A Bad Day Surfing is Better than a Good Day Working:
How to Revise a Total Preorder

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Abstract

Most approaches to iterated belief revision are accompanied by some motivation for the use of the proposed revision operator (or family of operators), and typically encode enough information for uniquely determining one-step revision. But in those approaches describing a family of operators, there is usually little indication of how to proceed uniquely after the first revision step. In this paper we take a step towards addressing that deficiency by providing a formal framework which goes beyond the first revision step. The framework is obtained by enriching the preference information starting from the following intuitive idea: we associate to each world \( x \) two abstract objects \( x^+ \) and \( x^- \), with the intuition that \( x^+ \) represents \( x \) “on a good day”, while \( x^- \) represents \( x \) “on a bad day”, and we assume that, in addition to preferences over the set of worlds, we are given preferences over this set of objects as well. The latter can be considered as meta-information which enables us to go beyond the first revision step of the revision operator being applied.

1 Introduction

Total preorders (hereafter \( tpos \)) are used to represent preferences in many contexts. In particular they are a common tool in belief revision [10, 14, 20]. In that setting they are taken to stand for plausibility orderings on the set of propositional worlds, which are used to encode the dispositions for change, or the conditional beliefs of an agent. The associated belief set is taken to be the set of those sentences true in all the minimal worlds. When new evidence \( \alpha \) comes in, the plausibility ordering is used to calculate the new belief set, usually by setting it to be the set of those sentences true in all the minimal models of \( \alpha \). This ensures a unique new belief set, but does not provide enough information to obtain a new tpo, which may then serve as the target for the next revision input. Thus the question of modelling the dynamics of \( tpos \) is of critical importance to the problem of iterated belief revision.

The past ten years has seen a flurry of activity in this area, with [7] and [17] being representative examples. Most approaches devote considerable effort to motivating the use of their proposed revision operator (or family of operators). But in those approaches describing a family of operators,

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there is usually little (or no) indication of how to choose among the available operators. In this paper we make a contribution towards overcoming that deficiency by providing a formal framework which obtains a unique tpo following one revision step, thereby going beyond just the belief set resulting from the revision input. The framework is obtained by enriching the preference information encoded in the tpo starting from the following intuitive idea: when we compare two different worlds \( x \) and \( y \) according to preference, often we are able to imagine different contingencies, according to whether all goes well in \( x \) and \( y \) or not. For example, given a choice between spending the day surfing at the beach and spending it in the office, we might think that even a bad day surfing is preferable to a good day working. Our idea is to associate to each world \( x \) two abstract objects \( x^+ \) and \( x^- \), with the intuition that \( x^+ \) represents \( x \) “on a good day”, while \( x^- \) represents \( x \) “on a bad day”, and we assume that, in addition to the given tpo \( \leq \) over the set of worlds, we are given a tpo \( \preceq \) over this set of objects.

This meta-information allows us to uniquely determine the new tpo: when new evidence \( \alpha \) comes in it casts a more favourable light on those worlds in which \( \alpha \) holds. Thus the evidence signals a “good day” for all those worlds satisfying \( \alpha \), and a “bad day” for the \( \neg \alpha \)-worlds. The revised tpo \( \leq^*_\alpha \) is obtained by setting \( x \leq^*_\alpha y \) iff \( x^\epsilon \preceq y^\delta \), where \( \epsilon, \delta \in \{+, -\} \) depending on whether \( x \), \( y \) satisfy \( \alpha \) or not.

As we will see, one commonly assumed rule from belief revision which will not generally hold for our revision operators is that the input \( \alpha \) is necessarily an element of the belief set associated to \( \leq^* \). Thus, at the belief set level, we are in the realm of so-called non-prioritised revision \([11, 12]\).

The plan of the paper is as follows. We begin in the next section by describing our enriched preference state. Then we show how to use this enrichment to define a unique tpo-revision operator, and we axiomatically characterise the resulting family of operators. Initially we describe the properties of this family on a semantic level, i.e., in terms of how the ordering of individual worlds \( x, y \) undergo change. In the following section we give an alternative, sentential formulation in terms of conditional beliefs, and introduce the notion of what it means for one sentence to overrule another in the context of a tpo-revision operator. After this we study some notions of strict preference which can be extracted from \( \preceq \) and show how these are closely related to the ‘overrules’ relation. Next we examine two known special cases of our family and give an example which shows how rigid use of either of these can sometimes lead to counter-intuitive results. In the penultimate section we describe and axiomatise an interesting sub-class of our family which remains general enough to include the two special cases, before concluding.

Preliminaries: We work in a propositional language \( L \) generated by finitely many propositional variables. We use \( \vdash \) and \( \equiv \) to denote classical logical consequence and classical logical equivalence respectively. We sometimes also use \( Cn \) to denote the operation of closure under classical logical consequence. \( W \) is the set of propositional worlds. Given \( \alpha \in L \), we denote the set of worlds which satisfy \( \alpha \) by \( [\alpha] \). Given any set \( S \subseteq W \) of worlds, \( Th(S) \) will denote the set of sentences true in all the worlds in \( S \). A tpo is a binary relation \( \leq \) which is both transitive and connected (for any \( x, y \) either \( x \leq y \) or \( y \leq x \)). In what follows we assume a fixed but arbitrary initial tpo \( \leq \) over \( W \) which we wish to revise. \( < \) will denote the strict part of \( \leq \), and \( \sim \) the symmetric closure of \( \leq \) (i.e. \( x \sim y \) iff both \( x \leq y \) and \( y \leq x \)). We are interested in functions \( * \) which, for each \( \alpha \in L \), return a new ordering \( \leq^*_\alpha \), and we will refer to any such \( * \) as a revision operator for \( \leq \).
2 Enriching the preference state

We let \( W^\pm = \{ x^\epsilon \mid x \in W \text{ and } \epsilon \in \{+,-\} \} \). We assume \( x^\epsilon = y^\delta \) only if both \( x = y \) and \( \epsilon = \delta \). We suppose, along with \( \leq \), we are given some relation \( \leq \) over \( W^\pm \). We expect some basic conditions on \( \leq \) and its interrelations with \( \leq \):

\[(\leq 1) \quad \leq \text{ is a tpo over } W^\pm\]
\[(\leq 2) \quad x^+ \preceq y^+ \iff x \leq y\]
\[(\leq 3) \quad x^- \preceq y^- \iff x \leq y\]
\[(\leq 4) \quad x^+ \preceq x^-\]

(\(\leq 2\)) and (\(\leq 3\)) say that the choice between two worlds both on a good day, resp. both on a bad day, should be precisely the same as that dictated by \( \leq \). (\(\leq 4\)) just says that given the choice between \( x \) on a good day and \( x \) on a bad day, we should choose \( x \) on a good day.

**Definition 1** Let \( \preceq \subseteq W^\pm \times W^\pm \). If \( \preceq \) satisfies (\(\leq 1\))–(\(\leq 4\)) we say \( \preceq \) is an \( \leq \)-faithful tpo (over \( W^\pm \)).

The following result shows that we could equivalently replace (\(\leq 4\)) in this definition by a seemingly stronger property:

**Proposition 1** Let \( \preceq \subseteq W^\pm \times W^\pm \) be any relation satisfying (\(\leq 1\)) and at least one of (\(\leq 2\)) and (\(\leq 3\)). Then \( \preceq \) satisfies (\(\leq 4\)) iff it satisfies the following rule:

\[(\leq 4') \quad x \leq y \text{ implies } x^+ \preceq y^-\]

**Proof:** Let \( \preceq \) be as stated. That (\(\leq 4'\)) \(\Rightarrow\) (\(\leq 4\)) is clear. For the converse direction suppose (\(\leq 4\)) is satisfied and suppose \( x \leq y \). If \( \preceq \) satisfies (\(\leq 2\)) then this gives \( x^+ \preceq y^+ \). We have \( y^+ \preceq y^- \) by (\(\leq 4\)), so putting these two together using (\(\leq 1\)) gives the required \( x^+ \preceq y^- \). If \( \preceq \) satisfies (\(\leq 3\)) rather than (\(\leq 2\)) then \( x \leq y \) yields \( x^- \preceq y^- \). We know \( x^+ \preceq x^- \) by (\(\leq 4\)) so putting these two together using (\(\leq 1\)) again gives \( x^+ \preceq y^- \).  

