* if $J \neq \emptyset$ and $p\big(\bigcup_{j\in J} X_j\big) > 0$.

Following [4], the partition $\mathcal{X}$ is called *p-full* if $t \geq 4$, every nonempty index set $I \subset \{1, \dots, t\}$ is p-positive, and $p(X_j) = 1$ for some $j \in \{1, \dots, t\}$. The maximum cardinality of a p-full partition is the *dimension* of p and is denoted by $\dim(p)$. If no p-full partition exists, then $\dim(p)$ is defined to be 0. A degree specification m is called *p-legal* if (10) is satisfied:

$$m(V) \geq 2(\dim(p) - 1). \tag{10}$$

The following lemma comes from Benczúr and Frank [4].

LEMMA 2 (see Benczúr and Frank [4]). *Let* $p : 2^V \to \mathbb{Z}$ *be a symmetric crossing supermodular set function.*

1. *A graph that covers* p *has at least* $\dim(p) - 1$ *edges.*
2. *If a partition* $\mathcal{X} = \{X_1, \dots, X_t\}$ *of* V *satisfies* $t \geq 4$, $p(X_1) = 1$, *and* $p(X_1 \cup X_i) > 0$ *for* $i = 2, \dots, t$, *then* $\mathcal{X}$ *is p-full.*

Let $G = (V, E)$ be a graph, p_0 a symmetric crossing supermodular set function on V, and m_0 a degree specification on V. Let us introduce the functions p_G and m_G, which will play an important role in this paper, as follows:

$$p_G(X) = p_0(X) - d_G(X) \quad \text{for all } X \subset V, \tag{11}$$

$$m_G(v) = m_0(v) - d_G(v) \quad \text{for all } v \in V. \tag{12}$$

By (1), the function $-d_G$ is symmetric crossing supermodular; hence so is p_G. Moreover, by (3) and (1) (respectively, by (4) and (2)), for crossing p_0-positive subsets X and Y of V, the following hold:

$$p_G(X) + p_G(Y) \leq p_G(X \cap Y) + p_G(X \cup Y) - 2d_G(X, Y), \tag{13}$$

$$p_G(X) + p_G(Y) \leq p_G(X \setminus Y) + p_G(Y \setminus X) - 2\overline{d}_G(X, Y). \tag{14}$$

It is useful to define the following surplus function s_G:

$$s_G = m_G - p_G. \tag{15}$$

OBSERVATION 3. *Note that m_G is p_G-admissible if and only if*

$$s_G(X) \geq 0 \quad \text{for all } \emptyset \neq X \subset V. \tag{16}$$

By modularity of m_G and (13) (respectively, (14)), for crossing p_0-positive subsets X and Y of V, the following hold:

$$s_G(X) + s_G(Y) \geq s_G(X \cap Y) + s_G(X \cup Y) + 2d_G(X, Y), \tag{17}$$

$$s_G(X) + s_G(Y) \geq s_G(X \setminus Y) + s_G(Y \setminus X) + 2(\overline{d}_G(X, Y) + m_G(X \cap Y)). \tag{18}$$

Operations. Let $m_G : V \to \mathbb{Z}_+$ be a p_G-admissible degree specification. An element v of V is called *m_G-positive* if $m_G(v) > 0$. The set of m_G-positive elements is denoted by $V_+(m_G)$. For an element $v \in V$, χ_v denotes the incidence vector of the set $\{v\}$. Let x, y be two different m_G-positive elements, and $uv \in E$ an edge of G that is not incident to either x or y. We will need the following operations :

1. *Splitting off* at x, y means replacing m_G by $m_{G_{xy}}$ and p_G by $p_{G_{xy}}$, where $G_{xy} = G + xy$.
2. *Unsplitting* uv is the reverse of splitting off: replace m by $m_{G^{uv}}$ and p_G by $p_{G^{uv}}$, where $G^{uv} = G - uv$. Note that $m_{G^{uv}}$ is $p_{G^{uv}}$-admissible.
3. The (uv, ux)*-flip* is defined as unsplitting uv and splitting off at x, u, that is, replacing m_G by $m_{G'}$ and p_G by $p_{G'}$, where $G' = G - uv + xu$. We will also call it *flipping uv for ux*.
4. *Improving uv to ux, vy* is defined as unsplitting uv and splitting off at x, u and at v, y, that is, replacing m_G by $m_{G''}$ and p_G by $p_{G''}$, where $G'' = G - uv + ux + vy$. *Improving uv by x and y* means improving uv to either xu, vy or xv, uy. The corresponding operation is an *improvement*.

Any of the above operations is called *p_G-admissible* if the new degree specification is admissible with the new set function. If the splitting off at x, y is p_G-admissible, then we say that the pair x, y is p_G-admissible.

OBSERVATION 4. $s_G(X) - s_{G_{xy}}(X) = 2$ *if both x and y belong to X, and* 0 *otherwise.*

Special sets. The following special sets will be used frequently in the paper. A set $X \subset V$ is called

1. *tight* if $m_G(X) = p_G(X)$, that is, if $s_G(X) = 0$;
2. *dangerous* if $m_G(X) \leq p_G(X) + 1$, that is, if $s_G(X) \leq 1$;
3. (uv, ux)*-perilous* if $p_G(X) = 0 = m_G(X) - 1$, $x, u \in X$, and $v \notin X$. A (uv, ux)- or (vu, vx)-perilous set is called (uv, x)*-perilous*.

A partition is called *tight* if its members are tight. For an m_G-positive element u, X_u and T_u denote the minimal and the maximal tight sets containing u. If u belongs to no tight set, then X_u is defined to be equal to V.

OBSERVATION 5. *Tight sets containing an m_G-positive element, dangerous sets containing two m_G-positive elements, and perilous sets are p_0-positive.*

Partition constraint. Let $\mathcal{P} = \{P_1, \ldots, P_r\}$ be a partition of V with $r \geq 2$. An element $v \in V$ that belongs to some P_i is said to be of color i. The notation $c(v) = i$ will also be used for $v \in P_i$. We say that the graph $G = (V, E)$ is *$\mathcal{P}$-partite* if every edge of G goes between two different classes of $\mathcal{P}$.

OBSERVATION 6. *Note that there always exists a $\mathcal{P}$-partite spanning tree on V. Indeed, let $u \in P_1$ and $v \in P_2$. Then the edge set $\{ux : x \in V \setminus P_1\} \cup \{vy : y \in P_1 \setminus \{u\}\}$ forms a spanning tree on V that is clearly $\mathcal{P}$-partite.*

A degree specification m is called $\mathcal{P}$-*feasible* if (19) and (20) are satisfied. If m is p_G-admissible and $\mathcal{P}$-feasible, then m is called $(p_G, \mathcal{P})$-*allowed.*

$$m(V) \text{ is even,} \tag{19}$$

$$m(P_i) \leq \frac{m(V)}{2} \quad \text{for all } P_i \in \mathcal{P}. \tag{20}$$

We call $P_i \in \mathcal{P}$ *dominating* if $m(P_i) = \frac{1}{2}m(V)$. A pair of m-positive elements is called *rainbow* if they are of different colors and any dominating color class contains one of them. A splitting off, a flip, or an improvement is called $(p_G, \mathcal{P})$-*allowed* if it is p_G-admissible and uses only rainbow pairs. We will simply write *allowed* for $(p_G, \mathcal{P})$-allowed, and we will specify the function to be considered when it differs from p_G. A *complete allowed splitting off* is a sequence of allowed splitting off that decreases $m(V)$ to zero. If the splitting off at x, y is allowed, then we say that the pair x, y is allowed.

Let m be a degree specification and $P \in \mathcal{P}$. A pair (X_1, X_2) of disjoint sets of V is called a P-*pair* if there exists a subpartition $\mathcal{X}_i$ of $X_i \cap P$ such that $\sum_{X \in \mathcal{X}_i} p(X) = p(X_i)$ for $i = 1, 2$, while it is called an (m, P)-*pair* if the m-positive elements of $X_1 \cup X_2$ are the m-positive elements of P. A subpartition $\mathcal{X}$ of V is called a P-*subpartition* if there exists a set $X' \subseteq X \cap P$ for every $X \in \mathcal{X}$ such that $p(X') = 1$, while it is called an (m, P)-*subpartition* if each $X \in \mathcal{X}$ contains an m-positive element of P.

Constructions, obstructions, obstacles. Let $p : 2^V \to \mathbb{Z}$ be a symmetric crossing supermodular function, $m : V \to \mathbb{Z}_+$ a degree specification, and $\mathcal{P} = \{P_1, \dots, P_r\}$ a partition of V.

DEFINITION 7. *A partition $\mathcal{A} = \{A_1, \dots, A_4\}$ of V is called a* C_4^*-obstacle for $(p, \mathcal{P}, m)$ *if the following hold:*

1. (a) $p(A_i) + p(A_{i+1}) - p(A_i \cup A_{i+1})$ *is odd for* $i = 1, \dots, 4$;
   (b) $p(A_{i-1} \cup A_i) + p(A_i \cup A_{i+1}) = p(A_{i-1}) + p(A_{i+1})$ *for* $i = 1, \dots, 4$;
   (c) *if* $p(A_i) = 1$ *for* $i = 1, \dots, 4$, *then* $p(A_1 \cup A_3) = p(A_2 \cup A_4) \leq 0$;
   (d) $p(A_1) + p(A_3) = p(A_2) + p(A_4) = \frac{1}{2}\sigma_p$.
2. *m is minimally p-admissible.*
3. *m is $\mathcal{P}$-feasible and there exist $\ell \in \{1, 2\}$ and $P \in \mathcal{P}$ such that $(A_\ell, A_{\ell+2})$ is an (m, P)-pair.*

A partition $\mathcal{A}$ is called a C_4^*-construction for p *(respectively, a* C_4^*-obstruction for (p, m)*) if it satisfies* 1 *(respectively,* 1 *and* 2*).* C_4^**-constructions,* C_4^**-obstructions, and* C_4^**-obstacles satisfying* $p(A_i) = 1$ *for* $i = 1, \dots, 4$ *are called* simple.

DEFINITION 8. *A partition $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \dots, B_t\}$ of V $(t \geq 1)$ is called a* C_5^*-obstacle for $(p, \mathcal{P}, m)$ *if the following hold:*

1. (a) $p(A_i) = 1$ *for* $i = 1, \dots, 4$;
   (b) $p(B_j) = 2$ *for* $j = 1, \dots, t$;
   (c) $p(A_i \cup B_j) = 1$ *for* $i = 1, \dots, 4$ *and* $j = 1, \dots, t$;
   (d) $p(A_i \cup A_{i+1}) = 1$ *for* $i = 1, \dots, 4$;
   (e) $p(A_i \cup A_{i+2}) \leq 0$ *for* $i = 1, 2$;
   (f) $\sigma_p = \sum_{X \in \mathcal{A}} p(X) = 2t + 4$.
2. *m is minimally p-admissible.*
3. *m is $\mathcal{P}$-feasible and*
   (a) *either there exist $\ell \in \{1, 2\}$ and $P \in \mathcal{P}$ such that $\{A_\ell, A_{\ell+2}, B_1, \dots, B_t\}$ is an (m, P)-subpartition,*
   (b) *or there exist $j_0 \in \{1, \dots, t\}$ and distinct $P_{k_1}, P_{k_2} \in \mathcal{P}$ such that for $i = 1, 2$, $\{A_i, A_{i+2}\} \cup \{B_j : j \neq j_0\}$ is an (m, P_{k_i})-subpartition.*

A partition $\mathcal{A}$ *is called a* C_5^*-construction for p *(respectively,* C_5^*-obstruction for (p, m)*) if it satisfies* 1 *(respectively,* 1 *and* 2*). A* C_5^**-obstacle is of* type 1 *(respectively,* type 2*) if* 3a *(respectively,* 3b*) is satisfied. For a* C_5^**-construction* $\mathcal{A}$*,* consecutive elements *of* $\mathcal{A}$ *means sets* A_i*,* A_{i+1}*, where the index* i *is considered modulo* 4.

DEFINITION 9. *A partition* $\mathcal{A} = \{A_1, \ldots, A_6\}$ *of* V *is called a* C_6^*-obstacle for $(p, \mathcal{P}, m)$ *if the following hold:*

1. (a) $p(A_i) = 1$ *for* $i = 1, \ldots, 6$;
   (b) $p(A_i \cup A_{i+1}) = 1$ *for* $i = 1, \ldots, 6$;
   (c) $p(A_i \cup A_j) \leq 0$ *for all nonconsecutive sets* A_i *and* A_j;
   (d) $\sigma_p = \sum_{i=1}^{6} p(A_i) = 6$.
2. m *is minimally* p*-admissible.*
3. m *is* $\mathcal{P}$*-feasible and there exist distinct* $P_{k_i} \in \mathcal{P}$ *such that* (A_i, A_{i+3}) *is a* (m, P_{k_i})*-pair for* $i = 1, 2, 3$.

A partition $\mathcal{A}$ *is called a* C_6^*-construction for p *(respectively,* C_6^*-obstruction for (p, m)*) if it satisfies* 1 *(respectively,* 1 *and* 2*).*

A *construction* (respectively, *obstruction*, *obstacle*) is a C_4^*- or a C_5^*- or a C_6^*-construction (respectively, obstruction, obstacle). Note that an obstruction is a special type of construction, and an obstacle is a special type of obstruction.

OBSERVATION 10. *If there exists a construction, then* σ_p *is even.*

OBSERVATION 11. *Two* m*-positive elements that belong to consecutive sets* A_i *and* A_{i+1} *of an obstacle have different colors.*

3. Preliminaries. In this section, $p_0 : 2^V \to \mathbb{Z}$ is a symmetric crossing supermodular function, $G = (V, E)$ is a graph, m_G is a p_G-admissible degree specification with $m_G(V) \geq 4$, and $\mathcal{P} = \{P_1, \ldots, P_r\}$ is a partition of V.

3.1. Positive sets.

CLAIM 12.

1. *If a family* $\{X_1, \ldots, X_k\}$ *of* p_0*-positive subsets of* V *satisfies that* X_j *crosses* $\bigcup_{i=1}^{j-1} X_i$ *and* $p_G(X_j \cap (\bigcup_{i=1}^{j-1} X_i)) \leq p_G(X_j)$ *for* $j = 2, \ldots, k$*, then* $p_G(X_1) \leq p_G(\bigcup_1^j X_i) \leq p_G(\bigcup_1^k X_i)$ *for* $j = 1, \ldots, k$.
2. *If a subpartition* $\{W_1, \ldots, W_k\}$ *of* V *satisfies* $\bigcup_1^k W_i \neq V$*, and for every* $j = 2, \ldots, k$ *there exists an* i_j *such that* $1 \leq i_j < j$*,* $p_G(W_{i_j}) = 1$*, and* $p_G(W_{i_j} \cup W_j) \geq 1$*, then* $p_G(\bigcup_1^k W_i) \geq 1$.

Proof.

1. We prove this point by induction on k. For $k = 1$ the inequalities hold (with equalities). Suppose that the inequalities hold for $k - 1$, that is, $p_G(X_1) \leq p_G(\bigcup_1^j X_i) \leq p_G(\bigcup_1^{k-1} X_i)$ for $j = 1, \ldots, k - 1$. To finish the proof we have to show that $p_G(\bigcup_1^{k-1} X_i) \leq p_G(\bigcup_1^k X_i)$. Applying (13) to X_k and $\bigcup_{i=1}^{k-1} X_i$, and using $p_G(X_k \cap (\bigcup_{i=1}^{k-1} X_i)) \leq p_G(X_k)$, gives the result.
2. Apply point 1 to $\{W_{i_j} \cup W_j : j = 2, \ldots, k\}$. ∎

CLAIM 13. *If a partition* $\mathcal{A} = \{A_1, \ldots, A_t\}$ *of* V $(t \geq 4)$ *satisfies* $p_G(A_i) = p_G(A_i \cup A_{i+1}) = 1$ *for* $i = 1, \ldots, t$*, then the following statements hold:*

1. *No edge of* G *connects nonconsecutive members of* $\mathcal{A}$.
2. $p_G(\bigcup_{j \in J} A_j) = 1$ *for all nonempty consecutive* $J \subset \{1, \ldots, t\}$.
3. *If* $p_G(\bigcup_{j \in J} A_j) \geq 1$ *for some nonconsecutive index set* J*,* $p_G(A_k \cup A_\ell) \geq 1$ *for all nonconsecutive pair* $\{k, \ell\}$ *such that* $k, \ell \notin J$*, then* $\{k + 1, \ldots, \ell - 1\}$

and $\{\ell+1,\dots,k-1\}$ both intersect J.

4. *There exists a p_G-positive nonconsecutive index set if and only if $\mathcal{A}$ is a p_G-full partition.*

Proof. Let $A_J := \bigcup_{j\in J} A_j$ for any index set $J \subset \{1,\dots,t\}$, and let $\overline{J} = \{1,\dots,t\}\setminus J$.

1. Suppose that there exists an edge of G between A_i and A_j, where $j \notin \{i-1, i+1\}$. By (14) applied to $A_{i-1}\cup A_i$ and $A_i\cup A_{i+1}$, we have $1+1 = p_G(A_{i-1}\cup A_i)+p_G(A_i\cup A_{i+1}) \le p_G(A_{i-1})+p_G(A_{i+1}) - 2\overline{d}(A_{i-1}\cup A_i, A_i\cup A_{i+1}) \le 1+1-2$, a contradiction.
2. Without loss of generality, we may assume that $J = \{1,\dots,j\}$ for some $j \le t-1$. Let $X_i = A_i \cup A_{i+1}$ for $i = 1,\dots,t-2$. Since $\mathcal{A}$ is a partition of V, we have $X_j \cap (\bigcup_{i=1}^{j-1} X_i) = A_j$ for $j = 2,\dots,t-1$; hence Claim 12.1 applies to $\{X_1,\dots,X_{t-2}\}$. It gives, since p_G is symmetric, $1 = p_G(X_1) \le p_G(\bigcup_{j\in J} A_j) \le p_G(\bigcup_{j=1}^{t-1} A_j) = p_G(A_t) = 1$, and we have equality.
3. By $p_G(A_i) = p_G(A_i \cup A_{i+1}) = 1$ for $i = 1,\dots,t$, there exist consecutive pairs $J_2,\dots J_r$ such that with $J_1 := J$ the corresponding sets satisfy the conditions of Claim 12.1 and $\bigcup_{j=1}^{r} A_{J_j} = \overline{A_k \cup A_\ell}$. Then, by Claim 12.1, $p_G(A_J) \ge 1$, and by the symmetry of p_G, the assertion follows.
4. We prove only the nontrivial direction. Suppose that $p_G(A_K) \ge 1$ for some nonconsecutive index set K. Then there exists a nonconsecutive pair $\{q,r\} \subseteq \overline{K}$ such that $\{q+1,\dots,r-1\}$ and $\{r+1,\dots,q-1\}$ both intersect K. By point 3 applied to K, we have $p_G(A_q \cup A_r) \ge 1$. Then by point 3 applied to $\{q,r\}$, we have $p_G(A_{q-1}\cup A_{q+1}) \ge 1$, and then point 3 applies to $\{q-1, q+1\}$, which gives $p_G(A_q \cup A_j) \ge 1$ for all $j \ne q-1, q+1$. Moreover, by assumption $p_G(A_q) = p_G(A_{q-1}\cup A_q) = p_G(A_q \cup A_{q+1}) = 1$; thus by Lemma 2.2, $\mathcal{A}$ is a p_G-full partition. □

3.2. Tight sets. The following properties of tight sets are well known. In this section, we will often implicitly use Observation 5.

CLAIM 14. *Let X and Y be p_0-positive tight sets. Then the following statements hold:*

1. *If $X\cap Y \ne \emptyset$ and $X\cup Y \ne V$, then $X\cap Y$ is tight and $X\cup Y$ is p_0-positive tight.*
2. *If $X\setminus Y \ne \emptyset$ and $Y\setminus X \ne \emptyset$, then $X\setminus Y$ and $Y\setminus X$ are p_0-positive tight and $m_G(X\cap Y) = 0$.*
3. *If an m_G-positive element v belongs to X and Y, then one of them contains the other one. Consequently, if an m_G-positive element v belongs to a tight set, then v belongs to a unique minimal and a unique maximal tight set.*

Proof.

1. By the tightness of X and Y, (17) and (16), we get $0+0 = s_G(X)+s_G(Y) \ge s_G(X\cap Y)+s_G(X\cup Y) \ge 0+0$, so equality holds everywhere and the assertion follows.
2. By the tightness of X and Y, (18) and (16), and the nonnegativity of m_G, we get $0+0 = s_G(X)+s_G(Y) \ge s_G(X\setminus Y)+s_G(Y\setminus X)+2m_G(X\cap Y) \ge 0+0+0$, so equality holds everywhere and the assertion follows.
3. Suppose indirectly that $X\setminus Y \ne \emptyset$ and $Y\setminus X \ne \emptyset$. Then, by point 2 and $v \in (X\cap Y)\cap V_+(m_G)$, we have $0 = m_G(X\cap Y) \ge m_G(v) > 0$, a contradiction. □

For an m_G-positive element u, the definitions of X_u and T_u—the minimal and the maximal tight sets containing u—are correct by Claim 14.3.

CLAIM 15. *Let D be a subset of the m_G-positive elements such that each element of D belongs to a tight set. Then we have the following:*

1. *The family $\{X_u : u \in D\}$ is laminar.*
2. *There exists a partition $\mathcal{X}$ of $\bigcup_{u\in D} X_u$ such that $\sum_{X\in\mathcal{X}} p_G(X) \geq m_G(D)$.*

Proof.

1. Suppose that, for some $u, v \in D$, none of $X_u - X_v, X_v - X_u$, and $X_u \cap X_v$ is empty. Then, by Claim 14.2, $X_u \setminus X_v \subset X_u$ is a tight set containing u, which contradicts the minimality of X_u.
2. Let $\mathcal{X}$ be the maximal sets of $\{X_u : u \in D\}$. Then, by point 1, $\mathcal{X}$ is a partition of $\bigcup_{u\in D} X_u$, and, by m_G being nonnegative and each element of $\mathcal{X}$ being tight, we have $m_G(D) \leq m_G(\bigcup_{u\in D} X_u) = \sum_{X\in\mathcal{X}} m_G(X) = \sum_{X\in\mathcal{X}} p_G(X)$. □

It is important to mention that a degree specification m can be modified without destroying p-admissibility, as follows.

CLAIM 16. *Let u be an m-positive element and $u' \in X_u$. If m is p-admissible, then so is $m' := m - \chi_u + \chi_{u'}$.*

Proof. Suppose that m' is not p-admissible; that is, there exists a set $Y \subset V$ such that $m'(Y) + 1 \leq p(Y)$. Then, by m being p-admissible and the definition of m', we have $p(Y) \leq m(Y) \leq m'(Y) + 1 \leq p(Y)$. Thus equality holds everywhere; that is, Y contains u but not u' and it was tight. Then $u \in X_u \cap Y$, so, by assumption, $m(X_u \cap Y) > 0$, and then, by $u' \in X_u \setminus Y$, Claim 14.2 implies $Y \setminus X_u = \emptyset$, that is, $Y \subset X_u$, which contradicts the minimality of X_u. □

CLAIM 17. *Suppose that m_G is minimally p_G-admissible. If the pair u, v is p_G-admissible, then $m_{G_{uv}}$ is minimally $p_{G_{uv}}$-admissible.*

Proof. The pair u, v being p_G-admissible, $m_{G_{uv}}$ is $p_{G_{uv}}$-admissible. Moreover, the splitting off decreases by 1 the p_G-value of at most two sets of any partition achieving σ_{p_G}. Then, by the minimality of m_G, we have $m_{G_{uv}}(V) \geq \sigma_{p_{G_{uv}}} \geq \sigma_{p_G} - 2 = m_G(V) - 2 = m_{G_{uv}}(V)$. Hence we have equality everywhere, in particular $m_{G_{uv}}(V) = \sigma_{p_{G_{uv}}}$; that is, $m_{G_{uv}}$ is minimally $p_{G_{uv}}$-admissible. □

CLAIM 18. *If $p_G \leq 1$ and $X \subset V$ crosses a p_G-positive tight set, then $p_G(X) \leq 0$.*

Proof. Assume that X and a tight set Y are p_G-positive and crossing. Then, by (13) for X and Y and $p_G \leq 1$, we have $1 + 1 \leq p_G(X) + p_G(Y) \leq p_G(X \cap Y) + p_G(X \cup Y) \leq 1 + 1$, and so $p_G(Y) = 1 = p_G(X \cap Y)$. By Y being tight, $m_G(Y) = p_G(Y) = 1$. Hence, by possibly complementing X, we may assume that $m_G(X \cap Y) = 0$. Then, by p_G being symmetric and m_G being p_G-admissible, we get $1 = p_G(X \cap Y) \leq m_G(X \cap Y) = 0$, a contradiction. □

3.3. Dangerous sets. We start this subsection by the characterization of admissible pairs; see [4]. In light of Lemma 19 below, it is natural to study the properties of dangerous sets.

LEMMA 19 (see [4]). *A pair of m_G-positive elements u, v is p_G-admissible if and only if no dangerous set contains both u and v.*

The following technical claims will be applied throughout the paper. From now on we suppose that $m_G(V)$ is even.

CLAIM 20. *For a dangerous set Y, the following statements hold:*

1. $m_G(Y) \le \frac{1}{2}m_G(V)$.
2. *If* Y *contains an* m_G*-positive element of a dangerous set* X*, then* $m_G(V \setminus Y \setminus X) \ge 1$.
3. *If* Y *intersects a tight set* X*,* X *and* Y *are* p_0*-positive, and* $X \cup Y \ne V$*, then* $Y \cup X$ *is dangerous.*

Proof.

