On The Number of Condoms at a Cheap Safe-Sex Orgy

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Abstract

Let M and F be finite sets. A straight orgy is a series of interactions between each pair in $M \times F$. Such an interaction is called *safe* if it is facilitated by a *condom complex*, a sequence of basic units called *condoms*. For the interaction to be safe, the condoms must fulfill some dynamic conditions which we formalize herein.
A straight orgy is called a safe-sex straight orgy if all the interactions are safe. We give an exact formula for the minimal number of condoms required to realize such an orgy, up to an additive factor of 1.

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1 Introduction

One of the best party riddles known to the authors is the following: Two straight couples are interested in having a safe-sex orgy. That is, each woman would like to have sex with each of the two men, using a condom. Condoms may be used more than once, but each participant may only touch a clean side of a condom, or one that is only stained by that participant's fluids. How can they do this with but two condoms? This is a good party riddle, since as the reader may readily verify, it is not very hard to solve, and serves as a potent ice breaker.

A vanilla riddle with the same flavor, is the following: Three surgeons need to operate on a patient (one after the other). Both the surgeons, and the patient, may be carrying a terrible disease, so they must use surgeon gloves. How can they operate on the patient if they have only two pairs of gloves?

Let us return to the obscene phrasing of the problem. Denote by $Con(f, m)$ the minimal number of condoms needed for a safe-sex orgy with f women and m men. What bounds can be given for this number? It is easy to see that:

$$\frac{1}{2}(m + f) \leq Con(f, m) \leq m + f,$$

and the clever reader might also see that $Con(f, m) \leq \min\{\lceil \frac{1}{2}m \rceil + f, m + \frac{1}{2}f\}$, but it turns out that both these bounds can be improved, as stated by the main theorem of this paper:

Theorem 1.1 *Let* $\alpha = \min\{\lceil \frac{2}{3}m \rceil + \lceil \frac{1}{2}f \rceil, \lceil \frac{2}{3}f \rceil + \lceil \frac{1}{2}m \rceil\}$, *then:*

$$\alpha - 1 \leq Con(m, f) \leq \alpha + 1$$

When there will be no way around that, we shall refer to a general participant as "she". This in no way reflects the authors' views on the gender of orgy participants. We would also like to clarify that we have no first-hand knowledge of the subject which inspires the mathematical discussion herein, nor wish to acquire one.

2 Definitions

Let $P = F \cup M$ be a finite set called *the participants*, where the subset F is called *the women*, and the subset M is called *the men*. Let C be a set called *the condoms*, where each member has two *sides*.

Definition 2.1 *A* condom complex *is an even-length sequence of condom sides, $(s_1, ..., s_l)$, such that for two sides in the sequence, s_j, s_k, there exists a $c \in C$ with $c = \{s_j, s_k\}$ iff there is some natural number i such that $j = 2i, k = 2i-1$. In such a condom complex we shall say that two condoms, c_1, c_2, are* touching, *if there exists an i such that $s_{2i} \in c_1, s_{2i+1} \in c_2$, or vice versa.*
A sexual act *is a sequence (p_1, S, p_2), where $p_1, p_2 \in P$, and S is a condom complex.*
An orgy *is a sequence of sexual acts. In this context we shall sometimes refer to a sexual act as an* orgy round*. Also, we shall say that the two participants in that round are* having sex*.*
A straight orgy *is an orgy where M and F are disjoint, and in each round one of the participants is in M, and one is in F.*

Condom sides also have a subset of P associated with them, called their *touch set*. These change as a result of sexual acts in which the condoms are involved, in the following manner:

Definition 2.2 *Let $(p_1, (s_1, ..., s_{2l}), p_2)$ be an orgy round, and let $T^r_{s_1}, ..., T^r_{s_{2l}}$ be the touch sets of $s_1, ..., s_{2l}$, respectively, before this round. Then the respective touch sets after this round, $T^{r'}_{s_1}, ..., T^{r'}_{s_{2l}}$ are defined as follows:*