A \( \leq \)-faithful tpo \( \preceq \) can be given a useful graphical representation. First recall that any tpo \( \preceq' \) can be equivalently represented as its linearly ordered set of *ranks*. The ranks of \( \preceq' \) are the equivalence classes \( [x] \) modulo the symmetric closure \( \sim' \) of \( \preceq' \), and they are ordered by the relation \( [x] \preceq' [y] \iff x \preceq' y \). By (\(\leq 2\)), resp. (\(\leq 3\)), if \( x \) and \( y \) are two worlds in the same \( \leq \)-rank, then \( x^+ \) and \( y^+ \), resp. \( x^- \) and \( y^- \), are in the same \( \leq \)-rank. Thus if \( R_1 < \cdots < R_m \) are the ranks of \( \leq \) we can represent \( \preceq \) as a \( 2 \times m \) table of numbers whose \( i^{th} \) column corresponds to rank \( R_i \), and whose top and bottom rows correspond to + and − respectively. Then \( x^\epsilon \preceq y^\delta \) iff the entry in \( (\epsilon, [x]) \) is less than or equal to the entry in cell \( (\delta, [y]) \). By (\(\leq 2\)) and (\(\leq 3\)) the numbers increase strictly monotonically from left to right, while (\(\leq 4\)) decrees they increase strictly monotonically from top to bottom. An example assuming just three \( \leq \)-ranks is shown below:

As this example shows, there is nothing to stop the same number appearing in *both* a cell in the “+” row and a cell in the “−” row. So in the above we see that if the rank of world \( x \) is \( R_1 \) and the rank of world \( y \) is \( R_3 \) then \( x^- \) and \( y^+ \) appear in the *same* \( \leq \)-rank. In other words, \( x \) on a bad day is *equally preferred* to \( y \) on a good day.
Revision operators defined from $\preceq$

Now given a $\preceq$-faithful tpo $\preceq$ over $W^\pm$ we want to use the information given by $\preceq$ to define a revision operator $* = *_{\preceq}$ for $\preceq$. The idea is that the evidence $\alpha$ casts a favourable light on those worlds satisfying $\alpha$. In other words, we consider worlds satisfying $\alpha$ to be having a “good day”, with those worlds inconsistent with the evidence having a “bad day”. We set, for any $\alpha \in L$ and $x \in W$:

$$r_\alpha(x) = \begin{cases} x^+ & \text{if } x \in [\alpha] \\ x^- & \text{if } x \in [\neg \alpha] \end{cases}$$

The revised tpo $\leq^*_\alpha$ is defined by setting, for each $x, y \in W$,

$$x \leq^*_\alpha y \text{ iff } r_\alpha(x) \preceq r_\alpha(y).$$

**Definition 2** For each $\preceq$-faithful tpo $\preceq$ over $W^\pm$, we call $*_{\preceq}$ as defined above the revision operator generated by $\preceq$.

**Example 1** Consider the propositional language generated by the atoms $p$ and $q$. We represent worlds as sequences of 0s and 1s, representing the valuations of $p$ and $q$ respectively (thus 01 represents a world where $p$ is false and $q$ is true). Let $\preceq$ be the ordering on worlds depicted in the following:

```
    10 11 01
   00
```

Let $\preceq$ be the $\preceq$-faithful tpo depicted in Figure 1. Revision by $q$ can be represented pictorially as follows:
In the table on the left, worlds satisfying $q$ are placed in the top row, with those not satisfying $q$ placed in the bottom row. The resulting ordering $\leq^*_q$, shown on the right, is obtained by reading the ranks from the corresponding cell in Figure 1. The resulting belief set, i.e., the set of sentences true in all the $\leq^*_q$-minimal worlds, is $Cn(p \land q)$. The revision of $\leq$ by $\neg p \land q$ can be similarly represented as follows:

<table>
<thead>
<tr>
<th>10</th>
<th>11</th>
<th>00</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>01</td>
</tr>
</tbody>
</table>

This time the resulting belief set associated with $\leq^*_\neg p \land q$ is $Cn(p \leftrightarrow \neg q)$. Since $\neg p \land q \not\in Cn(p \leftrightarrow \neg q)$, this example shows that new evidence is not always in the belief set associated to the new tpo.

What are the properties of $\leq^*$? Consider the following list:

1. $\leq^*_\alpha$ is a tpo over $W$
2. $\alpha \equiv \gamma$ implies $\leq^*_\alpha = \leq^*_\gamma$
3. If $x, y \in [\alpha]$ then $x \leq^*_\alpha y$ iff $x \leq y$
4. If $x, y \in [\neg \alpha]$ then $x \leq^*_\alpha y$ iff $x \leq y$
5. If $x \in [\alpha], y \in [\neg \alpha]$ and $x \leq y$ then $x \leq^*_\alpha y$
6. If $x \in [\alpha], y \in [\neg \alpha]$ and $y \leq^*_\alpha x$ then $y \leq^*_\gamma x$
7. If $x \in [\alpha], y \in [\neg \alpha]$ and $y \leq^*_\alpha x$ then $y \leq^*_\gamma x$

(*1) just says revising a tpo over $W$ should result in another tpo over $W$, while (*2) is a syntax-irrelevance property. The next three rules are all familiar from the literature on iterated belief change. (*3) and (*4) appear respectively as (CR1) and (CR2) in Darwiche and Pearl’s [7] well-known list of four postulates. They say that after revising by $\alpha$, the relative ordering between $\alpha$-worlds, respectively $\neg \alpha$-worlds, remains unchanged. (*5) was proposed independently by [5] and [13]. It is easily seen to be stronger than the other two rules in the Darwiche-Pearl list (which can be obtained by replacing $\leq$ by $\prec$ (CR3) and $\leq^*_\alpha$ by $\leq^*_\alpha$ (CR4) respectively). It says if an $\alpha$-world $x$ was considered at least as preferred as a $\neg \alpha$-world $y$ before receiving $\alpha$, then after revision it should be considered strictly more preferred. These three rules were considered characteristic of a family of operators called admissible revision operators [5].

So far each of our rules mention only one revision input sentence $\alpha$ (modulo logical equivalence). By analogy with the AGM postulates for belief set revision [1], we might consider them as the set of basic postulates for tpo-revision. One thing largely missing from the literature on iterated belief change is a serious study of supplementary rationality properties which bestow a certain amount of coherence on the results of revising $\leq$ by different sentences. The last couple of properties do this. First, suppose evidence $\alpha$ is received, and let $x \in [\alpha], y \in [\neg \alpha]$, but suppose $y \leq^*_\alpha x$. We propose that if $x$ is not more preferred than $y$, even after receiving evidence which clearly points more to $x$ being the case than it does to $y$, then there can be no evidence which will lead to $x$ being more preferred to $y$. This is expressed by (*6). Similarly (*7) says if $x$ is deemed
ensures that precisely one $\delta$ is itself connected. So suppose reduces to showing either $x \leq y$ after receiving $\alpha$ then $x$ must be strictly less preferred after receiving any input.

It turns out that these properties provide an exact characterisation of the revision operators we consider.

**Theorem 1** Let * be any revision operator for $\preceq$. Then * is generated from some $\preceq$-faithful tpo $\preceq$ over $W^{\pm}$ iff * satisfies (*1)–(*7).

**Proof:** **Soundness:** (*1) holds because of ($\preceq$1). (*2) holds because, as is easily seen, $\alpha \equiv \gamma$ implies $r_\alpha(x) = r_\gamma(x)$ for all $x \in W$. (*3) and (*4) hold as direct consequences of ($\preceq$2) and ($\preceq$3) respectively. (*5) holds as a consequence of ($\preceq$4'). For (*6) suppose $x \in [\alpha], y \in [\neg \alpha]$ and $y \preceq_\alpha x$. From the first two we know $r_\alpha(x) = x^+ \text{ and } r_\alpha(y) = y^-$. Using these with $y \preceq_\alpha x$ gives $y^- \preceq x^+$. From this and ($\preceq$4) we have

$$y^+ \prec y^- \preceq x^+ \prec x^-.$$  

Thus, we see that for any $\gamma \in L$, we will have $r_\gamma(y) \preceq r_\gamma(x)$, i.e., $y \preceq_\gamma x$ as required. (*7) is proved similarly.

To show the completeness part of Theorem 1, starting from any revision operator $*$ for $\preceq$ we can define an ordering $\preceq_*$ over $W^{\pm}$ as follows. Let $x, y \in W$ and $\delta, \epsilon \in \{+, -, 6\}$. If $\delta = \epsilon$ then we set

$$x^\delta \preceq_* y^\delta \text{ iff } x \leq y.$$  

This obviously ensures $\preceq_*$ complies with ($\preceq$2) and ($\preceq$3). Now suppose $\delta \neq \epsilon$. If $x = y$ then we simply set $x^+ \prec_* x^-$, to ensure compliance with ($\preceq$4). Otherwise we set

$$x^+ \preceq_* y^- \text{ iff } x \preceq_* y, \quad x^- \preceq_* y^+ \text{ iff } x \preceq_* y.$$  

Here, when we use a world $x$ as a subscript in $\preceq_*$, we are using it to denote any sentence $\alpha$ such that $[\alpha] = \{x\}$. Likewise, in the proofs which follow, when $x$ appears within the scope of a propositional connective, e.g., $x \lor y$, (note that if * satisfies (*2) the precise choice of $\alpha$ is irrelevant). Then if * satisfies (*1)–(*7) then $\preceq_*$ is a $\preceq$-faithful tpo and the revision operator generated from $\preceq_*$ is precisely *.

**Proof:** **Completeness:** We need to show two things: (a) $\preceq_*$ is a $\preceq$-faithful tpo, and then (b) the revision operator generated from $\preceq_*$ is precisely *.

(a) $\preceq_*$ is a $\preceq$-faithful tpo.