1. By Y being dangerous, p_G being symmetric, and m_G being p_G-admissible and modular, we have $m_G(Y) \le p_G(Y) + 1 = p_G(V \setminus Y) + 1 \le m_G(V \setminus Y) + 1 = m_G(V) - m_G(Y) + 1$, and then, by $m_G(V)$ being even, the assertion is satisfied.
2. By m_G being modular, point 1, and $m_G(Y \cap X) \ge 1$, we have $m_G(V \setminus Y \setminus X) = m_G(V) - m_G(Y) - m_G(X) + m_G(Y \cap X) \ge m_G(V) - \frac{1}{2}m_G(V) - \frac{1}{2}m_G(V) + 1 = 1$, so the assertion is satisfied.
3. By X being tight, Y being dangerous, (17) and (16), we have $0+1 \ge s_G(X) + s_G(Y) \ge s_G(X \cap Y) + s_G(Y \cup X) \ge 0 + s_G(Y \cup X)$, and the assertion is satisfied. ∎

CLAIM 21. *Let* $M = \max\{p_G(X) : X \subseteq V\}$*. If* W *is an inclusionwise minimal set satisfying* $p_G(W) = M$*,* X *is a dangerous set,* w, x *is a pair of* m_G*-positive elements,* $w \in W \cap X$*, and* $x \in X \setminus W$*, then the following hold:*

1. $W \subseteq X$*,*
2. $p_G(X) = M$*, and* $m_G(X \setminus W) = 1$*.*

Proof.

1. Suppose that $W \setminus X \ne \emptyset$. Then, by X being dangerous, $p_G(W) = M$, (14), (7), the minimality of W, and $w \in W \cap X$, we have $m_G(X) - 1 + M \le p_G(X) + p_G(W) \le p_G(X \setminus W) + p_G(W \setminus X) < m_G(X \setminus W) + M \le m_G(X) - 1 + M$, a contradiction, and the assertion follows.
2. By the definition of M, X being dangerous, point 1, the modularity of m_G, (7), $x \in (X \setminus W) \cap V_+(m_G)$, and $p_G(W) = M$, we have $M + 1 \ge p_G(X) + 1 \ge m_G(X) = m_G(W) + m_G(X \setminus W) \ge p_G(W) + 1 = M + 1$, so equality holds everywhere, and the assertion follows. ∎

LEMMA 22. *If* m_G *is* $\mathcal{P}$*-feasible and no* $(p_G, \mathcal{P})$*-allowed splitting off exists, then* $p_G(X) \le 1$ *for all* $X \subseteq V$*.*

Proof. Suppose that $M = \max\{p_G(X) : X \subseteq V\} \ge 2$. Let Y be an inclusionwise minimal set satisfying $p_G(Y) = M$. By the symmetry of p_G, we have $p_G(V \setminus Y) = M$, and let $Z \subseteq V \setminus Y$ be a minimal set satisfying $p_G(Z) = M$. By m_G being p_G-admissible, Y and Z contain m_G-positive elements.

Let $y \in Y$ and $z \in Z$ be m_G-positive elements. No dangerous set X contains y and z, since otherwise Claim 21 would imply $Y \cup Z \subseteq X$ and $1 = m_G(X \setminus Y)$, and then, by m_G being modular and nonnegative and by (7), we would have $1 = m_G(X \setminus Y) \ge m_G(Z) \ge p_G(Z) = M \ge 2$, a contradiction. It follows, by Lemma 19, that the pair y, z is p_G-admissible. Since there exists no allowed splitting off, the pair y, z is not rainbow.

Then, the set of the m_G-positive elements of $Y \cup Z$ is either a subset of some $P \in \mathcal{P}$ or is disjoint from a dominating color class $P' \in \mathcal{P}$. Since m_G is $\mathcal{P}$-feasible, there exists an m_G-positive element x of $V \setminus (Y \cup Z)$ that belongs to a dominating color class (if there exists one). Then, in the second case, $x \in P'$. For $y \in V_+(m_G) \cap Y$ and $z \in V_+(m_G) \cap Z$, x, y and x, z are rainbow pairs; hence there exist a dangerous set X containing x, y and a dangerous set X' containing x, z. Claim 21 applies to X and Y and also to X' and Z; hence $p_G(X) = p_G(X') = M$ and $m_G(X \cap X') = 1$. By

Claim 20.2, X and X' are crossing, and now (13) implies that $p_G(X \cap X') = M \geq 2 > m_G(X \cap X')$, which contradicts the p_G-admissibility of m_G. ∎

Claim 23. *Let $\mathcal{M} := \{M_1, M_2\}$ be a family of maximal dangerous sets. If $m_G(M_i^\star) \geq 1$ for $i = 0, 1, 2$, then the following hold:*
1. *$s_G(M_1) = s_G(M_2) = 1$, $M_1 \cap M_2$ is tight, and $s_G(M_1 \cup M_2) = 2$;*
2. *$M_1 \setminus M_2$ and $M_2 \setminus M_1$ are tight and $m_G(M_0^\star) = 1$;*
3. *$M_i^\star$ is a maximal tight set for $i = 0, 1, 2$.*

Proof. By Claim 20.2, $m_G(M_i^\star) \geq 1$ for $i = 0, 1, 2$, and by Observation 5, M_1 and M_2 are crossing p_0-positive sets. Thus (17) and (18) apply to M_1 and M_2.
1. By M_1 and M_2 being dangerous, (17), (16), and $M_1 \cup M_2$ not being dangerous, we have $1 + 1 \geq s_G(M_1) + s_G(M_2) \geq s_G(M_1 \cap M_2) + s_G(M_1 \cup M_2) \geq 0 + 2$, so equality holds everywhere and the assertion follows.
2. By M_1 and M_2 being dangerous, $M_i^\star \neq \emptyset$, (18), (16), and $m_G(M_0^\star) \geq 1$, we get $1 + 1 \geq s_G(M_1) + s_G(M_2) \geq s_G(M_1 \setminus M_2) + s_G(M_2 \setminus M_1) + 2m_G(M_1 \cap M_2) \geq 0 + 0 + 2$, so equality holds everywhere and the assertion follows.
3. By parts 1 and 2 of this claim, M_1 and M_2 being dangerous, and Claim 20.3, we have the assertion. ∎

Claim 24. *Let $\mathcal{M} = \{M_1, \ldots, M_\ell\}$ be a family of maximal dangerous sets with $\ell \geq 3$. If $m_G(M_i^\star) \geq 1$ for $i = 0, \ldots, \ell$, then the following hold:*
1. *$M_i^\star$ is a maximal tight set and $M_i = M_i^\star \cup M_0^\star$;*
2. *$m_G(M_i^\star) = 1$;*
3. *$s_G(M_j^\star \cup M_k^\star) = 1$ for $1 \leq j < k \leq \ell$.*

Proof.
1. By $m_G(M_i^\star) \geq 1$, there exists an m_G-positive element u_i in $M_i^\star$ for $i = 0, \ldots, \ell$. By applying Claim 23.3 to M_j, M_k $(1 \leq j < k \leq \ell)$, we get that $M_j \cap M_k = T_{u_0}$, $M_j \setminus M_k = T_{u_j}$, and $M_k \setminus M_j = T_{u_k}$. Then it follows that $M_i^\star = T_{u_i}$ and $M_i = M_i^\star \cup M_0^\star$ for $i = 0, \ldots, \ell$, so the assertion is satisfied.
2. Let i, j and k be three different indices between 1 and ℓ. By Claim 23.1, (18) applied to $M_i \cup M_j$ and $M_i \cup M_k$, point 1 of the current claim, modularity of m_G, (16), Claim 23.2, and $m_G(M_i^\star) \geq 1$, we have $2 + 2 = s_G(M_i \cup M_j) + s_G(M_i \cup M_k) \geq s_G(M_j^\star) + s_G(M_k^\star) + 2m_G(M_0^\star) + 2m_G(M_i^\star) \geq 0 + 0 + 2 + 2$, so equality holds everywhere and the assertion is satisfied.
3. By M_i being dangerous, Claim 23.1, (18) applied to M_i and $M_j \cup M_k$, point 1 of the current claim, (16), and Claim 23.2, we have $1 + 2 \geq s_G(M_i) + s_G(M_j \cup M_k) \geq s_G(M_i^\star) + s_G(M_j^\star \cup M_k^\star) + 2m_G(M_0^\star) \geq 0 + 1 + 2$, so equality holds everywhere and the assertion is satisfied. ∎

Using the above claims, we generalize a theorem of [6] on admissible edges. It will help us to find an allowed pair when no simple C_4^*-obstacle exists but an admissible pair exists.

For an m-positive element t, let S_t be the set of m-positive elements admissible with t.

Lemma 25. *Let $p : 2^V \to \mathbb{Z}$ be a symmetric crossing supermodular set function, and $m : V \to \mathbb{Z}_+$ a p-admissible degree specification with $m(V) \geq 4$ even. Suppose that an admissible pair exists. Then*
(i) *either there is an m-positive element t such that $m(S_t) \geq \frac{1}{2}m(V)$,*
(ii) *or there is a simple C_4^*-obstruction.*

Proof. By Lemma 19, let $\mathcal{M}_t = \{M_1, \dots, M_\ell\}$ be a minimal family of maximal dangerous sets such that $t \in M_0^\star$ and $V_+(m) \setminus S_t = V_+(m) \cap \bigcup_{i=1}^{\ell} M_i$. Suppose that (i) is violated, that is,

$$(*) \qquad m(S_t) \leq \frac{1}{2}m(V) - 1 \quad \text{for all } t \in V_+(m).$$

CLAIM 26. *For all $t \in V_+(m)$, we have $|\mathcal{M}_t| = 2$, $m(M_0^\star) = 1$, $M_i^\star$ a maximal tight set, and $M_i = M_i^\star \cup M_0^\star$ for all $M_i \in \mathcal{M}_t$.*

Proof. If for some $t \in V_+(m)$, $|\mathcal{M}_t| \leq 1$, then, by Claim 20.1, $m(S_t) \geq m(V) - m(M_1) \geq m(V) - \frac{1}{2}m(V) = \frac{1}{2}m(V)$, which contradicts $(*)$. Thus $|\mathcal{M}_t| \geq 2$ for all $t \in V_+(m)$. Suppose that, for some $t_0 \in V_+(m)$, $\ell = |\mathcal{M}_{t_0}| \geq 3$. By Claim 24 and Lemma 19, $S_{t_i} = S_{t_0}$ for all $t_i \in V_+(m) \setminus S_{t_0}$. The existence of an admissible pair implies that there exists $u \in S_{t_0}$. Since $t_0 \in S_u$, $\{t_0, t_1, \dots, t_\ell\} \subseteq S_u$. Then $(*)$ applied to u, Claim 24.2, and $(*)$ applied to t_0 imply that $\frac{1}{2}m(V) - 1 \geq m(S_u) \geq \ell + 1 = m(V \setminus S_{t_0}) = m(V) - m(S_{t_0}) \geq \frac{1}{2}m(V) + 1$, contradiction. Thus $|\mathcal{M}_t| = 2$ for all $t \in V_+(m)$, and, by Claim 23 applied to M_1 and M_2, the claim follows. ☐

By Claim 26, for all $t \in V_+(m)$ there exist $t_1, t_2 \in V_+(m)$ such that $\mathcal{M}_t = \{T_t \cup T_{t_1}, T_t \cup T_{t_2}\}$ and $m(T_t) = 1$. Then, by $(*)$ and $m(V) \geq 4$, for all $t \in V_+(m)$, $3 = m(T_t) + m(T_{t_1}) + m(T_{t_2}) = m(\bigcup \mathcal{M}_t) \geq \frac{1}{2}m(V) + 1 \geq 3$, and hence $m(V) = |V_+(m)| = 4$. Let $V_+(m) = \{a_1, a_2, a_3, a_4\}$ so that $\mathcal{M}_{a_1} = \{T_{a_1} \cup T_{a_2}, T_{a_1} \cup T_{a_4}\}$.

CLAIM 27. *$\mathcal{A} = \{T_{a_1}, T_{a_2}, T_{a_3}, T_{a_4}\}$ is a simple C_4^*-obstruction.*

Proof. By Claim 26 and $\mathcal{M}_{a_1} = \{T_{a_1} \cup T_{a_2}, T_{a_1} \cup T_{a_4}\}$, $\mathcal{M}_{a_3} = \{T_{a_2} \cup T_{a_3}, T_{a_3} \cup T_{a_4}\}$. Since $T_{a_1} \cup T_{a_2}$ is dangerous, so is $V \setminus (T_{a_1} \cup T_{a_2})$ by $m(V) = 4$. By maximality, $V \setminus (T_{a_1} \cup T_{a_2}) = T_{a_3} \cup T_{a_4}$, so $\{T_{a_1}, T_{a_2}, T_{a_3}, T_{a_4}\}$ is a partition of V. By $m(V) = 4$, m being p-admissible, the definition of σ_p, and Claim 26, we have $4 = m(V) \geq \sigma_p \geq \sum_{i=1}^{4} p(T_{a_i}) = 4$, which implies Definitions 7.2 and 7.1d for $\mathcal{A}$. Claim 23.1 implies that $p(T_{a_i}) = 1 = p(T_{a_i} \cup T_{a_{i+1}})$, so Definitions 7.1a and 7.1b hold for $\mathcal{A}$. Since there exists an admissible pair, Definition 7.1c also holds for $\mathcal{A}$. ☐

By Claim 27, (ii) of Lemma 25 is satisfied, and thus Lemma 25 is proved. ☐

COROLLARY 28. *There exists no p_G-admissible pair if and only if there exists a partition $\{V_1, \dots, V_\ell\}$ of V such that the following hold:*

1. $\ell \geq 4$;
2. *for $1 \leq i < j \leq \ell$, $m_G(V_i) = p_G(V_i) = p_G(V_i \cup V_j) = 1$ and V_i is a maximal tight set; that is, $\{V_1, \dots, V_\ell\} = \{T_w : w \in V_+(m_G)\}$;*
3. *for all $e = uv \in E$ there exists $1 \leq i_e \leq \ell$ such that $u, v \subseteq V_{i_e}$;*
4. *$p_G(\bigcup_{j \in J} V_j) = 1$ for all nonempty $J \subset \{1, \dots, \ell\}$.*

Proof. The sufficiency follows from point 2 and Lemma 19. Let us see the necessity. For an m_G-positive element t, by Lemma 19, let $\mathcal{M}_t = \{M_1, \dots, M_{\ell-1}\}$ be a minimal family of maximal dangerous sets containing t and covering all the m_G-positive elements.

1. By Claim 20.2, $\ell - 1 \geq 3$.
2. By Claim 24 applied to $\mathcal{M}_t$, $\{V_{i+1} := M_i^\star : i = 0, \dots, \ell - 1\}$ is a subpartition of V satisfying part 2. It is in fact a partition of V because if $Z := V \setminus \bigcup_i V_i \neq \emptyset$, then by the fact that $\mathcal{M}_t$ covers all the m_G-positive elements, m_G is p_G-admissible, p_G is symmetric, and M_i is p_G-positive for $i = 1, \dots, \ell - 1$, so by Claim 12.1 applied to $\mathcal{M}_t$, we have $0 = m_G(Z) \geq p_G(Z) = p_G(\bigcup_{i=1}^{\ell} V_i) = p_G(\bigcup_{i=1}^{\ell-1} M_i) \geq p_G(M_1) \geq 1$, a contradiction.

3.–4. Note that, by point 2, Claim 13 can be used for any order of the sets in $V_1, \dots, V_\ell$, and then, by Claims 13.1–2, the assertions in 3 and 4 follow. ∎

COROLLARY 29. *If there exists no p_G-admissible pair, $m_G(V) \geq 6$, and G' is obtained from G by a p_G-admissible improvement, then no $p_{G'}$-admissible pair exists.*

Proof. Suppose that G' is obtained from G by the p_G-admissible improvement of uv to ux, vy. Let $V_1, \dots, V_\ell$ be the partition of V provided by Corollary 28 applied for p_G. Then there exist $1 \leq i, j, k \leq \ell$ such that $x \in V_i$, $y \in V_j$, and $u, v \subseteq V_k$. Let $X = V_i \cup V_j \cup V_k$. By Corollary 28.2, modularity of m_G, the improvement being p_G-admissible, and Corollary 28.4, we have $1 = 1+1+1-2 = m_G(V_i)+m_G(V_j)+m_G(V_k)-2 = m_G(X) - 2 = m_{G'}(X) \geq p_{G'}(X) = p_G(X) = 1$, and then, by Corollary 28 for p_G and $m_G(V) \geq 6$, it follows that $\{V_1, \dots, V_\ell\} \setminus \{V_i, V_j, V_k\} \cup \{X\}$ satisfies Corollary 28.1–4 for $p_{G'}$, and hence, by Corollary 28, no $p_{G'}$-admissible pair exists. ∎

3.4. Perilous sets. In the previous section, the study of admissible pairs led to dangerous sets. Here, we are interested in admissible flips and improvements. This is where perilous sets come into play; see Lemma 31. We will often implicitly use the fact that a perilous set is p_0-positive; see Observation 5.

LEMMA 30. *Let x and y be two distinct m_G-positive elements and uv an edge of G. Then we have the following:*

(i) *Flipping uv for ux is p_G-admissible if and only if no dangerous set contains x and u but not v.*

(ii) *Improving uv to ux, vy is p_G-admissible if and only if both flipping uv for ux and flipping vu for vy are p_G-admissible and no dangerous set contains x, u, v and y.*

Proof.

(i) Recall that flipping uv for ux consists of first unsplitting uv and afterwards splitting off at x, u. Since unsplitting uv is p_G-admissible, $m_{G^{uv}}$ is $p_{G^{uv}}$-admissible. Then, flipping uv for ux is not p_G-admissible if and only if splitting off at u, x is not $p_{G^{uv}}$-admissible, which is equivalent, by Lemma 19, to the fact that there exists a dangerous set X with respect to $p_{G^{uv}}$ and $m_{G^{uv}}$ (that is, $s_{G^{uv}}(X) \leq 1$) containing x and u. Then, Observation 4 and $s_G(X) \geq 0$ imply $v \notin X$ and $s_G(X) = s_{G^{uv}}(X) \leq 1$; that is, X is dangerous with respect to p_G and m_G and contains x and u but not v.

(ii) Note that improving uv to ux, vy can be considered as flipping uv for ux and then splitting off at v, y. Let H (respectively, K) be the graph obtained from G after flipping uv for ux (respectively, improving uv to ux, vy).
   (a) To prove the necessity, suppose that improving uv to ux, vy is p_G-admissible; that is, m_K is p_K-admissible. Since unsplitting vy is p_K-admissible, m_H is p_H-admissible; that is, flipping uv for ux is p_G-admissible. Similarly, flipping vu for vy is p_G-admissible. If a dangerous set X of G contained x, u, v, and y, then, by Observation 4 and since m_K is p_K-admissible, we would have $1 \geq s_G(X) = s_K(X) + 2 \geq 2$, a contradiction.
   (b) To prove the sufficiency, suppose that improving uv to ux, vy is not p_G-admissible. If the (uv, ux)-flip or the (vu, vy)-flip is not p_G-admissible, then we are done; hence suppose they are both p_G-admissible. Since the splitting off at v, y is not p_H-admissible, there exists, by Lemma 19, a set X containing y and v which is dangerous with respect to p_H and m_H. Note that, since flipping vu for vy is p_G-admissible, X is not dangerous

with respect to $p_{G^{uv}}$ and $m_{G^{uv}}$. Hence we have $s_{G^{uv}}(X) \geq 2 > 1 \geq s_H(X)$. Since $G^{uv} = H^{ux}$, Observation 4 implies that X contains u and x. Therefore $s_G(X) = s_H(X)$; thus X is also dangerous with respect to p_G and m_G and contains x, u, v and y. ∎

As suggested by Lemma 22, it is reasonable to study the admissibility of flips and improvements when $p_G \leq 1$. The following lemma reveals how perilous sets arise in the process. In fact, perilous sets will always be studied when $p_G \leq 1$ and $\{T_w, w \in V_+(m_G)\}$ is a partition of V. We derive some of their properties in this situation. Note that, then, $T_x \cap T_y = \emptyset$ whenever x and y are distinct m_G-positive elements.

LEMMA 31. *Suppose that $p_G \leq 1$, $\{T_w, w \in V_+(m_G)\}$ is a partition of V, $x, y, z \in V_+(m_G)$, $uv \in E$, and $u, v \in T_z$.*

(i) *Flipping uv for ux is p_G-admissible if and only if there exists no (uv, ux)-perilous set.*

(ii) *Improving uv to ux, vy is p_G-admissible if and only if neither a (uv, ux)- nor a (vu, vy)-perilous set exists.*

Proof.

(i) The necessity comes from Lemma 30(i) and the definition of perilous sets. To see the sufficiency, by Lemma 30(i), we just have to show that a dangerous set X containing u and x but not v is a perilous set. Indeed, by $x \in V_+(m_G) \cap X$, m_G being modular and nonnegative, X being dangerous, and $p_G \leq 1$, we have $0 \leq m_G(x) - 1 \leq m_G(X) - 1 \leq p_G(X) \leq 1$. Then, by $m_G(V) \geq 4$, X crosses T_z; therefore, by Claim 18, we have $p_G(X) = 0$, and assertion (i) is proved.

(ii) We apply Lemma 30(ii). First, no dangerous set contains x, u, v, y: if X was such a set, since $p_G \leq 1$, we would have $2 \leq m_G(X) \leq p_G(X) + 1 \leq 2$. That would imply $p_G(X) = 1$ and $z \notin X$, and hence X and T_z would be crossing because $m_G(V) \geq 4$, contradicting Claim 18. Then, applying point (i) to the (uv, ux)- or the (vu, vy)-flip gives the assertion. ∎

The following results will be applied when either no admissible splitting off exists or a simple C_4^*-obstacle exists.

CLAIM 32. *Suppose that $p_G \leq 1$, $\{T_w, w \in V_+(m_G)\}$ is a partition of V, $x, z \in V_+(m_G)$, $x \neq z$, $uv \in E$, and $u \in T_z$. If X is a (uv, ux)-perilous set, then the following hold:*

1. $m_G(T_w) = 1$ *for all* $w \in V_+(m_G)$;
2. $m_G(X \cap T_z) = 0$, $p_G(X \cap T_z) = 0$, $p_G(X \cup T_z) = 1$, *and* $d(X, T_z) = 0$;
3. $p_G(X \setminus T_z) \geq 0$, $p_G(T_z \setminus X) \geq 0$, *and* $\overline{d}(X, T_z) = 0$;
4. $X \cup T_z = T_x \cup T_z$; *that is,* $X \setminus T_z = T_x$.

Proof. By $u \in X \cap T_z$, $x \in V_+(m_G) \cap (X \setminus T_z)$, $m_G(X) = 1$, and $m_G(V) \geq 4$, inequalities (13) and (14) apply to X and T_z.

1. By $w \in V_+(m_G) \cap T_w$, T_w being tight, and $p_G \leq 1$, we have $1 \leq m_G(T_w) = p_G(T_w) \leq 1$, and the assertion follows.
2. By (7), $m_G(X) = 1$, and $x \in V_+(m_G) \cap (X \setminus T_z)$, we have $p_G(X \cap T_z) \leq m_G(X \cap T_z) = 0$. Then, by X being perilous, Observation 5, (13) applied to X, and T_z and $p_G \leq 1$, we get $0 + 1 \leq p_G(X) + p_G(T_z) \leq p_G(X \cap T_z) + p_G(X \cup T_z) - 2d(X, T_z) \leq 0 + 1 - 0$, and the assertion follows.
3. By X being perilous, Observation 5, (14), and $p_G \leq 1$, we get $0 + 1 \leq p_G(X) + p_G(T_z) \leq p_G(X \setminus T_z) + p_G(T_z \setminus X) - 2\overline{d}(X, T_z) \leq 1 + 1 - 2\overline{d}(X, T_z)$, and the assertion follows.

4. Since $\{T_w, w \in V_+(m_G)\}$ partitions V, $m_G(V) \geq 4$ and, by m_G being modular, X being perilous, and by parts 1 and 2 of this claim, we have $m_G(X \cup T_z) = m_G(X) + m_G(T_z) - m_G(X \cap T_z) = 1 + 1 - 0 = 2$. Since, by part 2, $p_G(X \cup T_z) = 1$ and T_w is tight for $w \in V_+(m_G)$, Claim 18 implies that $X \cup T_z$ does not cross T_w. Then, by $m_G(X \cup T_z) = 2$, $X \cup T_z$ contains T_x and T_z and no other T_w; that is, $X \cup T_z = T_x \cup T_z$. Then, $T_x \cap T_z = \emptyset$ implies $X \setminus T_z = T_x$. ∎

Corollary 33. *Suppose that $p_G \leq 1$, $\{T_w, w \in V_+(m_G)\}$ is a partition of V, $z \in V_+(m_G)$, $uv \in E$, and $u \in T_z$. If $x \in V_+(m_G) \setminus \{z\}$ and $v \notin T_z$, then no (uv, ux)-perilous set exists.*

Proof. Indeed, if X is (uv, ux)-perilous for some $x \in V_+(m_G) \setminus \{z\}$, then, by Claim 32.4, we have $X \setminus T_z = T_x$. Moreover, by Claim 32.3, we have $\overline{d}(X, T_z) = 0$, and hence $v \in X \cup T_z$. Since, by definition, $v \notin X$, we get $v \in T_z$. However, by assumption, $v \notin T_z$. This contradiction proves the corollary. ∎

For $uv \in E$ and $z \in V_+(m_G)$ such that $u, v \in T_z$, let R_{uv}= $\{x \in V_+(m_G) \setminus \{z\}$: there exists a (uv, ux)-perilous set$\}$ and $\mathcal{R}_{uv}$ be the family of (uv, ux)-perilous sets for $x \in R_{uv}$. Note that $R_{uv} \neq R_{vu}$ and $\mathcal{R}_{uv} \neq \mathcal{R}_{vu}$.

Claim 34. *Suppose that $p_G \leq 1$ and $\{T_w, w \in V_+(m_G)\}$ is a partition of V, $uv \in E, z \in V_+(m_G), u, v \in T_z$. Then the following hold:*

1. *If X and X' are respectively (uv, ux)- and (uv, ux')-perilous sets for distinct $x, x' \in R_{uv}$, then $\overline{d}(X, X') = 1$, $X \cap X' = X \cap T_z = X' \cap T_z$, $p_G(X \cap X') = 0$, and $p_G(X \cup X') = 0$.*
2. *If X and X' are respectively (uv, ux)- and (vu, vx')-perilous sets for $x, x' \in R_{uv}$, then $X \cap X' = \emptyset$, $x \neq x'$, $p_G(T_z \setminus X) = p_G(T_z \setminus X') = 0$, and $p_G(T_z \setminus (X \cup X')) = 1$.*
3. *If $|R_{uv}| \geq 2$, then there exists a unique $X_{uv} \subset T_z$ such that $\mathcal{R}_{uv} = \{X_{uv} \cup T_x : x \in R_{uv}\}$.*
4. *If $|R_{uv}| \geq 2$ and all the edges of G are contained in members of the partition $\{T_w, w \in V_+(m_G)\}$ of V, then $d_G(X_{uv}) = 1$ and $p_0(X_{uv}) = 1$.*
5. *If $u'v' \in E$, $u', v' \in T_z$, $|R_{uv}| \geq 2$, $|R_{u'v'}| \geq 2$, and $R_{uv} \cap R_{u'v'} \neq \emptyset$, then $\{X_{uv}, X_{u'v'}\}$ is laminar.*

Proof.

