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Total unimodularity: Adding a row or a column to the incidence matrix of a directed graph

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A B S T R A C T

In this short note, we characterize when the addition of a single 0,±1 row or column to the incidence matrix of a directed graph yields a totally unimodular matrix. By considering the 0,±1 values as signs on the arcs or the vertices, the characterizations can be stated in graph theoretic terms only.

1. Introduction

A matrix is *totally unimodular* if all its subdeterminants are 0,±1. These matrices play a fundamental role in combinatorial optimization as they yield combinatorial min-max theorems, such as König's theorem about matchings in bipartite graphs [1], and more generally the MaxFlow-MinCut theorem of Ford and Fulkerson [2]. Examples of totally unimodular matrices are incidence matrices of bipartite graphs and of directed graphs, and those respectively give the two aforementioned results. Totally unimodular matrices are well understood thanks to a decomposition theorem of Seymour [3].

In this short note, we characterize the vectors whose addition to the incidence matrix of a directed graph, as a row or a column, yields a totally unimodular matrix. The proofs are short and combine standard matricial arguments with the structure of the underlying undirected graph. The characterizations can be stated in graph theory terms only. To the best of our knowledge, these statements do not appear in the literature.

Outline. After providing a few necessary definitions and useful results just below, we characterize the totally unimodular addition of a row in Section 2, and that of a column in Section 3.

Useful results. We will use the following well-known result of Camion [4].

Theorem 1 (Camion [4]). *A 0,±1 matrix is totally unimodular if and only if the sum of the entries in any square submatrix with even row and column sums is divisible by four.*

A matrix A is *minimally non-totally unimodular* if it is not totally unimodular, but every proper submatrix is totally unimodular. We will use the following consequence of Theorem 1 (see also [5, Theorem 6.6]): a minimally non-totally unimodular matrix is a square matrix of determinant ±2, has even row and column sums, and the sum of its entries is 2 (mod 4).

Given a directed graph $D = (V, A)$ and $U \subseteq V$, $\delta^+(U)$ denotes the set of arcs with their tail in U and their head in $V \setminus U$, $\delta^-(U) = \delta^+(V \setminus U)$, and $\delta(U) = \delta^+(U) \cup \delta^-(U)$. The incidence matrix of D is the $V \times A$ matrix M such that $M_{u,a} = -1$ if $a = uv$, 1 if $a = vu$, and 0 otherwise. The undirected graph associated with a directed graph is the graph obtained by replacing each arc by an edge. An *undirected circuit* is a connected undirected graph in which all the vertices have degree two. A directed graph is *connected* if the associated undirected graph is connected.

2. Adding a row

Let M be the incidence matrix of a directed graph $D = (V, A)$ and $c^\top \in \{-1, 0, 1\}^A$. Let M' be the matrix obtained from M by adding $c^\top$ as a new row. The coefficients of this new row $c^\top$ are used to classify the arcs of D as follows. An arc $a \in A$ is *positive* if $c_a = 1$, *negative* if $c_a = -1$, and *neutral* if $c_a = 0$. Let us call a set of neutral arcs an *arc-neutral uv-path* if forgetting the orientation of these arcs yields an undirected uv-path. We allow $u = v$, that is, the uv-path may have no edges or form an undirected cycle.

Let $P = a_1, \ldots, a_k$ be an arc-neutral uv-path, a a non-neutral arc incident to u and no other vertices of P, and $b \neq a$ a non-neutral arc incident

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0167-6377/

to v and no other vertices of P. Denote by $u = u_1, \dots, u_{k+1} = v$ the vertices of P. Then, the following is a submatrix of M', where in each column a_i one $\star$ equals 1 and the other -1, and in the columns a and b each $\star$ equals ± 1:

$$N = \begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ u_{k+1} \\ c^\top \end{array} \begin{array}{c} \begin{array}{ccccc} a & a_1 & \cdots & a_k & b \end{array} \\ \begin{bmatrix} \star & \star & 0 & 0 & 0 \\ 0 & \star & \ddots & 0 & 0 \\ \vdots & 0 & \ddots & \star & \vdots \\ 0 & 0 & 0 & \star & \star \\ \star & 0 & \cdots & 0 & \star \end{bmatrix} \end{array}. \tag{1}$$