To show this we need to show ($\preceq$1)–($\preceq$4) are satisfied. ($\preceq$2)–($\preceq$4) obviously hold by construction, so it remains to prove ($\preceq$1), i.e., $\preceq_*$ is a tpo. $\preceq_*$ is connected:

We need to show, for any $x, y \in W$ and $\epsilon, \delta \in \{+, -, 6\}$, either $x^\epsilon \preceq_* y^\delta$ or $y^\delta \preceq_* x^\epsilon$. If $\delta = \epsilon$ this reduces to showing either $x \leq y$ or $y \leq x$ by construction of $\preceq_*$, and this clearly holds since $\preceq$ is itself connected. So suppose $\delta \neq \epsilon$. Now if $x = y$ then the result holds since our construction ensures that precisely one of $x^\delta \preceq x^\epsilon$ and $x^\epsilon \preceq x^\delta$ holds (the former if $\delta = +$, the latter if $\epsilon = +$). So suppose also $x \neq y$. Then the construction tells us $x^\epsilon \preceq y^\delta$ iff $x \preceq A y$ and $y^\delta \preceq x^\epsilon$ iff $y \preceq A x$, so...
where \( A = x \) if \( \epsilon = + \) while \( A = y \) if \( \delta = + \). Whatever the value of \( A \) we know \( \leq_A \) is connected by \((\ast)\), thus at least one of \( x^\downarrow \leq y^\downarrow \) and \( y^\downarrow \leq x^\downarrow \) must hold as required.

\( \leq_A \) is transitive:

We need, for any \( x, y, z \in W \) and \( \delta, \epsilon, \nu \in \{+, -\} \),

\[
\text{if } x^\delta \leq_A y^\epsilon \text{ and } y^\epsilon \leq_A z^\nu \text{ then } x^\delta \leq_A z^\nu.
\]

For this proof let us denote these three by \( A, B, C \) respectively. Proving \( A + B \Rightarrow C \) is a tedious matter of individually going through all eight combinations of choices for \( \delta, \epsilon, \nu \). The easiest cases are when \( \delta = \epsilon = \nu = + \) or \( \delta = \epsilon = \nu = - \), for in these cases showing \( A + B \Rightarrow C \) reduces to showing that \( x \leq y \) and \( y \leq z \) implies \( x \leq z \), which clearly holds since \( \leq \) is itself transitive. Now let’s go through the other six cases:

(i) \( \delta = \epsilon = +, \nu = - \).

Firstly if \( x = y \) then \( B \) and \( C \) reduce to the same thing and so the result holds. Also if \( x = z \) then \( C \) holds by construction. So we assume \( x \neq y \) and \( x \neq z \). Then \( A \) becomes \( x \leq y \). We now split into two subcases according to whether \( y = z \). If \( y = z \) then the target consequent \( C \) becomes \( x \leq y \). But using \( x \leq y \) with our assumption \( x \neq y \) we may apply \((\ast)\) to deduce \( x <^z y \). Thus \( C \) certainly holds. Now suppose \( y \neq z \). Then \( A + B \Rightarrow C \) reduces to showing \( x \leq y + y \leq z \Rightarrow x \leq z \). Suppose for contradiction that \( A + B \) holds but \( C \) does not. If \( C \) doesn’t hold then \( z <^z x \) by \((\ast)\) so, since we assume \( z \neq x \), \( z <^x z \) by \((\ast)\). From \( x \leq y \) we get \( x \leq x \) by \((\ast)\) and so \( z < x \). Since we also assume \( z \neq y \) we may apply \((\ast)\) to this to obtain \( z <^x y \), contradicting \( y \leq z \). Hence the consequent must hold also in this case.

(ii) \( \delta = \epsilon = +, \nu = + \).

Now \( B \) reduces to \( y \leq z \), which means, by the already established \((\ast)\), we must have \( y \neq z \). Meanwhile \( C \) becomes \( x^+ \leq^+ z^+ \), i.e., \( x \leq z \). If \( x = z \) then \( C \) clearly holds. So we assume also \( x \neq z \). We now look at the two subcases according to which \( x = y \) or not. If \( x = y \) then \( B \) becomes \( x^\downarrow \leq^+ z^+ \), i.e., \( x \leq^+ z \) (since \( x \neq z \)). So \( B \Rightarrow C \) by \((\ast)\). If \( x \neq y \) then \( A \) is \( x \leq^+ y \) and \( B \) is \( y \leq^+ z \), so we must show \( x \leq^+ y + y \leq^+ z \Rightarrow x \leq z \). Assume for contradiction \( A + B \) holds but \( C \) doesn’t. From the latter \( z < x \), then \( z <^x x \) by \((\ast)\). Meanwhile, since \( y \neq z \), the assumption \( y \leq^+ z \) gives \( y \leq^+ z \) by \((\ast)\). Hence \( y \leq^+ z \) using \((\ast)\). Since we assume \( x \neq y \) we apply \((\ast)\) here to deduce \( y <^x x \), contradicting \( x \leq^+ y \). Hence the consequent must hold.

(iii) \( \delta = +, \epsilon = -, \nu = - \).

If \( x = z \) then \( C \) becomes \( x^+ \leq^+ x^- \), which already holds by \((\ast)\). Thus we assume \( x \neq z \) and so \( C \) is \( x \leq^+ z \). Meanwhile \( B \) reduces to \( y \leq z \). If \( x = y \) then this reduces in turn to \( x \leq z \), and so in this case we get \( B \Rightarrow C \) by \((\ast)\). If \( x \neq y \) then \( A \) is \( x \leq^+ y \) and so \( A + B \Rightarrow C \) reduces to \( x \leq^+ y + y \leq z \Rightarrow x \leq z \). Assume for contradiction \( A \) and \( B \) hold and \( C \) does not. Then \( z \leq^+ x \) from not\( C \) by \((\ast)\). Since we assume \( y \neq x \) we may apply \((\ast)\) to \( y \leq z \) to obtain \( y \leq^+ z \). Using this with \( z <^x x \) and \((\ast)\) yields \( x <^+ x \), contradicting \( A \). Hence \( C \) must follow from \( A \) and \( B \).

(iv) \( \delta = -, \epsilon = +, \nu = + \).

Here, \( A \) is \( x^- \leq^+ y^+ \), which implies \( x \neq y \) by \((\ast)\) and so gives \( x \leq_y y \). Meanwhile \( B \) is \( y^+ \leq^+ z^+ \), which gives \( y \leq z \). We first claim \( A + B \) implies \( x \neq z \). For if \( x = z \) then \( B \) would give \( y \leq x \). Since \( x \neq y \) this then yields \( y <_x y \) using \((\ast)\), contradicting the \( x \leq y \) we obtained
from $A$. Hence $x \neq z$ as claimed, and so given $A + B$, $C$ becomes $x \leq z$. We must now show $x \leq y + y \leq z \Rightarrow x \leq z$. But since $x \neq y$ we may use $(6)$ to get $x \leq y \lor z$ from $x \leq y$, while $y \leq z$ from $y \leq z$ using $(3)$. $x \leq y \lor z$ and $y \leq z$ together give $x \leq y \lor z$ using $(1)$. From this, since $y \neq x \neq z$, we may apply $(6)$ to deduce $x \leq z$ as required.

$(v)$ $\delta = -$, $\epsilon = +$, $\nu = -$.

As in the previous case, $A$ implies $x \neq y$ and $x \leq y$, while this time $C$ reduces to $x \leq z$. In the case $y = z$ this in turn becomes $x \leq y$, which is a consequence of $x \leq y$ by $(5)$. Thus in this case $A \Rightarrow C$. So suppose instead $y \neq z$. Then $B$ reduces to $y \leq z$, and so $A + B \Rightarrow C$ reduces to $x \leq y + y \leq z \Rightarrow x \leq z$. Suppose for contradiction $A + B$ holds but $C$ does not. Then this latter means $z < x$ which in turn gives $z < y$, $x$ using $(4)$ with the assumptions $x \neq y \neq z$. Using this with $x \leq y$ and $(1)$ gives $y < z$, contradicting $y \leq z$. Hence $A + B \Rightarrow C$ as required.

$(vi)$ $\delta = -$, $\epsilon = -$, $\nu = +$.

Now $B$ yields $y \neq z$ (by $(\leq 4)$) and $y \leq z$, while $A$ is equivalent to $x \leq y$. If $x = z$ were the case then this latter would become $z \leq y$ which would imply $z \leq y$ (since $y \neq z$ from $B$). But this contradicts the $y \leq z$ we obtained from $B$ and so we must have $x \neq z$. Hence, given $A + B$, $C$ reduces to $x \leq z$ and so we must show $x \leq y + y \leq z$ to $x \leq z$. But since $x \neq z \neq y$ we may use $x \leq y$ to deduce $x \leq z$ using $(4)$. From this and $y \leq z$ we obtain $x \leq z$ as required.

(b) the revision operator generated from $\leq x$ is precisely $\ast$.

Now let $\ast'$ be the revision operator generated from $\leq x$. We now need to show $\ast'$ is precisely $\ast$, i.e., for any $\alpha \in L$ and $x, y \in W$, $x \leq x y$ if $x \leq x y$. Since this latter is equivalent to $r_\alpha(x) \leq x r_\alpha(y)$, this means we need to show

$$x \leq x y \iff r_\alpha(x) \leq x r_\alpha(y).$$

We split into the three cases $x < x y$, $x \sim x y$ and $y < x$. (Using the $\leq x$-notation defined in Definition 3.)

Case $x < x y$

In this case $r_\alpha(x) = x^+$ and $r_\alpha(y) = y^-$. So we must show $x \leq x y$ if $x^+ \leq y^-$. Since $x < x y$ we must have $x \neq y$ so by construction of $\leq x$ the right-hand side is equivalent to $x \leq x y$. We will show $x \leq x y$ if $x \neq y$. By $(1)$ this is equivalent to showing $y \leq x$ if $y \neq x$. But by $(7)$ each side of this biconditional is equivalent to $[y \leq x$ for all $\gamma]$. Hence in this case the result holds.