- $\forall i = 1, ..., l-1,\ T^{r'}_{s_{2i}}, T^{r'}_{s_{2i+1}} = T^r_{s_{2i}} \cup T^r_{s_{2i+1}}$
- $T^{r'}_{s_1} = T^{r'}_{s_1} \cup \{p_1\}$
- $T^{r'}_{s_{2l}} = T^{r'}_{s_{2l}} \cup \{p_2\}$

We are now ready to define the kind of orgies we are interested in:

Definition 2.3 *A sexual act $(p_1, (s_1, ..., s_l), p_2)$ is a* safe sexual act *if $l > 0$, and at its beginning the touch set of s_1 is either empty, or $\{p_1\}$, and the touch set of s_l is either empty, or $\{p_2\}$.*
A straight orgy is a straight safe-sex orgy *if before the first round each touch set is empty, and each of the orgy rounds is a safe sexual act.*
$Con(m, f)$ is the minimal size of C, such that there exists a straight safe-sex orgy with $|F| = f, |M| = m$, and for each $(p_1, p_2) \in M \times \mathbb{F}$ there's an orgy round where the participants are p_1 and p_2..

3 Lower Bound

Let us first restrict the discussion to $Con(n, n)$. We'll need yet more definitions.

Definition 3.1 *A side of a condom is* clean *as long as its touch set is empty. Otherwise, it is* unclean*.*
A participant p owns *a condom's side s, if at some point, the touch set of s is $\{p\}$.*

For example, whenever a participant touches a clean side, she becomes the owner of that side.

Definition 3.2 *A participant is* modest *if she owns exactly one side of one condom.*
Two participants are a couple *if each owns one side of the same condom. The two members are called* partners *of each other.*
A couple is an f-couple *if both its members are female. It is an* m-couple *if both are male. Otherwise it is an* h-couple*.*
A couple is a modest couple *if both its members are modest.*

Note that a participant may be a member of more than one couple.

Definition 3.3 *A participant is* active *in an orgy round, if she has sex in that round. A participant is* passive *in an orgy round if she is not active, but one of her partners is.*

Note that during some rounds a participant is neither active nor passive.

Understanding the activity pattern of the (modest) participants is the key to bounding the number of condoms, and will also prove instrumental in designing the orgy for the upper bound.

Lemma 3.1 *Once a modest participant had been active, and then passive, she can not be active again.*

Proof: Assume for contradiction that p violates the lemma. Once she is active, her (unique) condom side becomes unclean, and no other participant may touch it. Once she is passive her condom is in use. If it touches an unclean condom side, then obviously p can't use it anymore. If it touches a clean condom side, then that condom side becomes unclean, giving ownership of it to p, in contradiction with her modesty. ∎

We shall need a couple of more definitions to get a few corollaries regarding the types of modest couples in a safe-sex orgy:

Definition 3.4 *Let (p_1, p_2) be a modest couple. Call the participant that is first to be active an* active member*, and call the other a* passive member*.*
Define $r(p)$ for modest pair members, as follows:
If p is a passive member, $r(p)$ is the number of the first round in which p is active.
If p is an active member, and her passive partner is q, $r(p) = r(q)$.

Corollary 3.1 *Let (p_1, p_2) be a modest couple, with p_1 the active member, and let $r = r(p_1) = r(p_2)$.*
If (p_1, p_2) is either an f-couple or an m-couple, then p_1 is never passive before r, and never active after it (and vice versa for p_2).
If $m, f > 1$ then this is also true when (p_1, p_2) is an h-couple.

Proof: First note that by definition p_2 is never active before r, so p_1 is never passive during this time. But as p_1 is the active member, she is active at some point before r.
If (p_1, p_2) is either an f-couple or an m-couple, then the pair members are never active on the same round. Thus, at r, p_2 is active and p_1 is passive. By the lemma, p_1 can not be active again, and thus from round r onward, p_1 is never active, and p_2 is never passive.
If (p_1, p_2) are an h-couple then it must be the case that both are active during round r. Otherwise, p_1 is passive, and by the lemma may not be active again. But if p_1 is never active after r, and p_2 is never active before r, when will they have sex with each other?
Furthermore, if $m, f > 1$, p_1 may not be active again. Consider the next round that one of them is active. If p_2 is active in it, then p_1 is passive, and, indeed, by the lemma, may not be active again. Otherwise the converse is

true, i.e. p_2 is never active again. But if p_2 is neither active after r, nor before it, when will she have sex with the other participants? ∎

