Extensions of Edge Graph Problems

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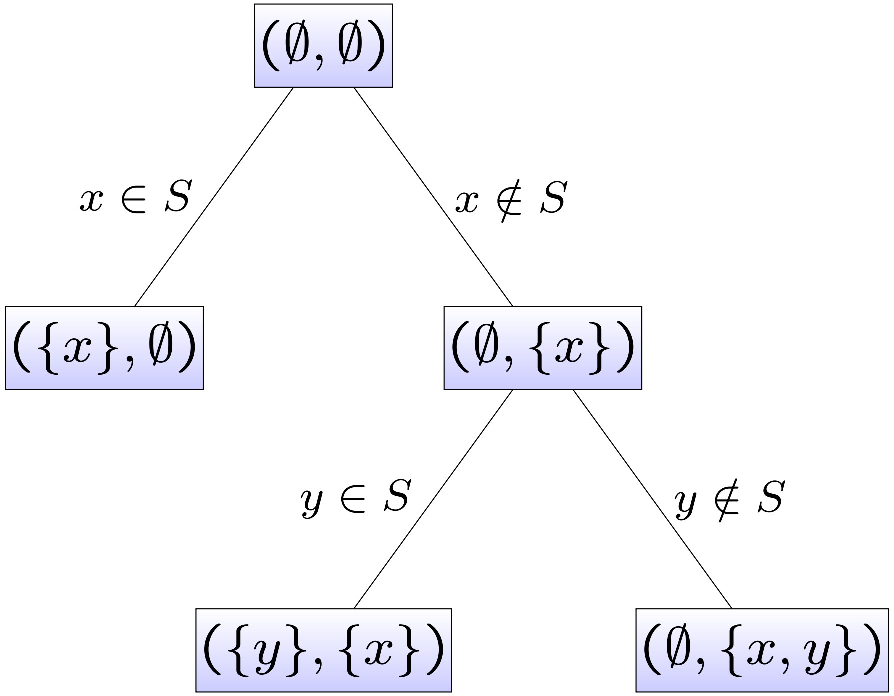
Overview

- The general setting:
  - What are extension problems?
  - Why are they interesting?
  - What about *edge graph problems*?
- Our results
  - NP-completeness results
  - poly-time cases
  - parameterized complexity
  - approximability
- A glimpse at some constructions
- Conclusions

Search Tree Approach for branching algorithms



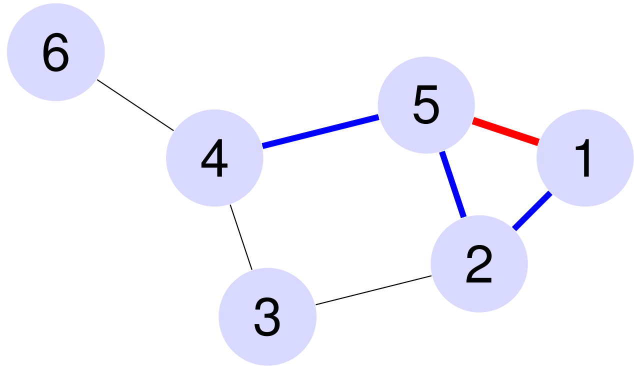
- Exploring the search space
- Typical technique for solving NP-hard problems
- Similar for counting
- Similar for enumeration
- Key question:
  When to stop branching?
- When is there no **promising** continuation?
- Flashlight algorithm
  Cut non-promising branches

When to stop? The flashlight idea

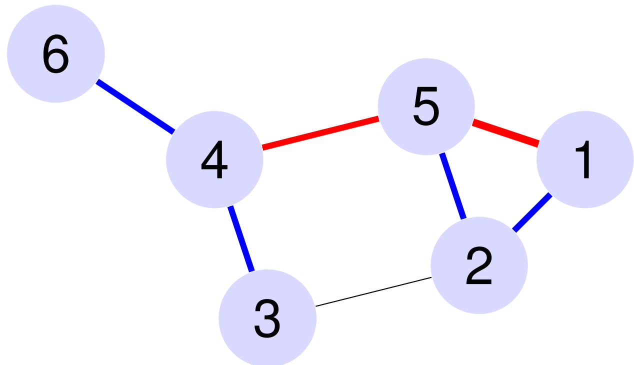
To be concrete, consider EDGE DOMINATION.

Assume that (only) the red edge is in the current (partial) solution.

The blue edges are dominated.



Can we add further edges to our partial solution in order to get an inclusion-wise minimal edge dominating set?