Case $x \sim x y$

In this case we show that both $x \leq x y$ and $r_\alpha(x) \leq x r_\alpha(y)$ are equivalent to $x \leq y$. That $x \leq x y$ if $x \leq y$ follows from either $(3)$ or $(4)$ (depending on whether $x, y \in [\alpha]$ or $x, y \in [\alpha]$ respectively). Meanwhile we have $r_\alpha(x) \leq x r_\alpha(x)$ if $x^\alpha \leq y^\alpha$ (where $\delta = +$ if $x, y \in [\alpha]$ and $\delta = -$ otherwise). By construction of $\leq x$ this latter is equivalent to $x \leq y$ as required.

Case $y < x$

In this case we show that both $x \leq x y$ and $r_\alpha(x) \leq x r_\alpha(y)$ are equivalent to saying $x \leq y$ for all $\gamma$. For $x \leq y$ this follows from $(6)$. Meanwhile $r_\alpha(x) \leq x r_\alpha(y)$ if $x^\alpha \leq y^\alpha$. Since $y < x$ we know $x \neq y$ so by construction of $\leq x$ this latter is equivalent to $x \leq y$. That this is equivalent to $[x \leq y$ for all $\gamma]$ follows once more from $(6)$. ■
3.1 Some social choice-like conditions

In this subsection we discuss some more properties satisfied by our revision operators. These properties are recognisable as versions of properties familiar from the theory of social choice, or preference aggregation [2]. The problem of preference aggregation is the problem of finding some function $f$ which, given any list of tpos (over some given set $X$ of alternatives) $\leq_1, \ldots, \leq_n$, with the $\leq_i$s representing the preferences over $X$ of the individuals in a group, will return a new single ordering $f(\leq_1, \ldots, \leq_n)$ over $X$ which adequately represents the preferences of the group as a whole. Now, we can think of our problem of determining $\leq_\alpha$ as a highly specialised case of this problem. To do this we need to repackage the new evidence $\alpha \in L$ into tpo-form. The simplest way to do this is as follows.

**Definition 3** For any $\alpha \in L$, the tpo $\leq_\alpha$ generated by $\alpha$ is the tpo over $W$ given by $x \leq_\alpha y$ iff $x \in [\alpha]$ or $y \in [\neg \alpha]$.

In other words $\leq_\alpha$ is the tpo over $W$ consisting of (at most) two ranks: the lower one containing all the $\alpha$-worlds and the upper one containing all the $\neg \alpha$-worlds. Then we can think of revision of $\leq$ by $\alpha$ as an aggregation of $\leq$ with $\leq_\alpha$. (This manoeuver is also carried out by [9] and [18]. An alternative way of generating tpos from sentences, based on the Hamming distance between two propositional worlds, is mentioned by [3].)

Many properties of preference aggregation operators have been proposed. One well-known property, known as the Pareto condition, is that, given two alternatives $x$ and $y$, if every individual prefers $x$ at least as much as $y$, and if at least one individual strictly prefers $x$ over $y$, then the group should strictly prefer $x$ over $y$. In our case, this condition translates into the following property:

(Pareto) If $x \leq y$ and $x \leq_\alpha y$, and at least one of these two inequalities is strict, then $x <^*_\alpha y$.

The case of the above rule where $\leq_\alpha$ is strict is nothing other than ($*5$), while the case where $x \sim_\alpha y$ and $x < y$ is easily seen to follow mainly from ($*3$) and ($*4$). Thus we have:

**Proposition 2** Every revision operator $*$ generated by some $\leq$-faithful tpo $\leq$ over $W^\pm$ satisfies (Pareto).

Another well-known property from preference aggregation is known as Independence of Irrelevant Alternatives, which states that for any two alternatives $x$ and $y$, the group preference between $x$ and $y$ should depend only on how each individual ranks $x$ and $y$. More precisely, if we were to replace individual $i$’s tpo $\leq_i$ by any other tpo $\leq'_i$ which ranks $x$ and $y$ in exactly the same way as $\leq$, then $x$ and $y$ would be ranked in exactly the same way in $f(\leq_1, \ldots, \leq'_i, \ldots, \leq_n)$ as in $f(\leq_1, \ldots, \leq_i, \ldots, \leq_n)$. It turns out that our family of operators satisfy a restricted version of this rule, which we call Independence of Irrelevant Alternatives in the Input. Given $\alpha, \gamma \in L$, and $x, y \in W$, let’s say $\alpha$ and $\gamma$ agree on $x$ and $y$ iff either both $x <^\alpha y$ and $x <^\gamma y$, or both $x \sim_\alpha y$ and $x \sim_\gamma y$, or both $y <^\alpha x$ and $y <^\gamma x$. In other words $\alpha$ and $\gamma$ both “say the same thing” regarding the relative plausibility of $x$ and $y$.

(I1A-Input) If $\alpha$ and $\gamma$ agree on $x$ and $y$ then $x \leq_\alpha y$ iff $x \leq^*_\gamma y$. 

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That this is a property of our family of tpo-revision operators can be straightforwardly shown by considering an arbitrary $\leq$-faithful tpo $\preceq$ over $W^\pm$. But in fact we can show the following:

**Proposition 3** Let * be any revision operator for $\leq$ which satisfies (*1) and (*3)–(*5). Then * satisfies (IIA-Input) iff * satisfies both (*6) and (*7).

**Proof:** Let * satisfy (*1) and (*3)–(*5).

(IIA-Input) $\Rightarrow$ (*6) + (*7).

First we show the following property, which will be useful:

If $x <^\alpha y$, $y \leq^* x$ and $\gamma$ and $\alpha$ do not agree on $x$ and $y$, then $y <^\gamma x$.

To see this, first note if $x <^\alpha y$ and $y \leq^* x$ then we must have $y < x$ by (*5). Also note if $\gamma$ and $\alpha$ do not agree on $x, y$ then, since $x <^\alpha y$, we must have either $x \sim^\gamma y$ or $y <^\gamma x$. In the first case we know $x \leq^\gamma y$ iff $x \leq y$ and $y \leq^\gamma x$ iff $y \leq x$ by (*3) and (*4). Using these with the already established $y < x$ gives $y <^\gamma x$ as required. In the case $y <^\gamma x$ we can use the fact $y < x$ to conclude $y <^\gamma x$ by (*5).

Now to show (*6) suppose $x <^\alpha y$ and $y \leq^* x$. If $\gamma$ and $\alpha$ do not agree on $x, y$ then $y <^\gamma x$ by the above property, so $y \leq^\gamma x$ as required. If $\gamma$ agrees with $\alpha$ on $x, y$ then we can conclude $y \leq^\gamma x$ from $y \leq^* x$ using (IIA-Input).

(*7) is proved similarly: Suppose $x <^\alpha y$ and $y <^\gamma x$. If $\gamma$ does not agree with $\alpha$ on $x, y$ then, since obviously $y <^\gamma x$ implies $y \leq^\gamma x$ we may apply the above proved property to conclude the required $y <^\gamma x$. If $\gamma$ agrees with $\alpha$ on $x, y$ then from (IIA-Input) we have $x \leq^\gamma y$ iff $x \leq^\alpha y$ and $y \leq^\gamma x$ iff $y \leq^\alpha x$. Hence we can conclude $y <^\gamma x$ from $y <^\alpha x$.

(*6) + (*7) $\Rightarrow$ (IIA-Input).

Suppose $\alpha$ and $\gamma$ agree on $x, y$. To show (IIA-Input) it suffices by symmetry to show $x \leq^\alpha y$ implies $x \leq^\gamma y$. First suppose $x <^\alpha y$, $x <^\gamma y$ and $x \leq^* y$. If it were not the case that $x \leq^\gamma y$ then we would have $y <^\gamma x$ by (*1). Using this with $x <^\gamma y$ and (*7) would then give $y <^\gamma x$, contradicting $x \leq^\gamma y$. Hence we must have $x \leq^\gamma y$ as required. Now look at the case in which both $x \sim^\alpha y$ and $x \sim^\gamma y$. In this case using these with (*3) or (*4) we get $x \leq^\alpha y$ iff $x \leq y$ iff $x \leq^\gamma y$. Hence $x \leq^\alpha y$ implies $x \leq^\gamma y$ (and conversely) as required. Finally we consider the case in which both $y <^\gamma x$ and $y <^\gamma x$. This time we get $x \leq^\alpha y$ implies $x \leq^\gamma y$ (and conversely) using (*6).

Thus, given the “basic” properties (*1)–(*5) for tpo-revision, requiring * to satisfy the two “supplementary” properties (*6) and (*7) amounts to enforcing (IIA-Input). Note this equivalence does not require the presence of the syntax-irrelevance property (*2). In fact, since sentences which are logically equivalent agree on all worlds $x$ and $y$, we see that (*2) actually follows from (IIA-Input). Consequently, we have established that in the list (*1)–(*7), property (*2) is redundant.

