Dealing with Interactivity Between Bi-polar Multiple Criteria Preferences in Outranking Methods

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Abstract

In this paper we introduce the modelling of specific interactions between criteria expressing positive and negative preferences considered as reasons in favor and reasons against the comprehensive preferences. We can call this method the *bi-polar approach* to MCDA. In order to take into account, specific interactions between criteria in this context, especially the power of the opposing criteria, multiple criteria positive and negative preferences are aggregated using the bi-polar Choquet integral. The bi-polar approach is applied to the two most well-known classes of outranking methods: ELECTRE and PROMETHEE. The final result is a new way to deal with the outranking approach, which permits to take into account some very important preferential information which could not be modelled before by the existing MCDA methodologies. Our approach is related with most of the current advanced research subjects in MCDA and more specifically with the following domains: outranking approach, fuzzy integral approach, four-valued logic approach, non-additive and non-transitive models of conjoint measurement, non-compensatory preference structures, interpretation of the importance of criteria, methods for assessing the non-additive weights, and the aggregation functions.

Key-words: Multiple criteria outranking methods, Interactive between criteria, Bi-polar Choquet integral, Bi-polar Choquet bi-integral.

1 Introduction

When dealing with decision problems, enumerating the reasons in favor and the reasons against of a certain action is an activity so common that it makes part of the main concerns of decision makers (DMs) and analysts. But what kind of decision problems do we actually have in real-world? There are so many... Let us start by a few but typical examples of decision problems. How shall we *choose* the best site for locating a power plant? How shall we establish a *ranking* of different companies asking for financial support in a bank? How shall we *assign* retail shops to different performance categories? These are barely some examples of complex decision problems involving a set of different actions (sites, companies, retail shops) evaluated on the basis of several and conflicting criteria. *Choosing*, *ranking* and *sorting* out actions into categories are thus the three major problem statements studied in multiple criteria decision analysis (MCDA) ([37]).

Over the last three decades, many MCDA methodologies have been developed to deal with this kind of multiple criteria decision problems. Very good references on this topic are the works by [37], [38], [46].

But, the problem of evaluating and comparing possible actions is not a recent issue and, even if in an informal way, some very interesting reflections have been proposed in the past centuries ([1], [21], [28], [33]). In this line, let us go back to the eighteenth century, and discuss the suggestion presented by Benjamin Franklin to one of his friends, Joseph Prestly, when dealing with a particular decision problem ([21]):

> London, Sept 19, l772
>
> Dear Sir,
>
> In the affair of so much importance to you, wherein you ask my advice, I cannot, for want of sufficient premises, advise you what to determine, but if you please I will tell you how. When those difficult cases occur, they are difficult, chiefly because while we have them under consideration, all the reasons pro and con are not present to the mind at the same time; but sometimes one set present themselves, and at other times another, the first being out of sight. Hence the various purposes or inclinations that alternatively prevail, and the uncertainty that perplexes us. To get over this, my way is to divide half a sheet of paper by a line into two columns; writing over the one Pro, and over the other Con. Then, during three or four days consideration, I put down under the different heads short hints of the different motives, that at different times occur to me, for or against the measure. When I have thus got them all together in one view, I endeavor to estimate their respective weights; and where I find two, one on each side, that seem equal, I strike them both out. If I find a reason pro equal to some two reasons con, I strike out the three. If I judge some two reasons con, equal to three reasons pro, I strike out the five; and thus proceeding I find at length where the balance lies; and if, after a day or two of further consideration, nothing new that is of importance occurs on either side, I come to a determination accordingly. And, though the weight of the reasons cannot be taken with the precision of algebraic quantities, yet when each is thus considered,

> separately and comparatively, and the whole lies before me, I think I can judge better, and am less liable to make a rash step, and in fact I have found great advantage from this kind of equation, and what might be called moral or prudential algebra.
>
> Wishing sincerely that you may determine for the best, I am ever, my dear friend, yours most affectionately.
>
> B. Franklin

It is quite clear that Benjamin Franklin's approach for dealing with this decision problem is based on the enumeration of the *reasons for* and *against* it. In line with his way of thinking, we can ask the following question: given a pair of actions *a* and *b* how can the current MCDA methods take into account the comparisons and weighting of the reasons in favor and against each of them?

In our opinion, and in view of the tremendous development in MCDA methodologies especially over the past two decades, some important aspects of the question above must still be considered. Let us give some more explanations on this topic.

From a methodological point of view, the preference model more concordant with Benjamin Franklin's suggestion is given by non-compensatory preference structures, introduced in the seventies ([19]). The definition considers a (strict) preference between actions *a* and *b* based on the comparison between the *importance of the criteria* in favor of *a*, on the one side, and the importance of the criteria in favor of *b*, on the other.

However, many MCDA methods which build a comprehensive relation of preference between pairs of actions (in particular the ELECTRE family methods, [36]), are based on another idea: *a* is considered to be "at least as good as" *b* if there are enough important criteria concordant with this assertion and there is no criterion among the discordant criteria for which *b* is so strongly preferred to *a* that it makes a veto to this assertion. In this second approach, we can observe that when evaluating if *a* is "at least as good" as *b* the arguments in favor are always considered, but the arguments against are only taken into account if they provoke and extreme negative effect on the above assertion.

A first observation is thus that, if we want to take into account the suggestion of Benjamin Franklin regarding the reasons in favor and the reasons against, when comparing actions *a* and *b*, we should consider in all the available methodologies both, i.e. the criteria in favor and the criteria against. This is also consistent with the suggestions of some old and recent works within MCDA focusing on the idea that the pairwise comparisons are based on the reasons in favor as well as on the reasons against ([20], [29], [34], [43], [44]).

However, the extension to all the MCDA methodologies of the comparison of the reasons in favor and against is important but it is only a part of the question. Indeed, after *recognizing the criteria in favor and the criteria against* of the preference of *a* over *b*, there is the very tricky question of comparing them. In this second step, some important observations must be taken into account.

One element that should be considered is the *synergy* or the *redundancy* of criteria in favor.

As an example of criteria which create a synergy effect, let us consider the case of risk evaluation of two firms *a* and *b*. Let us consider the case in which firm *a* is less indebted than *b* because its debt ratio, given by total debt on total assets, is smaller than the corresponding ratio of *b*. Moreover, the growth rate of *a* is also higher than the growth rate of *b*. In this case the two criteria, debt ratio (the less the better) and growth rate (the higher the better) are in favor of *b* and reinforce each one. Indeed, in general, there is no relation between the debt ratio and the growth rate of a firm. Therefore, intuitively, the preference of *a* over *b* is stronger than the sum of the preferences of *a* over *b* with respect to the debt ratio and the growth rate. This means that there is a synergy effect of debt ratio and growth rate.

As an example of redundant criteria consider the case of a car *a* having a maximum speed and acceleration better than a car *b*. Since one can expect that if the maximum speed of car *a* is higher than the maximum speed of car *b*, the acceleration of *a* is also higher than the acceleration of *b*. Intuitively, the preference of car *a* over car *b* is smaller than the sum of the preferences of *a* over *b* with respect to maximum speed and acceleration. This means that there is a redundancy effect of both criteria, the maximum speed and the acceleration.

Of course there are similar effects of synergy and redundancy regarding the criteria against the comprehensive preference of *a* over *b*.

To model the synergy and redundancy of criteria in favor and against, the typical tool which has been recently strongly developed within an MCDA framework are the non-additive integrals (called also fuzzy integrals). The most important of these integrals is the Choquet integral ([12]). The Choquet integral allows to calculate a specific weighted-sum which takes into account the possibility that the weight representing the importance of a set of criteria C is different from the sum of the weights of all the single criteria from C. Thus in the example given above, we can have 0.4 as the value of the weight of maximum speed and also as the weight of acceleration; however, the weight of maximum speed and acceleration together is 0.6, which is smaller than the sum of the weights of the two criteria considered separately. Similarly, in the other example we can have 0.3 as the value of the weight of both the criterion "debt ratio" and the criterion "growth rate". However, the weight of debt ratio and growth rate together is "0.8" which is larger than the sum of the two criteria considered separately.

The interactivity between criteria for the reasons in favor and the reasons against is very important for the construction of realistic preference models for multiple criteria comparisons. Anyway, there are other aspects of the comparison of criteria in favor and criteria against which must be examined.

One of these is the fact that we can have *different types of interactivity between criteria* in favor and criteria against.

Let us consider again the comparison of car a and car b with respect to maximum speed and acceleration. If car a performs worse than car b with respect to both maximum speed and acceleration, then the "negative preference" of a over b is larger than the sum of the negative preferences with respect to maximum speed and acceleration considered separately. This can be explained as follows: the negative preference with respect to maximum speed and acceleration reinforce each one, because in this case there is no possibility to compensate the loss of one criterion with a gain on the evaluation of the other. Therefore the final result of negative preferences of maximum speed and acceleration together is a synergy effect. What should be noticed in this case, is that while there is the perception of a redundancy regarding the positive preferences, there is, on the contrary, a synergy effect regarding the negative preferences.

In order to model these different interactions on the positive and the negative sides, we have to calculate two Choquet integrals, one for the positive preferences and the other for the negative preferences. To take into account the redundancy for the positive preferences and the synergy for the negative preferences we can continue to use the above weights of 0.4 for maximum speed as well as for the acceleration when considered separately. However, when maximum speed and acceleration are taken into account together for the positive preferences we consider a comprehensive weight of 0.6 (which is smaller than the sum of the weights of the two single criteria), while for the negative preferences we consider a comprehensive weight of 0.9 (which is larger than the sum of the weights of the two single criteria). This idea of considering different Choquet integrals for the positive and the negative sides is consistent with the Cumulative Prospect Theory ([45]), which is one of the most adopted models to represent decisions under uncertainty.

The above cases of interactions do not completely represent the field of all the possible interactions relative to comparisons of criteria in favor and criteria against.

Let us continue to consider the case of the comparison of car a and car b with respect to maximum speed and acceleration. Let us suppose that car a performs better than car b with respect to maximum speed but it is worse "with the same intensity" with respect to the acceleration. If we consider the above weights (0.4 for maximum speed as well as for the acceleration), then the fact that the two criteria are equally important when considered separately could suggest that the two cars are indifferent. However, the DM can feel that when there are opposite preferences with respect to maximum speed and acceleration, he/she chooses the car which is preferred with respect to acceleration. This type of interaction related to the effects of the opposing criteria cannot be represented using the two Choquet integrals, one for the positive preferences and the other for the negative preferences, because they do not take into account the fact that *the importance* *of criteria depends also on the criteria which are opposed* to them. In these cases a new, more general extension of the Choquet integral must be considered ([24], [25]). The bi-polar Choquet integral gives a specific weighted average where the criteria are weighted by means of a bi-polar capacity which gives a numerical evaluation of the comparison of the set of criteria in favor (expressing positive preferences) opposed to the set of criteria against (thus expressing negative preferences).

The use of the Choquet bi-polar integral in MCDA ([24]) is based on the idea of calculating a multiple criteria utility function that aggregates evaluations from different points of view which are under or above a neutral point. In our opinion, this specific use of the bi-polar Choquet integral has a quite problematic aspect: what is the neutral point? How to determine it? Is it reasonable to ask directly to the DM what the neutral point is?

The use of the bi-polar Choquet integral that we are proposing in this paper avoids the problem concerning the nature and the determination of the neutral point. Indeed, in multiple criteria pairwise comparisons of actions, the neutral point is clearly the indifference on the criterion considered, and this is natural and immediate for the DM. These considerations allow us to conclude that the context of the multiple criteria pairwise comparison of actions is the most adequate framework to the bi-polar Choquet integral. In this sense the bi-polar Choquet integral is a fundamental tool to deal with important aspects of the problem: how can we compare positive and negative reasons in multiple criteria decisions? A confirmation in this direction comes from a recent result ([27]) that proved that to consider a non-compensatory preference structure is equivalent to weighting criteria by means of a bi-polar capacity. From this point of view, the methodology we are proposing in this paper corresponds to an extension of the non-compensatory approach. Indeed, in the non-compensatory approach only the sign of the preferences is considered: positive, negative, or neutral. In this paper we consider not only the sign, but also the intensity of these preferences. From this point of view, our paper presents a methodology consistent with a very general non-additive and non-transitive model of conjoint measurement presented in ([7]). This model represents the multiple criteria comparisons of two alternatives as an aggregation of the positive and the negative preferences with respect to the considered criteria. Our methodology is a specific one, but a very general case of the non-additive and non-transitive model of conjoint measurement, in which the preferences on each single criterion are aggregated using the bi-polar Choquet integral. Let us also observe that our approach permits to decompose the comprehensive preferences into the difference of the comprehensive positive preferences and the comprehensive negative preferences. This is concordant with the non-transitive and non-additive conjoint measurement model of four-valued outranking presented in ([26]).

In this paper, besides the reformulation of some well-known methods (PROMETHEE and ELECTRE) in terms of a comparison of the criteria in favor and the criteria against by means of the bi-polar Choquet integral, we consider an important question related to the real-world applications: how is it possible to determine the bi-polar capacity, i.e. the non-additive weights, necessary to calculate the bi-polar Choquet integral? We know that even in the case of the additive weights, inferring these preferential parameters is a very complex and difficult task, because of the cognitive efforts that it is required from DMs. Moreover, the bi-polar capacity gives a weight to each pair of disjoint subsets of criteria. This means that if the number of criteria is n, then 3^n is the number of weights to be determined: this is a huge number. With respect to the use of Choquet integral a quite effective methodology to determine the weights has been proposed in ([30]). It is based on a linear programming model which determines non-additive weights which are concordant with some qualitative preferential information provided by the DM about the importance of and the interactivity between the criteria considered (the same kind of approach was applied to an outranking method, ELECTRE TRI [31]). An important aspect of this method of inferring non-additive weights is that a $2-$order capacity ([22]) is considered. From an intuitive point of view a $2-$order capacity means that only interaction between pairs of criteria is considered, while it is not taken into account the interaction between triplets, quadruplets and in general $k-$tuples of criteria with $k > 2$. A similar approach is also very important for the bi-polar Choquet integral. The problem, however, is the following: how can we define a 2-order bi-capacity? In this paper we propose a specific definition of a 2-order bi-capacity which permits a meaningful and manageable extension for determining non-additive weights to the bi-polar case of the method proposed in [30]. The specific decomposition of the bi-capacity derived from the considered 2-order bi-capacity is very meaningful and can be related with some interesting works on the notion of the relative importance of criteria ([39]).

This paper is organized as follows. Section 2 presents an illustrative example. Section 3 contains the elementary definitions and notation with respect to outranking methods. Sections 4 and 5 are devoted to PROMETHEE and ELECTRE methods. Section 6 is dedicated to the Choquet integral and several variants. Section 7 and 8 deal with two new variants of PROMETHEE and ELECTRE methods. Section 9 is devoted to the problem of the determination of the coefficients of the relative importance of criteria, the interaction between criteria and the power of the opposing criteria in PROMETHEE and ELECTRE methods according to the new concepts introduced in the previous sections. Section 10 concludes the paper and presents some avenues for future research.

2 Grading students: An illustrative example

In this section we present an example which is so common in the real-world of academics at Universities, Colleges, and High Schools. It was inspired on the example presented in [24].

In the example below, when comparing two actions *a* and *b*, if the action *a* is better than *b*, we also try to measure in some sense the intensity of that preference. This is very close to the MACBETH way of assessing preferences ([4], [5]), where the attractiveness between *a* and *b* can be modelled according to several different levels: *very weak, weak, moderate, strong, very strong, extreme*. In what follows, we will use mainly three levels: weak, moderate and strong.

2.1 Problem description

Let us consider the problem of evaluating High School students according to their grades in Mathematics, Physics and Literature.

The director thinks that scientific subjects (Mathematics and Physics) are more important than Literature. However, when students *a* and *b* are compared, if *a* is better than *b* both at Mathematics and Physics but *a* is much worse than *b* at Literature, then the director has some doubts about the comprehensive preference of *a* over *b*.

Mathematics and Physics are in some sense *redundant* with respect to the comparison of students, since usually students which are good at Mathematics are also good at Physics. As a consequence, if *a* is better than *b* at Mathematics, the comprehensive preference of the student *a* over the student *b* is stronger if *a* is better than *b* at Literature rather than if *a* is better than *b* at Physics.

According to this reasoning, the director considers the following two intuitive rules.

R1. Let us consider a student *a weakly* better than a student *b* with respect to Mathematics. In this case, to have a *moderate* comprehensive preference of the student *a* over the student *b*, the director wants a *strong* preference of *a* over *b* in Physics even if it is opposed to a *moderate* preference of *b* over *a* in Literature. On the contrary, a *moderate* preference in Physics even if it is opposed to a *weak* preference in Literature is not enough to have *moderate* comprehensive preference.

R2. Let us consider a student *a moderately* better than a student *b* with respect to Mathematics. In this case, to have a *strong* comprehensive preference of the student *a* over the student *b*, the director wants that *a* is not much worse than *b*, with respect to Literature. Thus, *a* is *strongly* preferred to *b* when there is *weak* preference of *b* over *a* in Literature, if there is also a *moderate* preference of *a* over *b* in Physics. On the contrary, a *strong* preference of *a* over *b* in Physics, if it is opposed to a *moderate* preference of b over a in Literature is not enough to have a *strong* comprehensive preference of a over b.

On the basis of these rules let us consider the students whose grades (belonging to the range $[0,20]$) are represented in Table 1.

Students	Mathematics	Physics	Literature
a_1	16	17	16
a_2	15	14	18
a_3	18	18	14
a_4	17	16	15
a_5	19	18	17
a_6	17	15	19
a_7	18	20	15
a_8	16	18	16

Table 1: Evaluations of the students

2.2 An intuitive representation of valued preferences

Let us consider the following formulation of the preference of a over b with respect to each criterion g_j, for all $j =$ (M) Mathematics, (Ph) Physics, (L) Literature.

$$P_j(a,b) = \begin{cases} 0 & \text{if} \quad g_j(b) \geq g_j(a) \\ (g_j(a) - g_j(b))/4 & \text{if} \quad 0 \leq g_j(a) - g_j(b) \leq 4 \\ 1 & \quad\ \text{otherwise} \end{cases}$$

Note that, according to the above formula if $P_j(a,b) > 0$, then $P_j(b,a) = 0$. From the values of the partial preferences $P_j(a,b)$ we obtain the positive and the negative partial preferences $P_j^B(a,b)$ with respect to each criterion g_j, for $j = M, Ph, L$. These preferences are defined as follows,

$$P_j^B(a,b) = \begin{cases} P_j(a,b) & \text{if} \quad P_j(a,b) > 0 \\ -P_j(b,a) & \text{if} \quad P_j(a,b) = 0 \end{cases}$$

Let us remark that,

1. $P_j^B(a,b) = -P_j^B(b,a)$;

2. If $P_j^B(a,b) > 0$, then we have a preference of a over b with respect to criterion g_j; and, $P_j^B(a,b)$ represents the value of this preference; this means that there is a *positive preference* of a over b whose value is $P_j^B(a,b)$;

3. If $P_j^B(a,b) < 0$, then we have a preference of b over a with respect to criterion g_j; and, $-P_j^B(b,a)$ is the value of this preference; this means that there is a *negative preference* of a over b whose value is $P_j^B(a,b)$;

4. If $P_j^B(a,b) = P_j^B(b,a) = 0$ then we have an indifference of a and b with respect to criterion g_j; this means that the preference of a over b is neither positive nor negative and thus its value is $P_j^B(a,b) = 0$; the criterion g_j is thus considered *neutral*.