We claim that N is minimally non-totally unimodular if and only if there are either one or three -1 among the $\star$ entries of the columns a and b. First, every proper submatrix of N is totally unimodular by Theorem 1. Moreover, $\det(N) = \pm 2$ if the total number of -1 entries in the columns a and b is odd, and $\det(N) = 0$ otherwise. Indeed, in the column a_i there is precisely one -1, for $i = 1, \dots, k$. Thus, the number of -1 entries in N equals k plus the number of -1 entries in the columns a and b. In particular, its parity differs from that of the size of N if and only if the number of -1 entries in the columns a and b is odd. Now, the claim follows from a well-known exercise [6, Lemma 2]: take the $n \times n$ incidence matrix of an undirected circuit, and change the sign of some entries: the resulting matrix A has $\det(A) = \pm 2$ if the parity of its number of -1 entries differs from the parity of n, and $\det(A) = 0$ otherwise.

Then, up to relabeling a as b and u as v, here are the four possible situations in which N is minimally non-totally unimodular:

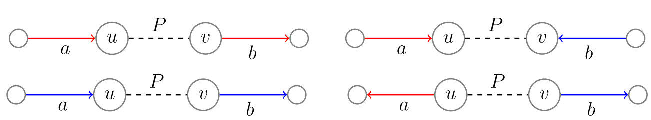

Fig. 1. The four possible situations where N is not totally unimodular: blue arcs are positive, red arcs are negative, and the dashes represent a path composed of neutral arcs.

A neutral arc that belongs to some arc-neutral path P associated with one of the possibilities of Fig. 1 is called *unsafe*. Other neutral arcs are *safe*.

Theorem 2. *The matrix M' obtained from the incidence matrix M of a connected directed graph $D = (V, A)$ by adding a row $c^\top \in \{-1, 0, 1\}^A$ is totally unimodular if and only if there exists $U \subseteq V$ such that $\delta^+(U)$ contains every positive arc, $\delta^-(U)$ contains every negative arc, and the neutral arcs of $\delta(U)$ are safe.*

Proof. Multiplying columns of M' by -1 does not impact its total unimodularity, hence we may assume $c^\top \in \{0, 1\}^A$, as it amounts to reversing negative arcs and making them positive. Note that M' is not totally unimodular if and only if M' contains a minimally non-totally unimodular matrix N.

Let us show that if such an N exists, it has the same form as in (1) up to row and column permutations, and the bottom-left situation of Fig. 1 occurs.

If such an N exists, then it is square with $\det(N) = \pm 2$, has even row and column sums, and has a row indexed by $c^\top$ because M is totally unimodular. Since M' has at most three non-zero entries in each column, N has exactly two non-zero entries in each column. Forgetting the signs in N yields the incidence matrix of an undirected graph in which all the degrees are even, with as many edges as vertices. By minimality of N, this graph is connected, hence is an undirected circuit. In particular, the row of N indexed by $c^\top$ has exactly two non-zero entries, whose indices correspond to two non-neutral arcs a and b, both positive because $c^\top \in \{0, 1\}^A$. Moreover, exactly one endpoint of a and one endpoint of b index a row of N. Thus, N has the same form as in (1).

These two endpoints cannot be both the tails of a and b, as otherwise the rows of N would sum up to zero, and we would have $\det(N) = 0$, contradicting that $\det(N) = \pm 2$. Neither can they be both the heads of a and b, as otherwise all the rows of N but $c^\top$ would sum up to $c^\top$, and we would have again $\det(N) = 0$. Without loss of generality, one of these ends is the head u of a and the other is the tail v of b. Given the structure of N, the only possible common endpoints of a and b are either $u = v$ or a vertex not indexing a row of N. Therefore, a and b are head-disjoint and tail-disjoint. Since a and b are both positive, we are in the bottom-left situation of Fig. 1, and N is indeed minimally non-totally unimodular as explained above Fig. 1.

Let us show that such an N exists if and only if, for every $U \subseteq V$, either $\delta^+(U)$ misses some positive arc or $\delta(U)$ contains some unsafe arc.

Suppose that such an N exists. Then, there are two cases. If $u = v$, then no $U \subseteq V$ can have both a and b in $\delta^+(U)$. If $u \neq v$, then the set of neutral arcs indexing columns of N forms an arc-neutral uv-path which connects the head and the tail of two positive arcs a and b, which are head-disjoint and tail-disjoint. Then, for every $U \subseteq V$ with $a, b \in \delta^+(U)$, $\delta(U)$ contains an arc of this path, which is unsafe by definition.