For more discussion on social choice-like conditions and their relevance to tpo-revision we refer the reader to the work of [9].
4 On the sentential level

So far all our properties of tpo-revision operators have been expressed on the “semantic level”, directly in terms of worlds. But there is also a sentential level on which we can recast our properties. For any tpo \( \leq' \) over \( W \) and any \( \beta \in L \) we let \( \min (\beta, \leq') \) denote the set of \( \leq' \)-minimal elements of \([\beta]\), i.e., \( \min (\beta, \leq') = \{ x \in [\beta] \mid \nexists y \in [\beta] \text{ s.t. } y <' x \} \). Then we define:

\[ \leq' \circ \beta = Th(\min (\beta, \leq')). \]

\( \leq' \circ \beta \) represents what is believed in \( \leq' \) on the supposition that \( \beta \) is the case. If \( \lambda \in \leq' \circ \beta \) then we might also say \( \beta \vdash \lambda \) is a conditional belief in \( \leq' \). Note that we do not necessarily assume this is the same thing as saying \( \lambda \) would be believed after receiving \( \beta \) explicitly as evidence. This is because we want to support non-prioritised revision, so in particular \( \beta \) itself might not necessarily be believed after receiving it as evidence (it might be simply too far-fetched). Nevertheless, new evidence will have some impact on the set of conditional beliefs. Note that this notation enables us to denote the belief set associated to \( \leq' \) by \( \leq' \circ \top \).

We can give all the properties (\( \ast 2 \))–(\( \ast 7 \)) an equivalent formulation in terms of \( \circ \), thus giving a set of sound and complete properties for our family of revision operators which has a different flavour:

(\( \circ 2 \)) If \( \alpha \equiv \gamma \) then \( \leq^*_\alpha \circ \beta = \leq^*_\gamma \circ \beta \)

(\( \circ 3 \)) If \( \beta \vdash \alpha \) then \( \leq^*_\alpha \circ \beta = \leq \circ \beta \)

(\( \circ 4 \)) If \( \beta \vdash \neg \alpha \) then \( \leq^*_\alpha \circ \beta = \leq \circ \beta \)

(\( \circ 5 \)) If \( \neg \alpha \not\in \leq \circ \beta \) then \( \alpha \in \leq^*_\alpha \circ \beta \)

(\( \circ 6 \)) If \( \alpha \not\in \leq^*_\alpha \circ \beta \) then \( \alpha \not\in \leq^*_\gamma \circ \beta \)

(\( \circ 7 \)) If \( \neg \alpha \in \leq^*_\alpha \circ \beta \) then \( \neg \alpha \in \leq^*_\gamma \circ \beta \)

(\( \circ 2 \)) just says revising by logically equivalent sentences yields the same set of conditional beliefs. (\( \circ 3 \)) and (\( \circ 4 \)) are essentially the well-known (C1) and (C2) of [7], while (\( \circ 5 \)) corresponds to rule (P) of [5], also referred to as Independence by [13]. The correspondences between these last three rules and their counterparts in the previous section were proved in those papers. (Although these papers all assume the “prioritised” setting for belief revision in which revision inputs are always believed after revision.) The last two rules are neatly explained with the help of the following terminology:

**Definition 4** Given any revision operator \( \ast \) for \( \leq \) and given \( \alpha, \beta \in L \), we shall say \( \beta \) overrules \( \alpha \) (relative to \( \ast \)) iff either \( \beta \) is inconsistent or \( \alpha \not\in \leq^*_\ast \circ \beta \). We shall say \( \beta \) strictly overrules \( \alpha \) (relative to \( \ast \)) iff \( \neg \alpha \in \leq^*_\ast \circ \beta \).

The inclusion of the clause “\( \beta \) is inconsistent” in the definition of “overrules” allows for a smoother exposition. This way we get the intuitively expected chain of implications \( \beta \vdash \neg \alpha \) implies \( \beta \) strictly overrules \( \alpha \), which implies \( \beta \) overrules \( \alpha \). If \( \ast \) satisfies (\( \circ 5 \)) then this in turn implies \( \neg \alpha \in \leq \circ \beta \). Now suppose that evidence \( \gamma \) is received and we then make a further supposition \( \beta \). (\( \circ 6 \)) says if \( \beta \) overrules \( \alpha \) and \( \beta \) is consistent then \( \alpha \) will not be believed, while (\( \circ 7 \)) says if \( \beta \) strictly overrules \( \alpha \) then \( \alpha \) will be rejected.
**Proposition 4** Let $*$ be a revision operator for $\leq$ which satisfies $(\ast 1)$. Then for each $i = 2, \ldots, 7,$ $*$ satisfies $(\ast 1)$ iff $*$ satisfies $(\circ 1)$. 

**Proof:** Suppose $*$ satisfies $(\ast 1)$, i.e., $\leq^*_\alpha$ is a tpo given any $\alpha$. Note that for any tpo $\leq'$ (in particular $\leq^*_\alpha$) and $x, y \in W$, we have 

$$x \leq' y \text{ iff } x \in \min(x \lor y, \leq')$$

(1)

where, recall, in the expression $x \lor y$, $x$ and $y$ stand for any sentences whose only model is $x$, respectively $y$, and so $[x \lor y] = \{x, y\}$. $(\ast 2) \iff (\circ 2)$

The “$\Rightarrow$” direction is obvious. For the “$\Leftarrow$” direction suppose $\alpha \equiv \gamma$. Then using (1) we know, given any $x, y, x \leq^*_\alpha y$ iff $x \in \min(x \lor y, \leq^*_\alpha)$ and $x \leq^*_\gamma y$ iff $x \in \min(x \lor y, \leq^*_\gamma)$. But by $(\ast 2)$ $\min(x \lor y, \leq^*_\alpha) = \min(x \lor y, \leq^*_\gamma)$. Hence $x \leq^*_\alpha y$ iff $x \leq^*_\gamma y$ for all $x, y$, i.e., $\leq^*_\alpha = \leq^*_\gamma$ as required.

$(\ast 3) \iff (\circ 3)$ and $(\ast 4) \iff (\circ 4)$

Proofs given already in [7]. (See Theorem 13 there.) $(\ast 5) \iff (\circ 5)$

Proof can be found in [5] (Prop. 2) or [13] (Theorem 4.2).

$(\ast 6) \iff (\circ 6)$

For the “$\Rightarrow$” direction suppose $(\ast 6)$ holds and suppose $\alpha \not\leq^*_\alpha \circ \beta$. Then there exists $y \in [-\alpha] \cap \min(\beta, \leq^*_\alpha)$. Assume for contradiction $\alpha \in \leq^*_\alpha \circ \beta$. Then $y \not\in \min(\beta, \leq^*_\alpha)$ so there exists $x \in \min(\beta, \leq^*_\alpha)$ such that $x <^*_\gamma y$. Since we assume $\alpha \in \leq^*_\alpha \circ \beta$ we must have $x \in [\alpha]$. Hence we may apply a contrapositive version of $(\ast 6)$ to obtain (with a little help from $(\ast 1)$) $x <^*_\alpha y$. But this contradicts $y \in \min(\beta, \leq^*_\alpha)$. Hence it must be the case that $\alpha \not\leq^*_\alpha \circ \beta$ as required.

For the converse direction suppose $(\circ 6)$ holds and let $x \in [\alpha], y \in [-\alpha]$ be such that $y \leq^*_\alpha x$. Then from (1) $y \in \min(x \lor y, \leq^*_\alpha)$. Hence, since $y \in [-\alpha], \alpha \not\leq^*_\alpha \circ (x \lor y)$. Using $(\circ 6)$ we infer $\alpha \not\leq^*_\alpha \circ (x \lor y)$. Since necessarily $\min(x \lor y, \leq^*_\alpha) \subseteq \{x, y\}$, the only way we can have $\alpha \not\leq^*_\alpha \circ (x \lor y)$ is if $y \in \min(x \lor y, \leq^*_\gamma)$, i.e., $y \leq^*_\gamma x$ as required to show $(\ast 6)$.

$(\ast 7) \iff (\circ 7)$

For the “$\Rightarrow$” direction suppose $(\ast 7)$ holds and suppose $\neg \alpha \in \leq^*_\alpha \circ \beta$. Suppose for contradiction $\neg \alpha \not\leq^*_\gamma \circ \beta$. Then there exists $x \in [\alpha] \cap \min(\beta, \leq^*_\alpha)$. Since $\neg \alpha \in \leq^*_\alpha \circ \beta$ we know $x \not\in \min(\beta, \leq^*_\alpha)$ so there exists $y \in \min(\beta, \leq^*_\alpha)$ such that $y <^*_\alpha x$. We know $y \in [-\alpha]$ since $\neg \alpha \in \leq^*_\alpha \circ \beta$, hence we may apply $(\ast 7)$ to deduce $y \leq^*_\gamma x$ contradicting $x \in \min(\beta, \leq^*_\alpha)$. Hence $\neg \alpha \not\leq^*_\gamma \circ \beta$ as required.

For the converse suppose $(\circ 7)$ holds and let $x \in [\alpha], y \in [-\alpha]$ such that $y <^*_\alpha x$. Then $\min(x \lor y, \leq^*_\alpha) = \{y\}$ and so, since $y \in [-\alpha], \neg \alpha \in \leq^*_\alpha \circ (x \lor y)$. Applying $(\circ 7)$ to this yields $\neg \alpha \in \leq^*_\gamma \circ (x \lor y)$ and so, since $x \in [\alpha], x \not\in \min(x \lor y, \leq^*_\gamma)$, i.e., $y \leq^*_\gamma x$ as required to show $(\ast 7)$. \[\Box\]

**Corollary 1** Let $*$ be a revision operator for $\leq$. Then $*$ is generated from some $\leq$-faithful tpo $\leq$ over $W^+$ iff $*$ satisfies $(\ast 1)$ and $(\circ 2)$–$(\circ 7)$.