Table 2 presents the values of the positive and negative preference of a over b, $P_j(a,b)$, for the following pairs of students from Table 1: (a_1,a_2), (a_3,a_4), (a_5,a_6) and (a_7,a_8). This Table presents also a verbal evaluation of the comprehensive preference of the above four pairs of students.

Students	Mathematics	Physics	Literature	Comprehensive Preferences
$P_j^B(a_1,a_2)$	0.25	0.75	-0.50	moderate
$P_j^B(a_3,a_4)$	0.25	0.50	-0.25	weak
$P_j^B(a_5,a_6)$	0.50	0.75	-0.50	moderate
$P_j^B(a_7,a_8)$	0.50	0.50	-0.25	strong

Table 2: Positive and negative preferences of the pairs (a,b)

The different levels of preference intensities, $P_j^B(a,b)$, can be interpreted as follows,

- $P_j^B(a,b) = 0$, means that a is *indifferent* to b w.r.t. the criterion g_j;
- $P_j^B(a,b) = 0.25$, means that a is *weakly preferred* to b w.r.t. the criterion g_j;
- $P_j^B(a,b) = 0.50$, means that a is *moderately preferred* to b w.r.t. the criterion g_j;

- $P_j^B(a,b) = 0.75$, means that a is *strongly preferred* to b w.r.t. the criterion g_j;
- $P_j^B(a,b) = 1$, means that a is *extremely preferred* to b w.r.t. the criterion g_j.

Note that the above relation between qualitative levels and numerical values is only used for a better understanding of the example.

2.3 Determining the comprehensive preferences

According to the above rule (**R1**), the director says that he has a comprehensive moderate preference of a_1 over a_2, while he only weakly prefers a_3 to a_4. Indeed, with respect to Mathematics, a_1 is weakly preferred to a_2 as well as a_3 is weakly preferred to a_4. Let us remark that the positive preference of a_1 over a_2 with respect to Physics is strong. In this case, even if there is also a moderate preference of a_2 over a_1 (i.e. a negative preference of a_1 over a_2) with respect to Literature, on the basis of (**R1**), the director clearly prefers a_1 to a_2. On the contrary, even if the preference of a_3 over a_4 (i.e. a negative preference of a_3 over a_4) with respect to Literature is only weak, there is also only a moderate preference of a_3 over a_4 with respect to Physics, which on the basis of (**R1**) determines only a weak preference of a_3 over a_4.

According to the above rule (**R2**), the director says that a_5 is comprehensively moderately preferred to a_6, but a_7 is comprehensively strong preferred to a_8. Indeed, with respect to Mathematics, the preference of a_5 over a_6 as well as the preference of a_7 over a_8 are moderate. However, the preference of a_6 over a_5 (i.e, a negative preference of a_5 over a_6) with respect to Literature is moderate, and therefore, on the basis of (**R2**), a_5 is considered moderately better but not strongly better than a_6, even if there is also a strong preference of a_5 over a_6 with respect to Physics. On the contrary, even if the preference of a_7 over a_8 with respect to Physics is only moderate, on the basis of (**R2**), the comprehensive preference of a_7 over a_8 is strong because with respect to Literature the preference of a_8 over a_7 (i.e. a negative preference of a_7 over a_8) is only weak.

Now we try to represent the preferences of the director using the following formulation which is concordant with the PROMETHEE method: for each pair of students (a,b), we have,

1. The positive preference of a over b,

$$P(a,b) = w_M P_M(a,b) + w_{Ph} P_{Ph}(a,b) + w_L P_L(a,b);$$

2. The negative preference of a over b,

$$P(b,a) = w_M P_M(b,a) + w_{Ph} P_{Ph}(b,a) + w_L P_L(b,a)$$

3. The comprehensive preference of a over b.

$$P^C(a,b) = -P^C(b,a) = P(a,b) - P(b,a)$$

where, w_j, $j = M, Ph, L$ is the weight of the considered criterion. Let us remark that w_j represents the *importance of the criterion* g_j and therefore the more important g_j the larger w_j. Moreover, we have that,

- $0 \leq w_j \leq 1$, for all $j = M, Ph, L$;
- $w_M + w_{Ph} + w_L = 1$.

Observe that, for all pairs of students a and b,

$$P^C(a,b) = w_M P^B_M(a,b) + w_{Ph} P^B_{Ph}(a,b) + w_L P^B_L(a,b)$$

Consequently, the comprehensive preferences of a_1 over a_2 is,

$$P^C(a_1,a_2) = w_M \times 0.25 + w_{Ph} \times 0.75 + w_L \times (-0.5)$$

Let us assume that $w_M = w_{Ph} > w_L$. Figure 1 gives a representation of the $P^C(a_1,a_2)$ as the sum of the areas of the two first rectangles (A and B), minus the area of the third one (C).

2.4 How this methodology works or does not work in our example?

This sub-section is divided into several paragraphs, which introduce in a didactic way the concepts developed later on, in a more formal way. We start by the most frequent case, where all the criteria are considered to be independent. Then, we will introduce the interactivity between criteria. Finally, we will consider the opposing power of the criteria.

2.4.1 A PROMETHEE like technique: The criteria are considered independent

To represent the preferences of the director, we should have that

(i) $P^C(a_1,a_2) > P^C(a_3,a_4)$

since a_1 is comprehensively moderately better than a_2, while a_3 is comprehensively weakly better than a_4

(ii) $P^C(a_5,a_6) < P^C(a_7,a_8)$

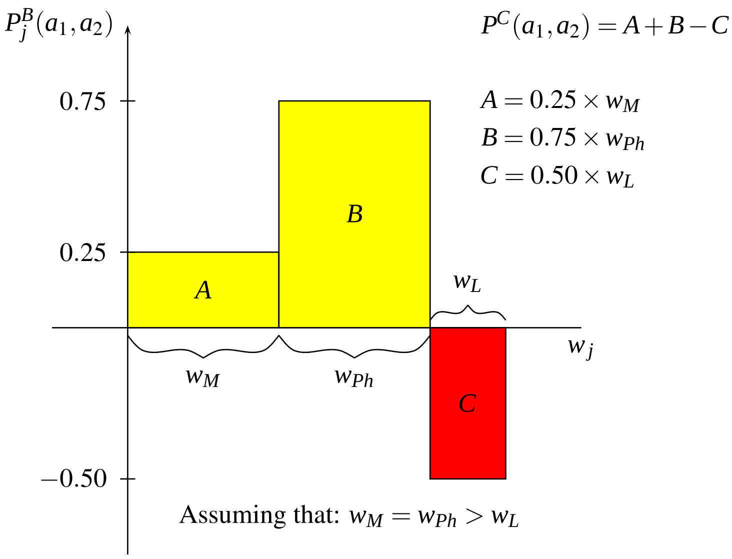

Figure 1: Geometric representation of preference aggregation with independent criteria

since a_5 is comprehensively moderately better than a_6, while a_7 is comprehensively strongly better than a_8.

From (i) we have that,

$$w_M P_M^B(a_1,a_2) + w_{Ph} P_{Ph}^B(a_1,a_2) + w_L P_L^B(a_1,a_2) >$$

$$> w_M P_M^B(a_3,a_4) + w_{Ph} P_{Ph}^B(a_3,a_4) + w_L P_L^B(a_3,a_4),$$

that is,

$$(iii) \quad w_M \times 0.25 + w_{Ph} \times 0.75 + w_L \times (-0.5) >$$
$$> w_M \times 0.25 + w_{Ph} \times 0.5 + w_L \times (-0.25).$$

which means that (iii) is equivalent to,

$$(iv) \quad w_{Ph} > w_L.$$

Analogously, from (ii) we obtain that,

$$(v) \quad w_{Ph} < w_L.$$

(iv) and (v) are clearly contradictory. Therefore (i) and (ii) cannot be satisfied for any possible values of w_M, w_{Ph}, and w_L. Thus the PROMETHEE like model cannot represent the preferences of the director. This can be explained with the observation that, according to rule **(R1)**, Physics is more important than Literature when the preference of the student a over the student b with respect to Mathematics is rather weak, while Literature is more important than Physics when the preference of student a over student b with respect to Mathematics is moderate.

2.4.2 On the interactivity between criteria

To represent the preference of the director, we can try to use the Choquet integral in order to represent the interaction between criteria. More particularly, for each pair of students (a,b), we calculate the Choquet integral of the,

- positive preferences (i.e. the positive preferences of a over b on each single subject);
- negative preferences (i.e. the negative preferences of a over b on each single subject).

This is equivalent to calculate the Sipos integral ([42]) of preferences $P_j^B(a,b)$.

The Choquet integral gives a specific weighted average in which weights relative to all possible subsets of criteria are considered. Thus, for example, $\mu(\{M,Ph\})$ is the weight expressing the importance of Mathematics and Physics when they are considered together, while $\mu(\{M\})$ and $\mu(\{Ph\})$ are the weights expressing the importance of Mathematics and Physics, respectively, when they are considered separately. In general, we can have that $\mu(\{M,Ph\}) \neq \mu(\{M\}) + \mu(\{Ph\})$. This is the case in the above example, because, due to the redundancy between Mathematics and Physics, one can expect that $\mu(\{M,Ph\}) < \mu(\{M\}) + \mu(\{Ph\})$.

The set of weights $\mu(.)$ considered to calculate the Choquet integral, are technically defined as a capacity.

We shall try to model the *importance of the criteria* and the *interaction between criteria* in the same way both for the positive and the negative preferences. This means that we shall use the same capacity for the Choquet integral of the positive preferences and the Choquet integral of the negative preferences.

To explain how the Choquet integral is calculated, let us consider the pair of student (a_1,a_2). Given a capacity μ, the positive preferences can be aggregated as follows. Let us start by ordering all the preferences from the smallest to the largest intensity, that is 0.25 for Mathematics and 0.75 for Physics. After, starting from the smallest, we observe that there is a preference of at least 0.25 for Mathematics and Physics. Therefore we multiply 0.25 by $\mu(\{M,Ph\})$ which is the weight of these two criteria considered together. Then, we can see that there is a preference of at least 0.75 only for Physics. We have, thus, an increasing of $0.75 - 0.25 = 0.50$ which is relative only to Physics. Therefore, we multiply 0.50 by $\mu(\{Ph\})$ which is the weight of Physics considered separately. Finally, we can add the two products and obtain the Choquet integral of the positive preferences of a_1 over a_2, which is,

$$\mu(\{M,Ph\}) \times 0.25 + \mu(\{Ph\}) \times 0.50$$

Now, let us calculate the Choquet integral of the negative preferences, which in our example gives in a very simple way,

$$\mu(\{L\}) \times 0.50$$

Finally, subtracting, from the Choquet integral of the positive preferences, the Choquet integral of the negative preferences, we obtain the comprehensive strength of the preferences of a_1 over a_2, which is (see also Figure 2),

$$P^C(a_1,a_2) = \Big(\mu(\{M,Ph\}) \times 0.25 + \mu(\{Ph\}) \times 0.50\Big) - \Big(\mu(\{L\}) \times 0.50\Big).$$

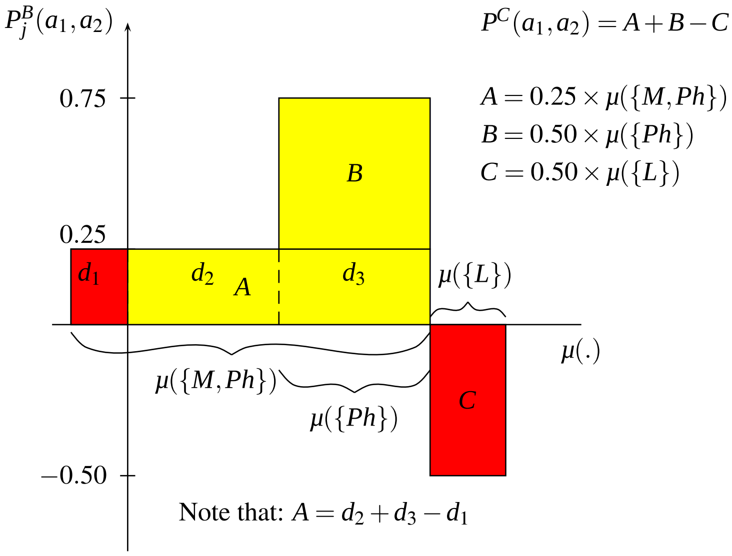

Figure 2: Preference aggregation with interaction of preferences of the same sign

The same kind of calculation can be made with respect to the other three pairs of students, then obtaining,

$$P^C(a_3,a_4) = \mu(\{M,Ph\}) \times 0.25 + \mu(\{Ph\}) \times 0.25 - \mu(\{L\}) \times 0.25;$$

$$P^C(a_5,a_6) = \mu(\{M,Ph\}) \times 0.5 + \mu(\{Ph\}) \times 0.25 - \mu(\{L\}) \times 0.50;$$

$$P^C(a_7,a_8) = \mu(\{M,Ph\}) \times 0.5 - \mu(\{L\}) \times 0.25.$$

In this context we have that the above (i), i.e.,

$$P^C(a_1,a_2) > P^C(a_3,a_4)$$

gives

$$(vi)\ \ \mu(\{M,Ph\}) \times 0.25 + \mu(\{Ph\}) \times 0.5 - \mu(\{L\}) \times 0.5 >$$
$$> \mu(\{M,Ph\}) \times 0.25 + \mu(\{Ph\}) \times 0.25 - \mu(\{L\}) \times 0.25$$

while (ii), i.e.,

$$P^C(a_5,a_6) < P^C(a_7,a_8)$$

gives

$$(vii)\ \ \mu(\{M,Ph\}) \times 0.5 + \mu(\{Ph\}) \times 0.25 - \mu(\{L\}) \times 0.5 <$$
$$< \mu(\{M,Ph\}) \times 0.5 - \mu(\{L\}) \times 0.25.$$

Form (vi) we obtain

$$(viii)\ \ \mu(\{Ph\}) > \mu(\{L\}).$$

while from (vii) we obtain

$$(ix)\ \ \mu(\{Ph\}) < \mu(\{L\})$$

It is easy to see that $(viii)$ and (ix) are clearly contradictory and this means that the Choquet integral with the same capacity on the positive preferences and on the negative preferences cannot represent the preferences of the director.

2.4.3 A more sophisticated technique is required

Therefore, we try to represent the preference of the director using the Choquet integral with different capacities for the positive and the negative preferences. In this way we try to represent a different importance and interaction of criteria for the positive and the negative preferences.

This means that we shall use the capacity μ^+ to calculate the Choquet integral of the positive preferences and the capacity μ^- to calculate the Choquet integral of the negative preferences.

For example, $\mu^+(\{M,Ph\})$ is the weight of Mathematics and Physics considered together when they express positive preferences, while $\mu^-(\{M,Ph\})$ is the weight of the same criteria when they express negative preferences. Thus, we have that

$$P^C(a_1,a_2) = \mu^+(\{M,Ph\}) \times 0.25 + \mu^+(\{Ph\}) \times 0.50 - \mu^-(\{L\}) \times 0.50;$$

$$P^C(a_3,a_4) = \mu^+(\{M,Ph\}) \times 0.25 + \mu^+(\{Ph\}) \times 0.25 - \mu^-(\{L\}) \times 0.25;$$

$$P^C(a_5,a_6) = \mu^+(\{M,Ph\}) \times 0.5 + \mu^+(\{Ph\}) \times 0.25 - \mu^-(\{L\}) \times 0.50;$$

$$P^C(a_7,a_8) = \mu^+(\{M,Ph\}) \times 0.5 - \mu^-(\{L\}) \times 0.25.$$

In this context we have that the above (i) gives,

$$(x) \quad \mu^+(\{M,Ph\}) \times 0.25 + \mu^+(\{Ph\}) \times 0.5 - \mu^-(\{L\}) \times 0.5 >$$
$$> \mu^+(\{M,Ph\}) \times 0.25 + \mu^+(\{Ph\}) \times 0.25 - \mu^-(\{L\}) \times 0.25$$

while (ii) gives

$$(xi) \quad \mu^+(\{M,Ph\}) \times 0.5 + \mu^+(\{Ph\}) \times 0.25 - \mu^-(\{L\}) \times 0.5 <$$
$$< \mu^+(\{M,Ph\}) \times 0.5 - \mu^-(\{L\}) \times 0.25$$

From (x) we obtain,

$$(xii) \quad \mu^+(\{Ph\}) > \mu^-(\{L\})$$

while from (xi) we obtain

$$(xiii) \quad \mu^+(\{Ph\}) < \mu^-(\{L\}).$$

And (xii) and $(xiii)$ are clearly contradictory and this means that the Choquet integral even with the capacity on the positive preferences different from the capacity on the negative preferences cannot represent the preferences of the director.

2.4.4 The power of the opposing criteria: Criteria in favor and against

Finally, let us consider the bi-polar Choquet integral using a bi-capacity $\hat{\mu}$.

The bi-polar Choquet integral gives a specific weighted average in which weights relative to all possible pairs of disjoint subsets of criteria are considered. Thus, for example, $\hat{\mu}(\{M,Ph\},\{L\})$ is the weight concerning the difference of the importance of Mathematics and Physics, expressing positive preferences, on one side, and Literature, expressing negative preferences, on the other side.

In this way we try to represent the importance and the interaction of criteria for positive and negative preferences which depend on the fact that the criteria are in favor or against to the comprehensive preference.

To explain how the bi-polar Choquet integral is calculated, let us consider again the pair of student (a_1,a_2). Given a bi-polar capacity $\hat{\mu}$, the preferences can be aggregated as follows. Let us start by ordering all the preferences from the smallest to the largest absolute value of their intensity, that is 0.25 for Mathematics, 0.50 for Literature and 0.75 for Physics. After, starting from the smallest absolute value, we observe that there is a positive preference of at least 0.25 for Mathematics and Physics against a negative preference of at least 0.25 for Literature. Therefore we multiply 0.25 by $\hat{\mu}(\{M,Ph\},\{L\})$. Then, we can see that there is a positive preference of at least 0.50 for Physics against a negative preference of at least 0.50 for Literature. We have, thus, an increasing of $0.50-0.25=0.25$ which is relative to a positive preference of Physics against a negative preference of Literature. Therefore, we multiply 0.25 by $\hat{\mu}(\{Ph\},\{L\})$. Now, we can see that there is a positive preference of 0.75 for Physics and that this positive preference is not counterbalanced by any negative preference of the same or larger intensity. We have, thus, an increasing of $0.75-0.50=0.25$ which is relative to a positive preference on Physics against no negative preference. Therefore, we multiply 0.25 by $\hat{\mu}(\{Ph\},\emptyset)$. Finally, we can add the three products and obtain the bi-polar Choquet integral of the preferences of a_1 over a_2, which is,

$$P^C(a_1,a_2)=\hat{\mu}(\{M,Ph\},\{L\})\times 0.25+\hat{\mu}(\{Ph\},\{L\})\times 0.25+\hat{\mu}(\{Ph\},\emptyset)\times 0.25$$

The calculation is similar for the remaining pairs of students.