Corollary 3.2 *There may not be both a modest m-couple, and a modest f-couple.*

Proof: Assume for contradiction that there is a modest m-couple (M_a, M_p), and a modest f-couple (F_a, F_p), with M_a and F_a being the active members. Consider the round t when M_a and F_p have sex. By Corollary 3.1, $r(F_p) \leq t < r(M_a)$. Similarly, by considering the round that F_a and M_p have sex, we get $r(M_p) < r(F_a)$. But as $r(F_p) = r(F_a)$, and $r(M_p) = r(M_a)$, we get a contradiction. ∎

Corollary 3.3 *There may not be more than 2 modest h-couples.*

Proof: Assume for contradiction that there are, then in particular we have that there are two couples where the active members are of the same gender. W.l.o.g. assume that they are women. Denote the couples by (F_1, M_1) and (F_2, M_2). Since our assumption implies that $m, f > 1$, by considering the round when F_1 and M_2 have sex, and the round when F_2 and M_1 have sex, we get a contradiction in the same manner as in Corollary 3.2. ∎

We are now ready to prove the lower bound:

Theorem 3.1

$$Con(n, n) \geq \lceil \frac{7}{6} n \rceil - 1$$

Proof: By Corollary 3.3 we know that there are at most 2 modest h-couple, so let us forget about them. Now, by Corollary 3.2 we may assume w.l.o.g. that the only modest couples are f-couples. In other words, modest men must have immodest partners.
Let each of the participants choose one of the condom sides they own, and call it their *chosen* side. It is enough to show that there are at least $\frac{1}{3}n$ sides which are unchosen, and we shall do just that.
Partition the men into gangs in the following manner. Each gang leader is an immodest man, and the other gang members are his modest men partners (if

there are any). A gang leader that owns k unchosen sides, has at most $k+2$ members in his gang. At best, the gang includes himself, his partner for the chosen side, and his k partners for the unchosen sides. Thus, the number of unchosen sides a gang leader owns is at least one third the size of his gang. Summing this up, we get that the number of unchosen sides is at least one third the number of men, or $\frac{1}{3}n$.
Recall now the 2 h-couples. We have actually shown that the number of unchosen sides is at least $\frac{n-2}{3}$, thus the number of condoms needed is at least $\lceil\frac{1}{2}(2n+\frac{n-2}{3})\rceil = \lceil\frac{7}{6}n-\frac{1}{3}\rceil \geq \lceil\frac{7}{6}n\rceil - 1$ ∎

Corollary 3.4

$$Con(m,f) \geq \min\{\lceil\frac{2}{3}m\rceil + \lceil\frac{1}{2}f\rceil, \lceil\frac{2}{3}f\rceil + \lceil\frac{1}{2}m\rceil\} - 1$$

Proof: The same proof as for Theorem 3.1 works, with the exception that we do loose generality by assuming that the only modest couples are f-couples (except for maybe two h-couples). This assumption is indeed to our disadvantage if $m \leq f$, but in the complementary case we may only assume that the only modest couples are m-couples, giving the bound in the corollary. It is an easy exercise to verify that the integral values are indeed as stated. ∎

4 Upper Bound

Let us now try to design an orgy that actually achieves this lower bound. Indeed, let us state it as a theorem, and then try to prove it:

Theorem 4.1

$$Con(m,f) \leq \min\{\lceil\frac{2}{3}m\rceil + \lceil\frac{1}{2}f\rceil, \lceil\frac{2}{3}f\rceil + \frac{1}{2}m\} + 1$$

Proof: The proof of Theorem 3.1 suggests how to design the parsimonious orgy we are looking for. We shall show how to use only $\lceil\frac{2}{3}m\rceil + \lceil\frac{1}{2}f\rceil + 1$ condoms. By exchanging the roles of men and women one gets $\lceil\frac{2}{3}f\rceil + \lceil\frac{1}{2}m\rceil + 1$, giving the stated bound.
We saw that the worst case in the proof above was when all the women were

Figure 1: An active, modest woman

modest, and two thirds of the men were modest. The immodest males were each leaders of a 3-men gang, that is, each had exactly two partners, and this was the sole source for unchosen condom sides.