This sentential reformulation is useful, since there are some interesting properties which can be formulated in sentential terms, but for which obvious semantic counterparts do not exist. For example:
Claim we know \( z < \gamma \) allows us to show (Disj2), for suppose both \( \lambda \) instead. Thus in either case we arrive at the required contradiction.

Thus we have both \( r \) equivalently (Disj1) and (Disj2).

Proposition 6 Let \( \ast \) be any revision operator for \( \leq \) satisfying (\ref{disj1}) and (\ref{disj2}).

We prove this result by considering an arbitrary \( \leq \)-faithful tpo \( \leq \) over \( W^\pm \).

Proof: Let \( \leq \) be a given \( \leq \)-faithful tpo over \( W^\pm \).

(Disj1): It suffices to show \( \min(\beta, \leq_{a\gamma}) \subseteq \min(\beta, \leq_a) \cup \min(\beta, \leq^*_a) \). So let \( x \in \min(\beta, \leq_{a\gamma}) \) and suppose for contradiction both \( x \notin \min(\beta, \leq^*_a) \) and \( x \notin \min(\beta, \leq^*_a) \). From these latter two we know there exist \( y_1 \in \min(\beta, \leq^*_a) \) and \( y_2 \in \min(\beta, \leq^*_a) \) such that \( y_1 < \gamma x \) and \( y_2 < \gamma x \). Equivalently \( r_a(y_1) < r_a(x) \) and \( r_a(y_2) < r_a(x) \). But since \( x \in \min(\beta, \leq_{a\gamma}) \) we know \( x \leq_{a\gamma} y_i \), equivalently \( r_{a\gamma}(x) \leq r_{a\gamma}(y_i) \), for \( i = 1, 2 \). Since \( r_{a\gamma}(y_i) = \min\{r_a(y_i), r_\gamma(y_i)\} \) this means we have both \( r_{a\gamma}(x) \leq r_a(y_1) \) and \( r_{a\gamma}(x) \leq r_a(y_2) \). But since \( r_{a\gamma}(x) = \min\{r_a(x), r_\gamma(x)\} \) we know \( r_{a\gamma}(x) \) is equal to either \( r_a(x) \) or \( r_\gamma(x) \). In the first case we get \( r_a(x) \leq r_a(y_1) \), contradicting \( r_a(y_1) < r_a(x) \). In the second case we obtain \( r_\gamma(x) \leq r_\gamma(y_2) \), contradicting \( r_\gamma(y_1) < r_\gamma(x) \). Thus in either case we arrive at the required contradiction.

(Disj2): We first claim the following: Given any pair of worlds \( y_1, y_2 \) such that \( y_1 \in \min(\beta, \leq^*_a) \) and \( y_2 \in \min(\beta, \leq^*_a) \), at least one of these worlds must be in \( \min(\beta, \leq_{a\gamma}) \). For suppose neither is an element of this set. Then there must exist \( z \in \min(\beta, \leq_{a\gamma}) \) such that \( z <_{a\gamma} y_i \), equivalently \( r_{a\gamma}(z) < r_{a\gamma}(y_i) \), for \( i = 1, 2 \). Since \( r_{a\gamma}(y_i) = \min\{r_a(y_i), r_\gamma(y_i)\} \) we obtain from this both \( r_{a\gamma}(z) < r_a(y_1) \) and \( r_{a\gamma}(z) < r_a(y_2) \). Then since \( r_{a\gamma}(z) = \min\{r_a(z), r_\gamma(z)\} \) we get from these either \( r_a(z) < r_a(y_1) \) or \( r_\gamma(z) < r_a(y_2) \). But in the former case we have \( z < \gamma y_1 \), contradicting \( y_1 \in \min(\beta, \leq^*_a) \), while similarly in the latter case, \( z < \gamma y_2 \), which contradicts \( y_2 \in \min(\beta, \leq^*_a) \). Hence no such \( z \) can exist and so the claim must be true. This then allows us to show (Disj2), for suppose both \( \lambda \notin \leq^*_a \circ \beta \) and \( \lambda \notin \gamma \circ \beta \). Then there must exist \( y_1 \in \min(\beta, \leq^*_a) \) and \( y_2 \in \min(\beta, \leq^*_a) \) such that \( y_i \in [-\lambda] \) for \( i = 1, 2 \). From the above claim we know \( y_i \in \min(\beta, \leq_{a\gamma}) \) for either \( i = 1 \) or \( i = 2 \). Either way we end up with some \( y \in \min(\beta, \leq_{a\gamma}) \) such that \( y \in [-\lambda] \), which is enough to prove \( \lambda \notin \leq_{a\gamma} \circ \beta \).

The next result shows that \( \leq^* \circ \beta \) falls neatly into one of three categories. Note we don’t need (\ref{disj6}) and (\ref{disj7}), nor (\ref{disj2}), for this.

Proposition 5 Every revision operator \( \ast \) generated from some \( \leq \)-faithful tpo \( \leq \) over \( W^\pm \) satisfies (Disj1) and (Disj2).
overrules relations be given relative to \( \ast \). Then for all \( \alpha, \beta \in L \),

\[
\leq^*_\alpha \beta = \begin{cases} 
\leq o(\alpha \land \beta) & \text{if } \beta \text{ doesn't overrule } \alpha \\
(\leq o(\alpha \land \beta)) \cap (\leq o\beta) & \text{if } \beta \text{ overrules } \alpha, \text{ but not strictly} \\
\leq o\beta & \text{if } \beta \text{ strictly overrules } \alpha 
\end{cases}
\]

**Proof:** We make use of the following standard properties, which hold for any tpo \( \leq' \) over \( W \) (note the assumption \((*1)\) is satisfied permits us to apply these properties to \( \leq^*_\alpha \)):

(i). If \( \alpha \leq' o\beta \) then \( \leq' o\beta = \leq o(\alpha \land \beta) \) (Cumulativity).

(ii). If \( \neg\alpha \not\leq' o\beta \) then \( \leq' o\beta \subseteq \leq' o(\alpha \land \beta) \) (Rational Monotony).

(iii). \( (\leq' o\beta_1) \cap (\leq' o\beta_2) \subseteq (\leq o(\beta_1 \lor \beta_2)) \) (Or).

Suppose \( \beta \) does not overrule \( \alpha \). We must show \( \leq^*_\alpha o\beta = \leq o(\alpha \land \beta) \). But if \( \beta \) does not overrule \( \alpha \) then \( \alpha \in \leq^*_\alpha o\beta \) so, using property (i) above, \( \leq^*_\alpha o\beta = \leq^*_\alpha o(\alpha \land \beta) \). Using \((\circ3)\) we conclude \( \leq^*_\alpha o\beta = \leq o(\alpha \land \beta) \) as required.

Now suppose \( \beta \) strictly overrules \( \alpha \). We must show in this case \( \leq^*_\alpha o\beta = \leq(o(\alpha \land \beta)) \). Firstly, if \( \beta \) is inconsistent then both these sets are equal to the entire set of sentences \( L \) and so the result clearly holds. So we assume \( \beta \) is consistent. We will in fact show both \( \leq^*_\alpha o\beta \) and \( \leq o\beta \) are equal to \( \leq o(\neg\alpha \land \beta) \). For the former, if \( \beta \) strictly overrules \( \alpha \) then \( \neg\alpha \in \leq^*_\alpha o\beta \) so, again using (i) above, \( \leq^*_\alpha o\beta = \leq^*_\alpha o(\neg\alpha \land \beta) \). Using \((\circ4)\) we then obtain \( \leq^*_\alpha o\beta = \leq o(\neg\alpha \land \beta) \) as required. Meanwhile from \( \neg\alpha \in \leq^*_\alpha o\beta \) and the assumption \( \beta \) is consistent we can infer \( \alpha \not\leq^*_\alpha o\beta \). From this and \((\circ5)\) we get \( \neg\alpha \in \leq o\beta \) and so, from (i) once more, also \( \leq o\beta = \leq o(\neg\alpha \land \beta) \) as required.

Finally we check the intermediate case where \( \beta \) overrules \( \alpha \), but not strictly, which means \( \neg\alpha \not\leq^*_\alpha o\beta \). We must show \( \leq^*_\alpha o\beta = (\leq o(\alpha \land \beta)) \cap (\leq o\beta) \). Looking back at the last sentence of the previous paragraph, we see we showed there that if \( \alpha \not\leq^*_\alpha o\beta \) then \( \leq o\beta = \leq o(\neg\alpha \land \beta) \). Hence we may equivalently formulate our target identity as \( \leq^*_\alpha o\beta = (\leq o(\alpha \land \beta)) \cap (\leq o(\neg\alpha \land \beta)) \). Applying \((\circ3)\) and \((\circ4)\), this in turn is the same as requiring \( \leq^*_\alpha o\beta = (\leq^*_\alpha o(\alpha \land \beta)) \cap (\leq^*_\alpha o(\neg\alpha \land \beta)) \). We can prove the right-to-left inclusion here by noting by property (iii) above that \((\leq^*_\alpha o(\alpha \land \beta)) \cap (\leq^*_\alpha o(\neg\alpha \land \beta)) \subseteq \leq^*_\alpha o((\alpha \land \beta) \lor (\neg\alpha \land \beta)) \Rightarrow \leq^*_\alpha o\beta \). For the converse direction note that from property (ii) we have \( \alpha \not\leq^*_\alpha o\beta \) implies \( \leq^*_\alpha o\beta \subseteq \leq^*_\alpha o(\neg\alpha \land \beta) \) and \( \neg\alpha \not\leq^*_\alpha o\beta \) implies \( \leq^*_\alpha o\beta \subseteq \leq^*_\alpha o(\alpha \land \beta) \). Thus the result holds.