In this context we have that the above (i) gives,

$$(xiv)\quad \hat{\mu}(\{M,Ph\},\{L\})\times 0.25+\hat{\mu}(\{Ph\},\{L\})\times 0.25+\hat{\mu}(\{Ph\},\emptyset)\times 0.25 >$$
$$> \hat{\mu}(\{M,Ph\},\{L\})\times 0.25+\hat{\mu}(\{Ph\},\emptyset)\times 0.25$$

while (ii) gives,

$$(xv)\quad \hat{\mu}(\{M,Ph\},\{L\})\times 0.5+\hat{\mu}(\{Ph\},\emptyset)\times 0.25 <$$

$< \hat{\mu}(\{M,Ph\},\{L\}) \times 0.25 + \hat{\mu}(\{M,Ph\},\emptyset) \times 0.25.$

From (xiv) we obtain,

(xvi) $\hat{\mu}(\{Ph\},\{L\}) > 0$

while from (xv) we obtain,

$(xvii)$ $\hat{\mu}(\{M,Ph\},\{L\}) + \hat{\mu}(\{Ph\},\emptyset) < b(\{M,Ph\},\emptyset)$

(xvi) and $(xvii)$ means that, differently from the previous cases, the bi-polar Choquet integral can represent the preferences of the director. For example, a bi-polar capacity such that

$$\hat{\mu}(\{Ph\},\{L\}) = 0.1$$

$$\hat{\mu}(\{M,Ph\},\{L\}) = 0.2$$

$$\hat{\mu}(\{Ph\},\emptyset) = 0.5$$

$$\hat{\mu}(\{M,Ph\},\emptyset) = 0.8$$

satisfies conditions (xiv) and (xv) and therefore, using $\hat{\mu}$, the Choquet bi-polar integral of the valued marginal preferences $P_j^B(a,b)$ permits to represent the preferences of the director of the school.

2.4.5 On the use of a decomposition procedure

The above bi-capacity, $\hat{\mu}$, can be decomposed in a meaningful way. This decomposition splits the values of the bi-capacity in three types of components:

1. A set of components *relative to the importance of each criterion* considered separately: thus a_M, a_{PH} and a_L is the importance of Mathematics, Physics and Literature, respectively, considered separately;

2. A set of components relative to the *interaction between pairs of criteria* expressing preferences of the same sign (both positive or both negative): thus, for example, $a_{M,Ph}$ is the interaction between Mathematics and Physics;

3. A set of components relative to the net result of the *opposition of pairs of criteria* expressing preferences of different sign (the first positive, the second negative): thus, for example, $a_{M|L}$ is the net result of the opposition of Mathematics expressing a positive preference and Literature expressing a negative preference. If $a_{M|L}$ is positive, this means that the positive preference of Mathematics reduce the importance of Literature more than the negative preference of Literature reduce the importance of Mathematics. We have an opposite interpretation if $a_{M|L}$ has a negative value.

In order to have an idea of how this decomposition works, let us consider the weight concerning the difference of the importance of Mathematics and Physics, expressing positive preferences, on one side, and Literature, expressing negative preferences, on the other side, i.e. $\hat{\mu}(\{M,Ph\},\{L\})$.

We can decompose $\hat{\mu}(\{M,Ph\},\{L\})$ as follows,

$$\hat{\mu}(\{M,Ph\},\{L\}) = a_M + a_{Ph} - a_L + a_{M,Ph} + a_{M|L} + a_{Ph|L},$$

that is,

1. The sum of the components relative to the importance of each criterion expressing positive preference ($a_M + a_{Ph}$);

2. Minus the sum of the components relative to the importance of each criterion expressing negative preference (a_L);

3. Plus the sum of all the components relative to the interaction between pairs of criteria expressing positive preferences ($a_{M,Ph}$);

4. Minus the sum of all the components relative to the interaction between pairs of criteria expressing negative preferences (in this example there is no such a kind of components);

5. Plus the components relative to the net result of the opposition of pair of criteria expressing preferences of different sign ($a_{M|L} + a_{Ph|L}$).

Figure 3 gives a representation of $\hat{\mu}(\{M,Ph\},\{L\})$.

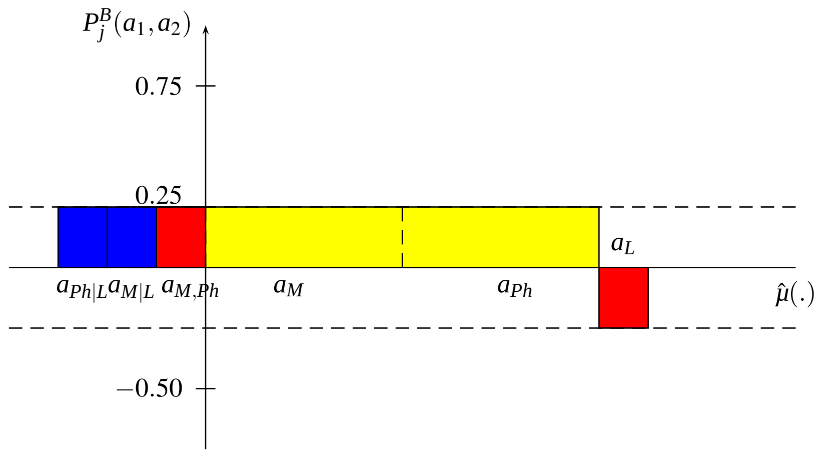

Figure 3: The power of opposing criteria: $\hat{\mu}(\{M,Ph\},\{L\})$

In this sense conditions (xvi) and $(xvii)$ can be rewritten as follows,

$(xviii)$ $\hat{\mu}(\{Ph\},\{L\}) = a_{Ph} - a_L + a_{Ph|aL} > 0$

(xix) $\hat{\mu}(\{M,Ph\},\{L\}) + \hat{\mu}(\{Ph\},\emptyset) - \hat{\mu}(\{M,Ph\},\emptyset) =$

$= (a_M + a_{Ph} + a_{M,Ph} - a_L + a_{Ph|aL} + a_{M|aL}) + a_{Ph} - (a_M + a_{Ph} + a_{M,Ph}) =$

$= -a_L + a_{Ph|aL} + a_{M|aL} + a_{Ph} > 0.$

On the basis of $(xviii)$ and (xix) we can conclude that a possible decomposition of the above bi-polar capacity $\hat{\mu}$ is the following:

$$a_M = a_{Ph} = 0.5, a_L = 0.2, a_{M,Ph} = a_{Ph|L} = a_{M|L} = -0.2$$

This decomposition gives an intuitive interpretation of the importance, the interactions and the power of opposing criteria of the director:

1. $a_M = a_{Ph} = 0.5$, $a_L = 0.2$ means that Mathematics and Physics are clearly more important than Literature;
2. $a_{M,Ph} = -0.2$ means that there is a certain redundancy of Mathematics and Physics;
3. $a_{Ph|L} = a_{M|L} = -0.2$ means that Literature has a certain opposition power with respect to Mathematics and Physics.

3 Elementary definitions and notation concerning outranking methods

The elementary data consist of a set of actions and a set of criteria and consequently an evaluation table or matrix where each action is evaluated on each criterion. Let,

- $A = \{a_1, \ldots, a_i, \ldots, a_m\}$, denote a finite set of actions with $|A| = m$;
- $F = \{g_1, \ldots, g_j, \ldots, g_n\}$, denote a finite set of criteria with $|F| = n$;
- $\mathcal{I}$, denote the set of the actions indices;
- $\mathcal{J}$, denote the set of the criteria indices;
- $g_j(a_i)$, denote the evaluation of action g_j on criterion g_j.

When comparing, in a comprehensive way, two actions a and b according to all the criteria considered together, several situations may occur,

1. a is *strictly preferred* to b, ($a\mathcal{P}b$), if there are enough reasons to justify the preference of a over b. An analogous interpretation can be applied when b is preferred to a.
2. a is *indifferent* to b, ($a\mathcal{I}b$), if there are enough reasons to justify an indifference between both actions.
3. a is *weakly preferred* to b, ($a\mathcal{Q}b$), if there is an hesitation between strictly preference and indifference.
4. a is *incomparable* to b, $a\mathcal{R}b$, if there are no reasons which permit us to define a preference (strict or weak) or an indifference of a over b or *vice-vera*.

It should be remarked that in outranking methods the expression *weakly preferred* means an hesitation while in the example presented in the previous section it was related with the intensity of preference of a over b.

In outranking methods and in particular in ELECTRE methods, when comparing a and b, it is frequent to define an outranking relation aSb, which means that a is ”at least as good” as b.

Definition 1 [outranking relation] (Roy, 1974 [35]).
An outranking relation is a binary relation defined in A such that aSb if, given what is known about the decision maker's preferences and given the quality of the performances of the actions and the nature of the problem, the arguments to decide that a is at least as good as b, while there is no essential argument to refute that statement.

In addition to the basic data, we need to introduce some preferential parameters. Let,

- w_j, denote the *relative importance coefficient* attached to criterion g_j;
- q_j (p_j), denote the *indifference (preference) threshold* for criterion g_j;
- v_j, the *veto threshold* with respect to criterion g_j.

This thresholds may be constant or vary with the evaluations of the actions along the scale of each criterion.

Outranking methods are decomposed in two main phases: construction of one or several outranking relation(s) followed by an exploitation procedure.

1. The construction of one or several outranking relation(s) aims at comparing in a comprehensive way each pair of actions.
2. The exploitation procedure is used to elaborate recommendations from the results obtained in the first phase. The nature of the recommendations obviously depends on the problem statement.

For more details on outranking methods the reader can consult the following references: [15], [37], [38], [46].

4 The PROMETHEE method

PROMETHEE was created in the ealier eighties. It was particularly designed for ranking actions from the best to the worst option ([8], [11]). Over the last two decades, PROMETHEE was applied to a huge variety of real-world decision making situations ([10]). It is perhaps the most simple and intuitive outranking method. A comprehensive description of PROMETHEE can be found in [9].

4.1 The construction of an outranking relation

As an outranking method PROMETHEE preference structures are based on pairwise comparisons. When comparing two actions a and b on criterion g_j the difference of evaluations or preferences between these two actions should be taken into account. Assuming that, $g_j(a) \geq g_j(b)$, the difference between the evaluations of a and b on criterion g_j can be stated as follows,

$$d_j(a,b) = g_j(a) - g_j(b)$$

When the difference $d_j(a,b)$ is very small and the DM can neglect it, there is no reason to say that a is preferred to b and so the actions are indifferent. The higher the value of d_j, the larger the preference $P_j(a,b)$ in favor of a over b, on criterion g_j. This preference can be defined through a function in the following way,

$$P_j(a,b) = f_j(d_j(a,b)), \ \forall \, a,b \, \in \, A$$

and we can assume that $P_j(a,b) \in [0,1]$ and if $P_j(a,b) > 0$, then $P_j(b,a) = 0$.

The pair $(g_j, P_j(a,b))$ is called a *generalized function* associated with criterion g_j, for all $j \in \mathcal{J}$. Several types of generalized functions can be provided ([9]). In each case we may define some intra-criterion preferential parameters. The most common generalized function is the following,

$$P_j(a,b) = f_j(d_j(a,b)) = \begin{cases} 0 & \text{if} \quad d_j(a,b) \leq q_j \\ \frac{d_j(a,b)}{p_j - q_j} & \text{if} \quad q_j < d_j(a,b) \leq p_j \\ 1 & \text{if} \quad d_j(a,b) > p_j \end{cases}$$

where,

- q_j, an indifference threshold;
- p_j, a preference threshold,

In Figure 4, we present an example of a generalized function of type 5 (see [9]) requiring the definition of both q_j and p_j.

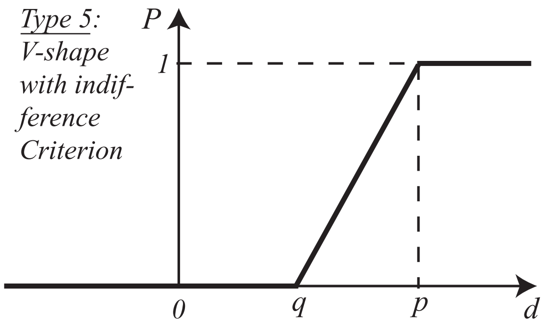

Figure 4: Generalized function

4.1.1 Comprehensive preferences

The preferences $P_j(a,b)$, for $j \in \mathcal{J}$ must be aggregated for all pairs of actions (a,b) belonging to A to obtain a comprehensive index $\pi(a,b)$ expressing the degree in which a is preferred to b, as follows,

$$\begin{cases} \pi(a,b) = \sum_{j \in \mathcal{J}} P_j(a,b) w_j \\ \pi(b,a) = \sum_{j \in \mathcal{J}} P_j(b,a) w_j \end{cases}$$

It is clear that $\pi(a,b) \in [0,1]$. When $\pi(a,b)$ is close to one, a strong preference in favor of a over b exists. On the contrary, when $\pi(a,b)$ is close to zero, a weak preference exists.

The following proprieties hold for all $a,b \in A$,

$$\begin{cases} \pi(a,a) = 0 \\ 0 \leq \pi(a,b) \leq 1 \\ 0 \leq \pi(b,a) \leq 1 \\ 0 \leq \pi(a,b) + \pi(b,a) \leq 1 \end{cases}$$

4.1.2 Positive, negative, and net flows

The fundamental idea underlying PROMETHEE methods is the quantification of how an action a *outranks* all the remaining $(m-1)$ actions and how a is *outranked* by the other $(m-1)$ actions. This idea leads to the definition of the *positive* and *negative outranking flows* as follows,

- *The positive outranking flow*:

$$\phi^{+}(a) = \frac{1}{m-1}\sum_{b\in A}\pi(a,b) = \frac{1}{m-1}\sum_{b\in A}\sum_{j\in\mathcal{J}} P_j(a,b)w_j$$

- *The negative outranking flow*:

$$\phi^{-}(a) = \frac{1}{m-1}\sum_{b\in A}\pi(b,a) = \frac{1}{m-1}\sum_{b\in A}\sum_{j\in\mathcal{J}} P_j(b,a)w_j$$

We can now define the net flow for each action $a \in A$,

$$\phi(a) = \phi^{+}(a) - \phi^{-}(a) = \frac{1}{m-1}\sum_{b\in A}\sum_{j\in\mathcal{J}}\Big(P_j(a,b) - P_j(b,a)\Big)w_j$$

4.1.3 Single criterion net flows and the profile of an action

For each action $a \in A$, it is obvious that we can also determine the net flow for each criterion separately. Let us define the net flow for criterion g_j as follows,

$$\phi_j(a) = \frac{1}{m-1}\sum_{b\in A}\Big(P_j(a,b) - P_j(b,a)\Big)$$

Consequently, the comprehensive net flow can also be defined in the following way,

$$\phi(a) = \sum_{j\in\mathcal{J}} \phi_j(a)w_j$$

The determination of $\phi_j(a)$, for all $j \in \mathcal{J}$ allow us to draw the profile of the action a.

The "quality" of a given action a can be appreciated by DMs through the definition of the profile of a. This profile is drawn from the single criterion net flow indices, $\phi_j(a)$, for all $j \in \mathcal{J}$ (see Figure 5). It expresses how an action a outranks ($\phi_j(a) > 0$) or is outranked ($\phi_j(a) < 0$) by the remaining $(m-1)$ actions in A.

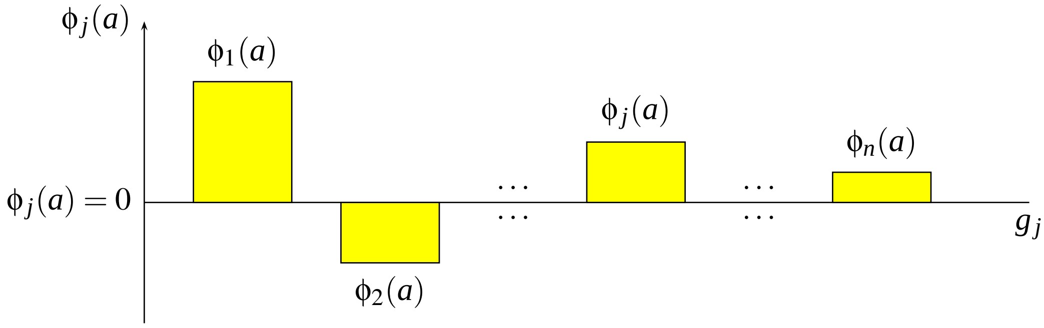

Figure 5: The single criterion net flows

4.2 The exploitation phase: Two variants of PROMETHEE

There are in fact two variants of PROMETHEE with respect to the output:

- PROMETHEE I was designed to establish a partial ranking over the set of the actions, allowing thus for the possibility to present some incomparabilities in the final ranking; and,
- PROMETHEE II was designed to determine a complete ranking.

The next two paragraphs present these two variants.

4.2.1 Identifying a partial pre-order: PROMETHEE I

The first variant of PROMETHEE, identifies a system of the form $(\mathcal{P}, I, \mathcal{R})$ comparing the actions in the following way,

$$\begin{cases} a\mathcal{P}b & \text{iff} \begin{cases} \phi^+(a) > \phi^+(b) \text{ and } \phi^-(a) < \phi^-(b), \text{ or} \\ \phi^+(a) = \phi^+(b) \text{ and } \phi^-(a) < \phi^-(b), \text{ or} \\ \phi^+(a) > \phi^+(b) \text{ and } \phi^-(a) = \phi^-(b) \end{cases} \\ aIb & \text{iff} \begin{cases} \phi^+(a) = \phi^+(b) \text{ and } \phi^-(a) = \phi^-(b) \end{cases} \\ a\mathcal{R}b & \text{iff} \begin{cases} \phi^+(a) > \phi^+(b) \text{ and } \phi^-(a) > \phi^-(b), \text{ or} \\ \phi^+(a) < \phi^+(b) \text{ and } \phi^-(a) < \phi^-(b) \end{cases} \end{cases}$$

4.2.2 Determining a complete pre-order: PROMETHEE II

This variant of PROMETHEE consists of the $(\mathcal{P}, I)$ complete ranking. It is the balance between the positive and the negative outranking flows. The higher the net flow, the better the action, so that:

$$\begin{cases} a\mathcal{P}b & \text{iff} \quad \phi(a) > \phi(b) \\ aIb & \text{iff} \quad \phi(a) = \phi(b) \end{cases}$$

In this case all the actions are comparable. No incomparabilities remain. The following properties hold:

$$\begin{cases} -1 \leq \phi(a) \leq 1 \\ \sum_{a \in A} \phi(a) = 0 \end{cases}$$

5 The ELECTRE III method

ELECTRE III was designed to improve ELECTRE II and so to deal with imperfect knowledge of data. This purpose was actually achieved and ELECTRE III was applied with success during the last two decades on a broad area of real-life applications. For a comprehensive description of ELECTRE methods the reader can consult [17], [38].