Can we easily conclude that $\{4, 5\}$ is out once $\{1, 5\}$ is in?

Extension Problems An interesting class of problems

2	5			3		9		1
	1				4			
4		7				2		8
		5	2					
				9	8	1		
	4				3			
			3	6			7	2
	7							3
9		3				6		4

Edge Minimization Problems

EXT r-EC
Input: A graph $G = (V, E)$ and $U \subseteq E$.
Question: Does there exist an edge set $S \subseteq E$ with $S \supseteq U$ such that the partial graph $G_S = (V, S)$ has minimum degree at least r and is minimal in G?

For $r = 1$, we arrive at the extension variant of EDGE COVER.

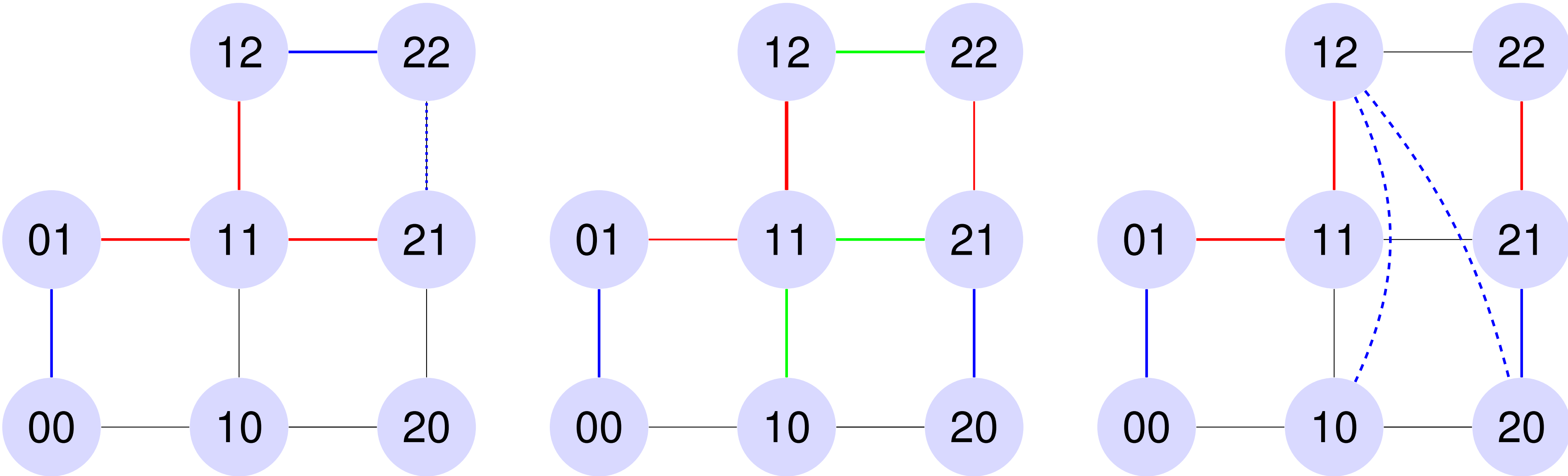
EXT r-EDS
Input: Given a simple graph $G = (V, E)$ and $U \subseteq E$.
Question: Is there a subset $S \subseteq E$ such that $S \supseteq U$ and S is a minimal r-eds?

An *r-edge dominating set* (r-eds) $S \subseteq E$ of $G = (V, E)$ is a set $S \subseteq E$ s.t. for any edge $e \in E$, at least r edges of S are incident to e.
For $r = 1$, we arrive at the extension variant of EDGE DOMINATING SET.

U: set of forced edges

Example instances for Ext EDS

forced EDS edges, determined private edges, problems



Edge Maximization Problems What should that mean??

EXT r-DCPS
Input: A graph $G = (V, E)$ and $U \subseteq E$.
Question: Does there exist an edge set $S \subseteq E$ with $S \subseteq U$ such that the partial graph G_S has maximum degree at most r and is maximal in G?

For $r = 1$, we arrive at the extension variant of MAXIMUM MATCHING.