Thus if \( \beta \) doesn’t overrule \( \alpha \) then making the supposition \( \beta \) after receiving \( \alpha \) as evidence is the same as supposing \( \alpha \) and \( \beta \) together in the initial tpo \( \leq \). If \( \beta \) strictly overrules \( \alpha \) then evidence \( \alpha \) is just ignored when making the further supposition \( \beta \). In the intermediate case where \( \beta \) overrules \( \alpha \), but not strictly, supposing \( \beta \) following evidence \( \alpha \) results in a mixture of these two.

In particular note what happens when \( \beta \equiv \top \). We see that \( \leq^*_\alpha o\top \) equals either (i) \( \leq o\alpha \), or (ii) \((\leq o\alpha) \cap (\leq o\top) \), or (iii) \( \leq o\top \). Thus either the evidence is fully incorporated into the belief set using the AGM revision operator corresponding to \( \leq \) [14] (case (i)), or the belief set remains unchanged (case (iii)), or there is an intermediate possibility ((ii)), which amounts to removing \( \neg\alpha \) from the initial belief set using the AGM contraction operator corresponding to \( \leq \). That is, we don’t commit to believing the evidence, but we leave open the possibility that it might hold. We will have more to say on these notions of overruling in the next section.
5 Notions of strict preference

In this section we shall assume a fixed $\leq$-faithful tpo $\preceq$ over $W^\pm$. Given $\leq$ we can define two more interesting preference orderings over $W$:

$$x \prec y \iff x^- \preceq y^+, \quad x \prec y \iff x^- \prec y^+$$

In other words, $x \prec y$, resp. $x \prec y$, is saying that $x$, even on a bad day, is at least as preferred as, resp. strictly preferred to, $y$. The next proposition collects some properties of these two orderings.

Proposition 7

(i) $\ll \subseteq \ll \subseteq <$ (where recall $<$ is the strict part of the initial tpo $\leq$).

(ii) $\ll$ and $\ll$ are both strict partial orders (i.e., irreflexive and transitive).

(iii) $\ll$ and $\ll$ both satisfy the filtered condition [8], i.e., for all $x, y \in W$ and $\beta \in L$, if $x, y \in [\beta] \setminus \min(\beta, \ll')$ then there exists $z \in [\beta]$ such that $z \ll x$ and $z \ll y$.

(Recall for a strict partial order $\ll'$, $\min(\beta, \ll') = \{x \in [\beta] | \beta y \in [\beta] \text{ s.t. } y \ll' x\}$.)

Proof: (i). The inclusion $\ll \subseteq \ll$ is immediate. The inclusion $\ll \subseteq \ll$ follows from ($\preceq 4'$).

(ii). The irreflexivity of $\ll$ follows from ($\preceq 4$). Since $\ll \subseteq \ll$ this means $\ll$ must be irreflexive as well. To show transitivity of the two relations, we actually show something stronger holds, namely

If $x \ll y$ and $y \ll z$ then $x \ll z$. (2)

This is true since if $x \ll y$ and $y \ll z$ then $x^- \preceq y^+$ and $y^- \preceq z^+$. Since $y^+ \prec y^-$ by ($\preceq 4$) we obtain $x^- \preceq y^+ \prec y^- \preceq z^+$, thus $x^- \prec z^+$, i.e., $x \ll z$ as claimed. (2) yields the transitivity of both $\ll$ and $\ll$ using the fact $\ll \subseteq \ll$.

(iii). To show $\ll$ satisfies the filtered condition let $x, y \in [\beta] \setminus \min(\beta, \ll)$. Since $x, y$ are not minimal there exist $z_1, z_2 \in [\beta]$ such that $z_1 \ll x$ and $z_2 \ll y$, i.e., $z_1^- \prec x^+$ and $z_2^- \prec y^+$. Since $\preceq$ is connected (since it is a tpo by ($\preceq 1$)) we know either $z_1^- \preceq z_2^+$ or $z_2^- \preceq z_1^-$. In the first case we obtain $z_1^- \prec y^+$ from $z_2^- \prec y^+$ and so there exists some $z \in [\beta]$ (namely $z_1$) such that both $z \ll x$ and $z \ll y$ as required. In the second case we obtain $z_2^- \prec x^+$ from $z_1^- \prec x^+$ and so again we find a $z$ (this time $z = z_2$) with the required properties. Hence $\ll$ satisfies the filtered condition. The case for $\ll$ is analogous.

By (i) we see $\ll$, $\ll$, and $\ll$ form progressively more stringent notions of strict preference. If we let $* = *_{\leq}$ then we see $x \ll y$ implies $r_\gamma(x) \prec r_\gamma(y)$ for all $\gamma \in L$, and so $x \ll^*_\gamma y$ for any $\gamma$. Thus $\ll$ can also be viewed as a set of core, or protected strict preferences in $<$ which are always preserved in any revision. Meanwhile we have $x \ll y$ implies $x \leq^*_\gamma y$ for any $\gamma$. Thus $\ll$ may be viewed as a set of weakly protected strict preferences, in the sense that if $x \ll y$ then no evidence will ever cause this preference to be reversed.

It turns out that these relations $\ll$ and $\ll$ are closely related to the notions of overruling and strict overruling from Definition 4.

Proposition 8 Let the overrules relations be given relative to $*_{\leq}$. Then (i) $\beta$ overrules $\alpha$ iff $\min(\beta, \ll) \subseteq [-\alpha]$. (ii) $\beta$ strictly overrules $\alpha$ iff $\min(\beta, \ll) \subseteq [-\alpha]$.
Proof: (i). We must show \( \text{min}(\beta, \leq) \subseteq [-\alpha] \) iff either \( \beta \) is inconsistent or \( \alpha \not\leq^* \beta \).

\[ \Rightarrow: \] Suppose \( \text{min}(\beta, \leq) \subseteq [-\alpha] \). If \( \beta \) is inconsistent we are done, so assume \( \beta \) is consistent. We must show \( \alpha \not\leq^* \beta \), i.e., \( \text{min}(\beta, \leq^*) \cap [-\alpha] \neq \emptyset \). Suppose \( \text{min}(\beta, \leq) = \{y_1, \ldots, y_k\} \). Since \( \text{min}(\beta, \leq) \subseteq [-\alpha] \) we know \( y_i \in [-\alpha] \) for all \( i = 1, \ldots, k \). We will show that at least one of these elements of \( \text{min}(\beta, \leq) \) must also be an element of \( \text{min}(\beta, \leq^*) \), which will suffice. Suppose for contradiction \( y_i \not\in \text{min}(\beta, \leq^*) \) for all \( i \). Then (since \( \leq^* \) is a tpo) there must be at least one element \( z \in [\beta] \) such that \( z <^* y_i \), equivalently \( r_\alpha(z) < r_\alpha(y_i) \), for all \( i \). Since \( r_\alpha(y_i) = y_i^- \) for all \( i \), this gives \( r_\alpha(z) < y_i^- \) for all \( i \). Clearly it cannot be the case that \( z = y_j \) for some \( j \) (since then we would have \( r_\alpha(y_j) < r_\alpha(y_j) \), which is impossible), hence \( z \not\in \text{min}(\beta, \leq) \). Hence it must be the case \( y_j < z \), i.e., \( y_j^- \leq z^+ \) for some \( j \). But this implies \( y_j^- \leq r_\alpha(z) \), contradicting \( r_\alpha(z) < r_\alpha(y_i) \), for all \( i \). Hence there must exist some \( j \) such that \( y_j < \min(\beta, \leq^*) \) as required.

\[ \Leftarrow: \] If \( \beta \) is inconsistent then \( \text{min}(\beta, \leq) = \emptyset \) and so the required conclusion \( \text{min}(\beta, \leq) \subseteq [-\alpha] \) holds true. So suppose \( \beta \) is consistent and \( \alpha \not\leq^* \beta \). Then there exists some \( y \in \text{min}(\beta, \leq^*) \). Suppose for contradiction \( \text{min}(\beta, \leq) \subseteq [-\alpha], \) so there exists \( x \in \text{min}(\beta, \leq) \). Using the minimality of \( y \) we get \( y \leq^* x \), i.e., \( r_\alpha(y) \leq r_\alpha(x) \). Since \( y \in [-\alpha] \) and \( x \in [\alpha] \) translates into \( y^- \leq x^+ \), i.e., \( y \leq x \). But this contradicts \( x \in \text{min}(\beta, \leq^*) \). Hence \( \text{min}(\beta, \leq) \subseteq [-\alpha] \) as required.