5.1 The construction of an outranking relation

The construction of an outranking relation requires the definition of a *credibility index* for the outranking relation aSb; let $\rho(aSb)$ denote this index. It is defined by using both the concordance index, $c(aSb)$, and a discordance index for each criterion g_j in F, that is, $d_j(aSb)$.

The concordance index can be presented as follows. Let us start by building the following two sets:

1. concerning the coalition of criteria in which aSb

$$\mathcal{J}^S = \left\{ j \in \mathcal{J} \, : \, g_j(a) + q_j\big(g_j(a)\big) \geq g_j(b) \right\}$$

2. concerning the coalition of criteria in which bQa

$$\mathcal{J}^Q = \left\{ j \in \mathcal{J} \, : \, g_j(a) + q_j\big(g_j(a)\big) < g_j(a) \leq g_j(b) + p_j\big(g_j(b)\big) \right\}$$

Let us define $\varphi_j(a,b)$ as follows,

$$\varphi_j(a,b) = \begin{cases} 1 & \text{if} \quad j \in \mathcal{J}^S \\ \frac{g_j(a)+p_j\big(g_j(a)\big)-g_j(b)}{p_j\big(g_j(a)\big)-q_j\big(g_j(a)\big)} & \text{if} \quad j \in \mathcal{J}^Q \\ 0 & \quad \text{otherwise} \end{cases}$$

the coefficient φ_j decreases linearly from 1 to 0, when g_j describes the range $[g_j(a) + q_j\big(g_j(a)\big),\ g_j(a) + p_j\big(g_j(a)\big)]$.

The concordance index will be defined in the following way,

$$c(aSb) = \sum_{j \in \mathcal{J}} w_j \varphi_j(a,b)$$

The discordance of a criterion g_j aims to take into account the fact that this criterion is more or less discordant with the assertion *aSb*. The discordance index reaches its maximal value when criterion g_j puts its veto to the outranking relation; it is minimal when the criterion g_j is not discordant with that relation. To define the value of the discordance index on the intermediate zone we simply admitted that this value grows in proportion to the difference $g_j(b) - g_j(a)$. This index can be presented as follows:

$$d_j(aSb) = \begin{cases} 1 \ \text{ if } \ g_j(b) > g_j(a) + v_j(g_j(a)) \\ 0 \ \text{ if } \ g_j(b) \le g_j(a) + p_j(g_j(a)) \\ \frac{g_j(b)-g_j(a)-p_j(g_j(a))}{v_j(g_j(a))-p_j(g_j(a))}, \ \text{ otherwise} \end{cases}$$

Now, we can define the credibility index,

$$\rho(aSb) = c(aSb) \prod_{\{j \in \mathcal{J}\ :\ d_j(aSb) > c(aSb)\}} \frac{1-d_j(aSb)}{1-c(aSb)}$$

Notice that, when $d_j(aSb) = 1$, it implies that $\rho(aSb) = 0$, since $c(aSb) < 1$.

5.2 The exploitation phase

In ELECTRE III, a partial pre-order Z is built as the intersection of two complete pre-orders, Z_1 and Z_2, which are obtained according to two variants of the same principle, both acting in an antagonistic way on the floating actions. The partial pre-order $\underline{Z}_1$ is defined as a partition of the set A into q ordered classes, $G_1, \ldots, G_h, \ldots, G_q$, where G_1 is the head-class in Z_1. Each class $\overline{G}_h$ is composed by *ex æquo* elements according to Z_1. The complete pre-order Z_2 is determined in a similar way, where A is partitioned into u ordered classes, $\mathrm{G}_1,\ldots,\mathrm{G}_h,\ldots,\mathrm{G}_u$, G_u being the head-class. Each one of these classes is obtained as a final distilled of a distillation procedure.

The procedure designed to compute Z_1 starts (first distillation) by defining an initial set $D_0 = A$; it leads to the first final distilled $\overline{G}_1$. After getting $\overline{G}_h$, in the distillation $h+1$, the procedure sets $D_0 = A\backslash(\overline{G}_1 \cup \ldots \cup \overline{G}_h)$. The actions in class $\overline{G}_h$ are, according to Z_1, preferable to those of class $\overline{G}_{h+1}$; for this reason, distillations that lead to these classes will be called as descendants.

The procedure leading to Z_2 is quite similar, but now the actions in $\overline{G}_{h+1}$ are preferred to those in class $\overline{G}_h$; these distillations will be called ascendants.

The partial pre-order Z will be computed as the intersection of Z_1 and Z_2.

6 The Choquet integral and other generalizations as aggregation functions

To represent the interaction between criteria we give, in this section, some elementary concepts concerning Choquet and bi-polar Choquet integrals. Moreover, we also introduce a new generalization of the bi-polar Choquet integral, the bi-polar Choquet bi-integral, which is a particular case of a new class of aggregation functions, the bi-aggregation functions. We recall a characterization of the Choquet and the bi-polar Choquet integrals. Finally, we give an axiomatic characterization of the bi-polar Choquet bi-integral.

6.1 The Choquet integral

Given a finite set $\mathcal{J} = \{1,2,\ldots,n\}$, a capacity, μ, on this set (also called fuzzy measure) is a set function of the form,

$$\mu \,:\, 2^{\mathcal{J}} \rightarrow [0,1]$$

such that,

1. $\mu(\emptyset) = 0$;
2. $\mu(\mathcal{J}) = 1$;
3. $\mu(C) \geq \mu(D)$ if $C \supseteq D, \;\forall\; C,D \in \mathcal{J}$.

The properties 1) and 2) are the boundary conditions, while the property 3) is the monotonicity condition.

Consider the $n-$dimensional vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n_+$ (all the components of the vector are positive)

The Choquet integral ([12]) of x with respect to μ can be stated as follows,

$$Ch(x,\mu) = \sum_{j \in \mathcal{J}} (x_{(j)} - x_{(j-1)})\mu(C_{(j)})$$

where,

- the index $(.)$ indicates a permutation of the elements of $\mathcal{J}$ such that,

$$x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(j)} \leq \dots \leq x_{(n)},$$

- $C_{(j)} = \{(j), \dots, (n)\}$, and
- $x_{(0)} = 0$.

The Choquet integral can also be re-written as follows,

$$Ch(x,\mu) = \sum_{j \in \mathcal{J}} x_{(j)} \Big(\mu(C_{(j)}) - \mu(C_{(j+1)})\Big)$$

where, $C_{(n+1)} = \emptyset$.

Example 1. The most interesting applications of the Choquet integral in real-world problems have been based on the $2-$order capacity model proposed in ([22]). It considers the class of capacities which can be represented in the following way,

1. $\mu(\{j\}) = a_j, \ \forall j \in \mathcal{J};$
2. $\mu(\{j,k\}) = a_j + a_k + a_{jk}, \ \forall \ \{j,k\} \subseteq \mathcal{J};$
3. $\mu(C) = \sum_{j \in C} a_j + \sum_{\{j,k\} \subseteq \mathcal{J}} a_{jk}, \ \forall C \subseteq \mathcal{J}, \ |C| \geq 2$

It is possible to prove that in this case the Choquet integral can be rewritten as,

$$Ch(x,\mu) = \sum_{j \in \mathcal{J}} a_j x_j + \sum_{\{j,k\} \subseteq \mathcal{J}} a_{jk} \min\{x_j, x_k\}, \ \forall x \in \mathbb{R}^n_+.$$

6.2 The bi-polar capacity

Let $P(\mathcal{J})$ denote a set of pairs of subsets of $\mathcal{J}$, defined as follows ([25]),

$$P(\mathcal{J}) = \{(C,D) \,:\, C \subseteq \mathcal{J},\, D \subseteq \mathcal{J},\, C \cap D = \emptyset\}$$

A bi-polar capacity, μ, on $\mathcal{J}$ is a function,

$$\mu \,:\, P(\mathcal{J}) \to [0,1] \times [0,1]$$

such that,

1. $\mu(C,\emptyset) = (c,0)$ and $\mu(\emptyset,D) = (0,d)$, with $c,d \in [0,1]$;
2. $\mu(\mathcal{J},\emptyset) = (1,0)$ and $\mu(\emptyset,\mathcal{J}) = (0,1)$.
3. For each (C,D), $(E,F) \in P(\mathcal{J})$, such that $C \supseteq E$ and $D \subseteq F$, we have $\mu(C,D) = (c,d)$ and $\mu(E,F) = (e,f)$, $c,d,e,f \in [0,1]$, with $c \geq e$ and $d \leq f$.

The properties 1) and 2) are the boundary conditions, while the property 3) is the monotonicity condition.

Given $(C,D) \in P(\mathcal{J})$ with $\mu(C,D) = (c,d)$, we use the following notation,

- $\mu^+(C,D) = c$;
- $\mu^-(C,D) = d$

In the following we consider also a bi-capacity, $\hat{\mu}$, on the set $\mathcal{J}$, being a function ([24]),

$$\hat{\mu} \,:\, P(\mathcal{J}) \to [-1,1]$$

such that,

1. $\hat{\mu}(\emptyset,\emptyset) = 0$;
2. $\hat{\mu}(\mathcal{J},\emptyset) = 1$ and $\hat{\mu}(\emptyset,\mathcal{J}) = -1$;
3. If $C \supseteq E$ and $D \subseteq F$, then $\hat{\mu}(C,D) \geq \hat{\mu}(E,F)$,

The properties 1) and 2) are the boundary conditions, while the property 3) is the monotonicity condition.

Let us observe that from each bi-polar capacity, μ, on $\mathcal{J}$, we can obtain a corresponding bi-capacity, $\hat{\mu}$, on $\mathcal{J}$, as follows,

$$\hat{\mu}(C,D) = \mu^+(C,D) - \mu^-(C,D), \;\; \forall \;\, (C,D) \in P(\mathcal{J}).$$

For a similar concept in cooperative game theory see [6]

6.3 The bi-polar Choquet integral

For each $x \in \mathbb{R}^n$ we use the following notation,

- $x^+ = \max\{x, 0\}$, is the positive part of x, for each $x \in \mathbb{R}$;
- $x^- = \max\{-x, 0\}$, is the negative part of x, for each $x \in \mathbb{R}$;
- $x^+ = (x_1^+, \ldots, x_n^+)$, is the positive part of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$;
- $x^- = (x_1^-, \ldots, x_n^-)$, is the negative part of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$;

Given $x \in \mathbb{R}^n$ let us consider a permutation $(.)$ of the elements of $\mathcal{J}$ such that,

$$|x_{(1)}| \leq |x_{(2)}| \leq \ldots \leq |x_{(j)}| \leq \ldots \leq |x_{(n)}|,$$

For each element $j \in \mathcal{J}$ let us consider the following two subsets of $\mathcal{J}$,

1. $C_{(j)} = \{i \in \mathcal{J} \; : \; x_i \geq |x_{(j)}|\}$;
2. $D_{(j)} = \{i \in \mathcal{J} \; : \; -x_i \geq |x_{(j)}|\}$.

Now, considering a bi-polar capacity, μ, on $\mathcal{J}$, and a vector $x \in \mathbb{R}^n$, we can define its bi-polar Choquet integral of the positive part in the following way ([25]),

$$\begin{aligned} Ch^+(x, \mu) &= \sum_{j \in \mathcal{J}^>} \Big(|x_{(j)}| - |x_{(j-1)}|\Big) \mu^+(C_{(j)}, D_{(j)}) = \\ &= \sum_{j \in \mathcal{J}^>} |x_{(j)}| \Big(\mu^+(C_{(j)}, D_{(j)}) - \mu^+(C_{(j+1)}, D_{(j+1)})\Big) \end{aligned}$$

where $\mathcal{J}^> = \{j \in \mathcal{J} \; : \; |x_j| > 0\}$.

Analogously, the bi-polar Choquet integral of the negative part can be defined as follows,

$$\begin{aligned} Ch^-(x, \mu) &= \sum_{j \in \mathcal{J}^>} \Big(|x_{(j)}| - |x_{(j-1)}|\Big) \mu^-(C_{(j)}, D_{(j)}) = \\ &= \sum_{j \in \mathcal{J}^>} |x_{(j)}| \Big(\mu^-(C_{(j)}, D_{(j)}) - \mu^-(C_{(j+1)}, D_{(j+1)})\Big) \end{aligned}$$

And, finally, the bi-polar Choquet integral of $x \in \mathbb{R}^n$ with respect to μ is defined as the difference between the above positive and negative bi-polar Choquet integrals,

$$Ch^B(x, \mu) = Ch^+(x, \mu) - Ch^-(x, \mu)$$

Let us remark that, if for each $(C,D),(C,E) \in P(\mathcal{J})$, $\mu^+(C,D) = \mu^+(C,E)$ and $\mu^-(D,C) = \mu^-(E,C)$, then there exists two capacities μ^+ and μ^-, on $\mathcal{J}$ such that for each $x \in \mathbb{R}^n$,

$$Ch^B(x,\mu) = Ch(x^+,\mu^+) - Ch(x^-,\mu^-)$$

The condition $\mu^+(C,D) = \mu^+(C,E)$ means that the value given by the bi-polar capacity μ on $\mathcal{J}$ to the subset C on the left, does not depend on the subsets on the right. A similar interpretation can be done for $\mu^-(C,D) = \mu^-(C,E)$.

Moreover, if for each, $(C,D),(E,C) \in P(\mathcal{J})$,

$$\mu^+(C,D) = \mu^-(E,C)$$

then, there exists only one capacity μ on $\mathcal{J}$ such that for each $x \in \mathbb{R}^n$ (see [42]),

$$Ch^B(x,\mu) = Ch(x^+,\mu) - Ch(x^-,\mu).$$

Condition $\mu^+(C,D) = \mu^-(E,C)$ means that the value given by the bi-polar capacity, μ, on $\mathcal{J}$, to the subset C does not depend on the other subsets, when it is placed on the left in μ^+ and on the right in μ^-.

Let us remark that $Ch^B(x,\mu)$ can be formulated as follows ([24]),

$$\begin{aligned} Ch^B(x,\mu) &= \sum_{j \in \mathcal{J}^>} \Big(|x_{(j)}| - |x_{(j-1)}|\Big) \hat{\mu}(C_{(j)}, D_{(j)}) = \\ &= \sum_{j \in \mathcal{J}^>} |x_{(j)}| \Big(\hat{\mu}(C_{(j)}, D_{(j)}) - \hat{\mu}(C_{(j+1)}, D_{(j+1)})\Big) \end{aligned}$$

where,

$$\hat{\mu}(C,D) = \mu^+(C,D) - \mu^-(C,D), \;\; \forall \;\; (C,D) \in P(\mathcal{J}).$$

6.4 The generalized bi-polar capacity

Let $P^*(\mathcal{J})$ denote a set of pairs of subsets of $\mathcal{J}$, defined as follows,

$$P^*(\mathcal{J}) = \{(C,D) \; : \; C \subseteq \mathcal{J}, \, D \subseteq \mathcal{J}\}$$

We define the generalized bi-polar capacity, μ^*, on $\mathcal{J}$ any function of the form ([23]),

$$\mu^* \; : \; P^*(\mathcal{J}) \rightarrow [0,1] \times [0,1]$$

such that,

1. $\mu^*(C,\emptyset) = (c,0)$ and $\mu^*(\emptyset,D) = (0,d)$, with $c,d \in [0,1]$;
2. $\mu^*(\mathcal{J},\emptyset) = (1,0)$ and $\mu^*(\emptyset,\mathcal{J}) = (0,1)$.
3. For each (C,D), $(E,F) \in P^*(\mathcal{J})$, such that $C \supseteq E$ and $D \subseteq F$, we have $\mu^*(C,D) = (c,d)$ and $\mu^*(E,F) = (e,f)$, $c,d,e,f \in [0,1]$, with $c \geq e$ and $d \leq f$.

The properties $1)$ and $2)$ are the boundary conditions, while the property $3)$ is the monotonicity condition.

Given $(C,D) \in P^*(\mathcal{J})$ with $\mu^*(C,D) = (c,d)$, $\mu^{*+}(C,D) = c$ and $\mu^{*-}(C,D) = d$.

In the following we consider also a generalized bi-capacity, $\hat{\mu}$, on $\mathcal{J}$, being a function

$$\hat{\mu} \,:\, P^*(\mathcal{J}) \rightarrow [-1,1]$$

such that,

1. $\hat{\mu}^*(\emptyset,\emptyset) = 0$;
2. $\hat{\mu}^*(\mathcal{J},\emptyset) = 1$, and $\hat{\mu}^*(\emptyset,\mathcal{J}) = -1$;
3. If $C \supseteq E$ and $D \subseteq F$, then $\hat{\mu}^*(C,D) \geq \hat{\mu}^*(E,F)$.

The properties $1)$ and $2)$ are the boundary conditions, while the property $3)$ is the monotonicity condition.

Let us observe that from the generalized bi-polar capacity, μ^*, on $\mathcal{J}$ we can obtain a corresponding generalized bi-capacity, $\hat{\mu}^*$, on $\mathcal{J}$, as follows,

$$\hat{\mu}^*(C,D) = \mu^{*+}(C,D) - \mu^{*-}(C,D), \;\; \forall \;\; (C,D) \in P^*(\mathcal{J}).$$

6.5 The bi-polar Choquet bi-integral

For each $(x^+,x^-) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$, $x^+ = (x^+_1,\ldots,x^+_n) \in \mathbb{R}^n_+$ and $x^- = (x^-_1,\ldots,x^-_n) \in \mathbb{R}^n_+$, we call x^+ the positive part of (x^+,x^-) while x^- is its negative part.

For each $(x^+,x^-) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$, let us consider the following one-to-one correspondence,

$$\{1,\ldots,2n\} \rightarrow \mathcal{J} = \{1^+,\ldots,n^+,1^-,\ldots,n^-\}$$

such that ,

$$x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(j)} \leq \ldots \leq x_{(n)} \leq \ldots \leq x_{(2n)},$$

where,

$$x_{(j)} = \begin{cases} x_i^+ & \text{if} \quad (j) = i^+ \\ \\ x_i^- & \text{if} \quad (j) = i^- \end{cases}$$

For each $j \in \mathcal{J}$ let us consider also the following two subsets of $\mathcal{J}$:

1. $C_{(j)} = \{i \in \mathcal{J} \; : \; x_i^+ \geq x_{(j)}\}$;
2. $D_{(j)} = \{i \in \mathcal{J} \; : \; x_i^- \geq x_{(j)}\}$.