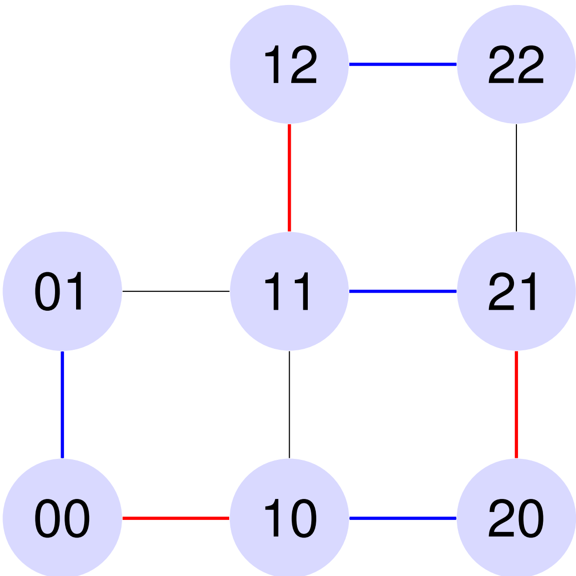
For any $r \geq 1$, MAX r-DCPS is known to be solvable in polynomial time even for the edge weighted version (Gabow, STOC 1983).

Focus on $r = 1$ in the presentation!

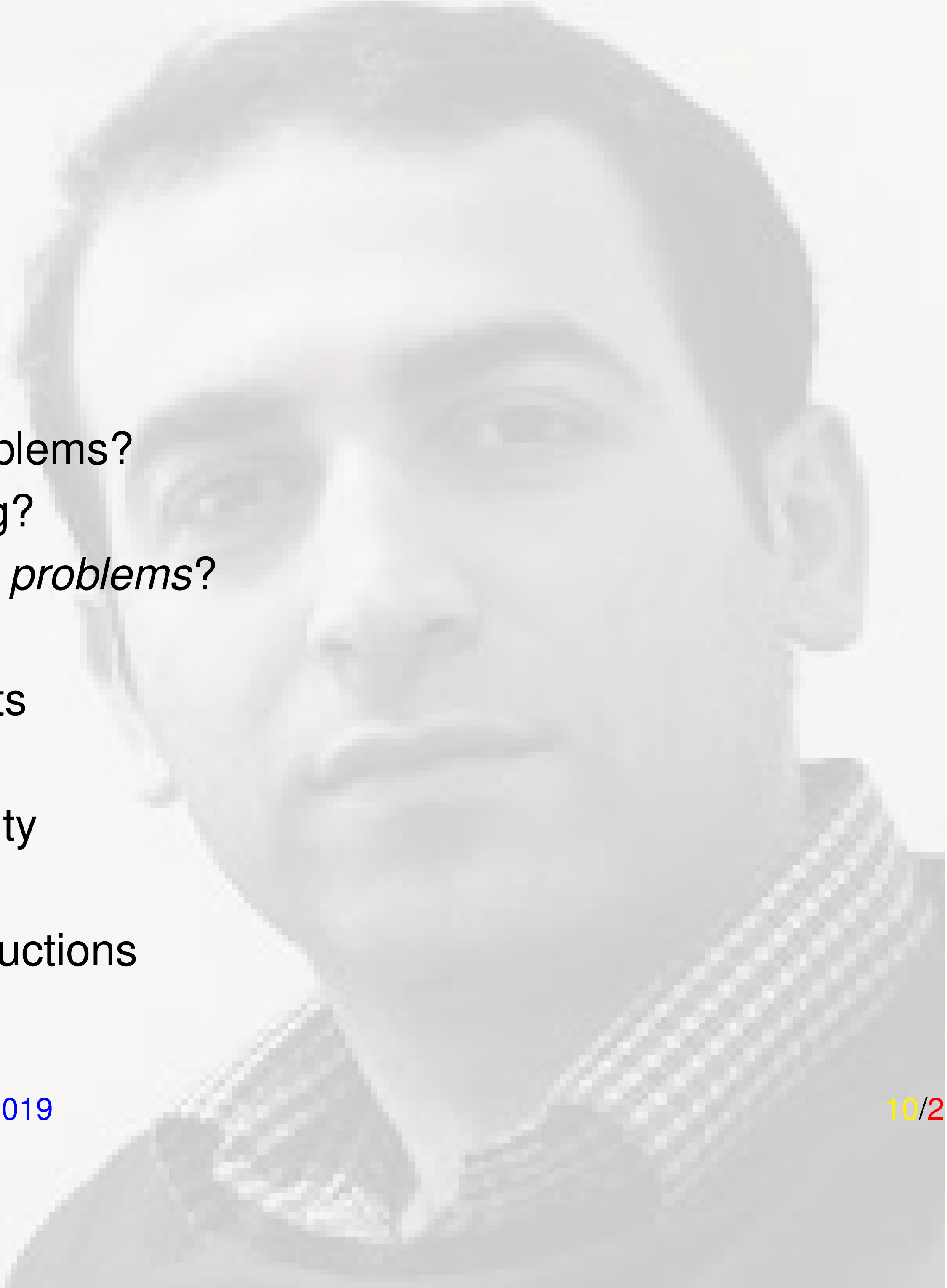
Justification: Partial order perspective (CFKMS 2019 ArXiv)