1. Since uv connects $X \cap X'$ and $V \setminus (X \cup X')$ and $p_G \leq 1$, applying (14) to the perilous sets X and X' gives $p_G(X \setminus X') = p_G(X' \setminus X) = 1$ and $\overline{d}(X, X') = 1$. By Claim 32.4, we have $T_x = X \setminus T_z$ and $T_x \cap X' = \emptyset$; hence $T_x \subseteq X \setminus X'$. Since $m_G(X \setminus X') = 1$, the maximality of T_x implies $X \setminus X' = T_x$. Similarly, we have $X' \setminus X = T_{x'}$. Thus $X \cap X' = X \cap T_z = X' \cap T_z$.
   Then, by Claim 32.2, $p_G(X \cap X') = 0$. By (13) applied to X and X' and Claim 18, we get $0+0 = p_G(X)+p_G(X') \leq p_G(X \cap X')+p_G(X \cup X') \leq 0+0$, and the assertion follows.
2. By Claim 32.1 and $x \in R_{uv}$, we have $x \in X \setminus T_z$. Note that $u \in (X \setminus X') \cap T_z$, $v \in (X' \setminus X) \cap T_z$, and, by Claim 32.2, $z \in T_z \setminus (X \cup X')$. If $X \cap X' \neq \emptyset$, then X and X' are crossing, and, by $m_G(V) \geq 4$ and Claim 32.1–4, $X \cup X'$ and T_z are also crossing. Then, by X and X' being perilous, (13) applied to X and X', uv connecting $X \setminus X'$ and $X' \setminus X$, $p_G \leq 1$, and Claim 18 applied for $X \cup X'$ and T_z, we have $0+0 = p_G(X)+p_G(X') \leq p_G(X \cap X')+p_G(X \cup X')-2d(X, X') \leq 1 + 0 - 2$, a contradiction.
   By $X \cap X' = \emptyset$, $x \in X$, and $x' \in X'$, we have $x \neq x'$.

Let $Y = T_z \setminus X$ and $Y' = T_z \setminus X'$. Claim 32.3 applied to T_z and X (and X') gives that $p_G(Y)$, $p_G(Y') \geq 0$. Hence, since uv enters Y and Y', they are p_0-positive. Moreover they cross; thus, by (13) applied to Y and Y', $p_G \leq 1$, and uv connecting $Y \setminus Y'$ and $Y' \setminus Y$, we have $0 = p_G(Y) = p_G(Y')$ and $1 = p_G(Y \cap Y')$, and the assertion follows.

3. By $|R_{uv}| \geq 2$, there exist (uv, ux)- and (uv, ux')-perilous sets X and X' for distinct $x, x' \in R_{uv}$. Defining $X_{uv} = X \cap X'$, point 1 gives the assertion.
4. By point 3, X_{uv} exists. By point 1 and the fact that all the edges of G are contained in members of the partition $\{T_w, w \in V_+(m_G)\}$ of V, $d_G(X_{uv}) = 1$ and $p_G(X_{uv}) = 0$. Then, by $p_0 = p_G + d_G$, the assertion follows.
5. Suppose indirectly that $\{X_{uv}, X_{u'v'}\}$ is not laminar. Then X_{uv} and $X_{u'v'}$ are crossing. Note that it is possible that $u'v' = vu$. By $|R_{uv}| \geq 2$, $R_{uv} \cap R_{u'v'} \neq \emptyset$, and point 3 of this claim, there exist two distinct $x, y \in V_+(m) \setminus \{z\}$ such that $X = X_{uv} \cup T_x$ is a (uv, ux)-perilous set, $X' = X_{uv} \cup T_y$ is a (uv, uy)-perilous set, and $Y = X_{u'v'} \cup T_y$ is a $(u'v', u'y)$-perilous set. Note that X and Y are crossing. By $X \cap Y \neq \emptyset$ and part 2 of this claim, $u'v' \neq vu$. Since $p_G \leq 1$, applying (14) to X and Y, and then Claim 18, gives that $p_G(X \setminus Y) = p_G(Y \setminus X) = 0$. We have three cases, as follows:
   i. If $u \notin X_{u'v'}$, then $X \setminus Y$ is a (uv, ux)-perilous set, so by point 3 of this claim, we have $X_{uv} \cup T_x = X \setminus Y \subset X_{uv} \cup T_x$, a contradiction.
   ii. If $u \in X_{u'v'}$ and $v \notin X_{u'v'}$, then Y is a (uv, uy)-perilous set, so by point 3 of this claim, we have $X_{uv} \cup T_y = Y = X_{u'v'} \cup T_y$, a contradiction.
   iii. If $u, v \in X_{u'v'}$, then the (vu, vy)-perilous set $Y \setminus X$ and the (uv, uy)-perilous set X' contradict point 2 of this claim. □

We reformulate an important part of Claim 34.2 as follows.

LEMMA 35. *Suppose that $p_G \leq 1$, $\{T_w, w \in V_+(m_G)\}$ is a partition of V, $x, z \in V_+(m_G)$, $x \neq z$, $uv \in E$, and $u, v \in T_z$. If there exists a (uv, ux)-perilous set, then no (vu, vx)-perilous set exists.*

Proof. Suppose that X and X' are (uv, ux)- and (vu, vx)-perilous sets. Then, by definition and by Claim 34.2, we have $x \in X \cap X' = \emptyset$, a contradiction. □

From now on, for $e = uv \in E$, we say that X_e *exists* whenever one of X_{uv} and X_{vu} exists; that is, one of $|R_{uv}|$ and $|R_{vu}|$ is greater than or equal to 2.

The following claim will be applied when no admissible splitting off exists.

CLAIM 36. *Suppose that m_G is allowed, no p_G-admissible pair exists, $z \in V_+(m_G)$, $e = uv \in E$, $u, v \in T_z$, and no allowed improvement exists for e. Then the following statements hold:*

1. *There exists a (uv, w)-perilous set for some $w \in V_+(m_G) \setminus \{z\}$.*
2. *If X is a (uv, ux)-perilous set for some $x \in V_+(m_G) \setminus \{z\}$, then $T_w \cup (X \cap T_z)$ is a (uv, uw)-perilous set for all $w \in V_+(m_G) \setminus \{z\}$.*
3. *X_e is well defined, and $p_0(X_e \cup T_w) = 1$ for all $w \in V_+(m_G) \setminus \{z\}$.*

Proof. By Lemma 22, we have $p_G \leq 1$. Moreover, by Corollary 28, we have $m_G(V) \geq 4$, that $\{T_w, w \in V_+(m_G)\}$ is a partition of V, and $p(T_x \cup T_y) = 1$ for all $x, y \in V_+(m_G)$. Suppose that $m_G(P_1) = \max\{m_G(P) : P \in \mathcal{P}\}$.

1. Without loss of generality there exist $x \in V_+(m_G) \cap P_1$ and $y \in V_+(m_G) \setminus P_1$ such that $x \neq z \neq y$. By possibly exchanging u and v, we can assume that $c(u) \neq c(x)$ and $c(v) \neq c(y)$. Then, since improving uv to ux, vy is not allowed, it is not p_G-admissible, and, by Lemma 31(ii), the assertion follows for $w = x$ or $w = y$.

2. Let $w \in V_+(m_G) \setminus \{x, z\}$. Since $m_G(V) \geq 4$, there exists $y \in V_+(m_G) \setminus \{x, w, z\}$. Apply (3) to X and $W = T_x \cup T_w$ to get that $p_G(X \cup W) \geq 0$. Now apply (4) to $X \cup W$ and $T_x \cup T_y$ to obtain $p_G(T_w \cup (X \cap T_z)) \geq 0$, and then, since this set crosses the p_G-positive tight set T_z, we get, by Claim 18, that $p_G(T_w \cup (X \cap T_z)) = 0$. Therefore, by Claim 32.1–2, $T_w \cup (X \cap T_z)$ is a (uv, uw)-perilous set and the assertion follows.
3. By points 1 and 2 of this claim, $m_G(V) \geq 4$, and Claim 34.3, X_{uv} or X_{vu} exists, so by Lemma 35, exactly one of them exists; thus X_e is well-defined. By point 2 of this claim, $X_e \cup T_w$ is perilous, so $p_G(X_e \cup T_w) = 0$. Then, by Corollary 28.3, we have $d_G(T_w) = 0$; by Claim 34.4, we have $d_G(X_e) = 1$; and hence, by $p_0(X_e \cup T_w) = p_G(X_e \cup T_w) + d_G(X_e \cup T_w) = 0 + 1 = 1$, the assertion follows. ∎

3.5. A special full partition. Let us introduce the following sets and families:

$$\begin{aligned} \mathcal{U} &= \{T_w : w \in V_+(m_G)\} \bigcup \{X_e : e \in E\}, \\ U^* &= U \setminus \bigcup \{U' : U' \in \mathcal{U}, U' \subset U\} \quad \text{for all } U \in \mathcal{U}, \\ \mathcal{U}^* &= \{U^* : U \in \mathcal{U}\}. \end{aligned}$$

LEMMA 37. *Suppose that m_G is allowed, $m_G(V) \geq 4$, and that neither a p_G-admissible splitting off nor an allowed improvement exists. Then the following statements hold:*

1. *$\{T_w, w \in V_+(m_G)\}$ is a tight partition of V and, for all $e = uv \in E$, there exists $z_e \in V_+(m_G)$ such that $u, v \subseteq T_{z_e}$ and $X_e \subset T_{z_e}$ exists.*
2. *$\mathcal{U}^*$ is a partition of V.*
3. *$U^* \neq \emptyset$ for all $U \in \mathcal{U}$.*
4. *$p_0(T_w \cup U) = 1$ for all $w \in V_+(m_G)$ and $U \in \mathcal{U}$.*
5. *$p_0(T_w \cup U^*) \geq 1$ for all $w \in V_+(m_G)$ and $U \in \mathcal{U}$.*
6. *$p_0(U^* \cup W^*) \geq 1$ for all $U, W \in \mathcal{U}$.*
7. *$\mathcal{U}^*$ is a p_0-full partition of V.*
8. *$|\mathcal{U}^*| = m_G(V) + |E|$.*
9. *m_0 is not p_0-legal.*
10. *There exists a $\mathcal{P}$-partite graph F on V that covers p_0 with $|E(F)| \leq \dim(p_0) - 1$.*

Proof. By Lemma 22, we have $p_G \leq 1$.

1. By Corollary 28.2–3, $\{T_w, w \in V_+(m_G)\}$ is a tight partition of V and, for all $e \in E$, there exists $z_e \in V_+(m_G)$ such that $e \subseteq T_{z_e}$. Let $e = uv \in E$. By Claim 36, we may assume that there exists a (uv, uw)-perilous set for all $w \in V_+(m_G) \setminus \{z_e\}$. Now, by $m_G(V) \geq 4$ and Claim 34.3, there exists a unique $X_e \subseteq T_{z_e} \setminus \{z_e\}$ such that $X_e \cup T_w$ is the (uv, uw)-perilous set for all $e \in E$ and $w \in V_+(m_G) \setminus \{z_e\}$.
2. By point 1 and Claims 36 and 34.5, the family $\mathcal{U}$ is laminar, and hence the assertion follows.
3. If $U \in \mathcal{U}$, then either $U = T_w$ for $w \in V_+(m_G)$ or $U = X_{uv}$ for $uv \in E$. In the former case, by Claim 32.2, $w \in U^*$, and in the latter case, by Claim 34.4, f is the only edge entering X_f for all $f \in E$, and hence $u \in U^*$.
4. By Corollary 28.2–3, this point is satisfied for all $U \in \{T_w, w \in V_+(m_G)\}$. By Claim 36.3, this point is satisfied for all $U \in \{X_e : e \in E\}$.
5. We may assume, by point 4, that $U \neq U^*$ and $U \subseteq T_x$, where $x \neq w$.

By $m_G(V) \geq 4$, there exists $y \in V_+(m_G) \setminus \{w, x\}$. Apply Claim 12.1 for $\{T_y \cup W : W$ maximal set of $\mathcal{U}$ strictly contained in $U\}$ and point 4 to deduce that $p_0(T_y \cup X) \geq 1$, where $X = \bigcup_{W \in \mathcal{U}, W \subsetneq U} W = U \setminus U^* \neq \emptyset$. Now, (4) applied to $T_w \cup U$ and $T_y \cup X$ gives, by point 4, that $p_0(T_w \cup U^*) \geq p_0(T_w \cup U) + p_0(T_y \cup X) - p_0(T_y) \geq 1 + 1 - 1 \geq 1$, as claimed.

6. By $m_G(V) \geq 4$, let $x \in V_+(m_G)$ be such that $U \cap T_x = \emptyset$ and $W \cap T_x = \emptyset$. Claim 12.1 applies to the family $\{T_x \cup X^* : X^* \in \mathcal{U}^* \setminus \{U^*, W^*\}\}$ and gives, by points 5 and 2 of this lemma, that $p_0(U^* \cup W^*) = p_0(V \setminus (U^* \cup W^*)) \geq 1$, as claimed.
7. If there are no edges, then this assertion comes from Corollary 28.4. Otherwise, a minimal $X_e \in \mathcal{U}$ belongs to $\mathcal{U}^*$ and satisfies $p_0(X_e) = 1$ by point 1 and Claim 34.4. Then, by points 2 and 6 of this lemma, Lemma 2.2 implies that $\mathcal{U}^*$ is a p_0-full partition.
8. By point 3, definition of $\mathcal{U}$, Corollary 28.2–3, Lemma 35, m_G being modular, and point 1, we have $|\mathcal{U}^*| = |\mathcal{U}| = |\{T_w : w \in V_+(m_G)\}| + |\{X_e : e \in E\}| = \sum_{w \in V_+(m_G)} m_G(T_w) + |E| = m_G(V) + |E|$.
9. By point 7, $\mathcal{U}^*$ is a p_0-full partition of V, and hence, by point 8, $\dim(p_0) \geq |\mathcal{U}^*| = m_G(V) + |E|$. Then, by $m_G(V) \geq 4$, we have $\frac{1}{2}m_0(V) = \frac{1}{2}m_G(V) + |E| \leq m_G(V) + |E| - 2 \leq \dim(p_0) - 2$; that is, m_0 is not p_0-legal.
10. By Corollary 28.2, $|V_+(m_G)| = m_G(V)$. By Observation 6, there exists a $\mathcal{P}$-partite spanning tree T on $V_+(m_G)$. Since T is a tree, we have $|E(T)| = m_G(V) - 1$. We claim that T covers p_G. Otherwise, by $p_G \leq 1$, there exists a set X such that $0 \leq d_T(X) < p_G(X) \leq 1$; that is, $0 = d_T(X)$ and $p_G(X) = 1$. Since T is connected and p_G is symmetric, we may suppose that X does not contain any vertex of $V_+(m_G)$. Then, by m_G being p_G-admissible, we have $0 = m_G(X) \geq p_G(X) = 1$, a contradiction. Let $F := (V, E(T) \cup E)$. Since T and G are both $\mathcal{P}$-partite, so is F. Since T covers p_G, we have $d_T(X) \geq p_G(X) = p_0(X) - d_G(X)$ for all $X \subset V$, so $d_F(X) \geq p_0(X)$ for all $X \subset V$, that is F covers p_0. Finally, by points 8 and 7, we have $|E(F)| = |E(T)| + |E| = m_G(V) - 1 + |E| = |\mathcal{U}^*| - 1 \leq \dim(p_0) - 1$.

4. Degree specified version.

In this section we solve the degree specified version of our problem: given a symmetric crossing supermodular function $p_0 : 2^V \to \mathbb{Z}$, a degree specification $m_0 : V \to \mathbb{Z}_+$, and a partition $\mathcal{P}$ of V, find a $\mathcal{P}$-partite graph on V covering p_0 and satisfying m_0. Note that finding such a graph is equivalent to finding a complete allowed splitting off.

4.1. Necessary conditions.

Here, we provide necessary conditions for the existence of a complete allowed splitting off.

LEMMA 38. *Let $p_0 : 2^V \to \mathbb{Z}$ be a symmetric crossing supermodular function, $m_0 : V \to \mathbb{Z}_+$ a degree specification, and $\mathcal{P}$ a partition of V. If there exists a complete allowed splitting off, then m_0 is $\mathcal{P}$-feasible, p_0-admissible, and p_0-legal.*

Proof. Let $G = (V, E)$ be a graph obtained by a complete allowed splitting off. Then G covers p_0 and satisfies m_0. Note that $m_0(V) = \sum_{v \in V} m_0(v) = \sum_{v \in V} d_G(v) = 2|E|$; hence (19) holds. Since every splitting off is rainbow, we get (20). Hence m_0 is $\mathcal{P}$-feasible. By (6) and (5), we have $m_0(X) = \sum_{x \in X} m_0(x) = \sum_{x \in X} d_G(x) \geq d_G(X) \geq p_0(X)$; hence we obtain (7), that is, m_0 is p_0-admissible. Since G covers p_0, G has, by Lemma 2.1, at least $\dim(p_0) - 1$ edges. By $m_0(V) = 2|E|$, we get (10); that is, m_0 is p_0-legal. ∎

4.2. Obstacles. It turns out that the conditions that appeared in section 4.1 are not sufficient to have a complete allowed splitting off. Exceptional structures must be forbidden in order to get the sufficiency. The description of these structures is given in section 2. An obstacle is a partition $\mathcal{A}$ of V satisfying two types of conditions. On the one hand, the p- and m-values of the sets in the partition $\mathcal{A}$ fulfill rigorous conditions. On the other hand, the partition $\mathcal{A}$ is closely related to the partition $\mathcal{P}$. We mention that the C_5^*-obstacle arises only for our abstract form of the problem; it does not exist in the framework of graphs or hypergraphs.

4.2.1. Properties of constructions. In this section we provide some basic properties of constructions.

CLAIM 39. *Let $\mathcal{A} = \{A_1, \dots, A_4\}$ be a C_4^*-construction for p_G. Then, for $i = 1, \dots, 4$, the following hold:*
1. $p_G(A_i) + p_G(A_{i+1}) - 1 \geq p_G(A_i \cup A_{i+1})$,
2. $p_G(A_i) \geq 1$,
3. $p_G(A_i \cup A_{i+1}) \geq 1$,
4. *$\mathcal{A}$ is a simple C_4^*-construction if and only if $p_G(A_i \cup A_{i+1}) = 1$ for $i = 1, \dots, 4$.*

Proof.
1. By Definition 7.1d and the definition of σ_{p_G}, we have $p_G(A_i) + p_G(A_{i+1}) \geq p_G(A_i \cup A_{i+1})$, and then, by Definition 7.1a, the assertion follows.
2. By Definition 7.1b and point 1, we have $p_G(A_{i-1}) + p_G(A_{i+1}) = p_G(A_{i-1} \cup A_i) + p_G(A_i \cup A_{i+1}) \leq p_G(A_{i-1}) + p_G(A_{i+1}) + 2p_G(A_i) - 2$, and the assertion follows.
3. By Definitions 7.1b and 7.1d, point 1, and the symmetry of p_G, we have $p_G(A_i \cup A_{i+1}) = \frac{1}{2}\sigma_{p_G} - p_G(A_{i-1} \cup A_i) \geq \frac{1}{2}\sigma_{p_G} - \frac{1}{2}((p_G(A_{i-1}) + p_G(A_i) - 1) + (p_G(A_{i+1}) + p_G(A_{i+2}) - 1)) = 1$, and the assertion follows.
4. By Definition 7.1b and points 2 and 3 of this claim, $p_G(A_i) = 1$ for $i = 1, \dots, 4$ if and only if $p_G(A_i \cup A_{i+1}) = 1$ for $i = 1, \dots, 4$, and the assertion follows. □

CLAIM 40. *If $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \dots, B_t\}$ is a C_5^*-construction for p_G, then for $J \subseteq \{1, \dots, t\}$ and $i = 1, \dots, 4$ the following hold:*
1. $p_G(A_i \cup A_{i+1} \cup \bigcup_{j \in J} B_j) = 1$,
2. $p_G(A_i \cup \bigcup_{j \in J} B_j) = 1$,
3. $p_G(\bigcup_{j \in J} B_j) = 2$ *if* $J \neq \emptyset$,
4. $p_G(A_i \cup A_{i+2} \cup \bigcup_{j \in J} B_j) \leq 0$.

Proof. We denote, for $I \subseteq \{1, \dots, 4\}$ and $J \subseteq \{1, \dots, t\}$, $\overline{I} := \{1, \dots, 4\} \setminus I$ and $\overline{J} := \{1, \dots, t\} \setminus J$, $A_I := \bigcup_{i \in I} A_i$, $B_J := \bigcup_{j \in J} B_j$, $A := \bigcup_1^4 A_i$, and $B := \bigcup_1^t B_j$. Let $i \in \{1, \dots, 4\}$ and $J \subseteq \{1, \dots, t\}$.
1. By Definitions 8.1d, 8.1a, and 8.1c, Claim 12.1 applies to $\{A_i \cup A_{i+1}\} \cup \{A_i \cup B_j : j \in J\} \cup \{A_i \cup B_j : j \in \overline{J}\}$. Let $I = \{i, i+1\}$. Then, using the symmetry of p_G and Definition 8.1d, we get $1 = p_G(A_I) \leq p_G(A_I \cup B_J) \leq p_G(A_I \cup B) = p_G(A_{\overline{I}}) = p_G(A_{i+2} \cup A_{i+3}) = 1$, and the assertion follows.
2. By Definitions 8.1d, 8.1a and (13) applied to $A_{i+1} \cup A_{i+2}$, and $A_{i+2} \cup A_{i+3}$, we have $p_G(A_{\overline{\{i\}}}) \geq 1$. Hence, by Definition 8.1c, Claim 12.1 applies to $\{A_{\overline{\{i\}}}\} \cup \{A_{i+1} \cup B_j : j \in \overline{J}\} \cup \{A_{i+1} \cup B_j : j \in J\}$. Then, using the symmetry of p_G, we get $1 \leq p_G(A_{\overline{\{i\}}}) \leq p_G(A_{\overline{\{i\}}} \cup B_{\overline{J}}) \leq p_G(A_{\overline{\{i\}}} \cup B) = p_G(A_i) = 1$, and we have equality everywhere. In particular, by the symmetry of p_G, we have $p_G(A_i \cup B_J) = p_G(A_{\overline{\{i\}}} \cup B_{\overline{J}}) = 1$.
3. Point 2 and the symmetry of p_G imply $p_G(A_{\overline{\{4\}}}) = p_G(A_{\overline{\{2\}}}) = 1$. Therefore,

(13) applied to these two sets and Definition 8.1e give $p_G(A) \geq p_G(A_{\overline{\{4\}}}) + p_G(A_{\overline{\{2\}}}) - p_G(A_1 \cup A_3) \geq 1 + 1 - 0 = 2$. Since $J \neq \emptyset$, there exists $k \in J$. By Definitions 8.1a and 8.1c, Claim 12.1 applies to $\{A\} \cup \{A_1 \cup B_j : j \in \overline{J}\} \cup \{A_1 \cup B_j : j \in J \setminus \{k\}\}$. Then, by the symmetry of p_G, we have $2 \leq p_G(A) \leq p_G(A \cup B_{\overline{J}}) \leq p_G(A \cup B_{\overline{\{k\}}}) = p_G(B_k) = 2$, and we have equality everywhere. In particular, by the symmetry of p_G, we have $p_G(B_J) = p_G(A \cup B_{\overline{J}}) = 2$.

4. Let $I = \{i, i+2\}$. If $p_G(A_I \cup B_J) > 0$, then, by Definitions 8.1a and 8.1c, Claim 12.1 applies to $\{A_I \cup B_J\} \cup \{A_i \cup B_j : j \in \overline{J}\}$. Then, by the symmetry of p_G and Definition 8.1e, we have $0 < p_G(A_I \cup B_J) \leq p_G(A_I \cup B) = p_G(A_{\overline{I}}) = p_G(A_{i+1} \cup A_{i+3}) \leq 0$, a contradiction. ∎

4.2.2. Properties of obstructions. We show now that obstructions are unique up to cyclically reordering their elements.

CLAIM 41. *Every set of an obstruction is tight.*

Proof. Let $\mathcal{A}$ be an obstruction for (p, m). Using that $\mathcal{A}$ is a partition of V, that m is p-admissible by Definitions 7.2, 8.2, and 9.2, then Definitions 7.1d, 8.1f, 9.1d, and finally Theorem 1, we have $m(V) = \sum_{X \in \mathcal{A}} m(X) \geq \sum_{X \in \mathcal{A}} p(X) = \sigma_p = m(V)$. Thus, there is equality everywhere, and every set of $\mathcal{A}$ is tight. ∎

Recall that for a C_5^*-construction $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\}$, two elements of $\mathcal{A}$ are distinct nonconsecutive if they are of the form A_i, A_{i+2} or A_i, B_j or B_i, B_j for $i \neq j$.