Let us look at the women first. Since they are all modest, they are partitioned into pairs. For each pair, call the active member an *active woman* (see Figure 1), and the passive a *passive woman*. Denote by F_A the set of active women, and by F_P the set of passive women. Let us now hazard a guess that there is some round r such that no passive woman is active before that round, and no active woman is active from that round on.

Now let us look at the men. They have to be active both before round r, and after it. By lemma 3.1 it must be the case that the modest men are (perhaps) passive at first, then active and then passive again. This implies that the immodest men are active, then passive, then active again (or they might start as passive, and then follow this pattern). In other words, the proof of theorem 3.1 suggests that the immodest men are first active with the active women, then give away their condoms to the modest men. These have sex with the active women, and then (after round r) with the passive women. Finally, the immodest men have sex with the passive women.

To see that this indeed works, denote by M_I the set of immodest men, and partition the modest men into M_1 and M_2 in such a way that no two men in the same set have a common partner (note that all three sets are of size $\frac{n}{3}$). We shall also need one extra condom. Give a condom to each woman in F_A, and to each man in M_I and in M_1. At first these have sex, and after

that each has a condom with one clean side, and one unclean side. Now have the men in M_I invert their condoms, and give them to the men in M_2. Now let the men in M_2 have sex with the women in F_A, using the extra condom in such a way that the condoms owned by F_A, maintain one clean side (the condom side previously used by M_I now becomes unusable, but not to worry, they have extra sides).
This brings us to round r. Now each woman inverts her condom, which still has one clean side, and gives it to her partner in F_P. These have sex with the men in M_2, which retain their condoms from the previous rounds. Now the men in M_1 have sex with F_P, using the condoms they previously used, and the extra condom to keep the other side of their condom clean. Finally M_1 invert their condoms, give them to M_I, which have sex with F_P, and the orgy is done (so is the proof). ∎

5 Orgy Graphs

In this problem we explored a straight safe-sex orgy where each possible straight couple have sex. Thinking of the participants as vertexes, and the sex acts as edges, we can say that a straight safe-sex orgy *realizes* the graph $K_{m,f}$. In general, a safe-sex orgy realizes a graph G, if the participants correspond to the vertexes of the graph, each orgy round is safe (the condom complex is non-empty, and the touch set of the sides at its end contain, are either empty, or the singletons corresponding to the participants on either end), and for each edge (u, v) in the graph there is an orgy round where the participants are u and v. Denote by $Con(G)$ the minimal number of condoms required to realize G as a safe-sex orgy. This paper shows that $Con(K_{m,f}) \approx \min\{\lceil\frac{2}{3}m\rceil + \lceil\frac{1}{2}f\rceil, \lceil\frac{2}{3}f\rceil + \lceil\frac{1}{2}m\rceil\}$.
Similar arguments to those used in the previous sections show that $\lceil\frac{2}{3}\rceil n-1 \leq Con(K_n) \leq \lceil\frac{2}{3}\rceil n+1$. For the lower bound, note that the proof of Corollary 3.2 implies that in a realization of K_n there can be only one modest couple. Partitioning the participant into gangs, as in the proof of Theorem 3.1, gives $Con(K_n) \geq \lceil\frac{2}{3}\rceil n-1$.
For the upper bound, divide the participants into three groups of size $\frac{n}{3}$, say M_I, M_1, M_2. Give a condom to those in M_I and in M_1. These can now realize the orgy among themselves, leaving one side clean. Next give the condoms used by M_I to M_2. M_2 can now realize the orgy among themselves, and with

the use of an extra condom, have sex with those in M_1, keeping on side of the M_1 condoms clean. Finally, the condoms from M_1 are given to M_I, and these have sex with the members of M_2.

We leave open the value of $Con(G)$ for other graphs (bipartite or otherwise), and the scope of applicability of the results herein.