Conversely, if $\delta(U)$ contains an unsafe arc for some $U \subseteq V$, then such an N exists by definition. Otherwise, $\delta^+(U)$ misses a positive arc for every $U \subseteq V$. In particular, for the set U composed of the tail of each positive arc, there exists a positive arc $a = uv \notin \delta^+(U)$. By construction, both u and v belong to U, thus v is the tail of some positive arc b. Hence, a and b form the bottom-left situation of Fig. 1 with P empty.

Negating both sides of the previous equivalence yields: M' is totally unimodular if and only if there exists $U \subseteq V$ such that $\delta^+(U)$ contains every positive arc and the neutral arcs of $\delta(U)$ are safe. Then, returning to the original signs of $c^\top$ yields the desired statement. □

Checking the conditions of Theorem 2 can be done in $\mathcal{O}(|A|)$. First, reverse negative arcs to have only positive arcs, and let U be the set of tails of positive arcs. As seen in the proof, M' is totally unimodular if and only if U satisfies the conditions of Theorem 2. We may assume that $\delta^+(U)$ contains every positive arc, as otherwise we are done. Now, if an unsafe arc exists, then there is one in $\delta(U)$, as the bottom-left situation of Fig. 1 occurs. Thus, contracting the neutral arcs which are not in $\delta(U)$ does not impact the existence of unsafe arcs. In the resulting graph, an arc is unsafe if and only if it connects the head and the tail of two disjoint positive arcs.

3. Adding a column

Let M be the incidence matrix of a directed graph $D = (V, A)$ and $c \in \{-1, 0, 1\}^V$. Let M' be the matrix obtained from M by adding c as a new column. The coefficients of this new column allow us to classify the vertices of D as follows. A vertex $v \in V$ is called *positive* if $c_v = 1$, *negative* if $c_v = -1$, and *neutral* if $c_v = 0$.

Given two non-neutral vertices u and v, consider a set of arcs such that forgetting their orientation yields an undirected uv-path whose internal vertices are all neutral. Such a set of arcs is a *balanced* uv-path if u and v have opposite signs, and *unbalanced* otherwise.

Three balanced uv_i-paths for $i = 1, 2, 3$ form a *long claw* if u is their only common vertex. Then, all the vertices v_i have the same sign, opposite to that of u. There are two types of long claws according to whether u is positive or negative. An illustration with u negative is given in Fig. 2.

We show below that unbalanced paths and long claws are the only structures to be forbidden to ensure total unimodularity. Given an undirected graph $G = (V, E)$ and $T \subseteq V$ with $|T|$ even, a subset $J \subseteq E$ is a *T-join* if T is the set of vertices of odd degree in (V, J). Each T-join is the edge-disjoint union of circuits and of $\frac{1}{2}|T|$ paths connecting disjoint pairs of vertices in T.

Theorem 3. *The matrix obtained from the incidence matrix of a connected directed graph by adding a $0, \pm 1$ column is totally unimodular if and only if there is neither an unbalanced path nor a long claw.*

Proof. Let M be the incidence matrix of a directed graph $D = (V, A)$, $c \in \{-1, 0, 1\}^V$, and M' the matrix obtained from M by adding c as a new column.

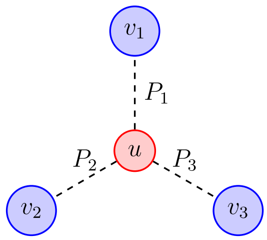

Fig. 2. A long claw: the red vertex is negative, the blue ones are positive, and the dashes represent paths whose internal vertices are neutral.

$$
\begin{array}{c|cccccccccc|}
 & a_1^{(1)} & \cdots & a_k^{(1)} & a_1^{(2)} & \cdots & a_k^{(2)} & a_1^{(3)} & \cdots & a_k^{(3)} & c \\
u & \star & 0 & 0 & \star & 0 & 0 & \star & 0 & 0 & -1 \\
u_2^{(1)} & \star & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & \ddots & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_1 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
u_2^{(2)} & 0 & 0 & 0 & \star & \star\ddots & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0 & 0 & 0 & \ddots & \star & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 & 1 \\
u_2^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star\ddots & 0 & 0 \\
\vdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \star & 0 \\
v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 1
\end{array}
$$

Fig. 3. The matrix associated with a long claw when u is negative: in each column, one $\star$ equals 1 and the other -1. The diagonals of dots ($\ddots$) contain only $\star$.