LEMMA 42. *If $\mathcal{A}$ is an obstruction for (p, m) and the m-positive elements x and y belong to distinct nonconsecutive elements of $\mathcal{A}$, then splitting off at x, y is p-admissible.*

Proof. By Claim 41, every set of $\mathcal{A}$ is tight. Suppose that the splitting off at x, y is not p-admissible. Then, by Lemma 19, there exists a maximal dangerous set Y containing x and y. By Observation 5, $p(Y) \geq 1$. By Claim 20.1, $Y \cup A \neq V$ for all $A \in \mathcal{A}$. Therefore, by Claim 20.3, Y is the union of elements of $\mathcal{A}$. We address three cases:

(a) $\mathcal{A}$ is a C_4^*-obstruction. Then, by nonconsecutiveness, $x \in A_i$ and $y \in A_{i+2}$ for some $i \in \{1, \ldots, 4\}$, and hence $A_i \cup A_{i+2} \subseteq Y$. By Claim 20.1, the modularity and nonnegativity of m, Claim 41, and Definitions 7.1d and 7.2, we have $\frac{1}{2}m(V) \geq m(Y) \geq m(A_i) + m(A_{i+2}) = p(A_i) + p(A_{i+2}) = \frac{1}{2}\sigma_p = \frac{1}{2}m(V)$; thus $m(Y) = \frac{1}{2}\sigma_p$, and, by Claims 41 and 39.2, $A_i \cup A_{i+2} = Y$. By Claim 39.3, $p(A_i \cup A_{i+1}) \geq 1$; therefore, (3) and (4) apply to Y and $A_i \cup A_{i+1}$. By Claim 39.3, (3) and (4), the symmetry of p, Definition 7.1d, and since Y is dangerous, we get $p(Y) + 1 \leq p(Y) + p(A_i \cup A_{i+1}) \leq \frac{1}{2}((p(A_i) + p(A_{i+3})) + (p(A_{i+1}) + p(A_{i+2}))) = \frac{1}{2}\sigma_p = m(Y) \leq p(Y) + 1$. Thus $p(A_i \cup A_{i+1}) = 1$. Similarly, $p(A_{i+1} \cup A_{i+2}) = 1$. Then, by Claim 39.4, $\mathcal{A}$ is a simple C_4^*-obstruction. Then, by Definition 7.1c, we have $0 \geq p(Y) \geq 1$, a contradiction.

(b) $\mathcal{A}$ is a C_5^*-obstruction. Let a and b be the number of A_i and B_j contained in Y. Note that $a + b \geq 2$. By Y being dangerous, m being modular, Claim 41, and Definitions 8.1a and 8.1b, we have $p(Y) \geq m(Y) - 1 = a + 2b - 1$. By Claim 40 and the symmetry of p, we get $2 \geq p(Y)$. Thus $b \leq 1$ and $a \leq 3$. Then, by $a + b \geq 2$, we have $1 \leq a \leq 3$. In this case, Claim 40 gives $1 \geq p(Y)$. It follows that $b = 0$ and $a = 2$. Then, by Definition 8.1e and the nonconsecutiveness of the two A_i's contained in Y, and since Y is dangerous, we get $0 \geq p(Y) \geq 1$, a contradiction.

(c) $\mathcal{A}$ is a C_6^*-obstruction. By Definition 9.1c, $\mathcal{A}$ is not p-full so, by Claim 13.4 and $p(Y) \geq 1$, Y is consecutive. Then, Claim 13.2 gives $p(Y) = 1$. Since Y is dangerous, $1 = p(Y) \geq m(Y) - 1 \geq 1$, and hence $m(Y) = 2$; that is, x and y belong to consecutive sets, a contradiction. ∎

LEMMA 43. *If $\mathcal{A}$ is an obstruction for (p_G, m_G), then $\mathcal{A}$ is the unique partition of V into maximal p_0-positive tight sets.*

Proof. Let X be an element of $\mathcal{A}$. By Claim 39.2, Definitions 8.1a–1b and 9.1a, and Claim 41, X is p_0-positive and tight. Let Y be a maximal tight set containing X, and suppose indirectly that $X \neq Y$; that is, since $\mathcal{A}$ is a partition of V, Y intersects some other $X' \in \mathcal{A}$. Since X is p_0-positive, so is Y. By Claim 14.1, we have $X' \subset Y$. By Y being tight and Lemmas 42 and 19, there exists an index i such that $Y = A_i \cup A_{i+1}$. Then, by m_G being modular and Claim 41, we have that $0 = m_G(A_i) + m_G(A_{i+1}) - m_G(A_i \cup A_{i+1}) = p_G(A_i) + p_G(A_{i+1}) - p_G(A_i \cup A_{i+1})$ is even, which contradicts Definitions 7.1a or 8.1a–1d or 9.1a–1b.

To see the uniqueness, let $\mathcal{A}'$ be a partition of V into maximal p_0-positive tight sets. Since the elements of $\mathcal{A}$ and $\mathcal{A}'$ are maximal p_0-positive tight sets, by Claim 14.1, the two partitions coincide. ∎

COROLLARY 44. *If there exists a simple C_4^*-, C_5^*-, or C_6^*-obstruction for (p_G, m_G), then it is the unique obstruction for (p_G, m_G), up to cyclically reordering its members.*

Proof. By Lemma 43, an obstruction is the unique partition of V into maximal p_0-positive tight sets. Since, by Definition 7.1, 8.1, or 9.1, in a simple C_4^*-, C_5^*-, or C_6^*-obstruction for (p_G, m_G), the union of any two sets of the obstruction has p_G-value 1 if they are consecutive and nonpositive p_G-value otherwise, we get the desired result. ∎

4.2.3. Inherited obstructions. In the next three claims we prove that splitting off a special admissible pair in an obstruction gives rise to another obstruction. These results will be used in three different ways. First, they will help us to show that the existence of an obstruction implies the existence of a complete p-admissible splitting off. Second, we use the result about C_5^*-obstructions to show that if a C_5^*-obstruction exists that is not a C_5^*-obstacle, then a complete allowed splitting off exists. Finally, these results will be applied in the next section about inherited obstacles.

We consider the three different obstructions separately.

CLAIM 45. *A C_4^*-obstruction $\mathcal{A} = \{A_1, \ldots, A_4\}$ for (p_G, m_G) is a C_4^*-obstruction for $(p_{\bar{G}}, m_{\bar{G}})$, where $\bar{G}$ is obtained from G by a p_G-admissible splitting off at $u \in A_j$ and $v \in A_{j+1}$ for some $j \in \{1, \ldots, 4\}$.*

Proof. By definition, $m_{\bar{G}}(X) = m_G(X) - \chi_X(\chi_u + \chi_v)$ and $p_{\bar{G}}(X) = p_G(X) - d_{uv}(X)$. We verify the conditions of Definition 7 one by one.

7.1a Since, by (1) applied to A_i and A_{i+1}, $d_{uv}(A_i) + d_{uv}(A_{i+1}) - d_{uv}(A_i \cup A_{i+1}) = 2d_{uv}(A_i, A_{i+1})$ is even, Definition 7.1a for p_G implies that Definition 7.1a holds for $p_{\bar{G}}$.

7.1b Since $d_{uv}(A_{i-1} \cup A_i) + d_{uv}(A_i \cup A_{i+1}) = 1 = d_{uv}(A_{i-1}) + d_{uv}(A_{i+1})$, Definition 7.1b for p_G implies that Definition 7.1b holds for $p_{\bar{G}}$.

7.1c If Definition 7.1c did not hold for $p_{\bar{G}}$ and $m_{\bar{G}}$, then $p_{\bar{G}}(A_i) = 1$ for $i = 1, \ldots, 4$ and $p_{\bar{G}}(A_j \cup A_{j+2}) \geq 1$. Then, $p_G(A_j \cup A_{j+2}) \geq 2$, $p_G(A_{j+2}) = p_G(A_{j+3}) = 1$, and $p_G(A_j) = p_G(A_{j+1}) = 2$. Since the splitting off is p_G-admissible, by Lemma 19, $A_j \cup A_{j+1}$ is not dangerous for m_G and p_G. Then, by Claim 41 and the modularity of m_G, $4 = 2 + 2 = p_G(A_j) + p_G(A_{j+1}) = m_G(A_j) + m_G(A_{j+1}) = m_G(A_j \cup A_{j+1}) \geq p_G(A_j \cup A_{j+1}) + 2$, and hence, by Definition 7.1b, $p_G(A_j \cup A_{j+3}) \geq 1$. Then, by (14), $1 + 2 \leq p_G(A_j \cup A_{j+3}) + p_G(A_j \cup A_{j+2}) \leq p_G(A_{j+2}) + p_G(A_{j+3}) = 1 + 1$, a contradiction.

7.1d By $m_{\bar{G}}$ being $p_{\bar{G}}$-admissible and the definition of σ_p, we have $\sigma_{p_{\bar{G}}} \leq m_{\bar{G}}(V) = m_G(V) - 2 = \sum_1^4 p_{\bar{G}}(A_j) \leq \sigma_{p_{\bar{G}}}$, and Definition 7.1d follows.

7.2 By Claim 17, $m_{\bar{G}}$ is minimally $p_{\bar{G}}$-admissible. ∎

CLAIM 46. *Let $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\}$ be a C_5^*-obstruction for (p_G, m_G), $\bar{G}$ obtained from G by a p_G-admissible splitting off at a vertex of A_i (respectively, B_i) and a vertex of B_j (respectively, $B_j \neq B_i$), and $\mathcal{A}'$ obtained from $\mathcal{A}$ by deleting B_j and replacing A_i (respectively, B_i) by $A_i \cup B_j$ (respectively, $B_i \cup B_j$).*

1. *If $t = 1$, then $\mathcal{A}'$ is a simple C_4^*-obstruction for $(p_{\bar{G}}, m_{\bar{G}})$.*
2. *If $t \geq 2$, then $\mathcal{A}'$ is a C_5^*-obstruction for $(p_{\bar{G}}, m_{\bar{G}})$.*

Proof. Since the splitting off is p_G-admissible, $m_{\bar{G}}$ is $p_{\bar{G}}$-admissible. The minimality of $m_{\bar{G}}$ follows from Claim 17. Hence Definition 7.2 if $t = 1$ (respectively, Definition 8.2 if $t \geq 2$) holds for $m_{\bar{G}}$ and $p_{\bar{G}}$.

1. Definition 7.1a–1d for $\mathcal{A}'$ and $p_{\bar{G}}$ follows from Definition 8.1a–1f for $\mathcal{A}$ and p_G and Claim 40. Note that $p(A'_i) = 1$ for all $A'_i \in \mathcal{A}'$ also follows.
2. Definition 8.1a–1e for $\mathcal{A}'$ and $p_{\bar{G}}$ follows from Definition 8.1a–1e and Claim 40 for $\mathcal{A}$ and p_G. By Claim 17, Theorem 1, and Definition 8.1f for $\mathcal{A}$ and p_G, we have $\sigma_{p_{\bar{G}}} = m_{\bar{G}}(V) = m_G(V) - 2 = \sigma_{p_G} - 2 = 2t + 4 - 2 = 2(t-1) + 4$; hence Definition 8.1f holds for $\mathcal{A}'$ and $p_{\bar{G}}$. ∎

CLAIM 47. *If $\mathcal{A}$ is a C_6^*-obstruction for (p_G, m_G) and $\bar{G}$ is obtained from G by splitting off at $u \in A_{i-1}, v \in A_{i+1}$ for some $i \in \{1, \ldots, 4\}$, then $\mathcal{A}' = \{A'_1, A'_2, A'_3, A'_4\} = \{A_{i-1} \cup A_i \cup A_{i+1}, A_{i+2}, A_{i+3}, A_{i+4}\}$ is a simple C_4^*-obstruction for $(p_{\bar{G}}, m_{\bar{G}})$.*

Proof. By Definition 9.1a and Claim 13.2 for $\mathcal{A}$, we have $p_{\bar{G}}(A'_j \cup A'_{j+1}) = p_G(A'_j \cup A'_{j+1}) = 1 = p_G(A'_j) = p_{\bar{G}}(A'_{j+1})$ for $j = 1, \ldots, 4$. Thus $\mathcal{A}'$ satisfies Definitions 7.1a and 7.1b. By Definition 9.1c applied to $A_{i+2} \cup A_{i+4}$ and the symmetry of p_G, $\mathcal{A}'$ satisfies Definition 7.1c. Since, by Lemma 42, the splitting off is p_G-admissible, $m_{\bar{G}}$ is $p_{\bar{G}}$-admissible. Then $\sigma_{p_{\bar{G}}} \leq m_{\bar{G}}(V) = m_G(V) - 2 = 6 - 2 = 4 = \sum_1^4 p_{\bar{G}}(A'_j) \leq \sigma_{p_{\bar{G}}}$, and Definitions 7.1d and 7.2 follow. ∎

LEMMA 48. *If $\mathcal{A}$ is an obstruction for (p, m), then there exists a complete p-admissible splitting off.*

Proof. Let us consider the following cases:

(a) If $\mathcal{A}$ is a simple C_4^*-obstruction, then, by Lemma 42, we are done.

(b) If $\mathcal{A}$ is a C_4^*-obstruction that is not simple, then, by Corollary 44, no simple C_4^*-obstruction exists. Moreover, by Claim 41 and Definitions 7.1d and 7.2, we have $m(A_i \cup A_{i+1}) = \frac{1}{2} m(V)$. Therefore, by Lemma 25, there exists an admissible pair u, v such that $u \in A_j$ and $v \in A_{j+1}$ for some $j \in \{1, \ldots, 4\}$. By Claim 45, after splitting off this pair, $\mathcal{A}$ remains a C_4^*-obstruction. By repeating this, we arrive at the first case, and then we are done.

(c) If $\mathcal{A}$ is a C_5^*-obstruction, then, by the repeated application of Lemma 42 and Claim 46, we arrive at the first case, and then we are done.

(d) If $\mathcal{A}$ is a simple C_6^*-obstruction, then, by Lemma 42 and Claim 47, we arrive at the first case, and then we are done. ∎

LEMMA 49. *Suppose that m_G is $\mathcal{P}$-feasible. If there exists a simple C_4^*-obstruction (respectively, C_5^*-obstruction) for (p_G, m_G) but no C_4^*-obstacle (respectively, C_5^*-obstacle)* *for $(p_G, \mathcal{P}, m_G)$, then there exists a complete $(p_G, \mathcal{P})$-allowed splitting off.*

Proof. We prove the two cases separately.

1. First, suppose that $\mathcal{A} = \{A_1, \ldots, A_4\}$ is a simple C_4^*-obstruction for (p_G, m_G) but not an obstacle for $(p_G, \mathcal{P}, m_G)$. Let $\{a_i\} = A_i \cap V^+(m_G)$ for $i = 1, \ldots, 4$. Then, by Lemma 42, splitting off at a_1, a_3 is p_G-admissible. Since Definition 7.3 does not hold, it is $\mathcal{P}$-feasible, hence $(p_G, \mathcal{P})$-allowed, and we are done.
2. Now, suppose that $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\}$ is a C_5^*-obstruction for (p_G, m_G) but not an obstacle for $(p_G, \mathcal{P}, m_G)$. We proceed by induction on t. Let $\{a_i\} = A_i \cap V^+$ for $i = 1, \ldots, 4$ and $\{b_j, b_j'\} = B_j \cap V^+$ for $j = 1, \ldots, t$. Without loss of generality, we may assume that a_1, b_1 is rainbow; thus splitting off at a_1, b_1 is $(p_G, \mathcal{P})$-allowed by Lemma 42. Let $p' := p_{G_{a_1 b_1}}$ and $m' := m_{G_{a_1 b_1}}$.
   Our approach is the following. First, split off at a_1, b_1 and let $\mathcal{A}' = \{A_1 \cup B_1, A_2, A_3, A_4, B_2, \ldots, B_t\}$ be the obstruction for (p', m') given by Claim 46. By Corollary 44, if an obstacle exists for $(p', \mathcal{P}, m')$, then it is $\mathcal{A}'$. If none exists, then we are done by the previous case or by induction. Otherwise, we will provide a $(p', \mathcal{P})$-allowed flip to get rid of the obstacle. We will perform either a (b_1a_1, b_1a_2)- or a (a_1b_1, a_1b_1')-flip; note that both are p'-admissible because they consist of an unsplitting and a splitting off which is p_G-admissible by Lemma 42.
   Distinguish the following cases, where we assume that $m'(P_1) \geq m'(P_2) \geq \cdots \geq m'(P_r)$, and say that the color of P_1 and P_2 are red and blue:
   (a) $\mathcal{A}'$ is a simple C_4^*-obstacle or a C_5^*-obstacle of type 1 for $(p', \mathcal{P}, m')$. By Definition 8.1a, we may assume that the color of b_j is red for $j = 2, \ldots, t$ and either i. a_2, a_4 or ii. b_1', a_3 are both red.
      i. Since $\mathcal{A}$ is neither a C_5^*-obstacle of type 1 nor a C_5^*-obstacle of type 2 for $(p_G, \mathcal{P}, m_G)$, none of b_1, b_1' is red, and $a_1, a_3, b_2', \ldots, b_t'$ are not all of the same color. Therefore, b_1, a_2 is a rainbow pair. As noted before, the (b_1a_1, b_1a_2)-flip is p'-admissible; thus it is $(p', \mathcal{P})$-allowed. Let $m'' = m_{Ga_2b_1}$ and $p'' = p_{G_{a_2 b_1}}$. By Claim 46, $\mathcal{A}'' = \{A_1, A_2 \cup B_1, A_3, A_4, B_2, \ldots, B_t\}$ is an obstruction for (p'', m''), and it is not an obstacle of type 2 for $(p'', \mathcal{P}, m'')$ because $c(b_1') \neq c(a_4)$, nor of type 1 because $a_1, a_3, b_2', \ldots, b_t'$ are not all of the same color.
      ii. Then, the red m'-positive elements are $a_3, b_1', b_2 \ldots, b_t$. Moreover, since $\mathcal{A}$ is not a C_5^*-obstacle of type 1 for $(p_G, \mathcal{P}, m_G)$, a_1 is not red. We may assume that $m'(P_2) < m'(V)/2$; that is, $a_2, a_4, b_2', \ldots, b_t'$ are not all of the same color, as we dealt with in the case above.
         A. If b_1 is red, then b_1, a_2 is a rainbow pair; hence the (b_1a_1, b_1a_2)-flip is $(p', \mathcal{P})$-allowed. Let $m'' = m_{Ga_2b_1}$ and $p'' = p_{G_{a_2 b_1}}$. Then, by Claim 46, $\mathcal{A}'' = \{A_1, A_2 \cup B_1, A_3, A_4, B_2, \ldots, B_t\}$ is an obstruction for (p'', m''), and it is not an obstacle for $(p'', \mathcal{P}, m'')$ because b_1' and a_3 are m''-positive elements with the same color that belong to distinct consecutive sets.
         B. If b_1 is not red, then, since $m'(P_2) < m'(V)/2$ and b_1' is red, a_1, b_1' is a rainbow pair; hence the (a_1b_1, a_1b_1')-flip is $(p', \mathcal{P})$-allowed. Since b_1 is not red but a_3 is red, and $a_2, a_4, b_2', \ldots, b_t'$ are not all of the same color, the obstruction $\mathcal{A}'$ for (p'', m'') is not an obstacle for $(p'', \mathcal{P}, m'')$.

(b) $\mathcal{A}'$ is a C_5^*-obstacle of type 2 (but not of type 1) for $(p', \mathcal{P}, m')$. Note that $m_G(P_i) \leq m'(P_i) + 1 = m'(V)/2 = m_G(V)/2 - 1$ for $i = 1, 2$, so we may assume that b_1' and a_3 are red and a_2 and a_4 are blue.

i. If b_1 is not blue, then b_1, a_2 is a rainbow pair; hence the (b_1a_1, b_1a_2)-flip is $(p', \mathcal{P})$-allowed. $\mathcal{A}'' = \{A_1, A_2 \cup B_1, A_3, A_4, B_2, \ldots, B_t\}$ is an obstruction for (p'', m''), yet not an obstacle for $(p'', \mathcal{P}, m'')$ since a_3 and b_1' are red m''-positive elements and belong to distinct consecutive sets of $\mathcal{A}''$.

ii. If b_1 is blue, then, since $\mathcal{A}$ is not a C_5^*-obstacle of type 2 for $(p_G, \mathcal{P}, m_G)$, a_1 is not red. Then a_1, b_1' is a rainbow pair, and hence the (a_1b_1, a_1b_1')-flip is $(p', \mathcal{P})$-allowed. Let $m'' = m_{G_{a_2b_1}}$ and $p'' = p_{G_{a_2b_1}}$, and note that $\mathcal{A}'$ is also an obstruction for (p'', m'') but not an obstacle for $(p'', \mathcal{P}, m'')$ because b_1 and a_2 are blue m''-positive elements that belong to distinct consecutive sets of $\mathcal{A}'$. ◻

4.2.4. Inherited obstacles. In this section, we will see that splitting off an allowed pair in an obstacle gives rise to another obstacle. This implies a link between obstacles and complete allowed splitting off; see Lemma 54. In this section, $p_0 : 2^V \to \mathbb{Z}$ is a symmetric crossing supermodular function, $G = (V, E)$ is a graph, and m_G is a p_G-admissible degree specification with $m_G(V) \geq 4$.

First let us see a result about an inherited simple C_4^*-obstacle after an allowed flipping.

CLAIM 50. *Let $\mathcal{A}$ be a simple C_4^*-obstacle for $(p_G, \mathcal{P}, m_G)$, $\{a_j\} = A_j \cap V^+(m_G)$ for $j = 1, \ldots, 4$, $uv \in E$, $u \in A_i$, $v \in A_{i+1}$, $c(a_{i+1}) = c(a_{i-1})$, and G' be obtained from G by the allowed (vu, va_i)-flip. Then $\mathcal{A}$ is a simple C_4^*-obstacle for $(p_{G'}, \mathcal{P}, m_{G'})$.*

Proof. Note that $p_G(A_i \cup A_j) = p_{G'}(A_i \cup A_j)$ for $1 \leq i \leq j \leq 4$. By Definition 7.2 for p_G, the fact that the flipping is allowed, the definition of $\sigma_{p_{G'}}$ and Definition 7.1d for p_G, we have $\sigma_{p_G} = m_G(V) = m_{G'}(V) \geq \sigma_{p_{G'}} \geq \sum_{i=1}^4 p_{G'}(A_i) = \sum_{i=1}^4 p_G(A_i) = \sigma_{p_G}$. Then Definitions 7.1–2 for p_G imply that Definitions 7.1–2 are satisfied for $p_{G'}$. Since $c(a_{i+1}) = c(a_{i-1})$, Definition 7.3 is also satisfied for $p_{G'}$. ◻

Now we show that an obstacle is inherited after an allowed splitting off. Let us see the three different obstacles separately.

CLAIM 51. *A C_4^*-obstacle $\mathcal{A}$ for $(p_G, \mathcal{P}, m_G)$ is a C_4^*-obstacle for $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$, where $\bar{G}$ is obtained from G by an allowed splitting off.*

Proof. By Definition 7.3, there exist $\ell \in \{1, 2\}$ and $P \in \mathcal{P}$ such that the m_G-positive elements of $A_\ell \cup A_{\ell+2}$ are the m_G-positive elements of P. Therefore, by Definitions 7.1d and 7.2, P is dominating; hence we may assume that the two m_G-positive elements u and v of V involved in the allowed splitting off satisfy $u \in A_j$ and $v \in A_{j+1}$ for some j. By Claim 45, $\mathcal{A}$ is a C_4^*-obstruction for $(p_{\bar{G}}, m_{\bar{G}})$. Definition 7.3 for m_G and the fact that splitting off is allowed immediately imply that Definition 7.3 holds for $m_{\bar{G}}$. ◻

CLAIM 52. *Let $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\}$ be a C_5^*-obstacle for $(p_G, \mathcal{P}, m_G)$, and let $\bar{G}$ be obtained from G by an allowed splitting off. Then there exists for $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$ a C_5^*-obstacle if $t \geq 2$ and a simple C_4^*-obstacle if $t = 1$.*

Proof. By Lemmas 19 and 42 and Definition 8.1d, the pair $\{u, v\}$ of elements that we split off is either (a) $\{a_i, b_j\}$ for some $a_i \in A_i$ $(1 \leq i \leq 4)$ and $b_j \in B_j$ $(1 \leq j \leq t)$, or (b) $\{b_j, b_k\}$ for some $b_j \in B_j$, $b_k \in B_k$ $(1 \leq j < k \leq t)$. In case (a), let $\mathcal{A}'$ be obtained from $\mathcal{A}$ by replacing A_i by $A_i \cup B_j$ and deleting B_j, and in case (b), let $\mathcal{A}'$ be obtained from $\mathcal{A}$ by replacing B_j and B_k by $B_j \cup B_k$. By Claim 46, $\mathcal{A}'$ is a simple C_4^*-obstruction or a C_5^*-obstruction for $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$. We show below that Definition 8.3 also holds for $\mathcal{A}'$ and $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$. There are two possible cases:

1. If $\mathcal{A}$ is a C_5^*-obstacle of type 1 for $(p_G, \mathcal{P}, m_G)$, then there exists $\ell \in \{1, 2\}$ and a color P, say red, such that $A_\ell, A_{\ell+2}$ and every B_j contains a red m_G-positive element. By Definitions 8.1f, 8.2, (20), and then 8.3a, we have $t + 2 = \frac{\sigma_p}{2} = \frac{m_G(V)}{2} \geq m_G(P) \geq t + 2$, and thus $m_G(P) = \frac{m_G(V)}{2}$; that is, red is dominating. Therefore, the splitting off being allowed, exactly one of u and v is red.
   - (a) If $i \in \{\ell, \ell + 2\}$, then $A_i \cup B_j$ contains one red $m_{\bar{G}}$-positive element, namely the red m_G-positive element contained in B_j. Otherwise, $A_\ell, A_{\ell+2}$ and every $B_{j'}$ $(j' \neq j)$ contains a red $m_{\bar{G}}$-positive element; that is, Definition 8.3a holds for $\mathcal{A}'$ and $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$.
   - (b) We may assume without loss of generality that b_j is red. Then $B_j \cup B_k$ contains one red $m_{\bar{G}}$-positive element, namely the red m_G-positive element contained in B_k; hence $A_\ell, A_{\ell+2}$ and every $B_{j'}$ $(j' \neq j, k)$ contains a red $m_{\bar{G}}$-positive element; that is, Definition 8.3a holds for $\mathcal{A}'$ and $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$.
2. If $\mathcal{A}$ is a C_5^*-obstacle of type 2 but not of type 1 for $(p_G, \mathcal{P}, m_G)$, then there exist $i' \in \{1, 2\}$, $j_0 \in \{1, \dots, t\}$ and two colors, say red and blue, such that $A_{i'}, A_{i'+2}$ and every $B_{j'}$ $(j' \neq j_0)$ contains a red m_G-positive element, $A_{i'+1}, A_{i'+3}$ and every $B_{j'}$ $(j' \neq j_0)$ contains a blue m_G-positive element, and B_{j_0} contains no red and no blue m_G-positive element. Note that the situation is symmetric for red and blue.
   - (a) Without loss of generality we may assume that a_i is red. If b_j is blue (that is, $j \neq j_0$), then $A_i \cup B_j$ contains one red $m_{\bar{G}}$-positive element, namely the red m_G-positive element contained in B_j. Then Definition 8.3b holds for $\mathcal{A}'$ and $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$. If b_j is not blue (that is, $j = j_0$), then $A_{i'+1}, A_{i'+3}$ and every $B_{j'}$ $(j' \neq j)$ contains a blue $m_{\bar{G}}$-positive element; that is, Definition 8.3a holds also for $\mathcal{A}'$ and $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$.
   - (b) Without loss of generality we may assume that b_j is red. If b_k is blue (that is $k \neq j_0$), then $B_j \cup B_k$ contains one red and one blue $m_{\bar{G}}$-positive element, namely the red (respectively, blue) m_G-positive element contained in B_k (respectively, in B_j). Then, Definition 8.3b holds for $\mathcal{A}'$ and $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$. If b_k is not blue (that is, $k = j_0$), then $A_{i'+1}, A_{i'+3}$ and every $B_{j'}$ $(j' \neq j, k)$ and $B_j \cup B_k$ contain a blue $m_{\bar{G}}$-positive element; that is, Definition 8.3a holds for $\mathcal{A}'$ and $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$.