$\overline{U}$: set of forbidden edges

A simple example instance for EXT 1-DCPS



- Red edges: forbidden to appear in maximal matching
  - $*$ they are not an induced matching
- Blue edges: maximal matching of the graph
  - $*$ they avoid using red edges



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Extending Edge Covers and Dominating Sets

Theorem 1. EXT EC *is NP-complete, even on (planar) subcubic bipartite graphs.*

This theorem remains true if the forced edges form a matching.

Theorem 2. EXT EDS *is NP-complete, even on (planar) subcubic bipartite graphs.*

Extending Matchings

Recall: $\bar{U}$ are the forbidden edges

Theorem 3. EXT 1-DCPS *is NP-complete, even on (planar) subcubic bipartite graphs.*

The theorem remains true if all connected components of $\bar{U}$ have ≤ 2 edges.
By way of contrast, we can show:

Theorem 4. EXT 1-DCPS *is poly-time solvable when the edges in $\bar{U}$ form an induced matching.*

Parameterized Complexity

Our standard parameter: $|U|$ for minimization, $|\bar{U}|$ for maximization problems.
Motivation: for parameter value zero, the problem becomes trivial.

Theorem 5. EXT EC *is in FPT.*

Theorem 6. EXT EDS *is W[1]-hard, even on bipartite graphs.*

Theorem 7. EXT 1-DCPS *is in FPT.*

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A typical NP-hardness proof

EXT 1-DCPS, i.e., EXT MATCHING, is NP-complete.

Start with 2-BALANCED 3-SAT, or $(3, B2)$-SAT.
Such a SAT-instance $(\mathcal{C}, \mathcal{X})$ satisfies:

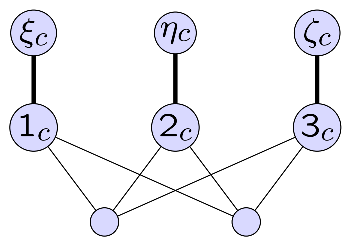
- each $c \in \mathcal{C}$ has exactly 3 literals,
- each $x \in \mathcal{X}$ occurs exactly twice as a negative and twice as a positive literal.

NP-hardness of $(3, B2)$-SAT by Berman, Karpinski, Scott 2003.

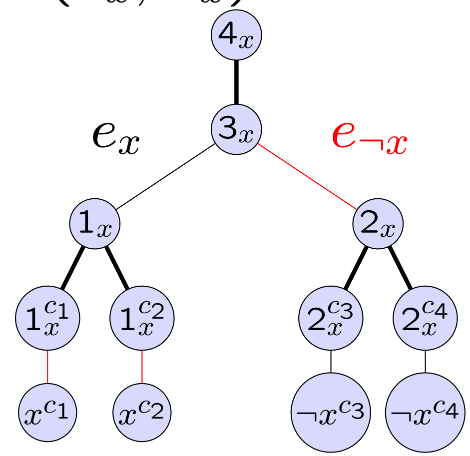
A typical NP-hardness proof

EXT MATCHING is NP-hard. Some gadgetry, with forbidden edges in bold:

For each $c = \xi \vee \eta \vee \zeta \in \mathcal{C}$, where ξ, η, ζ are literals, introduce $H(c) = (V_c, E_c)$.



For each variable $x \in \mathcal{X}$, introduce $H(x) = (V_x, E_x)$.



We interconnect $H(x)$ and $H(c)$ by adding edge $x_c x^c$ if x appears as a positive literal and edge $x_c \neg x^c$ if x appears as a negative literal.

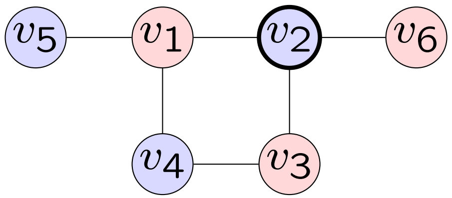
If $T(x) =$ *true*, avoid $\{e_x, 2_x^{c3}\neg x^{c3}, 2_x^{c4}\neg x^{c4}\} \cup \{x_c x^c\}$ in the matching.