Given a generalized bi-polar capacity, μ^*, on $\mathcal{J}$, and $(x^+, x^-) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$, we can define the positive part of the bi-polar Choquet bi-integral in the following way,

$$\begin{aligned} Ch^{*+}((x^+,x^-),\mu^*) &= \textstyle\sum_{j=1}^{2n} \Big(x_{(j)} - x_{(j-1)}\Big)\mu^{*+}(C_{(j)},D_{(j)}) = \\ &= \textstyle\sum_{j=1}^{2n} x_{(j)}\Big(\hat{\mu}^{*+}(C_{(j)},D_{(j)}) - \hat{\mu}^{*+}(C_{(j+1)},D_{(j+1)})\Big) \end{aligned}$$

Analogously, the negative part of the bi-polar Choquet bi-integral can be defined as follows:

$$\begin{aligned} Ch^{*-}((x^+,x^-),\mu^*) &= \textstyle\sum_{j=1}^{2n} \Big(x_{(j)} - x_{(j-1)}\Big)\mu^{*-}(C_{(j)},D_{(j)}) = \\ &= \textstyle\sum_{j=1}^{2n} x_{(j)}\Big(\hat{\mu}^{*-}(C_{(j)},D_{(j)}) - \hat{\mu}^{*-}(C_{(j+1)},D_{(j+1)})\Big) \end{aligned}$$

And, finally, the bi-polar Choquet bi-integral of $(x^+, x^-) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$, with respect to the generalized capacity μ^* is defined as follows,

$$Ch^{B*}((x^+,x^-),\mu^*) = Ch^{*+}((x^+,x^-),\mu^*) - Ch^{*-}((x^+,x^-),\mu^*)$$

Let us remark that $Ch^{B*}((x^+,x^-),\mu^*)$, can be formulated as follows,

$$\begin{aligned} Ch^{B*}((x^+,x^-),\mu^*) &= \textstyle\sum_{j=1}^{2n} \Big(x_{(j)} - x_{(j-1)}\Big)\hat{\mu}^{*}(C_{(j)},D_{(j)}) = \\ &= \textstyle\sum_{j=1}^{2n} x_{(j)}\Big(\hat{\mu}^{*}(C_{(j)},D_{(j)}) - \hat{\mu}^{*}(C_{(j+1)},D_{(j+1)})\Big) \end{aligned}$$

where,

$$\hat{\mu}^*(C,D) = \mu^{*+}(C,D) - \mu^{*-}(C,D), \;\; \forall \;\, (C,D) \in P^*(\mathcal{J}).$$

6.6 A characterization of the Choquet and the bi-polar Choquet integrals

In what follows we remember some properties which permit to characterize the Choquet and the bi-polar Choquet integrals.

Any function of the type,

$$F\ :\ \mathbb{R}^n \rightarrow \mathbb{R}$$

is called an aggregation function.

The following properties of an aggregation function F are useful to characterize the Choquet integral as well as the bi-polar Choquet integral.

1. F is *monotonic iff*, for each $x,y \in \mathbb{R}^n$, $x \geq y \Rightarrow F(x) \geq F(y)$, where $x \geq y$ means that $x_j \geq y_j$, for all $j \in \mathcal{J}$;

2. F is *idempotent iff*, for each $x \in \mathbb{R}^n$ such that $x = (\alpha, \ldots, \alpha)$, $F(x) = \alpha$;

3. F is *positively homogeneous iff* for each $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_+$, $F(\lambda x) = \lambda F(x)$;

4. F is *co-monotonic additive iff* for each $x,y \in \mathbb{R}^n$ such that the two vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are co-monotonic, i.e, there exists a permutation $(.)$ of the elements of $\mathcal{J}$ such that $x_{(1)} \leq \ldots \leq x_{(n)}$ and $y_{(1)} \leq \ldots \leq y_{(n)}$, we have that,
$$F(x+y) = F(x) + F(y).$$

5. F is *absolutely co-monotonic co-sign additive iff* for each $x,y \in \mathbb{R}^n$ such that,
   (a) the two vectors $(|x_1|, \ldots, |x_n|)$ and $(|y_1|, \ldots, |y_n|)$ are co-monotonic, i.e., there exists a permutation $(.)$ of the elements of $\mathcal{J}$ such that $|x_{(1)}| \leq \ldots \leq |x_{(n)}|$ and $|y_{(1)}| \leq \ldots \leq |y_{(n)}|$, and
   (b) x and y are co-signed, i.e., $x_j \times y_j \geq 0$ for each $j \in \mathcal{J}$, we have
$$F(x+y) = F(x) + F(y)$$

Theorem 1. (Dellacherie, 1970, [13]; and, Schmeidler, 1986, [40])
An aggregation function $F\ :\ \mathbb{R}^n_+ \rightarrow \mathbb{R}$ is monotonic, positively homogeneous, idempotent and co-monotonic additive iff there is a capacity μ on $\mathcal{J}$ such that for each $x \in \mathbb{R}^n$, $F(x) = Ch(x,\mu)$.

Theorem 2. (Greco *et al.*, 2002, [25])
An aggregation function $F\ :\ \mathbb{R}^n \rightarrow \mathbb{R}$ is monotonic, positively homogeneous, idempotent and absolutely co-monotonic co-sign additive iff there is a bi-capacity $\hat{\mu}$ on $\mathcal{J}$ such that for each $x \in \mathbb{R}^n$, $F(x) = Ch^B(x,\hat{\mu})$.

6.7 A characterization of bi-polar Choquet bi-integral

In this sub-section we introduce a characterization of the bi-polar Choquet bi-integral. We define any function,

$$F : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}$$

as a *bi-aggregation function* ([16]).

The following properties of a bi-aggregation function F are useful to characterize the bi-polar Choquet bi-integral:

1. F is *monotonic iff* for each (x^+, x^-), $(y^+, y^-) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$,

   $$(x^+, x^-) \geq^B (y^+, y^-) \Rightarrow F(x^+, x^-) \geq F(y^+, y^-),$$

   where $(x^+, x^-) \geq^B (y^+, y^-)$ means $x^+ \geq y^+$ and $x^- \leq y^-$;

2. F is *idempotent iff*

   (a) for each $(x, 0) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ such that $x = (\alpha, \ldots, \alpha)$ and $0 = (0, \ldots, 0)$, $F(x, 0) = \alpha$, and

   (b) for each $(0, x) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ such that $x = (\alpha, \ldots, \alpha)$ and $0 = (0, \ldots, 0)$, $F(0, x) = -\alpha$;

3. F is *positively homogeneous iff* for each $(x^+, x^-) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_+$, $F(\lambda x^+, \lambda x^-) = \lambda F(x^+, x^-)$

4. F is *bi-co-monotonic additive iff* for each $(x^+, x^-), (y^+, y^-) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ such that (x^+, x^-) and (y^+, y^-) are bi-co-monotonic, i.e., there exists a one-to-one correspondence as defined in sub-section 6.5, such that $x_{(1)} \leq \ldots \leq x_{(2n)}$ and $y_{(1)} \leq \ldots | y_{(2n)}$, we have

   $$F(x^+ + y^+, x^- + y^-) = F(x^+, x^-) + F(y^+, y^-)$$

Theorem 3. *A bi-aggregation function $F : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}$ is monotonic, positively homogeneous, idempotent and bi-co-monotonic additive iff there is a generalized bi-capacity $\hat{\mu}^*$ on $\mathcal{J}$ such that for each $(x^+, x^-) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$, $F(x^+, x^-) = Ch^{B*}((x^+, x^-), \hat{\mu}^*)$.*

PROOF. Let us consider the two vectors $\mathbf{1}_C$, $\mathbf{1}_D$, for $C, D \subseteq \mathcal{J}$, defined in $\mathbb{R}_+^n$ as follows. Let $\mathbf{1}_{Cj}$, denote the j^{th} component of $\mathbf{1}_C$.

$$\mathbf{1}_{Cj} = \begin{cases} 1 & \text{if} \quad j \in C \\ \\ 0 & \text{if} \quad j \notin C \end{cases}$$

A similar definition holds for $\mathbf{1}_{Dj}$.

Now, each vector $(x^+, x^-) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ can be expressed as a linear combination of pairs of vectors of type $(\mathbf{1}_C, \mathbf{1}_D)$ as follows,

$$(i) \quad (x^+, x^-) = \sum_{j=1}^{2n} (x_{(j)} - x_{(j-1)})(\mathbf{1}_{C_{(j)}}, \mathbf{1}_{D_{(j)}})$$

Let us remark that the bi-vectors $(\mathbf{1}_{C_{(j)}}, \mathbf{1}_{D_{(j)}})$ and $(\mathbf{1}_{C_{(j')}}, \mathbf{1}_{D_{(j')}})$, for $j, j' = 1, \ldots, 2n$, in (i) are bi-co-monotonic. Therefore, since F is bi-co-monotonic additive and positively homogeneous, we have that,

$$(ii) \quad F(x^+, x^-) = \sum_{j=1}^{2n} (x_{(j)} - x_{(j-1)}) F(\mathbf{1}_{C_{(j)}}, \mathbf{1}_{D_{(j)}})$$

Due to the monotonicity of F we have that for each $(C,D), (E,F) \in P^*(\mathcal{J})$ such that $C \supseteq E$ and $D \subseteq F$, we have,

1) $F(\mathbf{1}_C, \mathbf{1}_D) \geq F(\mathbf{1}_E, \mathbf{1}_F)$

Moreover, due to the property of idempotency we have that,

2) $F(\mathbf{1}_\emptyset, \mathbf{1}_\emptyset) = 0$
3) $F(\mathbf{1}_\mathcal{J}, \mathbf{1}_\emptyset) = 1$
4) $F(\mathbf{1}_\emptyset, \mathbf{1}_\mathcal{J}) = -1$

Now we can set,

$$F(\mathbf{1}_C, \mathbf{1}_D) = \mu^*(C, D)$$

where the property 1) ensures the monotonicity and the properties 2), 3) and 4) ensure the boundary conditions of a generalized bi-capacity, $\hat{\mu}^*$, on $\mathcal{J}$. Thus, $\hat{\mu}^*$, is a generalized bi-capacity and since on the basis of (ii) we can re-write

$$(ii) \quad F(x^+, x^-) = \sum_{j=1}^{2n} (x_{(j)} - x_{(j-1)}) \hat{\mu}^*(C_{(j)}, D_{(j)})$$

we can conclude that $F(x^+, x^-)$ is a bi-polar Choquet bi-integral.

□

7 New variants of PROMETHEE

This section will introduce some material which allows to build two new variants of PROMETHEE for ranking problems. We only define the positive, negative and net flows, that is, all the concepts we need to build an outranking relation. It is easy to see how to build a partial or a complete ranking as in the classical PROMETHEE method.

7.1 The bi-polar generalized preference function

Let us define the bi-polar generalized function, for each criterion, in the following manner,

$$P_j^B(a,b) = \begin{cases} P_j(a,b) & \text{if} \quad P_j(a,b) > 0 \\ \\ -P_j(b,a) & \text{if} \quad P_j(a,b) = 0 \end{cases}$$

It is easy to see that the above function may be re-written as follows,

$$P_j^B(a,b) = P_j(a,b) - P_j(b,a).$$

Let us remark that $P_j^B(a,b) = -P_j^B(b,a)$, for all j and for all possible pairs (a,b) and (b,a).

7.2 Determining comprehensive preferences

When computing the bi-polar generalized function $P_j^B(a,b)$ for all the criteria $j \in \mathcal{J}$, the absolute values of this function should be re-order in a non-decreasing way,

$$|P_{(1)}^B(a,b)| \leq |P_{(2)}^B(a,b)| \leq \ldots \leq |P_{(j)}^B(a,b)| \leq \ldots \leq |P_{(n)}^B(a,b)|$$

The comprehensive bi-polar Choquet integral for the pair (a,b) can now be determined,

$$Ch^B(P^B(a,b),\hat{\mu}) = \sum_{j \in \mathcal{J}^>} |P_{(j)}^B(a,b)| \Big[\hat{\mu}(C_{(j)},D_{(j)}) - \hat{\mu}(C_{(j+1)},D_{(j+1)})\Big]$$

where,

- $P^B(a,b) = \Big[P_j^B(a,b),\ j \in \mathcal{J}\Big]$;
- $C_{(n+1)} = D_{(n+1)} = \emptyset$;

- $\mathcal{J}^> = \{j \in \mathcal{J} \; : \; |P^B_{(j)}(a,b)| > 0\}$;
- $C_{(j)} = \{j \in \mathcal{J}^> \; : \; P^B_j(a,b) \geq |P^B_{(j)}(a,b)|\}$;
- $D_{(j)} = \{j \in \mathcal{J}^> \; : \; -P^B_j(a,b) \geq |P^B_{(j)}(a,b)|\}$.

The value $Ch^B(P^B(a,b),\hat{\mu})$ gives the comprehensive preference of a over b and it is equivalent to $\pi(a,b) - \pi(b,a) = P^C(a,b)$ in the classical PROMETHEE method. Let us remark that it is reasonable to expect that $P^C(a,b) = -P^C(b,a)$. This leads to the following *symmetry condition*,

$$Ch^B(P^B(a,b),\hat{\mu}) = -Ch^B(P^B(b,a),\hat{\mu}).$$

The following proposition gives a necessary and sufficient condition on the bi-capacity, which permits that the above symmetry condition holds.

Proposition 1. $Ch^B(P^B(a,b)) = -Ch^B(P^B(b,a))$, *for all possible* a,b, *iff* $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$ *for each* $(C,D) \in P(\mathcal{J})$.

PROOF. Let us prove that if $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$, then $Ch^B(P^B(a,b),\hat{\mu}) = -Ch^B(P^B(b,a),\hat{\mu})$. Since $P^B_j(a,b) = -P^B_j(a,b)$ for all $j \in \mathcal{J}$, we have that,

$$|P^B_{(j)}(a,b)| = |-P^B_{(j)}(b,a)| = |P^B_{(j)}(b,a)|$$

and

(*i*) $C_{(j)}(a,b) = \{j \in \mathcal{J}^> \; : \; P^B_j(a,b) \geq |P^B_{(j)}(a,b)|\} = \{j \in \mathcal{J}^> \; : \; -P^B_j(b,a) \geq |P^B_{(j)}(b,a)|\} = = D_{(j)}(b,a)$;

(*ii*) $D_{(j)}(a,b) = \{j \in \mathcal{J}^> \; : \; -P^B_j(a,b) \geq |P^B_{(j)}(a,b)|\} = \{j \in \mathcal{J}^> \; : \; P^B_j(b,a) \geq |P^B_{(j)}(b,a)|\} = = C_{(j)}(b,a)$.

From (*i*) and (*ii*) we have that

(*iii*) $$Ch^B(P^B(a,b),\hat{\mu}) = \sum_{j \in \mathcal{J}^>} |P^B_{(j)}(a,b)| \Big[\hat{\mu}(C_{(j)}(a,b),D_{(j)}(a,b)) - \hat{\mu}(C_{(j+1)}(a,b),D_{(j+1)}(a,b))\Big] = = \sum_{j \in \mathcal{J}^>} |P^B_{(j)}(b,a)| \Big[\hat{\mu}(D_{(j)}(b,a),C_{(j)}(b,a)) - \hat{\mu}(D_{(j+1)}(b,a),C_{(j+1)}(b,a))\Big].$$

If $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$, from (*iii*) we have that,

(iv) $Ch^B(P^B(b,a),\hat{\mu}) =$

$$= \sum_{j\in\mathcal{J}^>} |P^B_{(j)}(b,a)| \Big[\hat{\mu}(C_{(j)}(b,a),D_{(j)}(b,a)) - \hat{\mu}(C_{(j+1)}(b,a),D_{(j+1)}(b,a))\Big] =$$

$$= \sum_{j\in\mathcal{J}^>} |P^B_{(j)}(b,a)| \Big[-\hat{\mu}(D_{(j)}(b,a),C_{(j)}(b,a)) + \hat{\mu}(D_{(j+1)}(b,a),C_{(j+1)}(b,a))\Big]$$

$$= -Ch^B(P^B(a,b),\hat{\mu})$$

Let us now prove that if $Ch^B(P^B(a,b),\hat{\mu}) = -Ch^B(P^B(b,a),\hat{\mu})$, then $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$. Let us consider the pair (a,b) such that,

$$P^B_j(a,b) = 1 \text{ if } j \in C \text{ and } P^B_j(a,b) = -1 \text{ if } j \in D.$$

In this case we have that $Ch^B(P^B(a,b),\hat{\mu}) = \hat{\mu}(C,D)$ and $Ch^B(P^B(b,a),\hat{\mu}) = \hat{\mu}(D,C)$. Thus if $Ch^B(P^B(a,b),\hat{\mu}) = -Ch^B(P^B(b,a),\hat{\mu})$, from (iv) we obtain that $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$ and the proof is concluded.

□

The above redefinition in bi-polar terms of $\pi(a,b) - \pi(b,a)$ leads to the following bi-polar redefinition of the net flows,

$$\phi^B(a) = \frac{1}{m-1}\sum_{b\in A} Ch^B(P^B(a,b),\hat{\mu})$$

7.3 Positive, negative, and net flows

A complete ranking can now be determined over the set of actions. But, a question remains: are we able to design a procedure which distinguishes the positive from the negative bi-polar Choquet flows? In other words, what is the corresponding expression of $\pi(a,b)$ and $\pi(b,a)$ in the bi-polar case? $\pi(a,b)$ and $\pi(b,a)$ can be redefined in a bi-polar context considering a bi-polar capacity μ.

Using μ^+ we can compute the bi-polar comprehensive positive preference of a over b, $\pi^{B+}(a,b)$, as the positive bi-polar Choquet integral of $P^B(a,b)$ as follows,

$$\pi^{B^+}(a,b) = Ch^{B+}(P^B(a,b),\mu) = \sum_{j\in\mathcal{J}^>} |P^B_{(j)}(a,b)| \Big[\mu^+(C_{(j)},D_{(j)}) - \mu^+(C_{(j+1)},D_{(j+1)})\Big]$$

Analogously, using μ^- we can compute the bi-polar comprehensive positive preference of a over b, $\pi^{B-}(a,b)$, as the positive bi-polar Choquet integral of $P^B(a,b)$ as follows,

$$\pi^{B^-}(a,b) = -Ch^{B-}(P^B(a,b),\mu) = -\sum_{j\in\mathcal{J}^>} |P^B_{(j)}(a,b)| \Big[\mu^-(C_{(j)},D_{(j)}) - \mu^-(C_{(j+1)},D_{(j+1)})\Big]$$

It is reasonable to expect that $\pi^{B^+}(a,b) = -\pi^{B^-}(b,a)$ for all possible a and b. This leads to the following symmetry condition,

$$C^{B^+}(P^B(a,b),\mu) = -C^{B^-}(P^B(b,a),\mu)$$

The following proposition gives a necessary and sufficient condition on the bi-polar capacity, which permits that the above symmetry condition holds.