In the cases 1(a) and 2(a), if $t \geq 2$, then $\mathcal{A}'$ is a C_5^*-obstacle, and if $t = 1$, then, by Definition 8.1e, Definition 7.1c holds for $\mathcal{A}'$ and $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$, so $\mathcal{A}'$ is a simple C_4^*-obstacle for $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$. ∎

CLAIM 53. *If $\mathcal{A}$ is a C_6^*-obstacle for $(p_G, \mathcal{P}, m_G)$ and $\bar{G}$ is obtained from G by an allowed splitting off, then there exist $i \in \{1, \dots, 4\}$ and $a_{i-1} \in A_{i-1}$, $a_{i+1} \in A_{i+1}$ such that $\bar{G} = G_{a_{i-1}a_{i+1}}$ and $\{A_{i-1} \cup A_i \cup A_{i+1}, A_{i+2}, A_{i+3}, A_{i+4}\}$ is a simple C_4^*-obstacle for $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$.*

Proof. By Lemmas 19 and 42, and by Definitions 9.1b and 9.3, the only allowed pairs are a_{i-1}, a_{i+1} for all i. By Claim 47, $\mathcal{A}' = \{A'_1, A'_2, A'_3, A'_4\} = \{A_{i-1} \cup A_i \cup A_{i+1}, A_{i+2}, A_{i+3}, A_{i+4}\}$ is a simple C_4^*-obstruction for $(p_{\bar{G}}, \mathcal{P}, m_{\bar{G}})$. By $m_{\bar{G}}(V) = 4$ and Definition 9.3 for $\mathcal{A}$, $\mathcal{A}'$ satisfies Definition 7.3. ∎

The main result of this section, Lemma 54, motivates the definitions of obstacles by revealing the close link between them and the existence of a complete allowed splitting off.

LEMMA 54. *If there is an obstacle for $(p,\mathcal{P},m)$, then there exists no complete $(p,\mathcal{P})$-allowed splitting off.*

Proof. Suppose that there exists an obstacle for $(p,\mathcal{P},m)$. By Claims 51, 52, and 53, after any sequence of allowed splitting off there exists an obstacle, and then, by Claim 39.2, Definitions 8.1a and 9.1a, and Claim 41, the new degree specification satisfies $m'(V)>0$; that is, no complete allowed splitting off exists. ∎

4.2.5. Obstacles and split edges. An important subcase will occur when there is a simple C_4^*-obstacle $\mathcal{A}=\{A_1,\dots,A_4\}$ for $(p_G,\mathcal{P},m_G)$. Let us show some properties of such obstacles when G is not the edgeless graph. Let $\{a_i\}:=V_+(m_G)\cap A_i$ for $i=1,\dots,4$. For an edge $e=uv$, where u,v are contained in a member A_i of the C_4^*-obstacle, a *consecutive improvement for* e is an admissible improvement of e by a_{i-1} and a_{i+1}.

We start with some technical properties.

COROLLARY 55. *If $\mathcal{A}$ is a simple C_4^*-obstacle for $(p_G,\mathcal{P},m_G)$, then $p_G\le 1$.*

Proof. Since $\mathcal{A}$ is simple, Lemma 54 implies that there exists no allowed splitting off, and then, by Lemma 22, we have $p_G\le 1$. ∎

CLAIM 56. *Let $\mathcal{A}=\{A_1,\dots,A_4\}$ be a simple C_4^*-obstacle for $(p_G,\mathcal{P},m_G)$, $e=uv\in E$, $i\in\{1,\dots,4\}$, $u,v\in A_i$. Suppose that no allowed improvement exists. Then the following hold:*
1. *No (uv,a_{i+2})-perilous set exists.*
2. *There exists a (uv,a_j)-perilous set X_j for $j\in\{i-1,i+1\}$.*
3. *If $X_{i-1}\cap\{u,v\}=X_{i+1}\cap\{u,v\}$, then no edge connects distinct consecutive members of $\mathcal{A}$.*
4. *If $X_{i-1}\cap\{u,v\}=\{u\}$ and $X_{i+1}\cap\{u,v\}=\{v\}$, then $c(u)=c(a_{i+1})$, $c(v)=c(a_{i-1})$, and $c(a_i)=c(a_{i+2})$.*

Proof. By Corollary 55 and Lemma 43, we have $p_G\le 1$ and $T_{a_\ell}=A_\ell$ for all $\ell\in\{1,\dots,4\}$. We may assume that $i=1$.
1. By Claim 32.2–4 and Definition 7.1c, there is no (uv,a_3)-perilous set.
2. Suppose that, for some $j\in\{2,4\}$, there is no (uv,a_j)-perilous set. Then, by point 1 and Lemma 31(ii), the improvement of uv to ua_j,va_3 and that of uv to ua_3,va_j are p_G-admissible. Since $c(u)\neq c(v)$ and, by Observation 11, $c(a_j)\neq c(a_3)$, one of these improvements is allowed, contradicting our assumption.
3. By Claim 34.3, $X_j=X_e\cup A_j$ for $j=2,4$. By Claim 34.1, we have $p_G(X_2\cup X_4)=0$, and since e enters $X_2\cup X_4$, it is p_0-positive. Applying for A_1, X_2, and X_4 Claims 34.1 and 32.2–3, we get that no edge connects A_1 and $A_2\cup A_4$. By Claim 39.4, (13) applied for $X_2\cup X_4$ and $A_3\cup A_4$ (and $A_3\cup A_2$), $p_G\le 1$, and $p_G(X_2\cup X_4)=0$, we get that no edge connects A_3 and $A_2\cup A_4$, and the assertion follows.
4. By point 1 and Lemma 35, no (uv,a_3)-, (uv,ua_2)-, or (vu,va_4)-perilous sets exist. Then, by Lemma 31(ii), improving uv to ux,vy is p_G-admissible for $(x,y)=(a_2,a_3),(a_2,a_4),(a_3,a_4)$. If $c(a_2)\neq c(u)$, then, since the improvements of uv to ua_2,va_3 and that of uv to ua_2,va_4 are not $(p_G,\mathcal{P})$-allowed, and by Observation 11, we have $c(a_3)=c(v)=c(a_4)\neq c(a_3)$, a contradiction.

Hence $c(a_2) = c(u)$, and similarly $c(v) = c(a_4)$. Therefore, $c(a_2) \neq c(a_4)$, and then, by Definition 7.3 for $\mathcal{A}$, we have $c(a_1) = c(a_3)$. ◻

The following lemma will handle the case of C_4^*-obstacles.

LEMMA 57. *Let $\mathcal{A} = \{A_1, \dots, A_4\}$ be a simple C_4^*-obstacle for $(p_G, \mathcal{P}, m_G)$, $e = uv \in E$, $i \in \{1, \dots, 4\}$, $u \in A_i$, and $v \in A_{i+1}$.*

1. *Suppose that no allowed improvement exists for uv. Then either $c(u) = c(a_i) = c(a_{i+2})$ or $c(v) = c(a_{i-1}) = c(a_{i+1})$.*
2. *Suppose that no allowed improvement exists. If all the edges of G connect distinct members of $\mathcal{A}$ and there exists no C_4^*-obstacle for $(p_0, \mathcal{P}, m_0)$, then there exist two edges f and g such that there exists a complete $(p_{G^{f,g}}, \mathcal{P})$-allowed splitting off.*

Proof. By Corollary 55 and Lemma 43, we have $p_G \leq 1$ and $T_{a_h} = A_h$ for all $h \in \{1, \dots, 4\}$. We may assume that $i = 1$.

1. By Definition 7.3 for $\mathcal{A}$, we may assume $c(a_1) = c(a_3)$. Assume that $c(u) \neq c(a_1)$. Then, $u \neq a_1$. By Corollary 33, no (uv, ua_3)-, (vu, va_4)-, or (vu, va_1)-perilous sets exist. Therefore, by Lemma 31(ii), the improvement of uv to ua_3, va_1 and that of uv to ua_3, va_4 are p_G-admissible. Since these improvements are not allowed, we have, by Observation 11, $c(v) = c(a_1) \neq c(a_4) = c(v)$, a contradiction.
2. We will unsplit two edges and find a complete allowed splitting off by performing first an allowed flip and then an improvement that is allowed for the resulting functions.
   Since no allowed improvement exists, point 1 implies that for every edge $u'v'$ either $c(u') = c(a_1) = c(a_3)$ or $c(v') = c(a_2) = c(a_4)$. Then, since no C_4^*-obstacle exists for $(p_0, \mathcal{P}, m_0)$, there exist edges $f = u_1v_1$ and $g = u_2v_2$ such that $u_1, u_2 \in A_1 \cup A_3$, $c(u_1) \neq c(a_1) = c(a_3) = c(u_2)$, and $c(v_2) \neq c(a_2) = c(a_4) = c(v_1)$.
   Assume without loss of generality that $u_1 \in A_1$ and $v_1 \in A_2$. By Lemma 31(i) and Corollary 33, flipping v_1u_1 for v_1a_1 is p_G-admissible, and, by Observation 11, $c(a_1) \neq c(a_2) = c(v_1)$; thus the flipping is allowed. Let G' be the resulting graph. By Claim 50, $\mathcal{A}$ is a simple C_4^*-obstacle for $(p_{G'}, \mathcal{P}, m_{G'})$, and $u_1 = V_+(m_{G'}) \cap A_1$ is the new $m_{G'}$-positive element of A_1. Since $c(a_2) \neq c(v_2)$ and $c(u_1) \neq c(a_3)$, point 1 implies that there exists a $(p_{G'}, \mathcal{P})$-allowed improvement using the edge $u'v'$, which means that there exists a complete $(p_{G^{f,g}}, \mathcal{P})$-allowed splitting off. ◻

The following lemma considers the case of C_5^*-obstacles.

LEMMA 58. *Let $\mathcal{A} = \{A_1, \dots, A_4\}$ be a simple C_4^*-obstacle for $(p_G, \mathcal{P}, m_G)$, $e = uv \in E$, $i \in \{1, \dots, 4\}$, $u, v \in A_i$. Suppose that no allowed improvement exists.*

1. *If $e' = u'v' \in E$, $u', v' \in A_i$, $e' \neq e$, and no consecutive improvement exists for e and for e', then X_e and $X_{e'}$ exist and one of them contains the other one.*
2. *If all the edges of G are contained in members of $\mathcal{A}$ and no edge belongs to a consecutive improvement, then there exists a C_5^*-obstruction for $(p_0, \mathcal{P}, m_0)$.*

Proof. By Corollary 55 and Lemma 43, we have $p_G \leq 1$ and $T_{a_h} = A_h$ for all $h \in \{1, \dots, 4\}$. We may assume that $i = 1$.

1. By Claim 56.2, let $X_{\hat{e}}^j$ be an $(\hat{e}, a_j)$-perilous set for $\hat{u}\hat{v} = \hat{e} = e, e'$ and $j = 2, 4$. Without loss of generality we may assume that $X_{\hat{e}}^2 \cap \{\hat{u}, \hat{v}\} = \hat{u}$, that is, that $X_{\hat{e}}^2$ is a $(\hat{u}\hat{v}, \hat{u}a_2)$-perilous set. Then, by Lemma 35, no $(\hat{v}\hat{u}, \hat{v}a_2)$-perilous set

exists, so, since no consecutive improvement exists for $\hat{v}\hat{u}$, by Lemma 31(ii), there exists a $(\hat{u}\hat{v}, \hat{u}a_4)$-perilous set, and hence, by Lemma 35, no $(\hat{v}\hat{u}, \hat{v}a_4)$-perilous set exists, and then we have $X^2_{\hat{e}} \cap \hat{e} = \hat{u} = X^4_{\hat{e}} \cap \hat{e}$. Therefore, Claim 34.3 applies and defines a unique $X_{\hat{e}} \subset A_1$ such that $X_{\hat{e}} \cup A_j$ is the $(\hat{e}, a_j)$-perilous set $X^j_{\hat{e}}$ for $j = 2, 4$. By Claim 34.1, $p_G(Y_{\hat{e}}) = 0$, where $Y_{\hat{e}} = X_{\hat{e}} \cup A_2 \cup A_4$. Note that, since the edge $\hat{e}$ enters the set $Y_{\hat{e}}$, the latter implies that $Y_{\hat{e}}$ is p_0-positive for $\hat{e} = e, e'$. Note also that, since e' enters $X_{e'}$, Claim 34.1 implies that e does not enter $X_{e'}$.

Suppose that $X_e \cap X_{e'} = \emptyset$. Then applying (13) to Y_e and $Y_{e'}$ and Definition 7.1c implies $0 + 0 = p_G(Y_e) + p_G(Y_{e'}) \leq p_G(Y_e \cap Y_{e'}) + p_G(Y_e \cup Y_{e'}) = p_G(A_2 \cup A_4) + p_G(Y_e \cup Y_{e'}) \leq p_G(Y_e \cup Y_{e'})$. In particular, $Y_e \cup Y_{e'}$ is p_0-positive. Since $p_G \leq 1$, we have $p_G(X_e \cup X_{e'} \cup A_2) \leq -1$ because, by Claim 18, $X_e \cup X_{e'} \cup A_2$ is not tight, and, by e enters $X_e \cup X_{e'} \cup A_2$ and Claim 34.1, it is not (e, a_2)-perilous. Then, by $\mathcal{A}$ being a simple C_4^*-obstacle, Claim 39.4, (14) applied to $Y_e \cup Y_{e'}$, and $A_3 \cup A_4$, we have $0+1 \leq p_G(Y_e \cup Y_{e'}) + p_G(A_3 \cup A_4) \leq p_G(X_e \cup X_{e'} \cup A_2) + p_G(A_3) \leq -1 + 1 = 0$, a contradiction.

Thus, $X_e \cap X_{e'} \neq \emptyset$, and since, by Claims 56.2 and 34.5, $\{X_e, X_{e'}\}$ is laminar, the assertion follows.

2. Let $j \in \{1, \ldots, 4\}$. It follows, by point 1, that the edges contained in A_j can be ordered as $e_1, \ldots, e_\ell$ such that $1 \leq k < k' \leq \ell$ implies $X_{e_k} \subset X_{e_{k'}}$. Let $X_{e_{\ell+1}} = A_j$, $B_{e_k} = X_{e_{k+1}} \setminus X_{e_k}$ for $k = 1, \ldots, \ell$ and $A'_j = A_j \setminus \bigcup_{k=1}^{\ell} B_{e_k}$. Note that $A'_j = A_j$ if A_j contains no edge, and $A'_j = X_{e_1}$ otherwise. Thus, $\{A'_j, B_f : f \subseteq A_j\}$ partitions A_j, and, since $\mathcal{A}$ partitions V, we get that $\mathcal{A}' = \{A'_1, A'_2, A'_3, A'_4\} \cup \{B_f : f \in E\}$ is a partition of V.

We show that $\mathcal{A}'$ is a C_5^*-obstruction for $(p_0, \mathcal{P}, m_0)$. First, let us make a few observations. Recall that $p_0 = p_G + d_G$, $E_{\delta(\mathcal{A})}$ is empty, and, by Claim 34.4, f is the only edge entering X_f for all $f \in E$. In particular, no member of $\mathcal{A}'$ contains an edge, and $m_0(B_f) = m_G(B_f) + d_G(B_f) = 2$ and $d_G(A_j) = 0$, $d_G(A'_j) = 1$.

Let $f \in A_j$, say $f = e_k$ and $h \in \{j-1, j+1\}$. By Claims 34.4, 39.4, 56.2, and 34.3, we get $p_0(X_f) = p_0(A_j) = p_0(A_j \cup A_h) = p_0(X_f \cup A_h) = 1$. It follows that

$$p_0(A'_j) = p_0(A'_j \cup A_h) = 1. \tag{21}$$

Then, by applying (4) to $A_h \cup X_{e_{k+1}}$ and $A_{h+2} \cup X_{e_k}$, we get

$$p_0(A_h \cup B_f) \geq 1. \tag{22}$$

We next verify the conditions of Definition 8.

8.1a By (21), we have $p_0(A'_j) = 1$ for $j \in \{1, \ldots, 4\}$; hence Definition 8.1a follows.

8.1d By (21), we may suppose that $A'_j = X_f$ and $A'_{j+1} = X_g$ for $f, g \in E$. By (22), we may apply Claim 12.1 to $\{A_{j+2} \cup A_{j+3}\} \bigcup \{A_{j+3} \cup B_{f'} : f' \in A_j, f' \neq f\} \bigcup \{A_{j+2} \cup B_{g'} : g' \in A_{j+1}, g' \neq g\} \bigcup \{A_{j+2} \cup X_g\}$, and then, by Claim 39.4, the symmetry of p_0, and (21), we have $1 = p_0(A_{j+2} \cup A_{j+3}) \leq p_0(V \setminus (A'_j \cup A'_{j+1})) \leq p_0(V \setminus A'_j) = p_0(A'_j) = 1$; that is, equality holds everywhere and, by the symmetry of p_0, Definition 8.1d follows.

8.1c Note that the previous proof works for A'_j and B_g if $g \in A_{j-1} \cup A_{j+1}$. Suppose now that $g \in A_j$. By (22), we may apply Claim 12.1 to $\{V \setminus$

$A_j\} \bigcup \{A_{j+1} \cup B_{g'} : g' \in A_j, g' \neq g\} \bigcup \{A_{j+1} \cup B_g\}$, and then, by the symmetry of p_0 and (21), we have $1 = p_0(A_j) = p_0(V \setminus A_j) \leq p_0(V \setminus (A'_j \cup B_g)) \leq p_0(A'_j) = 1$; that is, equality holds everywhere and, by the symmetry of p_0, Definition 8.1c follows.

Finally, suppose that $g \in A_{j+2}$. By (22), we may apply Claim 12.1 to $\{A_{j+3} \cup B_{f'} : f' \in A_j\} \bigcup \{A_{j+3} \cup B_{g'} : g' \in A_{j+1}, g' \neq g\} \bigcup \{A_{j+1} \cup A'_{j+2}\} \bigcup \{A_{j+3} \cup B_g\}$, and then, by (22) and (21), we have $1 \leq p_0(A_{j+3} \cup B_{f'}) \leq p_0(V \setminus (A'_j \cup B_g)) \leq p_0(A'_j) = 1$; that is, equality holds everywhere and, by the symmetry of p_0, Definition 8.1c follows.

8.1b Let $f \in A_j$. By (21) and (3) applied to $A'_j \cup A_{j-1}$ and $A'_j \cup A_{j+1}$, we have $p_0(Y) \geq 1$, where $Y = A'_j \cup A_{j-1} \cup A_{j+1}$. Then, by (3) applied to Y and $V \setminus A_j$, the symmetry of p_0, and Definition 7.1c for $\mathcal{A}$, we get $p_0(Z) \geq 2$, where $Z = A'_j \cup (V \setminus A_j)$. By (22), we may apply Claim 12.1 to $\{Z\} \bigcup \{A_{j+1} \cup B_{f'} : f' \in A_j, f' \neq f\}$, and then, by the symmetry of p_0 and m_0 being p_0-admissible, we have $2 \leq p_0(Z) \leq p_0(V \setminus B_f) = p_0(B_f) \leq m_0(B_f) = 2$; that is, equality holds everywhere and Definition 8.1b follows.

8.1e If $p_0(A'_j \cup A'_{j+2}) \geq 1$, then Claim 12.1 applies to $\{A_j\} \bigcup \{A'_j \cup A'_{i+2}\} \bigcup \{A_{j+2}\}$, and we get, by Definition 7.1c, $1 \leq p_0(A_j \cup A_{j+2}) = p_G(A_j \cup A_{j+2}) \leq 0$, a contradiction.

8.1f and 8.2 Recall that m_0 is $(p_0, \mathcal{P})$-allowed. Moreover, no $A \in \mathcal{A}'$ contains an edge; thus the above results on d_G and p_0 imply $p_0(A) = m_0(A)$ for all $A \in \mathcal{A}'$. Therefore, we have $m_0(V) \geq \sigma_{p_0} \geq \sum_{A \in \mathcal{A}'} p_0(A) = \sum_{A \in \mathcal{A}'} m_0(A) = m_0(V)$. ∎

Finally, C_6^*-obstacles are treated by the following lemmas.

LEMMA 59. *Let $\mathcal{A} = \{A_1, \ldots, A_4\}$ be a simple C_4^*-obstacle for $(p_G, \mathcal{P}, m_G)$, $e = uv \in E$, $i \in \{1, \ldots, 4\}$, $u, v \in A_i$. Suppose that no allowed improvement exists.*

1. *If an edge connects distinct consecutive members of $\mathcal{A}$, then a consecutive improvement exists for e.*
2. *If a consecutive improvement exists for e, then there exists a C_6^*-obstacle for $(p_{G^e}, \mathcal{P}, m_{G^e})$.*

Proof. By Corollary 55 and Lemma 43, we have $p_G \leq 1$ and $T_{a_h} = A_h$ for all $h \in \{1, \ldots, 4\}$. We may assume that $i = 1$. By Claim 56.2, there exists a (uv, a_j)-perilous set X_j for $j \in \{2, 4\}$. Without loss of generality, X_2 is (uv, ua_2)-perilous. Note that, for both parts of the lemma, we have $X_2 \cap \{u, v\} \neq X_4 \cap \{u, v\}$; thus X_4 is (vu, va_4)-perilous. Indeed, for point 1 of the lemma, this relationship comes from Claim 56.3, and for point 2, it follows from Lemmas 35 and 31(ii) and the fact that e belongs to a consecutive improvement.

1. By Lemma 35, no (vu, va_2)- and no (uv, ua_4)-perilous sets exist. Thus, by Lemma 31(ii), improving uv to ua_4, va_2 is p_G-admissible, and the assertion follows.
2. By Claim 32.4, $X_j \setminus A_1 = A_j$ for $j = 2, 4$. By $\mathcal{A}$ being a partition of V and Claim 34.2, $\mathcal{A}' = \{A'_1, \ldots, A'_6\} = \{X_2 \cap A_1, A_2, A_3, A_4, X_4 \cap A_1, A_1 \setminus (X_4 \cup X_2)\}$ is a partition of V. To show that $\mathcal{A}'$ is a C_6^*-obstacle for $(p_{G^e}, \mathcal{P}, m_{G^e})$, we verify the conditions of Definition 9.

   9.1a–1b By Claim 34.2, we have $0 = p_G(A'_5 \cup A'_6) = p_G(A'_6 \cup A'_1)$ and $1 = p_G(A'_6)$. By Claim 32.2, $0 = p_G(A'_1) = p_G(A'_5)$. By $\mathcal{A}$ being a simple C_4^*-obstacle, Claim 39.4, and X_2 and X_4 being perilous, $1 = p_G(A'_2) = p_G(A'_3) = p_G(A'_4) = p_G(A'_2 \cup A'_3) = p_G(A'_3 \cup A'_4)$ and $0 = p_G(A'_1 \cup A'_2) =$

$p_G(A'_4 \cup A'_5)$. Then, by $p_G = p_{G^e} - d_e$, 9.1a–1b follow.

9.1c By $u \in A'_1$, $v \in A'_5$, m_G being p_G-admissible and modular, and Claim 32.2, we have $p_{G^e}(A'_1 \cup A'_5) = p_G(A'_1 \cup A'_5) \le m_G(A'_1 \cup A'_5) = m_G(X_2 \cap A_1) + m_G(X_4 \cap A_1) = 0$; by Claim 18 for $i = 2, 3, 4$, we have $p_{G^e}(A'_6 \cup A'_i) = p_G(A'_6 \cup A'_i) \le 0$; by Definition 7.1c, we have $p_{G^e}(A'_2 \cup A'_4) = p_G(A_2 \cup A_4) \le 0$; by Claims 18 and 56.1, no (uv, a_3)-perilous set exists, we have $p_{G^e}(A'_i \cup A'_3) = p_G(A'_i \cup A_3) + 1 \le -1 + 1$ for $i = 1, 5$; and, by Claim 18 and since no (uv, ua_4)- and no (vu, va_2)-perilous sets exist by Lemma 35, we have $p_{G^e}(A'_i \cup A'_{i+3}) = p_G(A_i \cup A'_{i+3}) + 1 \le -1 + 1$ for $i = 2, 4$.

9.1d–2 Since e was obtained by an allowed splitting off, m_{G^e} is p_{G^e}-admissible and $\mathcal{P}$-feasible. By Definition 7.1d for $\mathcal{A}$, the fact that m_{G^e} is p_{G^e}-admissible, the definition of $\sigma_{p_{G^e}}$, and Definition 9.1a for $\mathcal{A}'$, we have $6 = 4 + 2 = m_G(V) + 2 = m_{G^e}(V) \ge \sigma_{p_{G^e}} \ge \sum_{X \in \mathcal{A}'} p_{G^e}(X) = 6$ and Definitions 9.1d and 9.2 follow.

9.3 This is true by Claim 56.4. ∎

Lemma 60. *If $\mathcal{A}$ is a C_6^*-obstacle for $(p_G, \mathcal{P}, m_G)$ and e is an edge of G, then there exists a complete $(p_{G^e}, \mathcal{P})$-allowed splitting off.*

Proof. In each of the following cases, we will first perform a $(p_G, \mathcal{P})$-allowed splitting off and hence find, by Claim 53, a simple C_4^*-obstacle. If p' denotes the resulting function, then, by Corollary 55, we have $p' \le 1$. We will then find, by Lemma 31(ii), a $(p', \mathcal{P})$-allowed improvement involving e. This is equivalent to unsplitting e and then finding a complete $(p_{G^e}, \mathcal{P})$-allowed splitting off.