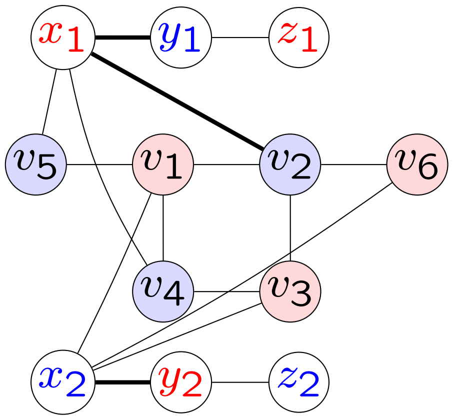
If $T(x) =$ *false*, avoid $\{e_{\neg x}, 1_x^{c1} x^{c1}, 1_x^{c2} x^{c2}\} \cup \{x_c \neg x^c\}$ in the matching.

How to prove W[1]-hardness

Aim: Prove W[1]-hardness of EXT EDS on bipartite graphs.

Recall: EXT VC W[1]-hard on bipartite graphs (CFKMS CIAC 2019)





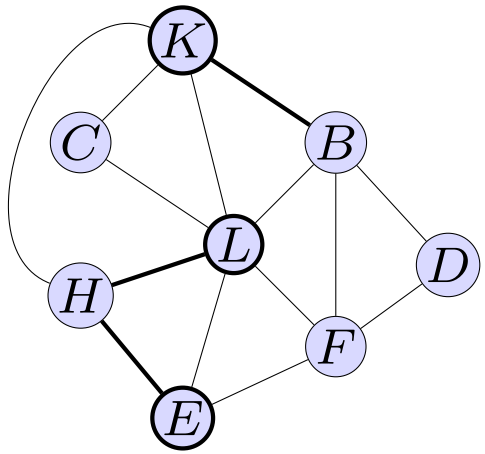
If $S \supseteq U$ is a minimal VC of G, then $S' = (S \times \{x_1, x_2\} \cap E') \cup \{x_1y_1, x_2y_2\}$ is a minimal EDS in G', extending $U' = (U \times \{x_1, x_2\} \cap E') \cup \{x_1y_1, x_2y_2\}$.

Conversely, the set S of vertices incident both to $S' \supseteq U'$ and to private edges of S' is a minimal VC in G, extending U.

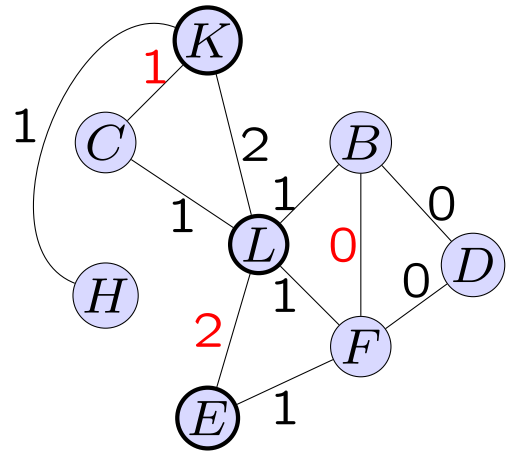
How to prove FPT-membership

Often based on nice combinatorial insights. . .

Theorem 8. *There is a maximal r-DCPS set S for G such that $S \cap \overline{U} = \emptyset$ if and only if there is a vertex cover V' of $G_{\overline{U}}$ such that there exists an r-DCPS set S' for the corresponding weighted graph G' such that $w'(S') \geq |V'| \times r$.*



$V' = \{L, E, K\}$



weighted graph (G', w')

Algorithm:

Cycle through all VCs V' in $G_{\overline{U}}$.

Compute maximum r-DCPS set S' in weighted graph G'.

$w'(S') \geq |V'| \times r$?

Weight of uv: $\chi_{V'}(u) + \chi_{V'}(v)$

Conclusions

We also defined related approximation problems.

MAX EXT EC

Input: A connected graph $G = (V, E)$ and a set of edges $U \subseteq E$.

Solution: Minimal edge cover S of G.

Output: Maximize $|S \cap U|$.

MAX EXT EDS

Input: A graph $G = (V, E)$ and a set of edges $U \subseteq E$.

Solution: Minimal edge dominating set S of G.

Output: Maximize $|S \cap U|$.

MIN EXT 1-DCPS or MIN EXT MATCHING

Input: A graph $G = (V, E)$ and a set of edges $U \subseteq E$.

Solution: Maximal matching S of G.

Output: Minimize $|U \cup S|$.

Thanks for your help! Tusind tak! . . . plus some announcements

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Thanks for your attention! Tusind tak!

Theoretische

Informatik Trier