Proposition 2. *$Ch^{B+}(P^B(a,b),\mu) = -Ch^{B-}(P^B(b,a),\mu)$, for all possible a,b, iff $\mu^+(C,D) = \mu^-(D,C)$ for each $(C,D) \in P(\mathcal{J})$. Moreover, in this case we also have that, $Ch^B(P^B(a,b),\mu) = -Ch^B(P^B(b,a),\mu)$.*

PROOF. Analogous to Proposition 1.

The positive and negative flows can now be determined, as follows:

1. The positive flows

$$\phi^{B+}(a) = \frac{1}{m-1}\sum_{b\in A} Ch^{B+}(P^B(a,b),\mu)$$

2. The negative flows

$$\phi^{B-}(a) = -\frac{1}{m-1}\sum_{b\in A} Ch^{B-}(P^B(b,a),\mu)$$

7.4 The bi-polar aggregation of the single criterion net flow

In this sub-section we propose a different formulation of the bi-polar net flow, which is concordant with the aggregation of the single criterion net flows of the classical PROMETHEE (see 4.1.3.).

According to this perspective, the net flow can be defined as follows,

$$\phi^{B'}(a) = Ch^B(\phi_j(a),\hat{\mu}) = Ch^B\Big(\frac{1}{m-1}\sum_{b\in A} P_j(a,b) - P_j(b,a),\hat{\mu}\Big)$$

Let us remark that, differently from the classic PROMETHEE, the bi-polar net flows $\phi^B(a)$ and $\phi^{B'}(a)$, in general, are different.

When computing the bi-polar generalized function $\phi_j^{B'}(a)$ for all the criteria $j \in \mathcal{J}$, the absolute values of this function should be re-ordered in a non-decreasing way.

$$|\phi_{(1)}^{B'}(a)| \leq |\phi_{(2)}^{B'}(a)| \leq \ldots \leq |\phi_{(j)}^{B'}(a)| \leq \ldots \leq |\phi_{(n)}^{B'}(a)|$$

The comprehensive bi-polar Choquet integral for the each a,

$$Ch^B(\phi^{B'}(a), \hat{\mu}) = \sum_{j \in \mathcal{J}^>} |\phi_{(j)}^{B'}(a)| \left[\hat{\mu}(C_{(j)}, D_{(j)}) - \hat{\mu}(C_{(j+1)}, D_{(j+1)})\right]$$

where,

- $\phi^{B'}(a) = \left[\phi_j^{B'}(a),\ j \in \mathcal{J}\right]$;
- $C_{(n+1)} = D_{(n+1)} = \emptyset$;
- $\mathcal{J}^> = \{j \in \mathcal{J}\ :\ |\phi_{(j)}^{B'}(a)| > 0\}$;
- $C_{(j)} = \{j \in \mathcal{J}^>\ :\ \phi_j^{B'}(a) \geq |\phi_{(j)}^{B'}(a)|\}$;
- $D_{(j)} = \{j \in \mathcal{J}^>\ :\ -\phi_j^{B'}(a) \geq |\phi_{(j)}^{B'}(a)|\}$.

8 A new variant of ELECTRE III

In this section we propose a new concordant index based on the concepts developed in Section 5 which leads to a new variant of ELECTRE III, but it can also be applied other ELECTRE methods, for example, ELECTRE IS, for choice problems, and ELECTRE TRI, for sorting problems. The exploitation phase is the same as in the classical ELECTRE method.

8.1 The bi-polar outranking relation

A modification of the definition of outranking in a bi-polarity context can be stated as follows.

Definition 2 [bi-polar outranking relation].
A bi-polar outranking relation is a binary relation defined in A such that aSb if, given what is known about the decision maker's preferences and given the quality of the performances of the actions and the nature of the problem, the arguments to decide that a is at least as good as b compared to the arguments to decide that a is not at least as good as b are enough, while there is no essential argument to refute that statement.

In the above definition we underlined the fragments of the definition of the bi-polar outranking relation which differs from the definition given by B. Roy (1974).

8.2 Redefining the concordance index

On the basis of the above definition, the bi-polar concordance index can be modified. Let us start by building the following four sets:

1. concerning the coalition of criteria which constitutes the arguments totally in favor of *aSb*
$$\mathcal{J}^{S^+} = \Big\{ j \in \mathcal{J} \,:\, g_j(a) + q_j\big(g_j(a)\big) \geq g_j(b) \Big\}$$
2. concerning the coalition of criteria which constitutes the arguments totally against of *aSb*
$$\mathcal{J}^{S^-} = \Big\{ j \in \mathcal{J} \,:\, g_j(a) + p_j\big(g_j(a)\big) < g_j(b) \Big\}$$
3. concerning the coalition of criteria which constitutes the arguments partially in favor of *aSb*
$$\mathcal{J}^{Q^+} = \Big\{ j \in \mathcal{J} \,:\, g_j(a) + q_j\big(g_j(a)\big) < g_j(a) \leq g_j(b) + p_j\big(g_j(b)\big) \Big\}$$
4. concerning the coalition of criteria which constitutes the arguments partially against of *aSb*
$$\mathcal{J}^{Q^-} = \Big\{ j \in \mathcal{J} \,:\, g_j(a) + q_j\big(g_j(a)\big) < g_j(a) \leq g_j(b) + p_j\big(g_j(b)\big) \Big\}$$

Let us define $\varphi_j^+(a,b)$ and $\varphi_j^+(a,b)$ as follows,

$$\varphi_j^+(a,b) = \begin{cases} 1 & \text{if} \quad j \in \mathcal{J}^{S^+} \\ \frac{g_j(a)+p_j\left(g_j(a)\right)-g_j(b)}{p_j\left(g_j(a)\right)-q_j\left(g_j(a)\right)} & \text{if} \quad j \in \mathcal{J}^{Q^+} \\ 0 & \quad \text{otherwise} \end{cases}$$

$$\varphi_j^-(a,b) = \begin{cases} 1 & \text{if} \quad j \in \mathcal{J}^{S^-} \\ 1 - \frac{g_j(a)+p_j\left(g_j(a)\right)-g_j(b)}{p_j\left(g_j(a)\right)-q_j\left(g_j(a)\right)} & \text{if} \quad j \in \mathcal{J}^{Q^-} \\ 0 & \quad \text{otherwise} \end{cases}$$

The following figure give us an idea of the representation of each function of type $\phi_j(a,b)$.

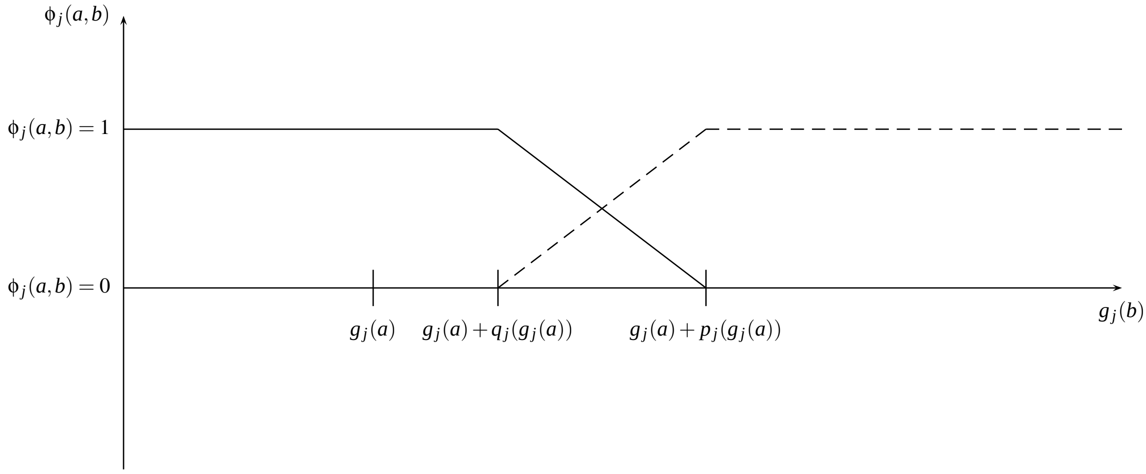

Figure 6: Reasons in favor and against w.r.t. g_j

It should be remarked, however, that in the above Figure 6 and previous definitions of ϕ_j^+ and ϕ_j^- the sets $\mathcal{J}^{Q^+}$ and $\mathcal{J}^{Q^-}$ are the same because when the arguments partially in favor of *aSb* decrease the arguments against of the same assertion increase in the same proportion. But, this is not always the case as it can be seen by observing Figure 7. We will not present more material on this topic and in what follows, for the sake of simplicity, we assume that the first situation occurs.

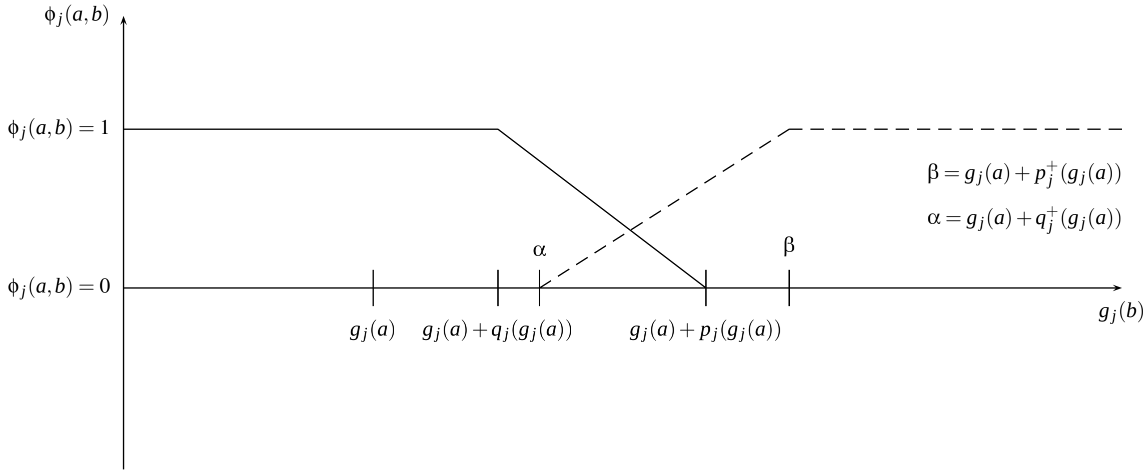

Figure 7: A more complex situation of the reasons in favor and against w.r.t. g_j

Contrary to PROMETHEE method, in ELECTRE method $\varphi_j^+(a,b)>0$ does not imply that $\varphi_j^-(a,b)=0,\ \ j\in\mathcal{J}$. Therefore, to aggregate the value $\phi_j^+(a,b)$ and $\phi_j^-(a,b)$ in a bi-polar concordance index, the bi-polar Choquet integral is not enough powerful, because it considers only a vector of values, which can be positive or negative. On the contrary, in the current situation we know that $\phi_j^+(a,b)\geq 0$ and $\phi_j^-(a,b)\geq 0$, for each $j\in\mathcal{J}$, that is, each criterion g_j can be, at the same moment, partially in favor and partially against the outranking relation.

The concordance index can now be defined as a bi-polar Choquet bi-integral,

$$c^B(a,b)=Ch^B\Big(\varphi^+(a,b),\varphi^-(a,b),\hat{\mu}^*\Big)$$

where,

$$\phi^+(a,b)=\Big[\phi_j^+(a,b),\ \ j\in\mathcal{J}\Big];$$

$$\phi^-(a,b)=\Big[\phi_j^-(a,b),\ \ j\in\mathcal{J}\Big].$$

Also with respect to ELECTRE III it can be useful to calculate the negative and the positive part, $c^{B^+}(a,b)$ and $c^{B^-}(a,b)$, of the concordance index $c^B(a,b)$. The values of $c^{B^+}(a,b)$ and $c^{B^-}(a,b)$ can be calculated as the positive and the negative parts of the bi-polar bi-integral of $[\phi^{B^+}(a,b),\phi^{B^-}(a,b)]$ with respect to the generalized bi-polar bi-capacity μ^* such that,

$$\mu^{*+}(C,D) - \mu^{*-}(C,D) = \hat{\mu}^*(C,D), \ \forall \ (C,D) \in P^*(\mathcal{J}).$$

Thus we have

$$c^{B^+}(a,b) = Ch^{B^+}\Big(\phi^+(a,b), \phi^-(a,b), \mu^*\Big)$$

and,

$$c^{B^-}(a,b) = Ch^{B^-}\Big(\phi^+(a,b), \phi^-(a,b), \mu^*\Big)$$

of course we have,

$$c^B(a,b) = c^{B^+}(a,b) - c^{B^-}(a,b).$$

In this case we do not expect any symmetry condition, because the outranking of a over b does not create any constraint on the outranking of b over a.

8.3 Comprehensive outranking and bi-aggregation functions

Let us remark that $c^B(a,b)$ is a bi-aggregation of the marginal outranking $\phi_j^+(a,b)$ and $\phi_j^-(a,b)$, for all $j \in \mathcal{J}$. The analytical formulation of $c^B(a,b)$ can shed light on this point.

Let us now consider the definition of $\phi_{(j)}(a,b)$ as follows,

$$\varphi_{(j)}(a,b) = \begin{cases} \phi_i^+(a,b) & \text{if} \quad (j) = i^+ \\ \\ \phi_i^-(a,b) & \text{if} \quad (j) = i^- \end{cases}$$

where, (j) is a permutation of $\{1^+, 2^+, \ldots, n^+, 1^-, 2^-, \ldots, n^-\}$. And, thus we have,

$$\varphi_{(1)}(a,b) \leq \varphi_{(2)}(a,b) \leq \ldots \leq \varphi_{(j)}(a,b) \leq \ldots \leq \varphi_{(n)}(a,b) \leq \ldots \leq \varphi_{(2n)}(a,b).$$

The bi-polar Choquet bi-integral can be calculated as follows,

$$Ch^B\Big(\phi^+(a,b), \phi^-(a,b), \ \mu^*\Big) = \sum_{j \in \mathcal{J}} \phi_{(j)}(a,b)\Big[\hat{\mu}^*(C_{(j)}, D_{(j)}) - \hat{\mu}^*(C_{(j+1)}, D_{(j+1)})\Big]$$

where,

1. $C_{(j)} = \{i \in \mathcal{J} \ : \ \phi_i^+(a,b) \geq \phi_{(j)}^+(a,b)\}$;
2. $D_{(j)} = \{i \in \mathcal{J} \ : \ -\phi_i^-(a,b) \geq \phi_{(j)}^+(a,b)\}$.

9 Determining the importance, the interaction and the power of the opposing criteria

Several studies dealing with the determination of the relative importance of criteria were proposed over the past two decades. A recent work [18] presents a very simple and intuitive technique (the so called "play cards technique", firstly created by J. Simos, [41]) for determining the weights of criteria for outranking methods. In this work, the criteria are considered independent. But, a recent extension has been proposed to deal with the interactivity between criteria [3]. The question of the interactivity between criteria was also studied in the context of MAUT methods ([30]). In this, section we present a quite similar technique for outranking methods, which takes into account also the power of the opposing criteria. Our method will be applied to PROMETHEE and ELECTRE, but it can also be easily extended to the MAUT like methods.

9.1 The case of PROMETHEE method

The use of the bi-polar Choquet integral is based on a bi-polar capacity which assigns numerical values to each element $P(\mathcal{J})$. Let us remark that the number of elements of $P(\mathcal{J})$ is 3^n. This means that the definition of a bi-polar capacity requires a rather huge and unpractical number of parameters. Moreover, the interpretation of these parameters is not always simple for the DM. Therefore, the use of bi-polar Choquet integral in real-world decision making problems requires some methodology assisting the DMs in assessing the preferential parameters (bi-polar capacity). Thus, in the following we consider $2-$order decomposable capacities, a particular class of bi-polar capacity.

9.1.1 Defining a manageable and meaningful bi-polar capacity measure

We define a $2-$order decomposable, the bi-polar capacity, such that

- $\mu^{+}(C,D) = \sum\limits_{j \in C} a^{+}(\{k\},\emptyset) + \sum\limits_{\{j,k\} \subseteq C} a^{+}(\{j,k\},\emptyset) + \sum\limits_{j \in C,\, k \in D} a^{+}(\{j\},\{k\})$
- $\mu^{-}(C,D) = \sum\limits_{j \in D} a^{-}(\emptyset,\{j\}) + \sum\limits_{\{j,k\} \subseteq D} a^{-}(\emptyset,\{j,k\}) + \sum\limits_{j \in D,\, k \in C} a^{-}(\{k\},\{j\})$

The interpretation of each $a^{\pm}(.)$ is the following:

- $a^{+}(\{j\},\emptyset)$, represents the power of the criterion g_j by itself; this value is always positive.

- $a^+(\{j,k\},\emptyset)$, represents the interaction between g_j and g_k, when they are in favor of the preference of a over b; when its value is zero there is no interaction; on the contrary, when the value is positive there is a synergy effect when putting together g_j and g_k; a negative value means that the two criteria are redundant.
- $a^+(\{j\},\{k\})$, represents the power of the criterion g_k against the criterion g_j, when the criterion g_j is in favor of and g_k is against to the preference of a over b; this provokes always a reduction or no effect on the value of μ^+ since this value is always non-positive.

Analogous interpretation can be applied to the value of $a^-(\emptyset,\{j\})$, $a^-(\emptyset,\{j,k\})$, and $a^-(\{k\},\{j\})$.

In what follows, for the sake of simplicity, we will use a_j^+, a_{jk}^+, $a_{j|k}^+$, instead of $a^+(\{j\},\emptyset)$, $a^+(\{g_j,g_k\},\emptyset)$, and, $a^+(\{j\},\{k\})$, respectively; and a_j^-, a_{jk}^-, $a_{j|k}^-$, instead of $a^-(\emptyset,\{j\})$, $a^-(\emptyset,\{j,k\})$, and $a^-(\{k\},\{j\})$, respectively.

The above $2-$order decomposable bi-polar capacities cannot represent all the possible preferential information relative to the interaction and the power of the opposing criteria. For example, a parameter $a_{jk|p}^+$ meaning the power of criterion g_p against criteria $\{g_j,g_k\}$ when considered together cannot be represented by these $2-$order decomposable bi-capacity, μ. However, the DM would be serious cognitive difficulties to understand the meaning of such an additional parameter. This is the reasons which limited us to use the above $2-$order of decomposition, neglecting more sophisticated, but unpractical decompositions. Anyway, the complete decomposition of the bi-polar capacity has been developed in [14].