Let $\mathcal{A} = \{A_1, \ldots, A_6\}$ and $\{a_i\} = A_i \cap V_+(m_G)$ for every $i = 1, \ldots, 6$. Denote $e = uv$. We may assume, by Definitions 9.1a–1b and Claim 13.1, that $u \in A_1$ and either $v \in A_1$ or $v \in A_2$.

Let $G_i = G_{a_i a_{i+2}}$ and $\mathcal{A}_i = \{A_i \cup A_{i+1} \cup A_{i+2}, A_{i+3}, A_{i+4}, A_{i+5}\}$ for $i = 2, 3, 5$. By Definition 9.3 for $\mathcal{A}$, $c(a_1) \ne c(a_5) \ne c(a_3)$ and $c(a_6) \ne c(a_2) \ne c(a_4)$. Then, by Lemma 42, the pair a_i, a_{i+2} is $(p_G, \mathcal{P})$-allowed, so m_{G_i} is $(p_{G_i}, \mathcal{P})$-allowed, and, by Claim 53, $\mathcal{A}_i$ is a simple C_4^*-obstacle for $(p_{G_i}, \mathcal{P}, m_{G_i})$ for $i = 2, 3, 5$. By Corollary 55, we have $p_{G_i} \le 1$ for $i = 2, 3, 5$. We consider two cases:

1. If $v \in A_1$, by Claim 56.1 for $\mathcal{A}_2$ (respectively, $\mathcal{A}_3$),

$$\text{no } (uv, a_5)\text{- (respectively, } (uv, a_4))\text{-perilous set exists} \tag{23}$$
$$\text{with respect to } p_{G_2} \text{ and } m_{G_2} \text{ (respectively, } p_{G_3} \text{ and } m_{G_3}).$$

By Claim 56.2 for $\mathcal{A}_3$, there exists for $j = 2, 6$ a (uv, a_j)-perilous set X_j with respect to p_{G_3} and m_{G_3} and, by Claim 32.4, with respect to p_G and m_G. We may assume that X_2 is (uv, ua_2)-perilous.

We show that $X' = X_2 \cup A_3 \cup A_4$ is a (uv, ua_3)-perilous set with respect to p_{G_2} and m_{G_2}. By Claim 32.4, we have $X_2 \setminus A_1 = A_2$. By X_2 being perilous; by Claim 13.2 applied for $J = \{2, 3, 4\}$; by (13) applied for p_G, X_2, and $A_2 \cup A_3 \cup A_4$; by Definition 9.1a; and by Claim 18 applied for the crossing sets X' and A_1, we have $p_G(X') = 0$. Note that $p_G(X') = p_{G_2}(X')$ and, by m_G being modular, X being perilous, Claim 41, and Definition 9.1a, we have $m_{G_2}(X') = m_G(X') - 2 = m_G(X_2) + m_G(A_3) + m_G(A_4) - 2 = 1 + 1 + 1 - 2 = 1$, so X' is (uv, ua_3)-perilous for p_{G_2} and m_{G_2}.

Hence, by Lemma 35,

$$\text{no } (vu, va_3)\text{-perilous set exists with respect to } p_{G_2} \text{ and } m_{G_2}. \tag{24}$$

(a) If X_6 is (vu, va_6)-perilous, we may suppose, by Claim 56.4 for $\mathcal{A}_3$, that $c(u) = c(a_6)$, $c(v) = c(a_2)$, and $c(a_1) = c(a_4)$. Now, by (23), (24), and Lemma 31(ii), improving uv to va_3, ua_5 is p_{G_2}-admissible, hence $(p_{G_2}, \mathcal{P})$-allowed, and we are done.

(b) If X_6 is (uv, ua_6)-perilous, then we have the following:

i. Suppose that $c(u) \neq c(a_4)$. Note that there exists $j \in \{2, 6\}$ such that $c(v) \neq c(a_j)$, and that, by Lemma 35, there is no (vu, va_j)-perilous set with respect to p_{G_3} and m_{G_3}. Then, by (23) and Lemma 31(ii), improving uv to ua_4, va_j is p_{G_3}-admissible, hence $(p_{G_3}, \mathcal{P})$-allowed, and we are done.

ii. Otherwise, $c(u) = c(a_4)$, and if necessary reverse the order of the sets $A_2, \ldots, A_6$ to assume that $c(a_3) \neq c(v)$. Hence, by (23), (24), and Lemma 31(ii), improving uv to ua_5, va_3 is $(p_{G_2}, \mathcal{P})$-allowed, and this finishes the proof of the lemma.

2. If $v \in A_2$, note that uv connects consecutive sets of $\mathcal{A}_5$ and $\mathcal{A}_3$. Then, for $i = 5$ or $i = 3$, depending on whether $c(u) = c(a_1)$ or $c(u) \neq c(a_1)$, Lemma 57.1 applied for $(p_{G_i}, \mathcal{P}, m_{G_i})$, $\mathcal{A}_i$, and uv implies that there exists a $(p_{G_i}, \mathcal{P})$-allowed improvement involving uv. ∎

4.3. Splitting off theorem. We will now prove our new splitting off result. First, we state a version that we will need for the algorithm, and then we state it as a characterization of the existence of a complete $(p_0, \mathcal{P})$-allowed splitting off.

THEOREM 61. *Let $p_0 : 2^V \to \mathbb{Z}$ be a symmetric crossing supermodular set function, $\mathcal{P}$ a partition of V, and $m_0 : V \to \mathbb{Z}_+$ a $(p_0, \mathcal{P})$-allowed degree specification. Then one of the following exists:*

(a) *a complete $(p_0, \mathcal{P})$-allowed splitting off,*

(b) *a p_0-full partition that shows that m_0 is not p_0-legal and a $\mathcal{P}$-partite graph F on V that covers p_0 with $|E(F)| \leq \dim(p_0) - 1$,*

(c) *an obstacle for $(p_0, \mathcal{P}, m_0)$.*

Proof. We outline an algorithm that outputs one of the above possibilities. First, we perform allowed splitting off as long as possible; second, we perform allowed improvements as long as possible. When we get stuck, unsplitting edges is necessary. Depending on the position of the edges, distinct cases occur. The description of the complete algorithm can be found in Figure 1.

Now we show the correctness of the SPLITTING OFF ALGORITHM.

Step 1. Perform arbitrary allowed splitting off as long as possible, and let G be the resulting graph. If $m_G(V) \leq 2$, then if necessary, perform a final allowed splitting off, and then we are done.

From now on, we suppose that $m_G(V) \geq 4$ and that *no allowed splitting off exists.*

Step 2. Suppose that no p_G-admissible splitting off exists. If no allowed improvement exists, then, by Lemma 37.9–10, we have the required partition and graph. Otherwise, there exists an allowed improvement, and by Corollary 29, after performing an allowed improvement, we can repeat it. Thus let us perform an arbitrary sequence of allowed improvements as long as there exists one, and let G' be the resulting graph. If $m_{G'}(V) \geq 4$, then we can stop with the required partition and graph provided by Lemma 37.9–10. Otherwise, $m_{G'}(V) \leq 2$ and hence, performing a final allowed splitting off if necessary, we are done.

From now on, we assume that *a p_G-admissible splitting off exists.*

Step 3. By (20) for m_G and Lemma 25, there exists a simple C_4^*-obstruction $\mathcal{A} =$

Splitting off algorithm.

Input : A symmetric crossing supermodular set function $p_0 : 2^V \to \mathbb{Z}$, a $(p_0, \mathcal{P})$-allowed degree specification $m_0 : V \to \mathbb{Z}_+$, and a partition $\mathcal{P}$ of V.

Output: Either a complete $(p_0, \mathcal{P})$-allowed splitting off, or a p_0-full partition that shows that m_0 is not p_0-legal and a $\mathcal{P}$-partite graph F on V that covers p_0 with $|E(F)| \le \dim(p_0)$ -1, or an obstacle for $(p_0, \mathcal{P}, m_0)$.

Step 1. Perform an arbitrary sequence of allowed splittings off as long as there exists one. Let G, m_G, and p_G be the resulting graph, degree specification, and function.

Step 2. If there exists no p_G-admissible splitting off, then perform an arbitrary sequence of allowed improvements as long as there exists one, and let G' be the resulting graph.

(a) If $m_{G'}(V) \ge 4$, then stop with the required partition and graph provided by Lemma 37.9–10.

(b) Otherwise, perform, if necessary, a final allowed splitting off and stop.

Step 3. Otherwise, by Lemmas 25 and 49, find a simple C_4^*-obstacle $\mathcal{A}$ for $(p_G, \mathcal{P}, m_G)$.

Step 4. If there exists an allowed improvement, then perform it, and then perform a final allowed splitting off and stop.

Step 5. If all the edges of G connect distinct members of $\mathcal{A}$, then, by Lemma 57.2, find

(a) either a C_4^*-obstacle for $(p_0, \mathcal{P}, m_0)$ and stop,

(b) or two edges such that after their unsplitting there exists a complete allowed splitting off and stop.

Step 6. If all the edges of G are contained in members of $\mathcal{A}$ but no consecutive improvement exists, then, by Lemma 58.2, there exists a C_5^*-obstruction for $(p_0, \mathcal{P}, m_0)$.

(a) If it is a C_5^*-obstacle for $(p_0, \mathcal{P}, m_0)$, then stop.

(b) Otherwise, Lemma 49 provides a complete $(p_0, \mathcal{P})$-allowed splitting off and stop.

Step 7. Otherwise, by Lemma 59.1, there exists a consecutive improvement for an edge e, and then, by Lemma 59.2, there exists a C_6^*-obstacle for $(p_{G^e}, \mathcal{P}, m_{G^e})$.

(a) If $G^e = G$, then stop with the C_6^*-obstacle for $(p_0, \mathcal{P}, m_0)$.

(b) Otherwise, G^e contains an edge e' and then, by Lemma 60 applied for e', there exists a complete $(p_{G^{e,e'}}, \mathcal{P})$-allowed splitting off and stop.

Fig. 1. *Splitting off algorithm.*

$\{A_1, \dots, A_4\}$ for $(p_G, \mathcal{P}, m_G)$. Since no splitting off is allowed, Lemma 49 implies that $\mathcal{A}$ satisfies Definition 7.3; hence $\mathcal{A}$ *is a simple* C_4^**-obstacle* for $(p_G, \mathcal{P}, m_G)$.

Step 4. Suppose that there exists an allowed improvement. Then perform it and then, since $m_G(V) = 4$, perform a final allowed splitting off, and we are done.

Thus we assume that *no allowed improvement exists*.

Step 5. Suppose that all the edges of G connect distinct members of $\mathcal{A}$. Then, by Lemma 57.2, we can find either a C_4^*-obstacle for $(p_0, \mathcal{P}, m_0)$, or two edges e and f such that a complete $(p_{G^{e,f}}, \mathcal{P})$-allowed splitting off exists, and we are done.

Step 6. Suppose that all the edges of G are contained in members of $\mathcal{A}$ but no consecutive improvement exists. Then, by Lemma 58.2, there exists a C_5^*-obstruction for $(p_0, \mathcal{P}, m_0)$. By Lemma 49, either it is a C_5^*-obstacle for $(p_0, \mathcal{P}, m_0)$, or we have a complete $(p_0, \mathcal{P})$-allowed splitting off and we are done.

We may assume, by Lemma 59.1, that *a consecutive improvement exists* for an edge e.

Step 7. By Lemma 59.2, there exists a C_6^*-obstacle for $(p_{G^e}, \mathcal{P}, m_{G^e})$. If $G^e = G$, then we have a C_6^*-obstacle for $(p_0, \mathcal{P}, m_0)$, and we are done. Otherwise, G^e contains an edge e', so Lemma 60 applies for e' and we are done. □

The previous theorem gives at once the main result of the section.

THEOREM 62. *Let $p_0 : 2^V \to \mathbb{Z}$ be a symmetric crossing supermodular set function, $m_0 : V \to \mathbb{Z}_+$ a degree specification, and $\mathcal{P}$ a partition of V. There exists a complete $(p_0, \mathcal{P})$-allowed splitting off if and only if m_0 is $(p_0, \mathcal{P})$-allowed and p_0-legal, and there exists no obstacle for $(p_0, \mathcal{P}, m_0)$.*

Proof. The necessity of the conditions follows by Lemmas 38 and 54. To show the sufficiency, let us suppose that m_0 is $(p_0, \mathcal{P})$-allowed and p_0-legal, and there exists no obstacle for $(p_0, \mathcal{P}, m_0)$. Then, by Theorem 61, there exists a complete $(p_0, \mathcal{P})$-allowed splitting off. □

5. Minimization version. In this section, we are given a symmetric crossing supermodular function p on V and a partition $\mathcal{P} = \{P_1, \dots, P_r\}$ of V. We show how to find algorithmically a $\mathcal{P}$-partite graph covering p with a minimum number of edges.

First, we provide the lower bound. Second, we explain how to find a minimum $(p, \mathcal{P})$-allowed degree specification. Then, we describe the instances for which the lower bound may not be achieved. Finally, we prove our main result; see Theorem 71.

5.1. Lower bound. Let $OPT(p, \mathcal{P})$ be the minimum number of edges of a $\mathcal{P}$-partite a graph that covers p.

DEFINITION 63. *Let Φ be the maximum of the following values:*

$$\alpha_p = \max\left\{\left\lceil \frac{1}{2}\sum_{X\in\mathcal{X}} p(X)\right\rceil : \mathcal{X} \text{ subpartition of } V\right\},$$

$$\beta_p = \max\left\{\sum_{Y\in\mathcal{Y}} p(Y) : \mathcal{Y} \text{ subpartition of } P, P \in \mathcal{P}\right\},$$

$$\dim(p) - 1 = \max\{|\mathcal{V}| : \mathcal{V} \text{ } p\text{-full partition of } V\} - 1.$$

LEMMA 64. *$OPT(p, \mathcal{P}) \geq \Phi$.*

Proof. Let $G = (V, E)$ be a $\mathcal{P}$-partite graph that covers p and that has $OPT(p, \mathcal{P})$ edges. First, let $\mathcal{X}$ be a subpartition of V such that $\alpha_p = \lceil \frac{1}{2}\sum_{X\in\mathcal{X}} p(X)\rceil$. Since an edge of E connects at most two sets of $\mathcal{X}$, applying (5) gives $|E| \geq \lceil \frac{1}{2}\sum_{X\in\mathcal{X}} d_G(X)\rceil \geq \lceil \frac{1}{2}\sum_{X\in\mathcal{X}} p(X)\rceil = \alpha_p$. Second, let $\mathcal{Y}$ be a subpartition of some $P \in \mathcal{P}$ such that $\beta_p = \sum_{Y\in\mathcal{Y}} p(Y)$. Since G is $\mathcal{P}$-partite, an edge of E enters at most one set of $\mathcal{Y}$; thus applying (5) gives $|E| \geq \sum_{Y\in\mathcal{Y}} d_G(Y) \geq \sum_{Y\in\mathcal{Y}} p(Y) = \beta_p$. By the above inequalities and by Lemma 2.1, $OPT(p, \mathcal{P}) = |E| \geq \max\{\alpha_p, \beta_p, \dim(p) - 1\} = \Phi$. □

5.2. Extension. We start this section with some algorithmic arguments. Since we do not want to rely on the results of Benczúr and Frank [4], we do not know how to calculate $\dim(p)$. On the other hand, we want to present an algorithmic proof of the result, so we will not use Φ, which depends on $\dim(p)$, in the extension phase.

A degree specification m is called an *extension for* $(p, \mathcal{P})$ if m is p-admissible, $\mathcal{P}$-feasible, and satisfies

$$\frac{1}{2}m(V) = \max\{\alpha_p, \beta_p\}. \tag{25}$$

An extension always exists, and we describe how to find one in Figure 2. The algorithm is formulated in a way such that any extension can be its output (with appropriate

EXTENSION ALGORITHM.
INPUT : A symmetric crossing supermodular function $p : 2^V \to \mathbb{Z}$ and a partition $\mathcal{P}$ of V.
OUTPUT: An extension m for $(p, \mathcal{P})$.

Step 1. Pick a p-admissible degree specification m that minimizes $m(V)$.

Step 2. If $m(V)$ is odd, then let $m = m + \chi_u$ for some $u \in V$.

Step 3. If some $P \in \mathcal{P}$ satisfies $m(P) > \frac{m(V)}{2}$, then

3a. If $X_u \not\subseteq P$ for some $u \in P \cap V_+$, then let $m = m - \chi_{\{u\}} + \chi_{\{u'\}}$ for some $u' \in X_u \setminus P$, Repeat 3.

3b. Otherwise, let $v \in V \setminus P$ and $m = m + (2m(P) - m(V))\chi_{\{v\}}$.

Step 4. Stop.

FIG. 2. *Extension algorithm.*

choices). Recall that, for $u \in V$, X_u is the minimal tight set containing u (if u is not contained in a tight set, then $X_u := V$).

LEMMA 65. *The* EXTENSION ALGORITHM *outputs an extension; see Figure* 2.

Proof. Let m_i be the degree specification obtained after Step i in the above algorithm.

- By Step 1, m_1 (and then m_2) is p-admissible. Hence, by Claim 16, m_{3a} is p-admissible, and then so is m_4.
- By Step 2, $m_2(V)$ is even. If m_2 satisfies (20), then $m_4 = m_2$ is $\mathcal{P}$-feasible. Otherwise, there exists a $P \in \mathcal{P}$ such that $m_2(P) > \frac{1}{2}m_2(V)$. Then either $m_2(P)$ decreases to $\frac{1}{2}m_2(V)$ in Step 3a—that is, $m_4(P) = m_{3a}(P) = \frac{1}{2}m_2(V) = \frac{1}{2}m_4(V)$—or $m_2(V)$ increases to $2m_{3a}(P)$ in Step 3b—that is, by $v \in V \setminus P$, $m_4(P) = m_{3b}(P) = m_{3a}(P) = \frac{1}{2}m_{3b}(V) = \frac{1}{2}m_4(V)$. In both cases, m_4 is $\mathcal{P}$-feasible.
- It remains to show that (25) is satisfied. For some subpartition $\mathcal{Y}$ of some $P \in \mathcal{P}$, by m_4 being p-admissible and $\mathcal{P}$-feasible, we have $\beta_p = \sum_{Y \in \mathcal{Y}} p(Y) \leq \sum_{Y \in \mathcal{Y}} m_4(Y) \leq m_4(P) \leq \frac{1}{2}m_4(V)$. By parity, Theorem 1, and (9), $\frac{1}{2}m_2(V) = \lceil \frac{1}{2}m_1(V) \rceil = \lceil \frac{1}{2}\sigma_p \rceil = \alpha_p$. It follows that $\max\{\alpha_p, \beta_p\} \leq \frac{1}{2}m_4(V)$.
  If $m_4(V) = m_2(V)$, then $\max\{\alpha_p, \beta_p\} \leq \frac{1}{2}m_4(V) = \frac{1}{2}m_2(V) = \alpha_p \leq \max\{\alpha_p, \beta_p\}$, and we are done.
  Otherwise, for some $P' \in \mathcal{P}$, $m_4(V) = m_{3b}(V) = m_{3a}(V) + (2m_{3a}(P') - m_{3a}(V)) = 2m_{3a}(P')$. Note that the set D of the m_{3a}-positive elements of P' is precisely the set of the m_{3b}-positive elements of P'. We have, by definition, $m_{3a}(D) = m_{3a}(P')$. Since the algorithm executed Step 3b, every element of D belongs to a tight set. Hence, by Claim 15.2 applied to D, there exists a subpartition $\mathcal{Z}$ of P' such that $\sum_{Z \in \mathcal{Z}} p(Z) \geq m_{3a}(D)$. By the definition of β_p, we have $\beta_p \geq \sum_{Z \in \mathcal{Z}} p(Z)$. Then $\max\{\alpha_p, \beta_p\} \geq \beta_p \geq \sum_{Z \in \mathcal{Z}} p(Z) \geq m_{3a}(D) = m_{3a}(P') = \frac{1}{2}m_4(V) \geq \max\{\alpha_p, \beta_p\}$, and we are done. ∎

Note that if at least one of the conditions of Step 2 or 3b holds, then $m(V) > \sigma_p$; therefore there is no obstacle for $(p, \mathcal{P}, m)$.

5.3. Configurations. In this section we describe the functions and the partitions for which the lower bound may not be achieved. They may be classified into three different types of structures, called configurations. Two of these are natural generalizations of the configurations arising for graphs and hypergraphs. A new kind of configuration arises, which exists only in the abstract form of the problem. A configuration is a partition $\mathcal{A}$ of V satisfying two kinds of conditions. First, the p-values of the sets in the partition $\mathcal{A}$ and the lower bound Φ satisfy strict conditions. Second, the partition $\mathcal{A}$ is intimately related to the partition $\mathcal{P}$.

DEFINITION 66. *A partition* $\mathcal{A} = \{A_1, \dots, A_4\}$ *of* V *is a* C_4^*-configuration for $(p, \mathcal{P})$ *if the following hold:*
1. $\mathcal{A}$ *is a* C_4^**-construction for* p,
2. $\alpha_p = \max\{\alpha_p, \beta_p\}$,
3. *there exist* $\ell \in \{1, 2\}$ *and* $P \in \mathcal{P}$ *such that* $(A_\ell, A_{\ell+2})$ *is a* P*-pair.*

DEFINITION 67. *A partition* $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \dots, B_t\}$ *of* V $(t \geq 1)$ *is a* C_5^*-configuration for $(p, \mathcal{P})$ *if the following hold:*
1. $\mathcal{A}$ *is a* C_5^**-construction for* p,
2. $\alpha_p = \max\{\alpha_p, \beta_p\}$,
3. (a) *either there exist* $\ell \in \{1, 2\}$ *and* $P \in \mathcal{P}$ *such that* $\{A_\ell, A_{\ell+2}, B_1, \dots, B_t\}$ *is a* P*-subpartition,*
   (b) *or there exist* $j_0 \in \{1, \dots, t\}$ *and distinct* $P_{k_1}, P_{k_2} \in \mathcal{P}$ *such that for* $i = 1, 2$, $\{A_i, A_{i+2}\} \cup \{B_j : j \neq j_0\}$ *is a* P_{k_i}*-subpartition.*

A C_5^**-configuration is of* type 1 *(respectively,* type 2*) if point* 3a *(respectively,* 3b*) is satisfied.*

DEFINITION 68. *A partition* $\mathcal{A} = \{A_1, \dots, A_6\}$ *of* V *is a* C_6^*-configuration for $(p, \mathcal{P})$ *if the following hold:*
1. $\mathcal{A}$ *is a* C_6^**-construction for* p,
2. *there exist distinct* $P_{k_i} \in \mathcal{P}$ *such that* (A_i, A_{i+3}) *is a* P_{k_i}*-pair for* $i = 1, 2, 3$.

Definitions 66.3, 67.3, and 68.2 will also be called *color conditions.* A *configuration* is a C_4^*- or a C_5^*- or a C_6^*-configuration.

The following lemma explains why the condition $\alpha_p = \max\{\alpha_p, \beta_p\}$ does not exist for C_6^*-configurations and why the third lower bound $\dim(p) - 1$ does not appear in the definition of configurations.

LEMMA 69. *If a configuration exists for* $(p, \mathcal{P})$*, then we have the following:*
1. $\alpha_p = \max\{\alpha_p, \beta_p\}$,
2. $\alpha_p = \Phi$.

Proof. Let $\mathcal{A}$ be a configuration for $(p, \mathcal{P})$.
1. If $\mathcal{A}$ is a C_4^*- or a C_5^*-configuration, then Definition 66.2 or 67.2 implies the assertion. Let $\mathcal{A}$ be a C_6^*-configuration. By (9) and Definitions 68.1 and 9.1d, $\alpha_p = \lceil \frac{1}{2}\sigma_p \rceil = 3$. By Definitions 68.2 and 9.1a, for $i = 1, 2, 3$, there exist distinct $P_{k_i} \in \mathcal{P}$ and subpartitions $\mathcal{X}_j$ of $A_j \cap P_{k_i}$ for $j = i, i+3$ such that $\sum_{X \in \mathcal{X}_i \cup \mathcal{X}_{i+3}} p(X) = p(A_i) + p(A_{i+3}) = 2$. Let $P \in \mathcal{P}$ and $\mathcal{Y}$ subpartition of P be such that $\beta_p = \sum_{Y \in \mathcal{Y}} p(Y)$. Without loss of generality we may suppose that $P_{k_1} \neq P \neq P_{k_2}$. Then $\mathcal{Z} := \mathcal{X}_1 \cup \mathcal{X}_4 \cup \mathcal{X}_2 \cup \mathcal{X}_5 \cup \mathcal{Y}$ is a subpartition of V, and hence $6 = \sigma_p \geq \sum_{Z \in \mathcal{Z}} p(Z) = \sum_{X \in \mathcal{X}_1 \cup \mathcal{X}_4} p(X) + \sum_{X \in \mathcal{X}_2 \cup \mathcal{X}_5} p(X) + \sum_{Y \in \mathcal{Y}} p(Y) = 2 + 2 + \beta_p$, that is, $\alpha_p = 3 > 2 \geq \beta_p$, and the assertion follows.
2. By Theorem 1, let m be a minimal p-admissible degree specification. Then, by Definition 66.1 or 67.1 or 68.1, $\mathcal{A}$ is an obstruction for (p, m). By Lemma 48, there exists a complete p-admissible splitting off; let E be the resulting set of edges. By Lemma 2.1 and Observation 10, we have $\dim(p) - 1 \leq |E| = \frac{1}{2}m(V) = \frac{1}{2}\sigma_p = \alpha_p$. Then, by point 1, $\alpha_p \geq \Phi = \max\{\alpha_p, \beta_p, \dim(p) - 1\} \geq$

α_p, as required. ∎

There is a strong relation between configurations and obstacles, which is shown in the following lemma.

LEMMA 70. *A partition $\mathcal{A}$ is an obstacle for $(p, \mathcal{P}, m)$ for every extension m for $(p, \mathcal{P})$ if and only if $\mathcal{A}$ is a configuration for $(p, \mathcal{P})$.*

Proof of sufficiency. Let $\mathcal{A}$ be a configuration and m an extension for $(p, \mathcal{P})$. To show that $\mathcal{A}$ is an obstacle, we verify the three conditions separately.