First, let us prove that the existence of an unbalanced path or a long claw yields a non-totally unimodular submatrix of M'.

Let P be a unbalanced uv-path. Let N be the submatrix of M' whose rows are indexed by the vertices of P and whose columns are indexed by c and by the arcs of P. Since u and v are either both positive or both negative, N has even row and column sums and the sum of the entries of N is $2 \pmod 4$. By Theorem 1, N is not totally unimodular.

Let P_1, P_2, and P_3 be three balanced paths forming a long claw, whose common vertex is u, and respective other endpoints v_1, v_2, and v_3. Let N be the submatrix of M' whose rows are indexed by the vertices of P_1, P_2, and P_3, and whose columns are indexed by c and by the arcs of P_1, P_2, and P_3. By construction, N has exactly two non-zero entries in each row and column, except in the row u and the column c where there are four, see Fig. 3. Therefore, N has even row and column sums. In each column of N except c, there is precisely one 1 and one -1, thus the sum of the entries of N equals the sum of its entries in column c. Since the sign of u is opposite to the sign of v_1, v_2, and v_3, the latter sum equals $\pm 2 = 2 \pmod 4$. By Theorem 1, N is not totally unimodular.

To prove the remaining direction, suppose that M' is not totally unimodular. Then, M' contains a minimally non-totally unimodular matrix N which is square, has even row and column sums, and the sum of the entries of N equals $2 \pmod 4$. Since M is totally unimodular, N has a column indexed by c, and its other columns are arcs of D with both extremities indexing a row of N. Forgetting the orientation of these arcs yields a set of edges J. By the parity of the sum of each row of N, the vertices incident to an even number of edges of J are neutral, and the other ones are non-neutral. In other words, J is a T-join, where T is the set of non-neutral vertices indexing a row of N. Moreover, J is a forest. Indeed, suppose J contains an undirected circuit C and let C' be the corresponding circuit in D. The square submatrix P of N whose rows and columns are respectively the vertices and the arcs of C' has $\det(P) = 0$, since its rows sum up to zero. Since P contains all the nonzero coefficients of the columns of N associated with the arcs of C', these columns are linearly dependent, contradicting $\det(N) \neq 0$.

In the subgraph associated with N, the number of positive and negative vertices differ. Indeed, if they were equal, then each column of N would have as many -1 as $+1$ entries, hence the rows of N would sum up to zero, contradicting that the sum of the entries of N is $2 \pmod 4$.

Therefore, there exists a connected component F of J in which the number of positive and negative vertices differ. Recall that F is a tree, and note that the leaves of F are non-neutral. If F has exactly two leaves, then all the non-leaf vertices of F have even degree, and hence are neutral. Then, as the number of positive and negative vertices in F differ, the two leaves of F have the same sign, hence F is an unbalanced path. Otherwise, let ℓ_1, ℓ_2, and ℓ_3 be three leaves of F. Let u be the unique vertex that is common to the three $\ell_i\ell_j$-paths in F for $i, j \in \{1, 2, 3\}$ with $i \neq j$. For $i \in \{1, 2, 3\}$, let $v_i \neq u$ be the non-neutral vertex of the $u\ell_i$-path which is the closest to u. If u is neutral, then F contains an unbalanced path as two of the v_i have the same sign. Otherwise, u is non-neutral, and there are two cases. If u has the same sign as one of the v_i, then F contains an unbalanced path. If u has a sign opposite to that of all the v_i, then F contains a long claw. $\square$

Checking the condition of Theorem 3 can be done in $\mathcal{O}(|A|)$. First, contract arcs incident to a neutral vertex, the sign of the resulting vertex being the sign of the other extremity of the arc. This operation does not impact the existence of unbalanced paths and long claws. The resulting graph contains no unbalanced path and no long claw if and only if all its connected components are paths and cycles with no arcs between vertices of the same sign.

CRediT authorship contribution statement

Roland Grappe: Writing – original draft; **Jules Nicolas-Thouvenin:** Writing – original draft.

Data availability

No data was used for the research described in the article.

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