And now, we can define $\mu(C,D)$ as follows,

$$\mu(C,D) = \mu^+(C,D) - \mu^-(C,D) = \sum_{j\in C} a_j^+ - \sum_{j\in D} a_j^- + \sum_{\{j,k\}\subseteq C} a_{jk}^+ - \sum_{\{j,k\}\subseteq D} a_{jk}^- + \sum_{j\in C,\, k\in D} a_{j|k}$$

where, $a_{j|k} = a_{j|k}^+ - a_{j|k}^-$.

There are two monotonicity conditions that should be fulfilled.

Monotonicity conditions

1. $\mu^+(C,D) \leq \mu^+(C \cup \{j\},D), \ \ \forall\, j \in \mathcal{J},\, \forall (C \cup \{j\},D) \in P(\mathcal{J})$

$$\sum_{h\in C} a_h^+ + \sum_{\{h,k\}\subseteq C} a_{hk}^+ + \sum_{h\in C,k\in D} a_{h|k}^+ \leq \sum_{h\in C\cup\{j\}} a_h^+ + \sum_{\{h,k\}\subseteq C\cup\{j\}} a_{hk}^+ + \sum_{h\in C\cup\{j\},k\in D} a_{h|k}^+$$

$$\Longleftrightarrow$$

$$a_j^+ + \sum_{k\in C} a_{jh}^+ + \sum_{k\in D} a_{j|k}^+ \geq 0, \ \ \forall\, j \in \mathcal{J},\, \forall (C \cup \{j\},D) \in P(\mathcal{J})$$

2. $\mu^+(C,D) \geq \mu^+(C,D \cup \{j\}), \ \ \forall (C,D \cup \{j\}) \in P(\mathcal{J})$

$$\sum_{h\in C} a_h^+ + \sum_{\{h,k\}\subseteq C} a_{hk}^+ + \sum_{h\in C,k\in D} a_{h|k}^+ \geq \sum_{h\in C} a_h^+ + \sum_{\{h,k\}\subseteq C} a_{hk}^+ + \sum_{h\in C,k\in D\cup\{j\}} a_{h|k}^+$$

$$\Longleftrightarrow$$

$$\sum_{h\in C} a_{h|j}^+ \leq 0, \ \ \forall\, j \in \mathcal{J},\, \forall (C,D \cup \{j\}) \in P(\mathcal{J})$$

Conditions 1) and 2) are clearly equivalent to the general monotonicity for μ^+, i.e.,

$$\forall\ (C,D),\, (E,F)\ \in\ P(\mathcal{J})\ \text{ such that }\ C \supseteq E,\ D \subseteq F,\ \mu^+(C,D) \geq \mu^+(E,F).$$

As can be seen condition 2 is always satisfied because g_k provokes always a negative effect on g_j.

The same kind of monotonicity should be satisfied for μ^-.

3. $\mu^-(C,D) \leq \mu^-(C \cup \{j\},D), \ \ \forall\, j \in \mathcal{J},\, \forall (C \cup \{j\},D) \in P(\mathcal{J})$

$$\sum_{h\in C} a_h^- + \sum_{\{h,k\}\subseteq C} a_{hk}^- + \sum_{h\in C,k\in D} a_{h|k}^- \leq \sum_{h\in C\cup\{j\}} a_h^- + \sum_{\{h,k\}\subseteq C\cup\{j\}} a_{hk}^- + \sum_{h\in C\cup\{j\},k\in D} a_{h|k}^-$$

$$\Longleftrightarrow$$

$$a_j^- + \sum_{k\in C} a_{jh}^- + \sum_{k\in D} a_{j|k}^- \geq 0, \ \ \forall\, j \in \mathcal{J},\, \forall (C \cup \{j\},D) \in P(\mathcal{J})$$

4. $\mu^-(C,D) \geq \mu^-(C,D\cup\{j\}),\ \forall (C,D\cup\{j\}) \in P(\mathcal{J})$

$$\sum_{h\in C} a_h^- + \sum_{\{h,k\}\subseteq C} a_{hk}^- + \sum_{h\in C,k\in D} a_{h|k}^- \geq \sum_{h\in C} a_h^- + \sum_{\{h,k\}\subseteq C} a_{hk}^- + \sum_{h\in C,k\in D\cup\{j\}} a_{h|k}^-$$

$$\Longleftrightarrow$$

$$\sum_{h\in C} a_{h|j}^- \leq 0,\ \ \forall\ j\in\mathcal{J},\ \forall (C,D\cup\{j\}) \in P(\mathcal{J})$$

Conditions 3) and 4) are equivalent to the general monotonicity for μ^-, i.e.,

$$\forall\ (C,D),\ (E,F)\ \in\ P(\mathcal{J})\ \text{ such that }\ C\supseteq E,\ D\subseteq F,\ \mu^-(C,D)\leq\mu^-(E,F).$$

Conditions 1), 2), 3) and 4) together ensure the monotonicity of the bi-capacity, $\hat{\mu}$, on $\mathcal{J}$, obtained as the difference of the above bi-capacities, μ^+ and μ^-, that is,

$$\forall\ (C,D),\ (E,F)\ \in\ P(\mathcal{J})\ \text{ such that }\ C\supseteq E,\ D\subseteq F,\ \hat{\mu}(C,D)\geq\hat{\mu}(E,F).$$

Boundary conditions

1. $\mu^+(\mathcal{J},\emptyset) = 1$, i.e., $\sum_{j\in\mathcal{J}} a_j^+ + \sum_{\{j,k\}\subseteq\mathcal{J}} a_{jk}^+ = 1$
2. $\mu^-(\emptyset,\mathcal{J}) = 1$, i.e., $\sum_{j\in\mathcal{J}} a_j^- + \sum_{\{j,k\}\subseteq\mathcal{J}} a_{jk}^- = 1$

We have also the symmetry condition,

Symmetry condition $\mu^+(C,D) = \mu^-(D,C)$ which means that

1. $\forall j\in\mathcal{J},\ a_j^+ = a_j^-$;
2. $\forall \{j,k\}\subseteq\mathcal{J},\ a_{jk}^+ = a_{jk}^-$;
3. $\forall j,k\in\mathcal{J},\ a_{j|k}^+ = a_{j|k}^-$.

9.1.2 The 2-order bi-polar Choquet integral

The following theorem expresses the bi-polar Choquet integral in terms of the above 2-order decomposition.

Theorem 4 *If the bi-polar capacity* μ *is* $2-$*order decomposable, then for all* $x \in \mathbb{R}^n$

$$\begin{aligned} Ch^B(x,\mu) &= \sum_{j\in\mathcal{J},x_j>0} a_j^+ x_j + \sum_{j\in\mathcal{J},x_j<0} a_j^- x_j + \\ &+ \sum_{j,k\in\mathcal{J},j\neq k,x_j,x_k>0} a_{jk}^+ \min\{x_k,x_j\} + \sum_{j,k\in\mathcal{J},j\neq k,x_j,x_k<0} a_{jk}^- \max\{x_k,x_j\} + \\ & \sum_{j,k\in\mathcal{J},x_j>0,x_k<0} a_{j|k}^+ \min\{x_j,-x_k\} + \sum_{j,k\in\mathcal{J},x_j>0,x_k<0} a_{j|k}^- \max\{-x_j,x_k\} \end{aligned}$$

PROOF. If the bi-polar capacity μ is $2-$order decomposable, then

$$\begin{aligned} Ch^B(x,\mu) &= \sum_{j\in\mathcal{J}^>} |x_{(j)}| \left[\mu(C_{(j)},D_{(j)}) - \mu(C_{(j+1)},D_{(j+1)})\right] = \\ &= \sum_{j\in\mathcal{J}^>} |x_{(j)}| \Bigg[\Bigg(\sum_{k\in\mathcal{J}^>,x_k\geq|x_{(j)}|} a_k^+ - \sum_{k\in\mathcal{J}^>,-x_k\geq|x_{(j)}|} a_k^- + \\ &+ \sum_{h,k\in\mathcal{J}^>,h\neq k,x_h,x_k\geq|x_{(j)}|} a_{hk}^+ - \sum_{h,k\in\mathcal{J}^>,h\neq k,-x_h,-x_k\geq|x_{(j)}|} a_{hk}^- + \\ & \sum_{h,k\in\mathcal{J}^>,x_h,-x_k\geq|x_{(j)}|} a_{h|k}^+ - \sum_{h,k\in\mathcal{J}^>,x_h,-x_k\geq|x_{(j)}|} a_{h|k}^- \Bigg) - \\ &- \Bigg(\sum_{k\in\mathcal{J}^>,x_k\geq|x_{(j+1)}|} a_k^+ - \sum_{k\in\mathcal{J}^>,-x_k\geq|x_{(j+1)}|} a_k^- + \\ &+ \sum_{h,k\in\mathcal{J}^>,h\neq k,x_h,x_k\geq|x_{(j+1)}|} a_{hk}^+ - \sum_{h,k\in\mathcal{J}^>,h\neq k,-x_h,-x_k\geq|x_{(j+1)}|} a_{hk}^- + \\ & \sum_{h,k\in\mathcal{J}^>,x_h,-x_k\geq|x_{(j+1)}|} a_{h|k}^+ - \sum_{h,k\in\mathcal{J}^>,x_h,-x_k\geq|x_{(j+1)}|} a_{h|k}^- \Bigg)\Bigg] = \end{aligned}$$

$$
\begin{aligned}
\chi) \quad &= \sum_{j\in\mathcal{J}^>} |x_{(j)}| \Big[\Big(\sum_{k\in\mathcal{J}^>, x_k\geq|x_{(j)}|} a_k^+ - \sum_{k\in\mathcal{J}^>, x_k\geq|x_{(j+1)}|} a_k^+ \Big) - \\
&- \Big(\sum_{k\in\mathcal{J}^>, -x_k\geq|x_{(j)}|} a_k^- - \sum_{k\in\mathcal{J}^>, -x_k\geq|x_{(j+1)}|} a_k^- \Big) + \\
&+ \Big(\sum_{h,k\in\mathcal{J}^>, h\neq k, x_h, x_k\geq|x_{(j)}|} a_{hk}^+ - \sum_{h,k\in\mathcal{J}^>, h\neq k, x_h, x_k\geq|x_{(j+1)}|} a_{hk}^+ \Big) - \\
&- \Big(\sum_{h,k\in\mathcal{J}^>, h\neq k, -x_h, -x_k\geq|x_{(j)}|} a_{hk}^- - \sum_{h,k\in\mathcal{J}^>, h\neq k, -x_h, -x_k\geq|x_{(j+1)}|} a_{hk}^- \Big) + \\
&+ \Big(\sum_{h,k\in\mathcal{J}^>, x_h, -x_k\geq|x_{(j)}|} a_{h|k}^+ - \sum_{h,k\in\mathcal{J}^>, x_h, -x_k\geq|x_{(j+1)}|} a_{h|k}^+ \Big) - \\
&- \Big(\sum_{h,k\in\mathcal{J}^>, x_h, -x_k\geq|x_{(j)}|} a_{h|k}^- - \sum_{h,k\in\mathcal{J}^>, x_h, -x_k\geq|x_{(j+1)}|} a_{h|k}^- \Big) \Big]
\end{aligned}
$$

Let us remark that,

$$
a) \quad \Big(\sum_{k\in\mathcal{J}^>, x_k\geq|x_{(j)}|} a_k^+ - \sum_{k\in\mathcal{J}^>, x_k\geq|x_{(j+1)}|} a_k^+ \Big) = \begin{cases} a_{(j)}^+ & \text{if} \quad x_{(j)} > 0 \\ \\ 0 & \text{otherwise} \end{cases}
$$

$$
b) \quad \Big(\sum_{k\in\mathcal{J}^>, x_k\geq|x_{(j)}|} a_k^- - \sum_{k\in\mathcal{J}^>, x_k\geq|x_{(j+1)}|} a_k^- \Big) = \begin{cases} a_{(j)}^- & \text{if} \quad x_{(j)} < 0 \\ \\ 0 & \text{otherwise} \end{cases}
$$

$$
c) \quad \Big(\sum_{h,k\in\mathcal{J}^>, h\neq k, x_h, x_k\geq|x_{(j)}|} a_{hk}^+ - \sum_{h,k\in\mathcal{J}^>, h\neq k, x_h, x_k\geq|x_{(j+1)}|} a_{hk}^+ \Big) = \begin{cases} \sum_{k\in\mathcal{J}^>, x_k\geq|x_{(j)}|} a_{(j)k}^+ & \text{if} \quad x_{(j)} > 0 \\ \\ 0 & \text{otherwise} \end{cases}
$$

$$
d) \quad \Big(\sum_{h,k\in\mathcal{J}^>, h\neq k, x_h, x_k\geq|x_{(j)}|} a_{hk}^- - \sum_{h,k\in\mathcal{J}^>, h\neq k, x_h, x_k\geq|x_{(j+1)}|} a_{hk}^- \Big) = \begin{cases} \sum_{k\in\mathcal{J}^>, -x_k\geq|x_{(j)}|} a_{(j)k}^- & \text{if} \quad x_{(j)} < 0 \\ \\ 0 & \text{otherwise} \end{cases}
$$

$$e)\quad \Big(\sum_{h,k\in\mathcal{J}^>,x_h,-x_k\geq|x_{(j)}|} a^+_{h|k} - \sum_{h,k\in\mathcal{J}^>,x_h,-x_k\geq|x_{(j+1)}|} a^+_{h|k}\Big) = \begin{cases} \sum_{h\in\mathcal{J}^>,x_h\geq|x_{(j)}|} a^+_{h|(j)} & \text{if} \quad x_{(j)}<0 \\ \\ \sum_{k\in\mathcal{J}^>,-x_k\geq|x_{(j)}|} a^+_{(j)|k} & \text{if} \quad x_{(j)}>0 \\ \\ 0 & \text{otherwise} \end{cases}$$

$$f)\quad \Big(\sum_{h,k\in\mathcal{J}^>,x_h,-x_k\geq|x_{(j)}|} a^-_{h|k} - \sum_{h,k\in\mathcal{J}^>,x_h,-x_k\geq|x_{(j+1)}|} a^-_{h|k}\Big) = \begin{cases} \sum_{h\in\mathcal{J}^>,x_h\geq|x_{(j)}|} a^-_{h|(j)} & \text{if} \quad x_{(j)}<0 \\ \\ \sum_{k\in\mathcal{J}^>,-x_k\geq|x_{(j)}|} a^-_{(j)|k} & \text{if} \quad x_{(j)}>0 \\ \\ 0 & \text{otherwise} \end{cases}$$

With respect to $\chi)$ and $a)$ we have that,

$$|x_{(j)}|\Big(\sum_{k\in\mathcal{J}^>,x_k\geq|x_{(j)}|} a^+_k - \sum_{k\in\mathcal{J}^>,x_k\geq|x_{(j+1)}|} a^+_k\Big) = \begin{cases} x_{(j)}a^+_{(j)} & \text{if} \quad x_{(j)}>0 \\ \\ 0 & \text{otherwise} \end{cases}$$

and thus,

$$\sum_{j\in\mathcal{J}^>} |x_{(j)}|\Big(\sum_{k\in\mathcal{J}^>,x_k\geq|x_{(j)}|} a^+_k - \sum_{k\in\mathcal{J}^>,x_k\geq|x_{(j+1)}|} a^+_k\Big) = \sum_{j\in\mathcal{J}^>,\,x_j>0} x_{(j)}a^+_{(j)}$$

Considering in the same way $b)$, $c)$, $d)$, $e)$, and $f)$, we obtain the thesis of this theorem.

□

Proposition 4. *If $\hat{\mu}$ is decomposable then $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$ for each $(C,D) \in P(\mathcal{J})$ iff*

1. *for each $j \in \mathcal{J}$, $a^+_j = a^-_j$*
2. *for each $\{j,k\} \subseteq \mathcal{J}$, $a^+_{jk} = a^-_{jk}$*

3. *for each* $j,k \in \mathcal{J}$, $j \neq k$, $a^+_{j|k} = a^-_{k|j}$.

PROOF. First, let us prove that

(i) $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$

implies 1), 2) and 3). For each $j \in \mathcal{J}$,

(ii) $\hat{\mu}(\{j\},\emptyset) = a^+_j$ and $\hat{\mu}(\emptyset,\{j\}) = a^-_j$

From (i) and (ii) we have,

$$a^+_j = \hat{\mu}(\{j\},\emptyset) = -\hat{\mu}(\emptyset,\{j\}) = -a^-_j$$

which is 1).
For each $\{j,k\} \subseteq \mathcal{J}$ we have that,

(iii) $\hat{\mu}(\{i,j\},\emptyset) = a^+_j + a^+_k + a^+_{jk}$ and $\hat{\mu}(\emptyset,\{i,j\}) = a^-_j + a^-_k + a^-_{jk}$

On the basis of 1) we have that $a^+_j = a^-_j$ and $a^+_k = a^-_k$. Thus from (iii) we have that for each $\{j,k\} \subseteq \mathcal{J}$, $a^+_{jk} = a^-_{jk}$, i.e. 2).

Now, let us prove that 1), 2), and 3) implies (i). We have

(iv) $\hat{\mu}(C,D) = \sum_{j\in C} a^+_j - \sum_{j\in D} a^-_j + \sum_{\{j,k\}\subseteq C} a^+_{jk} - \sum_{\{j,k\}\subseteq D} a^-_{jk} + \sum_{j\in C,k\in D} a^+_{j|k} - \sum_{j\in C,k\in D} a^-_{j|k}$

and,

(iv) $\hat{\mu}(D,C) = \sum_{j\in D} a^+_j - \sum_{j\in C} a^-_j + \sum_{\{j,k\}\subseteq D} a^+_{jk} - \sum_{\{j,k\}\subseteq C} a^-_{jk} + \sum_{j\in D,k\in C} a^+_{j|k} - \sum_{j\in D,k\in C} a^-_{j|k}$

From 1), 2), and 3) and (iv) and (v) we have,

$$\hat{\mu}(C,D) = -\hat{\mu}(D,C)$$

which is what we wanted to prove.

□

9.1.3 Assessing the preferential information

On the basis of the above $2-$order decomposition, we propose the following methodology which simplifies the assessment of preferential information. We consider the following information given by the DM and their representation in terms of linear constraints:

1. *Comparing pairs of actions*. The constraints represent some pairwise comparisons on a set of training sample actions. As extreme case, this comparison can be a complete pre-order over this sample set, but it can be also a simple partial pre-order. Given two actions a and b, DMs may prefer a to b, b to a or indifferent to both actions.

   (a) The linear constraint associated with $a\mathcal{P}b$ is $Ch^B(P^B(a,b),\hat{\mu}) > 0$;

   (b) The linear constraint associated with aIb is $Ch^B(P^B(a,b),\hat{\mu}) = 0$.