1. By Definitions 66.1, 67.1, and 68.1, $\mathcal{A}$ is a construction for p of the same type, and hence Definitions 7.1, 8.1, and 9.1 are satisfied.
2. Since m is an extension for $(p, \mathcal{P})$, m is p-admissible and $\frac{1}{2}m(V) = \max\{\alpha_p, \beta_p\}$. By Lemma 69.1, $\max\{\alpha_p, \beta_p\} = \alpha_p$. By Lemma 69.2, $\alpha_p = \Phi$. By Lemma 2.1 and Observation 10, $\alpha_p = \frac{1}{2}\sigma_p$. It follows that $\sigma_p = m(V)$, so m is minimal and also $m(V) = 2\Phi$. Thus, Definitions 7.2, 8.2, and 9.2 are satisfied.
3. Since m is an extension for $(p, \mathcal{P})$, m is $\mathcal{P}$-feasible. We consider the three different configurations separately.
   (a) $\mathcal{A}$ is a C_4^*-configuration. Then, there exist $\ell \in \{1, 2\}$ and $P \in \mathcal{P}$ such that $(A_\ell, A_{\ell+2})$ is a P-pair; that is, there exists a subpartition $\mathcal{X}_i$ of $A_i \cap P$ such that $\sum_{X \in \mathcal{X}_i} p(X) = p(A_i)$ for $i = \ell, \ell + 2$. By Lemma 69.2; Definitions 66.2, 7.1d, and 66.3; m being p-admissible, nonnegative, and $\mathcal{P}$-feasible; and $m(V) = 2\Phi$, we have $\Phi = \frac{1}{2}\sigma_p = p(A_\ell) + p(A_{\ell+2}) = \sum_{X \in \mathcal{X}_\ell \cup \mathcal{X}_{\ell+2}} p(X) \leq \sum_{X \in \mathcal{X}_\ell \cup \mathcal{X}_{\ell+2}} m(X) \leq m(P \cap (A_\ell \cup A_{\ell+2})) \leq m(P) \leq \frac{1}{2}m(V) = \Phi$. It follows that the m-positive elements of $A_\ell \cup A_{\ell+2}$ are the m-positive elements of P, so Definition 7.3 is satisfied.
   (b) $\mathcal{A}$ is a C_5^*-configuration.
      i. If it is of type 1, that is, there exist $\ell \in \{1, 2\}$ and $P \in \mathcal{P}$ such that $\mathcal{X} = \{A_\ell, A_{\ell+2}, B_1, \ldots, B_t\}$ is a P-subpartition, then, by Definitions 67.2, 8.1f, 67.3a, m being p-admissible and $\mathcal{P}$-feasible, and $m(V) = 2\Phi$, we have $\Phi = \frac{1}{2}\sigma_p = t+2 = \sum_{X \in \mathcal{X}} 1 = \sum_{X \in \mathcal{X}} p(X') \leq \sum_{X \in \mathcal{X}} m(X') \leq m(P \cap (\bigcup_{X \in \mathcal{X}} X)) \leq \frac{1}{2}m(V) = \Phi$. It follows that each set of $\mathcal{X}$ contains an m-positive element of P, so Definition 8.3a is satisfied.
      ii. If it is of type 2, that is, there exist $j_0 \in \{1, \ldots, t\}$ and distinct $P_{k_1}, P_{k_2} \in \mathcal{P}$ such $\mathcal{X}_1 = \{A_1, A_3\} \cup \{B_j : j \neq j_0\}$ and $\mathcal{X}_2 = \{A_2, A_4\} \cup \{B_j : j \neq j_0\}$ are P_{k_1}- and P_{k_2}-subpartitions, then, by Definitions 67.2, 8.1f, 67.3b, m being p-admissible, Definition 8.1b, and $m(V) = 2\Phi$, we have $2\Phi - 2 = \sigma_p - 2 = 2t + 2 = \sum_{X \in \mathcal{X}_1} 1 + \sum_{X \in \mathcal{X}_2} 1 = \sum_{X \in \mathcal{X}_1} p(X') + \sum_{X \in \mathcal{X}_2} p(X') \leq \sum_{X \in \mathcal{X}_1} m(X') + \sum_{X \in \mathcal{X}_2} m(X') \leq m(P_{k_1} \cap (\bigcup_{X \in \mathcal{X}_1} X)) + m(P_{k_2} \cap (\bigcup_{X \in \mathcal{X}_2} X)) \leq m(V - B_{j_0}) = m(V) - 2 = 2\Phi - 2$. It follows that A_i, A_{i+2}, B_j for $j \in \{1, \ldots, t\} \setminus j_0$ contains an m-positive element of P_{k_i} for $i = 1, 2$, so Definition 8.3b is satisfied.
   (c) $\mathcal{A}$ is a C_6^*-configuration. Then there exist, for $i = 1, 2, 3$, distinct $P_{k_i} \in \mathcal{P}$ such that (A_i, A_{i+3}) is a P_{k_i}-pair; that is, for $j = i, i + 3$ there exists a subpartition $\mathcal{X}_j$ of $A_j \cap P_{k_i}$ such that $\sum_{X \in \mathcal{X}_j} p(X) = p(A_j)$. By Lemma 69.2, Definitions 9.1d and 68.2, m being p-admissible, and $m(V) = 2\Phi$, we now have $2\Phi = \sigma_p = \sum_{i=1}^{3}(p(A_i) + p(A_{i+3})) = \sum_{i=1}^{3} \sum_{X \in \mathcal{X}_i \cup \mathcal{X}_{i+3}} p(X) \leq \sum_{i=1}^{3} \sum_{X \in \mathcal{X}_i \cup \mathcal{X}_{i+3}} m(X) \leq \sum_{i=1}^{3} m(P_{k_i} \cap (A_i \cup A_{i+3})) \leq \sum_{i=1}^{3} m(A_i \cup A_{i+3}) = m(V) = 2\Phi$. It follows that the

m-positive elements of $A_i \cup A_{i+3}$ are the m-positive elements of P_{k_i} for $i = 1, 2, 3$, so Definition 9.3 is satisfied.

Proof of necessity. Let us suppose that no configuration exists for $(p, \mathcal{P})$. By Lemma 65, there exists an extension m for $(p, \mathcal{P})$. If no obstacle exists for $(p, \mathcal{P}, m)$, then the lemma is proved. Suppose that there is an obstacle $\mathcal{A}$ for $(p, \mathcal{P}, m)$.

1. By Definitions 7.1, 8.1, and 9.1, $\mathcal{A}$ is a construction for p of the same type, and hence Definitions 66.1, 67.1, and 68.1 are satisfied.
2. Since, by Definitions 7.2, 8.2, and 9.2, the extension m is minimally p-admissible, we have, by Observation 10, $\max\{\alpha_p, \beta_p\} = \frac{1}{2}m(V) = \frac{1}{2}\sigma_p = \alpha_p$, so Definitions 66.2 and 67.2 are satisfied.
3. Since no configuration exists for $(p, \mathcal{P})$, Definition 66.3 (respectively, 67.3 and 68.2) does not hold for $\mathcal{A}$.

   Throughout this proof, we will very often replace m by $m' = m - \chi_{\{u\}} + \chi_{\{u'\}}$ for $u \in A \in \mathcal{A}$ and $u' \in X_u$. We use that, by Claim 16, m' is p-admissible and, by Lemma 43 and Claim 14.3, $X_u \subseteq A$, so $m'(A_i) = m(A_i)$ for every $A_i \in \mathcal{A}$, and hence $\mathcal{A}$ is an obstruction for (p, m').

   Below, we treat each obstacle separately. We will replace the extension m by an extension m' so that no obstacle exists for $(p, \mathcal{P}, m')$, arguing as follows. Suppose that an obstruction exists for (p, m'). If $\mathcal{A}$ fits the hypothesis of Corollary 44, that is, $\mathcal{A}$ is any obstacle but a nonsimple C_4^*-obstacle for $(p, \mathcal{P}, m)$, then $\mathcal{A}$ is the unique obstruction for (p, m) and hence, by construction, the unique one for (p, m'). In this case we will choose m' so that $\mathcal{A}$ does not satisfy the color condition for $(p, \mathcal{P}, m')$, ensuring that no obstacle exists for $(p, \mathcal{P}, m')$. Otherwise, $\mathcal{A}$ is a C_4^*-obstacle that is not simple for $(p, \mathcal{P}, m)$, and if an obstruction exists for (p, m'), it is either $\mathcal{A}$ or $\mathcal{A}' = \{A_1, A_3, A_2, A_4\}$. We will choose m' so that $\mathcal{A}$ is not an obstacle for $(p, \mathcal{P}, m')$, and show that $\mathcal{A}'$ is not an obstruction for (p, m') (see case (a)). Again, no obstacle will exist for $(p, \mathcal{P}, m')$.

   (a) If $\mathcal{A}$ is a C_4^*-obstacle, then Definition 66.3 does not hold. Note that, by Definition 7, there exist at least one and at most two dominating colors. We show that for every dominating $P \in \mathcal{P}$, there exist an m-positive element $u \in P$ and an element $u' \in X_u \setminus P$. Suppose indirectly that for some dominating $P \in \mathcal{P}$ and for all m-positive elements $u \in P$ we have $X_u \subseteq P$. Then, by Definition 7.3, there exists $\ell \in \{1, 2\}$ such that the m-positive elements of $A_\ell \cup A_{\ell+2}$ are the m-positive elements of P. Hence for $i = \ell, \ell + 2$ and for the subpartition $\mathcal{X}_i$ of A_i defined by the maximal elements of the laminar family $\{X_u : u \in V_+(m) \cap A_i\}$ we have, by Lemma 43, $\sum_{X \in \mathcal{X}_i} p(X) = \sum_{X \in \mathcal{X}_i} m(X) = m(A_i) = p(A_i)$; that is, $A_\ell \cup A_{\ell+2}$ is a P-pair, so Definition 66.3 holds for $\mathcal{A}$, a contradiction.

   For every dominating $P \in \mathcal{P}$, let us do the following: choose an m-positive element $u \in P$ and an element $u' \in X_u \setminus P$ and replace m by $m' = m - \chi_{\{u\}} + \chi_{\{u'\}}$. Without loss of generality we may assume that we made a replacement in A_1 with $u \in P \cap A_1$ for some dominating $P \in \mathcal{P}$, $u' \in A_1 \setminus P$, and the m-positive elements of $A_1 \cup A_3$ are exactly the m-positive elements of P.

   We show that no obstacle exists for $(p, \mathcal{P}, m')$. Suppose indirectly that an obstacle $\mathcal{A}'$ exists for $(p, \mathcal{P}, m')$. Since Definition 7.3 does not hold for $\mathcal{A}$ and m', $\mathcal{A}' \neq \mathcal{A}$.

   By Lemma 43 and Claim 14.3, $\mathcal{A}$ is the unique partition of V into maximal tight sets for (p, m) and also for (p, m'). Since $\mathcal{A}' \neq \mathcal{A}$, we may suppose that $\mathcal{A}' = \{A_1, A_3, A_2, A_4\}$.

We show that $m(A_1) = m(A_4) = 1$. By Claim 41 and Definitions 7.1d and 7.2 for $\mathcal{A}, m$ and $\mathcal{A}', m'$, we have $m(A_1) + m(A_3) = \frac{1}{2}m(V) = \frac{1}{2}m'(V) = m'(A_4) + m'(A_3) = m(A_4) + m(A_3)$; thus $m(A_1) = m(A_4)$. Suppose that $m(A_1) = m(A_4) \geq 2$. Since $m(A_1) \geq 2$, after the replacement in A_1 there will be m'-positive elements of two different colors in $A_1 \cup A_2$. By assumption, no replacement occurred in A_3; hence, whether a replacement occurred in A_4 or not, by $m(A_4) \geq 2$, Lemma 43, Claim 39.2, and Observation 11, there will be m'-positive elements of two different colors in $A_3 \cup A_4$. This contradicts Definition 7.3 for $\mathcal{A}'$ and m'.

Then, by Lemma 43 for (p, m), $A_1 \cup A_4$ is not tight with respect to m, and so, by Definition 7.1b for m, Claim 41 for m, and Lemmas 42 and 19, we have $2 = m(A_1 \cup A_4) > p(A_1 \cup A_4) = p(A_1) + p(A_3) - p(A_3 \cup A_4) = m(A_1 \cup A_3) - p(A_3 \cup A_4) = m(A_3 \cup A_4) - p(A_3 \cup A_4) \geq 2$, a contradiction.

(b) If $\mathcal{A}$ is a C_5^*-obstacle, then let R and B be two of the color classes maximizing $m(P)$ over $P \in \mathcal{P}$. We will say that $u \in R$ (respectively, B) is red (respectively, blue). We suppose in this case that m is chosen in such a way that $m(R) + m(B)$ is minimum. Then if u is a red or a blue m-positive element, we have $X_u \subseteq R \cup B$. There are four cases, depending on $m(R)$ and $m(B)$:

i. $m(R) = \frac{m(V)}{2} - 1 = m(B)$. Then $\mathcal{A}$ is of type 2, and by Definition 8.3b for m, every $B_j \neq B_{j_0}$ has exactly one red and one blue m-positive element, and B_{j_0} has no red and no blue m-positive element. Note that the situation is exactly the same for red and for blue. Since Definition 67.3b does not hold, we may assume that there exists a red m-positive element u and a blue element $u' \in X_u$. Replace m by m'. Then none of Definitions 8.3a and 8.3b hold for m' because B became dominating and contains no m'-positive element of B_{j_0}.

ii. $m(R) = \frac{m(V)}{2} > m(B) + 1$. Then $\mathcal{A}$ is of type 1, and by Definition 8.3a for m, every B_j contains a red m-positive element. Since Definition 67.3a does not hold, there exist a red m-positive element u and a blue element $u' \in X_u$. Replace m by m'. Then $m'(P) < \frac{m'(V)}{2}$ for all $P \in \mathcal{P}$, so Definition 8.3a does not hold for m'. If Definition 8.3b holds for m', then the two colors involved in it are red and blue. However, each set B_j contains either a red or a blue m'-positive element; thus Definition 8.3b does not hold for m'.

iii. $m(R) = \frac{m(V)}{2} = m(B) + 1$. Then $\mathcal{A}$ is of type 1 with R. Since Definition 67.3a does not hold, there exist a red m-positive element u and a blue element $u' \in X_u$. Replace m by m'. Then $m'(R) + 1 = \frac{m(V)}{2} = m'(B)$, and the only dominating set for m' is B.

A. If u is in some A_i, then the m'-positive element u' of A_i is blue, and, by Definition 8.3a for m, the m'-positive element of A_{i+2} is red. Therefore, since B is dominating, neither Definition 8.3a nor Definition 8.3b holds for m'.

B. Otherwise, u is in some B_k. If Definition 8.3a or 8.3b holds for m', then by Definition 8.3a for m, both A_ℓ and $A_{\ell+2}$ contain a red m'-positive element, both $A_{\ell+1}$ and $A_{\ell+3}$ contain a blue m'-positive element, every $B_j \neq B_k$ contains exactly one red and one blue m'-positive element, and the m'-positive element in B_k different from u' is neither blue nor red. Note that the situation is exactly the same for m' and blue and for m and red. Since Definition 67.3b does not hold, we may assume that there exist a red m-positive element $v \notin B_k$ and a blue element $v' \in X_v$. Repeating case iii for m with v and v' instead of u and u', we get an extension m'' such that B_k has two blue m''-positive elements, and since B is dominating, we are done.

iv. $m(R) = \frac{m(V)}{2} = m(B)$. Then $\mathcal{A}$ is of type 1 both for B and R, and by Definition 8.3a for m, both A_ℓ and $A_{\ell+2}$ contain a red m-positive element, both $A_{\ell+1}$ and $A_{\ell+3}$ contain a blue m-positive element, and every B_i contains exactly one red and one blue m-positive element. Since Definition 67.3a does not hold for R and for B, there exist a red m-positive element u, a blue element $u' \in X_u$, a blue m-positive element v, and a red element $v' \in X_v$. Replace m by $m'' = m - \chi_u + \chi_{u'} - \chi_v + \chi_{v'}$. Note that $m''(R) = m''(B) = \frac{m''(V)}{2}$.

A. If u or v is in some A_i, then either A_{i-1} and A_i or A_i and A_{i+1} contain m''-positive elements of the same color, and thus neither Definition 8.3a nor Definition 8.3b holds for m''.

B. Otherwise, since Definition 67.3b does not hold, we may assume that $u \in B_i$, $v \in B_j$ with $i \neq j$. Then B_i contains two blue m''-positive elements and B is dominating; hence neither Definition 8.3a nor Definition 8.3b holds for m''.

(c) If $\mathcal{A}$ is a C_6^*-obstacle, then Definition 68.2 does not hold. By Definition 9.3, there exist distinct $P_{k_i} \in \mathcal{P}$ such that the m-positive elements of $A_i \cup A_{i+3}$ are the m-positive elements of P_{k_i} for $i = 1, 2, 3$. If for $i = 1, 2, 3$, for $j = i, i+3$, and for all m-positive elements $u \in A_j$, we have $X_u \subseteq P_{k_i}$, then $p(X_u) = m(X_u) = m(A_j) = p(A_j)$; that is, $A_i \cup A_{i+3}$ is a P_{k_i}-pair, so Definition 68.2 holds for $\mathcal{A}$, a contradiction. Thus there exist an m-positive element $u \in P$ and an element $u' \in X_u \setminus P$ for some $P \in \mathcal{P}$. Replace m by m'. Then Definition 9.3 does not hold for m'.

After these modifications, Definition 7.3 (respectively, 8.3 and 9.3) does not hold; hence $\mathcal{A}$ is not an obstacle for $(p, \mathcal{P}, m')$. When $\mathcal{A}$ is any obstacle but a nonsimple C_4^*-obstacle for $(p, \mathcal{P}, m)$, then $\mathcal{A}$ is the unique obstruction for (p, m); hence, by construction, $\mathcal{A}$ is the unique obstruction for (p, m'), and thus no obstacle exists for $(p, \mathcal{P}, m')$. When $\mathcal{A}$ is a C_4^*-obstacle that is not simple for $(p, \mathcal{P}, m)$, we use ad hoc arguments to show that no obstacle exists for $(p, \mathcal{P}, m')$; see case (a). In conclusion, m' is the desired extension. ∎

5.4. Main theorem. By exploiting the relations between configurations and obstacles and by applying our splitting off result, we may now prove our main theorem. It states that the lower bound Φ, defined in section 5.1, may always be achieved unless there exists a configuration, in which case one more edge is needed.

THEOREM 71. *Let $p : 2^V \to \mathbb{Z}$ be a symmetric crossing supermodular set function and $\mathcal{P}$ a partition of V. Then the minimum number of edges of a $\mathcal{P}$-partite graph that covers p is Φ unless a configuration exists, in which case it is $\Phi + 1$.*

Proof. The following lemmas prove the theorem.

LEMMA 72. $OPT(p,\mathcal{P}) \geq \Phi$. *If there exists a configuration for* $(p,\mathcal{P})$*, then the inequality is strict.*

Proof. By Lemma 64, $OPT(p,\mathcal{P}) \geq \Phi$.

Suppose there exists a configuration for $(p,\mathcal{P})$ and that the inequality is not strict; that is, $OPT(p,\mathcal{P}) = \Phi$. Let F be a minimum set of edges such that (V,F) covers p and satisfies the partition constraint, and let m be the degree specification obtained from $m := 0$ by unsplitting every edge of F. Note that m is $(p,\mathcal{P})$-allowed and there exists a complete $(p,\mathcal{P})$-allowed splitting off. By the construction of m, the minimality of F, $OPT(p,\mathcal{P}) = \Phi$, and Lemma 69.2–1, $\frac{1}{2}m(V) = |F| = OPT(p,\mathcal{P}) = \Phi = \alpha_p = \max\{\alpha_p, \beta_p\}$, so m is an extension for $(p,\mathcal{P})$. Since there is a configuration for $(p,\mathcal{P})$, by Lemma 70, there is an obstacle for $(p,\mathcal{P},m)$. But now Lemma 54 contradicts the existence of a complete $(p,\mathcal{P})$-allowed splitting off. ∎

LEMMA 73. $OPT(p,\mathcal{P}) \leq \Phi + 1$. *If there exists no configuration for* $(p,\mathcal{P})$*, then the inequality is strict.*

Proof. If there exists no configuration for $(p,\mathcal{P})$, then, by Lemma 70, there exists an extension m for $(p,\mathcal{P})$ such that no obstacle exists for $(p,\mathcal{P},m)$. Hence $\frac{1}{2}m(V) = \max\{\alpha_p, \beta_p\}$. By Theorem 61, there exists a $\mathcal{P}$-partite graph (V,F) that covers p_0 with either $|F| \leq \frac{1}{2}m(V) = \max\{\alpha_p, \beta_p\}$ or $|F| \leq \dim(p_0) - 1$. In both cases $OPT(p,\mathcal{P}) \leq |F| \leq \Phi$, and the strict inequality follows.

If there exists a configuration for $(p,\mathcal{P})$, then let m be an extension for $(p,\mathcal{P})$. By Lemma 69, $m(V) = 2\Phi$. This implies that m is p-legal. Replace m by $m' := m+\chi_u+\chi_v$ for some u,v without violating $m'(P) \leq \frac{m'(V)}{2}$ for every $P \in \mathcal{P}$. Then $m'(V) = 2\Phi+2$, m' is p-legal, and no set containing u or v is tight. By Claim 41, there exists no obstacle for $(p,\mathcal{P},m')$. Then, by Theorem 62, there exists a complete $(p,\mathcal{P})$-allowed splitting off, and the inequality follows. ∎

6. Applications.

6.1. Covering of a symmetric crossing supermodular function by a graph. Our main result, Theorem 71, implies the theorem of Benczúr and Frank [4]. Indeed, let $\mathcal{P}$ be the partition of V consisting of the singletons. Then no configuration exists and $\Phi = \max\{\alpha_p, \dim(p) - 1\}$.

6.2. Partition constrained global edge-connectivity augmentation of a hypergraph. We show in this section that our main result, Theorem 71, implies the theorem of Bernáth, Grappe, and Szigeti [6] about partition constrained global edge-connectivity augmentation of a hypergraph. We mention that the proof of Theorem 76 given in [6] is considerably shorter than that of the present paper.

Let $\mathcal{G} = (V,\mathcal{E})$ be a hypergraph. For a vertex set X, we denote by $\delta_{\mathcal{G}}(X)$ the set of hyperedges intersecting both X and $V \setminus X$ and $d_{\mathcal{G}}(X) = |\delta_{\mathcal{G}}(X)|$. Let us denote by $d_0(X,Y)$ (respectively, $d_1(X,Y)$) the number of hyperedges intersecting $X \setminus Y$ and $Y \setminus X$ and none of $X \cap Y$ and $V \setminus (X \cup Y)$ (respectively, exactly one of $X \cap Y$ and $V \setminus (X \cup Y)$).

For an integer k, let $p = k - d_{\mathcal{G}}$. It is well known that $d_{\mathcal{G}}$ satisfies (26) for all subsets X and Y of V. By (26), for all crossing subsets X and Y of V, p satisfies (27):

$$d_{\mathcal{G}}(X) + d_{\mathcal{G}}(Y) = d_{\mathcal{G}}(X \cap Y) + d_{\mathcal{G}}(X \cup Y) + 2d_0(X,Y) + d_1(X,Y), \tag{26}$$

$$p(X) + p(Y) = p(X \cap Y) + p(X \cup Y) - 2d_0(X,Y) - d_1(X,Y). \tag{27}$$

Let us recall the following two lower bounds : $\Phi' = \max\{\alpha_{\mathcal{G}}, \beta_{\mathcal{G}}, \omega_{\mathcal{G}} - 1\}$, where $\alpha_{\mathcal{G}}$, $\beta_{\mathcal{G}}$, and $\omega_{\mathcal{G}}$ are defined in the introduction, and $\Phi = \max\{\alpha_p, \beta_p, \dim(p) - 1\}$, where $p = k - d_{\mathcal{G}}$.

DEFINITION 74. *A partition $\mathcal{A} = \{A_1, \ldots, A_4\}$ of V is called a $\mathcal{C}_4$-configuration of $\mathcal{G}$ if the following hold:*

1. $\Phi' = k - d_{\mathcal{G}}(A_1) + k - d_{\mathcal{G}}(A_3) = k - d_{\mathcal{G}}(A_2) + k - d_{\mathcal{G}}(A_4)$;
2. *there exists $\mathcal{F} \subseteq \mathcal{E}$ such that*
   (a) $\mathcal{F} = \delta(A_1) \cap \delta(A_3) = \delta(A_2) \cap \delta(A_4)$,
   (b) *$k - |\mathcal{F}|$ is odd;*
3. *there exist $P \in \mathcal{P}$ and $\ell \in \{1, 2\}$ such that $(A_\ell, A_{\ell+2})$ is a P-pair.*

DEFINITION 75. *A partition $\mathcal{A} = \{A_1, \ldots, A_6\}$ of V is called a $\mathcal{C}_6$-configuration of $\mathcal{G}$ if the following hold:*

1. $\Phi' = 3$;
2. (a) *$k - d_{\mathcal{G}}(A_i) = 1$ for $i = 1, \ldots, 6$;*
   (b) *$k - d_{\mathcal{G}}(A_i \cup A_{i+1}) = 1$ for $i = 1, \ldots, 6$;*
   (c) *there exists $\mathcal{F} \subseteq \mathcal{E}$ such that*
      i. *$\mathcal{F} = \delta(A_j) \cap \delta(A_\ell)$ for all distinct non consecutive A_j and A_ℓ,*
      ii. *$k - |\mathcal{F}|$ is odd;*
3. *there exist distinct $P_{k_i} \in \mathcal{P}$ such that (A_i, A_{i+3}) is a P_{k_i}-pair for $i = 1, 2, 3$.*

A *$\mathcal{C}$-configuration* is a $\mathcal{C}_4$-configuration or a $\mathcal{C}_6$-configuration.