2. *Comparison of the intensity of preferences between pairs of actions*. This comparison can be stated as follows,

$$Ch^B(P^B(a,b),\hat{\mu}) > Ch^B(P^B(c,d),\hat{\mu}) \quad \text{if} \quad (a,b)\mathcal{P}(c,d)$$

   where, $(a,b)\mathcal{P}(c,d)$ means that the comprehensive preference of a over b is larger than the comprehensive preference of c over d.

3. *Importance of criteria*. A partial ranking over the set of criteria $\mathcal{J}$, both in the positive and the negative part of the bi-polar scales may be provided by DMs . A lot of different situations can occur, for example:

   (a) Criterion g_j is more important than criterion g_k when comparing positive preferences, which leads to the constraint $a_j^+ \geq a_k^+$;

   (b) Criterion g_j is more important when it expresses positive preferences rather than when it expresses negative preferences; and so we can define the constraint $a_j^+ \geq a_j^-$.

4. *Interaction between pairs of criteria*. DMs can provide some information about interaction between criteria. Let us present two examples of such a kind of preferential information.

   (a) If DMs feel that when considering positive preferences, interaction between g_j and g_k is more important than the interaction between g_p and g_q, the constraint should be defined as follows: $a_{jk}^+ > a_{pq}^+$;

(b) If DMs feel that interaction between g_j and g_k, when they express negative preferences, is more important than the interaction between the same criteria, when they express negative preferences, the constraint will be the following: $a^+_{jk} > a^-_{jk}$.

5. *The sign of interactions*. DMs may be able, for certain cases, to provide the sign of some interactions. For example, if there is a synergy effect when criterion g_j interacts with criterion g_k in the positive part of the scale, the following constraint should be added to the model: $a^+_{jk} > 0$. It is not difficult to imagine other situations.

6. *The power of the opposing criteria*. Concerning the power of the opposing criteria several situations may occur. For example:

   (a) When the opposing power of g_k is larger than the opposing power of g_h, with respect to g_j, which expresses a positive preference, we can define the following constraint: $a^+_{j|k} > a^+_{j|h}$;

   (b) But, if the opposing power of g_k, expressing negative preferences, is larger with g_j rather than with g_h, the constraint will be $a^+_{j|k} > a^+_{h|k}$.

9.1.4 A linear programming model

All the constraints presented in the previous section along with the boundary and monotonicity conditions, and possibly symmetry condition, can now be put together and form a system of linear constraints. Strict inequalities can be converted into inequalities adding a variable ε. It is well-know that such a system has a feasible solution if and only if when maximizing ε, its value is strictly positive ([30]). The linear programming model can be stated as follows (where, $j^+\mathcal{P}k^+$ means that criterion g_j is more important than criterion g_k in the positive part of the scale; the remaining relations have similar interpretation):

Max z $=$ ε

$$\left.\begin{array}{lll} Ch^B(P^B(a,b)) \geq \varepsilon & \text{if} & a\mathcal{P}b \\ Ch^B(P^B(a,b)) = 0 & \text{if} & aIb \end{array}\right\}\text{Pairwise comparisons}$$

$$\left.\begin{array}{lll} Ch^B(P^B(a,b)) \geq Ch^B(P^B(c,d)) + \varepsilon & \text{if} & (a,b)\mathcal{P}(c,d) \\ Ch^B(P^B(a,b)) = Ch^B(P^B(c,d)) & \text{if} & (a,b)I(c,d) \end{array}\right\}\text{Comparing pairs of actions}$$

$$\left.\begin{array}{lll} a_j^+ - a_k^+ \geq \varepsilon & \text{if} & j^+\mathcal{P}k^+ \\ a_j^+ = a_k^+ & \text{if} & j^+Ik^+ \\ a_j^- - a_k^- \geq \varepsilon & \text{if} & j^-\mathcal{P}k^- \\ a_j^- = a_k^- & \text{if} & j^-Ik^- \\ a_j^+ - a_k^- \geq \varepsilon & \text{if} & j^+\mathcal{P}k^- \\ \vdots & & \vdots \end{array}\right\}\text{Ranking of criteria}$$

$$\left.\begin{array}{lll} a_{jk}^+ - a_{pq}^+ \geq \varepsilon & \text{if} & \{j^+,k^+\}\mathcal{P}\{p^+,q^+\} \\ a_{jk}^+ = a_{pq}^+ & \text{if} & \{j^+,k^+\}I\{p^+,q^+\} \\ a_{jk}^- - a_{pq}^- \geq \varepsilon & \text{if} & \{j^-,k^-\}\mathcal{P}\{p^-,q^-\} \\ a_{jk}^- = a_{pq}^- & \text{if} & \{j^-,k^-\}I\{p^-,q^-\} \\ a_{jk}^+ - a_{pq}^- \geq \varepsilon & \text{if} & \{j^+,k^+\}\mathcal{P}\{p^-,q^-\} \\ \vdots & & \vdots \end{array}\right\}\text{Ranking of pairs of criteria}$$

$$\left.\begin{array}{lll} a_{jk}^+ \geq \varepsilon & \text{if} & a_{jk}^+ > 0 \\ a_{jk}^+ = 0 & \text{if} & a_{jk}^+ = 0 \\ a_{jk}^+ \leq -\varepsilon & \text{if} & a_{jk}^+ < 0 \\ a_{jk}^- \geq \varepsilon & \text{if} & a_{jk}^- > 0 \\ a_{jk}^- = 0 & \text{if} & a_{jk}^- = 0 \\ a_{jk}^- \leq -\varepsilon & \text{if} & a_{jk}^- < 0 \end{array}\right\}\text{Sign of the interactions}$$

$$\left.\begin{array}{l} a_{j|k}^+ - a_{j|p}^+ \geq \varepsilon \\ a_{j|k}^- - a_{j|p}^- \geq \varepsilon \\ a_{j|k}^+ - a_{p|k}^+ \geq \varepsilon \\ a_{j|k}^- - a_{p|k}^- \geq \varepsilon \\ a_{j|k}^+ - a_{j|p}^- \geq \varepsilon \\ \vdots \quad\quad \vdots \end{array}\right\}\text{Power of the opposing criteria}$$

$$\left.\begin{array}{l} a_j^+ = a_j^-, \ \forall\, j \in \mathcal{J} \\ a_{jk}^+ = a_{jk}^-, \ \forall\, \{j,k\} \subseteq \mathcal{J} \\ a_{j|k}^+ = a_{j|k}^-, \ \forall\, j,k \in \mathcal{J} \end{array}\right\} \text{Symmetric conditions}$$

$$\left.\begin{array}{l} \sum_{j\in\mathcal{J}} a_j^+ + \sum_{\{j,k\}\subseteq\mathcal{J}} a_{jk}^+ = 1 \\ \sum_{j\in\mathcal{J}} a_j^- + \sum_{\{j,k\}\subseteq\mathcal{J}} a_{jk}^- = 1 \\ a_j^+,\, a_j^-,\, \geq 0 \ \forall\, j \in \mathcal{J} \\ a_{j|k}^+,\, a_{j|k}^-,\, \leq 0 \ \forall\, j,k \in \mathcal{J} \end{array}\right\} \text{Boundary conditions}$$

$$\left.\begin{array}{l} a_j^+ + \sum_{k\in C} a_{jk}^+ + \sum_{k\in D} a_{j|k}^+ \geq 0, \ \forall\, j \in \mathcal{J}, \forall (C\cup\{j\},D) \in P(\mathcal{J}) \\ a_j^- + \sum_{k\in C} a_{jk}^- + \sum_{k\in D} a_{k|j}^- \geq 0, \ \forall\, j \in \mathcal{J}, \forall (C,D\cup\{j\}) \in P(\mathcal{J}) \end{array}\right\} \text{Monotonicity conditions}$$

$$\left.\begin{array}{l} \mu^+(C,D) = \sum_{j\in C} a_j^+ + \sum_{\{j,k\}\subseteq C} a_{jk}^+ + \sum_{j\in C,\, k\in D} a_{j|k}^+, \ \forall\, (C,D) \in P(\mathcal{J}) \\ \mu^-(C,D) = \sum_{j\in C} a_j^- + \sum_{\{j,k\}\subseteq C} a_{jk}^- + \sum_{j\in C,\, k\in D} a_{j|k}^-, \ \forall\, (C,D) \in P(\mathcal{J}) \\ \hat{\mu}(C,D) = \mu^+(C,D) - \mu^-(C,D), \ \forall\, (C,D) \in P(\mathcal{J}) \\ Ch^B(P^B(a,b),\hat{\mu}) = \sum_{j\in\mathcal{J}^>} |P_{(j)}^B(a,b)| \Big[\mu(C_{(j)},D_{(j)}) - \mu(C_{(j+1)},D_{(j+1)})\Big] \end{array}\right\} \text{Definitions}$$

9.1.5 Restoring PROMETHEE

The two conditions which allows to restore PROMETHEE are the following:

1. $\forall j \in \mathcal{J},\ a_j^+ = a_j^-$;
2. $\forall j,k \in \mathcal{J},\ a_{jk}^+ = a_{jk}^- = a_{j|k}^+ = a_{j|k}^- = 0$.

If $a_{jk}^+ = a_{jk}^-$ does not satisfies the constraint, then the comprehensive preference of a over b is calculated as the difference between the Choquet integral of the positive preference and the Choquet integral of the negative preference, with a common capacity for the positive and the negative preferences. We shall call this type of aggregation of preferences, the Choquet integral PROMETHEE method.

9.1.6 A constructive learning preferential information elicitation process

The previous conditions 1) and 2) suggest a proper way to deal with the linear programming model in order to assess the interactive bi-polar criteria coefficients. Indeed, it is very wise to try before to elicit weights concordant with the classic PROMETHEE method. If this is not possible, one can consider a PROMETHEE method which aggregates positive and negative preferences using the Choquet integral. If, by proceeding in this way, we are not able to represent the DM's preferences, we can take into account a more sophisticated aggregation procedure by using the bi-polar Choquet integral. This way to progress from the simplest to the most sophisticated models can be outlined in a four step procedure as follows,

1. Solve the linear programming model adding the constraints related to the previous conditions 1) and 2). If the model has a feasible solution with $\varepsilon > 0$, the obtained preferential parameters are concordant with the classical PROMETHEE method. Otherwise,

2. Solve the linear programming model adding only condition 2). If there is a solution with $\varepsilon > 0$, the information is concordant with the Choquet integral PROMETHEE method. Otherwise,

3. Solve the problem without conditions 1) and 2. A solution with $\varepsilon > 0$ means that the preferential information is concordant with the bi-polar Choquet integral PROMETHEE method. Otherwise,

4. We can try to help the DM by providing some information about inconsistent judgments, when it is the case, by using a similar constructive learning procedure proposed in [32].

In fact, in the linear programming model some of the constraints cannot be relaxed, that is, the basic properties of the model (conditions and definitions). The remaining constraints can lead to an unfeasible linear system which means that DMs provided inconsistent information about their preferences. The methods proposed in [32] can then be used in this context. And, thus provide to the DM some useful information about inconsistent judgements.

With respect to the elicitation of a bi-polar capacity a further support for DMs can be obtained from a procedure using the playing cards method of Simos' in the bi-polar context (see [2]).

9.2 The case of ELECTRE method

A specific case of our bi-polar extension of ELECTRE III method is the Choquet integral ELECTRE method. This is the case in which the comprehensive outranking of a over b is calculated as the difference between the Choquet integral of the negative marginal outranking ϕ_j^+ and the Choquet integral of the negative marginal outranking ϕ_j^-, with a common capacity for the positive and the negative preferences.

9.2.1 Assessing the preferential information

The process of eliciting preferential information from the DM is very similar to the technique developed in the previous sub-section concerning the PROMETHEE method. The following constraints remain unchanged:

1. *Pairwise comparisons*. In the case where we assume that all the other preferential parameters (indifference, preference and veto thresholds) are known.

2. *Comparing the intensity of preference between pairs of actions*

3. *Ranking of criteria*

4. *Ranking of pairs of criteria*

5. *Sign of the interactions*

6. *Power of the opposing criteria*

Concerning the conditions the following changes should be taken into account:

1. The *symmetry conditions* should be removed. Contrary, to PROMETHEE method, ELECTRE does not requires symmetry.

2. The *boundary conditions* remain the same, but we should consider $P^*(\mathcal{J})$ instead of $P(\mathcal{J})$.

3. The modifications of the *monotonicity conditions* are the same as in the previous item.

The modifications of the definitions are very simple. We only should consider $P^*(\mathcal{J})$ instead of $P(\mathcal{J})$ and the expression of

$$Ch^{B*}\Big(\phi^{+}(a,b), \phi^{-}(a,b),\ \hat{\mu}^*\Big)$$

instead of

$$Ch^{B}\Big(P^{B}((a,b), \hat{\mu})\Big).$$

9.2.2 Restoring the Choquet ELECTRE method

The Choquet integral ELECTRE method is restored only in the case where no criterion has an opposing power,

1. $\forall j,k \in \mathcal{J},\ a^{+}_{j|k} = a^{-}_{j|k} = 0.$

9.2.3 Restoring ELECTRE

The classical ELECTRE method is restored when,

$$\forall j,k \in \mathcal{J},\quad a^{-}_{j} = a^{-}_{jk} = a^{+}_{j|k} = a^{-}_{j|k} = a^{+}_{jk} = a^{-}_{jk} = 0,$$

that is, only when $a^{+}_{j} \neq 0$, for each $j \in \mathcal{J}$.

10 Concluding remarks

In this paper we introduced the modelling of specific interactions, between criteria expressing positive and negative preferences, which are considered the pro and cons of comprehensive preferences. We defined this methodology as the bi-polar approach to MCDA. Taking into account the specific interaction between criteria in this context, especially the power of the opposing criteria, the multiple criteria, positive and negative preferences, are aggregated using the bi-polar Choquet integral. We believe that the proposed approach is in the main stream of the most current advanced research on MCDA. In what follows, we briefly remember the subjects of this research, and we explain the links with the work presented in this paper.

1. *The outranking approach*. We took into account the two most well-known classes of outranking methods: ELECTRE and PROMETHEE, and considered them from the point of view of the bi-polar approach to MCDA. The final result was a new way to deal with the outranking approach. It allows for the representation of some very important preferential information, one that could not be modeled by the existing MCDA methodologies. This crucial preferential information, is the interaction between criteria expressing preferences of the same sign (synergy and redundancy), or opposite sign (the power of the opposing criteria).

2. *The fuzzy integral approach*. To aggregate positive and negative multiple criteria preferences, we used the bi-polar Choquet integral. This is an aggregation function, recently introduced in the literature pertaining to non-additive integrals. Our interest in bi-polar Choquet integral is due to the fact that, when aggregating positive and negative preferences, there is a very natural "zero", which is the indifference between two actions. Let us remark again that this is not the case with MAUT like approaches. Indeed, in the latter case, the "zero" is defined as a neutral level which is often meaningless for DMs. We believe that our approach is the most appropriate, and natural application of the bi-polar Choquet integral within an MCDA framework.

3. *The four-valued logic approach*. The proposed approach is based on the aggregation of positive and negative preferences. They are respectively, the reasons in favor and against of comprehensive preferences. This is concordant with the four-valued logic approach for MCDA. From this point of view, our approach seems to be a very promising tool, in dealing with the particular representation of preferences of this theory. Thus, rendering it effective when applied to real-world problems.

4. *The non-additive and non-transitive models of conjoint measurement*. Since our approach considers outranking methods, it does not require the transitivity of the

comprehensive preference relations. Moreover, the bi-polar Choquet integral is a very general non-additive aggregation function. Therefore, our approach is concordant with the most advanced models of conjoint measurement. It is then, interesting from two points of view:

(a) From the point of view of the MCDA preference modelling, because this gives a perspective for further extension of the model. In other words, what type of extensions are possible between the use of bi-polar Choquet integral and the absolutely universal generality of the non-transitive and non-additive models of conjoint measurement?

(b) From the point of view of the conjoint measurement, because our approach opens the problem of the proper characterization of some models of conjoint measurement which are specific cases of the very general non-additive and non-transitive model. In other words, what are the axiomatic basis of the aggregation of positive and negative preferences using the bi-polar Choquet integral?

5. *The non-compensatory preference structures*. These structures consider only preferences, clearly and definitively in favor and, clearly and definitively against, excluding the possibility to introduce any graduation of the preferences. From this point of view, our approach can be seen as a generalization of the non-compensatory preferences, pertaining to modelling the degrees of preferences. The formal properties and the effective application of this approach, is a new domain of research within an MCDA framework.

6. *The interpretation of the importance of criteria*. The decomposition of bi-capacity proposed in our approach, gives a very meaningful interpretation of the preferential parameters, relating to the importance of the criteria. In MCDA there is a large interest in a correct interpretation of the importance of criteria. Within this line of research, our approach represents clearly the different influences of one criterion, on the relative importance of other criteria. From a theoretical point of view, this allows progression to a deeper understanding of the concept of the importance of criteria. Moreover, it provides effective tools for real-world applications. It gives a better explanation to DMs of the preferential parameters of the model, and allows DMs to learn more about their preferences during the preferential parameters elicitation processes.

7. *The procedures for eliciting non-additive weights*. Our approach permits also to represent a lot of very delicate preferential information, relative to interaction of criteria. However, the price to pay is a huge number of preferential parameters,

which is not reasonable to assess directly from DMs. Therefore, the potential of our approach in real-world problems crucially depends, on the efficiency of some methods to assess the preferential parameters (bi-polar capacity), on the basis of the preferential information given by DMs. From this point of view, our approach requires re-interpretation in this new context of the methods to assess the weights representing the importance of criteria. In the paper, we considered a generalization of a well-known method, to infer the capacity of a ”mono-polar” Choquet integral. But, we are planning future research on the extension of another well-known methods, to determine weights based on the so called ”Simos' play cards technique”.

8. *The aggregation function.* Our approach is based on a specific type of aggregation functions, which consider positive and negative values. Let us remark, that more research has been recently devoted to the properties and the characterization of aggregation functions. In this direction our approach suggests some line of research for the future. To accomplish this in MCDA, it is important to develop a theory of bi-polar aggregation functions, which allows a better understanding, and generalization of the bi-polar Choquet integral. Moreover, let us remark that the definition of the bi-polar concordance index of ELECTRE III, which we proposed, is based on the concept of bi-integral and bi-aggregation. We can as well, use the same kind of concordance index for other ELECTRE methods, i.e. ELECTRE IS and ELECTRE TRI. Let us point out, that the bi-aggregation is based on the idea that a criterion can express, at the same time, a partial reason in favor of and partial reason against a comprehensive preference. It is our view, that it is a very realistic interpretation of situations, often encountered in real-world problems. Until now, there are not appropriate tools to deal with this kind of situations. We believe that this will become an important issue for the development of the MCDA methodologies.

At the end of this paper, let us observe that Benjamin Franklin's letter quoted above, still remains an interesting perspective for MCDA. We hope that this paper, will give some contribution to the application of the idea of Benjamin Franklin to MCDA. We also think that many open problems remain to be dealt with in this direction.

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