THEOREM 76 (Bernáth, Grappe, and Szigeti [6]). *Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph, $\mathcal{P}$ a partition of V, and k an integer. Then the minimum number of graph edges between different members of $\mathcal{P}$ whose addition to $\mathcal{G}$ results in a k-edge-connected hypergraph is $\Phi' = \max\{\alpha_{\mathcal{G}}, \beta_{\mathcal{G}}, \omega_{\mathcal{G}} - 1\}$ unless $\mathcal{G}$ contains a $\mathcal{C}$-configuration, in which case it is $\Phi' + 1$.*

Proof. We will apply Theorem 71 for $p = k - d_{\mathcal{G}}$. First we will observe that the lower bound Φ is equal to Φ'. Then, we will show that C_4^*- and C_6^*-configurations for p specialize to $\mathcal{C}_4$- and $\mathcal{C}_6$-configurations for $\mathcal{G}$, and finally we will show that no C_5^*-configuration exists for p.

CLAIM 77. $\Phi = \Phi'$.

Proof. Note that $\alpha_{\mathcal{G}} = \alpha_p$ and $\beta_{\mathcal{G}} = \beta_p$.

First we show that $\Phi \geq \Phi'$. If $\omega_{\mathcal{G}} - 1 < \Phi'$, then $\Phi' = \max\{\alpha_{\mathcal{G}}, \beta_{\mathcal{G}}\} = \max\{\alpha_p, \beta_p\} \leq \Phi$. If $\omega_{\mathcal{G}} - 1 \geq \Phi'$, then let $\mathcal{F}$ be a set of $k-1$ hyperedges whose deletion results in a hypergraph with $\omega_{\mathcal{G}}$ connected components. Let $\mathcal{X} = \{X_1, \ldots, X_{\omega_{\mathcal{G}}}\}$ be the vertex sets of these components. Then $\mathcal{X}$ is a p-full partition. Indeed, if Y is the union of some of the X_i's, then $p(Y) = k - d_{\mathcal{G}}(Y) = k - d_{\mathcal{F}}(Y) \geq k - (k-1) = 1$, and there is an index i such that $p(X_i) = 1$; otherwise, we have $\omega_{\mathcal{G}} - 1 \geq \Phi' \geq \alpha_{\mathcal{G}} \geq \frac{1}{2}\sum_{i=1}^{\omega_{\mathcal{G}}}(k - d_{\mathcal{G}}(X_i)) = \frac{1}{2}\sum_{i=1}^{\omega_{\mathcal{G}}} p(X_i) \geq \frac{1}{2}\sum_{i=1}^{\omega_{\mathcal{G}}} 2 = \omega_{\mathcal{G}}$, a contradiction. Then $\Phi \geq \dim(p) - 1 \geq \omega_{\mathcal{G}} - 1 \geq \Phi'$.

Now we show that $\Phi \leq \Phi'$. If $\dim(p) - 1 < \Phi$, then $\Phi = \max\{\alpha_p, \beta_p\} = \max\{\alpha_{\mathcal{G}}, \beta_{\mathcal{G}}\} \leq \Phi'$. If $\dim(p) - 1 \geq \Phi$, then, by Lemma 6.3 of [4], $\omega_{\mathcal{G}} \geq \dim(p)$, and hence $\Phi' \geq \omega_{\mathcal{G}} - 1 \geq \dim(p) - 1 \geq \Phi$.

By the above arguments, the claim follows. ∎

CLAIM 78. *A partition $\mathcal{A} = \{A_1, \ldots, A_4\}$ of V is a C_4^*-configuration for p if and only if it is a $\mathcal{C}_4$-configuration for $\mathcal{G}$.*

Proof. Definition 66.3 is the same as Definition 74.3.

By (27) applied to $A_i \cup A_{i-1}$ and $A_i \cup A_{i+1}$, Definition 7.1b is equivalent to $d_0(A_i \cup A_{i-1}, A_i \cup A_{i+1}) = d_1(A_i \cup A_{i-1}, A_i \cup A_{i+1}) = 0$, that is, to $\delta(A_1) \cap \delta(A_3) = \delta(A_2) \cap \delta(A_4)$, which is Definition 74.2a. Let $\mathcal{F}$ be this set of hyperedges. Then, by (26) applied to A_i and A_{i+1}, Definition 7.1a is equivalent to saying that $k - |\mathcal{F}| = k - d_{\mathcal{G}}(A_i) - d_{\mathcal{G}}(A_{i+1}) + d_{\mathcal{G}}(A_i \cup A_{i+1}) + 2d_0(A_i, A_{i+1}) = p(A_i) + p(A_{i+1}) - p(A_i \cup A_{i+1}) + 2d_0(A_i, A_{i+1})$ is odd, that is, to Definition 74.2b.

Suppose that Definition 74.1 is satisfied. Then, by the definition of Φ, Claim 77, Definition 74.1, and the definition of $\alpha_{\mathcal{G}}$, we have $\max\{\alpha_{\mathcal{G}}, \beta_{\mathcal{G}}, \dim(p)-1\} = \Phi = \Phi' = \frac{1}{2}(k-d_{\mathcal{G}}(A_1)+k-d_{\mathcal{G}}(A_3)+k-d_{\mathcal{G}}(A_2)+k-d_{\mathcal{G}}(A_4)) \le \alpha_{\mathcal{G}}$. This implies that Definitions 7.1d and 66.2 hold. Suppose that $1 = p(A_i) = k - d_{\mathcal{G}}(A_i)$ for $i = 1, \dots, 4$. It also follows that $\dim(p) - 1 \le \frac{1}{2}(k - d_{\mathcal{G}}(A_1) + k - d_{\mathcal{G}}(A_3) + k - d_{\mathcal{G}}(A_2) + k - d_{\mathcal{G}}(A_4)) = 2$. Then, by $|\mathcal{A}| = 4$, $\mathcal{A}$ is not a p-full partition. Since $1 = p(A_i)$ for $i = 1, \dots, 4$ and, by Claim 39.4, $p(A_i \cup A_{i+1}) = 1$ for $i = 1, \dots, 4$, it follows, by the symmetry of p, that $p(A_1 \cup A_3) = p(A_2 \cup A_4) \le 0$, and Definition 7.1c follows.

Now suppose that Definitions 7.1c, 7.1d, and 66.2 are satisfied. Then, by Definition 7.1d, Lemma 69.2, and Claim 77, we have $p(A_1) + p(A_3) = p(A_2) + p(A_4) = \frac{1}{2}\sigma_p = \Phi = \Phi'$; so, by $p(A_i) = k - d_{\mathcal{G}}(A_i)$, Definition 74.1 follows. ∎

We note that in [6] there was a fourth condition for a $\mathcal{C}_4$-configuration, namely $k - d_{\mathcal{G}}(A_i) > 0$ for $i = 1, \dots, 4$. However, by Claim 39.2, this condition is implied by the others.

CLAIM 79. *There exists no C_5^*-configuration for p.*

Proof. Let us suppose for a contradiction that $\mathcal{A} = \{A_1, \dots, A_4, B_1, \dots, B_t\}$ ($t \ge 1$) is a C_5^*-configuration for p. Let $B = \bigcup_{j=1}^t B_j$, $X_i = A_i \cup B$, and $Y_i = A_i \cup A_{i+1}$. By Claim 40, $p(B) = 2$, $p(X_i) = 1$, $p(Y_i \cup X_{i+1}) = p(X_i \cup X_{i+1}) = 1$, and $p(X_i \cup X_{i+2}) \le 0$. By (27) applied to X_1 and X_2, we have $1 + 1 = p(X_1) + p(X_2) = p(B) + p(X_1 \cup X_2) - 2d_0(X_1, X_2) - d_1(X_1, X_2) = 2 + 1 - 2d_0(X_1, X_2) - d_1(X_1, X_2)$, that is, $d_1(X_1, X_2) = 1$; thus there exists a hyperedge e of $\mathcal{G}$ that enters A_1, A_2 and exactly one of B and $A_3 \cup A_4$. If e enters B, then $d_1(Y_1, X_2) \ge 1$, and hence, by (27) applied to Y_1 and X_2, we have $1 + 1 = p(Y_1) + p(X_2) = p(A_2) + p(Y_1 \cup X_2) - 2d_0(Y_1, X_2) - d_1(Y_1, X_2) \le 1 + 1 - 0 - 1$, a contradiction. Otherwise, e enters A_3 or A_4, say A_3. Then $d_1(X_1, X_3) \ge 1$, and hence, by (27) applied to X_1 and X_3, we have $1+1 = p(X_1)+p(X_3) = p(B)+p(X_1 \cup X_3) - 2d_0(X_1, X_3) - d_1(X_1, X_3) \le 2+0-0-1$, a contradiction. ∎

CLAIM 80. *A partition $\mathcal{A} = \{A_1, \dots, A_6\}$ of V is a C_6^*-configuration for p if and only if it is a $\mathcal{C}_6$-configuration for $\mathcal{G}$.*

Proof. First we show that Definition 75.2c is a corollary of Definitions 75.2a–2b.

Suppose indirectly that there exists a hyperedge $F \in \mathcal{E}$ that intersects nonconsecutive sets A_j and A_ℓ but does not intersect A_i. Without loss of generality we may suppose that there exists an index k such that $1 \le j < k < \ell < i = 6$. Let $X = \bigcup_{t=1}^k A_t$ and $Y = \bigcup_{t=k}^5 A_t$. By Definitions 75.2a–2b, (27) applied to X and Y, and Claim 13.2, $1+1 = p(X)+p(Y) = p(A_k)+p(\bigcup_{t=1}^5 A_t) - 2d_0(X,Y) - d_1(X,Y) \le 1+1+0+0$. It follows that $d_0(X,Y)+d_1(X,Y) = 0$. Since $A_j \subseteq X \setminus Y$, $A_\ell \subseteq Y \setminus X$ and $A_i = V \setminus (X \cup Y)$, F intersects $X \setminus Y$ and $Y \setminus X$ but not $V \setminus (X \cup Y)$, so $d_0(X,Y)+d_1(X,Y) \ge 1$. By this contradiction Definition 75.2ci follows. Note that $|\mathcal{F}| = d_1(A_i, A_j)$ for all $i \ne j$. Then, by (26) applied to A_i and A_{i+1}, and by Definitions 75.2a–2b, we have that $k - |\mathcal{F}| = k - d_{\mathcal{G}}(A_i) - d_{\mathcal{G}}(A_{i+1}) + d_{\mathcal{G}}(A_i \cup A_{i+1}) + 2d_0(A_i, A_{i+1}) = 1 + 1 - 1 + 2d_0(A_i, A_{i+1})$ is odd; that is, Definition 75.2cii is satisfied.

Definitions 9.1a–1b and 68.2 are the same as Definitions 75.2a–2b and 75.3.

Suppose that Definition 75.1 is satisfied. Then, by Claim 77, we have $3 = \Phi' = \Phi = \max\{\alpha_p, \beta_p, \dim(p) - 1\}$. This implies that $\alpha_p \leq 3$ and then, $6 = \sum_{i=1}^{6} p(A_i) \leq \sigma_p \leq 2\alpha_p \leq 6$, and Definition 9.1d follows. It also implies $\dim(p) \leq 4$. Since $|\mathcal{A}| = 6$, $\mathcal{A}$ is not a p-full partition, and then, by Claim 13.3, Definition 9.1c follows.

Now suppose that $\mathcal{A}$ is a C_6^*-configuration. Then, by Definition 9.1d, Lemma 69.2, and Claim 77, we have $3 = \frac{1}{2}\sigma_p = \Phi = \Phi'$, so Definition 75.1 follows. □

By Claims 78, 79, and 80, Theorem 71 implies Theorem 76. □

6.2.1. Global edge-connectivity augmentation of a hypergraph in a subset of vertices. For an integer k and a subset T of V, a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is called k-edge-connected in T if $d_{\mathcal{H}}(X) \geq k$ for all $X \subset V$ such that $X \cap T$ and $T - X$ are nonempty. The problem of making a given hypergraph k-edge-connected in T by adding a minimum set of edges was solved by Benczúr and Frank [4]. Theorem 71 solves the following partition constrained version of this problem: given a hypergraph $\mathcal{H} = (V, \mathcal{E})$, a subset T of V, a partition $\mathcal{P}$ of T, and an integer k, find a graph $G = (T, E)$ with a minimum number of edges to be added to $\mathcal{H}$ between distinct members of $\mathcal{P}$ such that the resulting hypergraph is k-edge-connected in T. Indeed, if we define the function $p : T \to \mathbb{Z}$ with $p(X) = \max\{k - d_{\mathcal{H}}(X \cup Y) : \ Y \subseteq V - T\}$ for any nonempty $X \subset T$ and $p(\emptyset) = p(T) = 0$, then it was shown in [4] that p is symmetric and crossing supermodular, so Theorem 71 can be applied.

7. Algorithm and complexity. In this section, we describe an algorithm that, given a symmetric crossing supermodular set function $p : 2^V \to \mathbb{Z}$ and a partition $\mathcal{P}$ of V, finds a $\mathcal{P}$-partite a graph that covers p having a minimum number of edges. We then explain in which settings the subroutines needed for the algorithm are polynomial. Finally, we sketch why the algorithm itself is polynomial in these settings.

Throughout, G will denote a graph, $p_G = p - d_G$, and m_G a p_G-admissible degree specification.

7.1. Augmentation algorithm. Given a symmetric crossing supermodular set function $p : 2^V \to \mathbb{Z}$ and a partition $\mathcal{P}$ of V, the augmentation algorithm finds a $\mathcal{P}$-partite a graph that covers p having a minimum number of edges. It consists of three major steps: extension, then splitting off, and finally determining whether a configuration exists. See Figure 3 for details of this algorithm.

7.2. Subroutines. In this section, we give the framework in which our algorithms run in polynomial time, together with the basic bricks needed throughout. These bricks mostly concern crossing supermodular functions, operations such as splitting off, tight sets, and partition constraints.

Some of the arguments used below come from [4], adapted to the partition constrained version when necessary.

7.2.1. Crossing supermodular functions.

Minimization oracle. We assume that the function p is given with an evaluation oracle and a polynomial *minimization oracle*. The last oracle outputs, in polynomial time, a subset of V that is a solution of $\min_{\emptyset \neq X \subset V}\{m(X) - p_G(X)\}$ for any modular function m and for any graph G. Note that checking whether a degree specification m is p_G-admissible can be done in polynomial time using these oracles.

When p is *fully crossing supermodular*, that is, when (3) is satisfied for all crossing pairs (the positivity condition is dropped), then a polynomial minimization oracle can be implemented from an evaluation oracle. Indeed, in this case, the minimization of

Augmentation Algorithm.
Input : A symmetric crossing supermodular function $p : 2^V \to \mathbb{Z}$ and a partition $\mathcal{P}$ of V.
Output: A $\mathcal{P}$-partite graph that covers p of minimum size.

Step 1. Find an extension m for $(p, \mathcal{P})$ applying the Extension Algorithm described in Figure 2 (section 5.2).

Step 2. Apply the Splitting off algorithm described in Figure 1 (section 4.3).

Step 3. If it stops with a complete $(p, \mathcal{P})$-allowed splitting off, then we have found a $\mathcal{P}$-partite graph that covers p having $\frac{m(V)}{2} = \max\{\alpha_p, \beta_p\}$ edges, and stop.

Step 4. If it stops with a $\mathcal{P}$-partite graph that covers p having at most $\dim(p) - 1$ edges, then stop.

Step 5. Otherwise, it stops with an obstacle $\mathcal{A}$ for $(p, \mathcal{P}, m)$; then apply the proof of Lemma 70 to $\mathcal{A}$.

Step 6. If it finds another extension m' for $(p, \mathcal{P})$ such that no obstacle exists for $(p, \mathcal{P}, m')$, then go to Step 2.

Step 7. Otherwise, it finds a configuration for $(p, \mathcal{P})$. The algorithmic proof of Lemma 73 provides the desired graph with $\frac{m'(V)}{2} = \Phi + 1$ edges, and stop.

Fig. 3. *Augmentation algorithm.*

$m - p_G$ can be reduced to the minimization of a fully submodular function (see Theorem 10.3.11 in [8]), which can be solved in polynomial time. For example, [9] provides a minimizer in $O((|V|^5\gamma + |V|^6)\log|V|)$, where γ is the time needed to call the function evaluation oracle.

Note that, in our application for hypergraphs, evaluating the function can be done in polynomial time. In fact, in this case the minimization problem can be solved with network flow techniques [3].

Subpartition lower bound and minimal degree specification. Thanks to the minimization oracle, a greedy algorithm computes σ_p and a minimal degree specification m in Theorem 1; see [7]. As a consequence, the subpartition lower bound α_p can be computed in polynomial time.

7.2.2. Operations.

Splitting off. Deciding whether splitting off at x, y is p_G-admissible is equivalent to checking whether $\min_{\emptyset \neq X \subset V}\{m_{G'}(X) - p_{G'}(X)\} \geq 0$, where G' is obtained from G by splitting off at x, y. Therefore, this can be done in polynomial time, calling the minimization oracle once.

Flips and improvements. Since flips and improvements can also be described by a fixed number of unsplittings and splittings off, checking their admissibility is also polynomial.

7.2.3. Tight sets.

Minimal tight sets. Here, we explain how to find, in polynomial time, a minimal tight set containing a given positive element, if one exists.

First, note that, given an m_G-positive element $x \in V$ and $y \in V \setminus x$, one can find, if it exists, a tight set containing x but not y by calling the minimization oracle for $m' + m_G - p_G$, where $m'(x) = -M$, $m'(y) = M$, $m'(z) = 0$ otherwise, and M is a suitable big number. Indeed, due to the p_G-admissibility of m_G and the choice of m'

and M, we have $\min_{\emptyset \neq X \subset V}\{m'(X) + m_G(X) - p_G(X)\} \geq -M$, and any solution to this minimization problem contains x but not y. Moreover, such a solution S is tight if and only if $m'(S) + m_G(S) - p_G(S) = -M$.

Then, given an m_G-positive element x, one can find a minimal tight set containing x by applying the above remark to x and y, for all $y \in V \setminus x$, and then by taking the intersection of the solutions which are tight sets. By Claim 14, this provides a minimal tight set containing x.

Maximal tight sets. Arguments similar to those above allow one to find a maximal tight set containing a given m_G-positive element in polynomial time, if one exists.

First, given distinct $x, y \in V$, one can find a tight set containing both x and y, if one exists, by calling the minimization oracle for $m'' + m_G - p_G$, where $m''(x) = m''(y) = -M$, and 0 otherwise. Then, given $x \in V_+(m_G)$, one can find a maximal tight set containing x by applying the above remark to x and y, for all $y \in V \setminus x$, and then by taking the union of the solutions which are tight sets. By Claim 14, this provides a maximal tight set containing x.

Obstructions. We explain how to find an obstruction for (p_G, m_G) in polynomial time, if one exists.

Recall that, by Lemma 43, an obstruction for (p_G, m_G) is the unique partition of V into maximal tight sets, in which, by Claim 39.2 and Definitions 8.1a and 9.1a, every member contains an m_G-positive element. Applying at most $|V|$ times the algorithm that finds the maximal tight set containing a given element, one can find such a partition of V, if one exists. Then, since Definitions 7.1, 8.1, and 9.1 involve single sets or pairs of sets of the partition, it is straightforward to check whether one holds. Finally, by the remarks on the lower bounds of section 7.2.1, one can check whether Definition 7.2, 8.2, or 9.2 holds. Therefore, if an obstruction exists for (p_G, m_G), then it can be found in polynomial time.

7.2.4. Partition constraints.

Color condition. Note that it is immediate to check whether (20) holds.

Allowed operations. Since checking the p_G-admissibility of any operation can be done in polynomial time, the above remark implies that checking whether an operation is allowed is polynomial.

Obstacles. If an obstacle exists for $(p, \mathcal{P}, m)$, then one can find it by, first, finding the corresponding obstruction and, then, checking whether this obstruction is an obstacle (that is, Definition 7.3, 8.3, or 9.3 holds). The first part is done in polynomial time by results of section 7.2.3, and the second is immediate from the definitions.

Sequences of allowed splitting off. We show how to perform, in polynomial time, arbitrary allowed splitting off until there are none of the allowed splittings left.

(a) Suppose there is a dominating color P. Then, every allowed splitting off involves exactly one m_G-positive element of P.
Let x be such an element. Given $y \in V_+(m_G) \setminus P$, the arguments of section 7.2.3 about minimal tight sets allows one to compute $\omega_{x,y} = \min_{x,y \in X \subset V}\{m_G(X) - p_G(X)\}$ in polynomial time. Note that the maximum number of p_G-admissible (hence allowed) splittings off at x, y is $\min\{\frac{1}{2}\omega_{x,y}, m_G(x), m_G(y)\}$. Repeating this for all $y \in V_+(m_G) \setminus P$, one can perform in polynomial time allowed splittings off involving x, until there are none left.
Repeating this for every m_G-positive element of P, one can perform allowed splittings off until there are none left, in polynomial time.

(b) Suppose there is no dominating color. Let $x \in V_+(m_G)$, and apply the argument of (a) to perform arbitrary allowed splitting off involving x: it stops either because a dominating color appears, and then we can apply (a), or because x no longer belongs to an allowed splitting off, in which case we repeat (b) with another positive element.

Since (b) is repeated at most $|V|$ times, and both (a) and (b) are done in polynomial time, we performed allowed splitting off until there were none left, in polynomial time.

7.3. Complexity. The aim of this section is to sketch why the augmentation algorithm runs in polynomial time, provided the function p is given with an evaluation oracle and a minimization oracle. Before doing so, we first sketch why the extension and splitting off algorithms are polynomial.

7.3.1. Extension algorithm. We sketch why each step of the extension algorithm can be done in polynomial time, provided the function p is given with a minimization oracle.

Step 1 is polynomial; see the Subpartition lower bound paragraph of section 7.2.1.

Step 2 is immediate.

If we execute Step 3 as it is given in Figure 2, then the algorithm is not polynomial. Indeed, Step 3 can be repeated $|V|p_{max}$ times, where p_{max} is the maximum value of the function p. To turn Step 3 polynomial we have to use the tricks explained in section 6.6 of [2]. One has to use the minimal degree specification algorithm of [7] for different starting values; for details, see [2].

Step 3b is immediate.

Lower bounds. As we have already mentioned in section 5.2, we cannot calculate $\dim(p)$. On the other hand, the extension algorithm, as it is described above, provides in polynomial time $\max\{\alpha_p, \beta_p\}$.

7.3.2. Splitting off algorithm. We sketch why each step of the splitting off algorithm can be done in polynomial time, provided the function p is given with a minimization oracle.

Step 1 is polynomial by the remarks of section 7.2.4 about sequences of allowed splitting off.

Step 2 is polynomial. Indeed, by Corollary 28, we have $m_G(V) \leq |V|$. Then, since performing an improvement decreases $m_G(V)$ by 2, the number of improvements in the sequence is at most $m_G(V)/2 \leq |V|/2$. There are at most $\binom{|V|}{4}$ possible improvements at each of the $|V|/2$ steps, and checking whether an improvement is allowed is polynomial.

- If the algorithm stops at (a), then we apply Lemma 37. Let us see why it provides the required partition and graph in polynomial time. By Corollary 28.2, $\{T_w, w \in V_+(m_G)\}$ is a partition of V into maximal tight sets and hence can be found by applying at most $|V|$ times the maximal tight sets subroutine. The set X_e for each $e \in E$ can be found in polynomial time because it is defined as the intersection of two perilous sets. These last sets can be found in polynomial time because we can decide if the degree specification after a flip is admissible or not in polynomial time. Finally, having $\mathcal{U}$ in hand, we get $\mathcal{U}^*$ in polynomial time.
- Otherwise, the algorithm stops at (b), and the final step is clearly polynomial.

Step 3 consists of finding a C_4^*-obstacle, which is polynomial by results about obstacles of section 7.2.4.

Step 4 is polynomial, by the same reasoning by which Step 2 is.

Step 5 finds in polynomial time, by the proof of Lemma 57.2, either a C_4^*-obstacle for $(p, \mathcal{P}, m)$ or two edges e and f of G such that in $G^{e,f}$ there exists a complete $(p_{G^{e,f}}, \mathcal{P})$-allowed splitting off. Since $m_{G^{e,f}}(V) = 8$, this complete splitting off can be found in polynomial time.

Step 6 first finds a C_5^*-obstruction for (p, m) (that exists by Lemma 58.2) in polynomial time by subroutine obstructions. Then in (a) it is checked in polynomial time whether this obstruction is a C_5^*-obstacle. If not, then we apply in (b) the proof of Lemma 49. This can be done in polynomial time, the first part by the results of section 7.2.4, and the second because the proof of Lemma 49 is inductive and needs, at each of the $t + 2 \leq |V| + 2$ steps, to find one allowed splitting off and at most one allowed flip.

Step 7 first finds a consecutive improvement, unsplits the edge e involved, and then finds a C_6^*-obstacle in G^e. This can be done in polynomial time by the results of sections 7.2.2 and 7.2.4. Then, Step 7(b) unsplits an arbitrary edge e' of G^e and finds a complete allowed splitting off. Since $m_{G^{e,e'}}(V) = 8$, this can be done in polynomial time.

7.3.3. Augmentation algorithm. We sketch why each step of the augmentation algorithm can be done in polynomial time, provided the function p is given with a minimization oracle.

Steps 1–4 can be done in polynomial time by the arguments of sections 7.3.1 and 7.3.2.

Step 5 follows the proof of Lemma 70, in which the main algorithmic ingredients are checking color conditions and finding one or two suitable tight sets. By the remarks of sections 7.2.3 and 7.2.4, this can be done in polynomial time. We mention that in the proof of Lemma 70 we suppose that the extension m minimizes $m(R) + m(B)$. This assumption is made in order to have a shorter proof. We note that this assumption is not essential. When this assumption is not satisfied, then we will change the extension at most twice, and hence the proof can be modified so that Step 5 becomes polynomial.

Step 6 is similar to Step 2. Note that in this case the algorithm will stop either in Step 3 or in Step 4.

Step 7 first increases $m(V)$ by two so that a complete allowed splitting off exists, and then applies the splitting off algorithm, which is polynomial.

8. Conclusion. In this paper we proposed an abstract form for the problem of partition constrained global edge-connectivity augmentation of a hypergraph. We provided a minimax theorem for this problem, and we sketched a polynomial algorithm to find an optimal solution when the function is given with a minimization oracle. This theorem implies the main theorems of [4] and [6], and consequently the results in [2], [3], and [11]. Our abstract form also provides a new application, given in section 6.

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