(ii). We must show \( \text{min}(\beta, \leq) \subseteq [-\alpha] \) iff \( \neg \alpha \leq^* \beta \).

\[ \Rightarrow: \] As mentioned above, just after the proof of Proposition 7, we have \( \leq \subseteq \leq^* \). This implies \( \text{min}(\beta, \leq^*) \subseteq \text{min}(\beta, \leq^*) \). Hence if \( \text{min}(\beta, \leq) \subseteq [-\alpha] \) then also \( \text{min}(\beta, \leq^*) \subseteq [-\alpha] \), i.e., \( \neg \alpha \leq^* \beta \) as required.

\[ \Leftarrow: \] Suppose \( \neg \alpha \leq^* \beta \), i.e., \( \text{min}(\beta, \leq^*) \subseteq [-\alpha], \) and suppose for contradiction \( \text{min}(\beta, \leq) \not\subseteq [-\alpha]. \) Then there exists \( x \in \text{min}(\beta, \leq) \cap [\alpha] \). Since \( x \in [\alpha] \) and \( \text{min}(\beta, \leq) \not\subseteq [-\alpha] \) this implies \( x \not\in \text{min}(\beta, \leq^*) \), so there exists \( y \in \text{min}(\beta, \leq^*) \) such that \( y \leq^* x \), i.e., \( r_\alpha(y) \leq r_\alpha(x) \). Since \( y \in [\alpha] \) and \( \text{min}(\beta, \leq^*) \subseteq [-\alpha] \) we know \( y \in [-\alpha] \) and so \( r_\alpha(y) = y^- \). Hence since \( x \in [\alpha] \) we know \( r_\alpha(x) = x^+ \). Hence \( r_\alpha(y) < r_\alpha(x) \) translates into \( y^- < x^+ \), i.e., \( y \leq^* x \), which contradicts \( x \in \text{min}(\beta, \leq^*) \). Hence if \( \text{min}(\beta, \leq) \subseteq [-\alpha] \) as required.

For each of the two overrules relations we may consider an interdefinable inference relation. We define:

\[ \beta \Rightarrow \alpha \text{ iff } \beta \text{ overrules } \neg \alpha \]

\[ \beta \Rightarrow \alpha \text{ iff } \beta \text{ strictly overrules } \neg \alpha \]

Using fundamental results by [8] and [15], classifying various families of nonmonotonic inference relations, Proposition 8 together with the properties of \( \leq \) and \( \leq^* \) now allows us to deduce many properties of \( \Rightarrow \) and \( \Rightarrow \), and thereby of the overrules relations:

Corollary 2 The binary relations \( \Rightarrow \) and \( \Rightarrow \) are both (consistency-preserving) preferential inference relations, in the sense of [15]. Furthermore they both satisfy the rule of Disjunctive Rationality, i.e., if \( \beta \lor \gamma \Rightarrow \alpha \) then either \( \beta \Rightarrow \alpha \) or \( \gamma \Rightarrow \alpha \).

The first part is a consequence of the fact that \( \leq \) and \( \leq^* \) are strict partial orders [15]. In particular it implies \( \Rightarrow \) and \( \Rightarrow \) both satisfy the following rules (among others):
Switching things around in terms of the corresponding overrules relations, Right Weakening implies if \( \beta \) (strictly) overrules \( \alpha \) then \( \beta \) (strictly) overrules every sentence logically stronger than \( \alpha \). The And-rule tells us that if \( \beta \) (strictly) overrules both \( \alpha \) and \( \gamma \) separately, then it (strictly) overrules their disjunction. While Cautious Monotony translates into the rule that if \( \beta \) (strictly) overrules \( \alpha \), then so does \( \beta \land \neg \gamma \), provided \( \beta \) (strictly) overrules \( \gamma \).

The second part of Corollary 2 follows from results by [8] and Proposition 7(iii). It implies a disjunction \( \beta \lor \gamma \) cannot (strictly) overrule \( \alpha \) without at least one of its disjuncts doing so. However it’s possible for neither \( \Rightarrow \) nor \( \land \) to satisfy the well-known rule Rational Monotony [15] (and thus also Monotony). I.e., if \( \beta \Rightarrow \alpha \) and \( \beta \not\Rightarrow \neg \gamma \) then \( \beta \land \gamma \Rightarrow \alpha \). This is because it can be shown that the relations \( \ll \) and \( \lll \) are not in general modular, i.e., they do not verify the property \( x' < y \) implies there exists \( z \) such that either \( x <' z \) or \( z <' y \). (A counterexample for \( \ll \) can be found by taking the initial tpo from Example 1 with the \( \preceq \) defined earlier in Figure 1, and then by taking \( x = 10 \) and \( y = 01 \).)

6 Limiting cases

In this section we investigate some special limiting cases of our family of revision operators. Firstly, suppose we insist on the following strengthening of property \((\preceq 4)\): 

\[(\preceq L) \quad x^+ \ll y^- .\]

In other words, given a choice between any world on a good day and any world on a bad day, we choose the world on a good day every time. This is equivalent to the limiting case where \( \ll = \emptyset \) (thus also \( \lll = \emptyset \)). Hence this condition can be thought of as expressing “minimal confidence” behind the initial tpo \( \preceq \). Note that adding this rule to \((\preceq 2)\) and \((\preceq 3)\) is enough to specify a unique tpo over \( W^\pm \), thus causing \((\preceq 1)\) to become redundant. Indeed we are left with the tpo defined by,

| \( x, y \in W \) and \( \delta, \epsilon \in \{+, -, \} \) | \( x^\delta \preceq^\epsilon y^\epsilon \) iff either (\( \delta = + \) and \( \epsilon = - \)) or (\( \delta = \epsilon \) and \( x \preceq y \)). In terms of the graphical representation of \( \preceq \), this corresponds to the case where every number in the “+” row is strictly less than every number in the “−” row:

| + | 1 | 2 | ⋯ | \( n \) |
| − | \( n + 1 \) | \( n + 2 \) | ⋯ | \( 2n \) |
The revision operator $*_{L}$ defined by this $\preceq$ then reduces to:

$$x \leq_{\alpha} y \text{ iff either } x <^{\alpha} y \text{ or } (x \sim^{\alpha} y \text{ and } x \leq y)$$

This is the well-known lexicographic revision operator studied and axiomatised in the context of iterated belief revision [9, 21, 17]. It amounts to $\leq_{\alpha}$ being refined by $\leq$. We can characterise $*_{L}$ within our family in the following way:

**Proposition 9** If $*$ is generated from some $\preceq$-faithful tpo over $W^\pm$ satisfying ($\preceq L$) then $*$ satisfies:

($*$L) If $x \in [\alpha]$ and $y \in [-\alpha]$ then $x <^{*}_{\alpha} y$.

Furthermore if $*$ is any revision operator for $\preceq$ which satisfies ($* L$) then the $\preceq$-faithful tpo $\preceq^{*}$ defined right after Theorem 1 satisfies ($\preceq L$).

**Proof:** Suppose $\preceq$ satisfies ($\preceq L$) and let $* = *_{\preceq}$. Let $x \in [\alpha]$ and $y \in [-\alpha]$. Then $r_{\alpha}(x) = x^{+}$ and $r_{\alpha}(y) = y^{-}$. By ($\preceq L$) $r_{\alpha}(x) < r_{\alpha}(y)$, i.e., $x <^{\alpha} y$ as required to show ($* L$).

Conversely suppose $*$ is a revision operator for $\preceq$ satisfying ($\preceq L$) and let $\preceq^{*}$ be as defined after Theorem 1. We must show $x^{+} < y^{-}$ for all $x, y$. If $x = y$ then $x^{+} < x^{-}$ directly by construction. So suppose $x \neq y$. We need to show $x^{+} \leq y^{-}$ and $y^{-} \notin x^{+}$. By construction these are equivalent to $x <^{*}_{\alpha} y$ and $y \notin^{*}_{\alpha} x$ respectively, i.e., $x <^{\alpha} y$. But by ($* L$) $x <^{\alpha}_{z} z$ for all $z \neq x$. Hence $x <^{\alpha}_{x} y$ as required.

From this result we see that $*_{L}$ is axiomatically characterised by ($1$)$–$(7) plus ($L$). However it is easy to see that ($L$) implies ($5$)$–$(7). ($1$) also becomes redundant, since ($3$), ($4$) and ($L$) are enough to force the unique tpo $\leq^{*}_{\alpha}$, and we already established after Proposition 3 that ($2$) can be removed. Hence ($3$), ($4$) and ($L$) form a sound and complete axiomatisation for $*_{L}$. The sentential counterpart of ($L$) is the rule Recalcitrance of [17], i.e.,

($\circ L$) If $\beta \not\vdash \neg \alpha$ then $\alpha \in \leq^{*}_{\alpha} \circ \beta$.

Note also that new evidence is always believed after lexicographic revision. A characterisation of $*_{1}$ in terms of social choice-like conditions was given by [9], who referred to it as “J-revision”.

At the other extreme, suppose instead we insist on

($\leq P$) $x < y$ implies $x^{-} < y^{+}$.

This rule is equivalent to saying $\llll = <$. (Thus also $\llll = <$.) This property expresses maximal confidence behind the initial tpo $\leq$, or skepticism towards new evidence. Adding this rule to ($\leq 2$)$–($5) is again enough to specify $\leq$ completely. It is not difficult to show this time we are left with $x \delta \leq y^{e}$ iff either $x < y$ or $[x \sim y$ and $(\delta = +$ or $\epsilon = -)]$:

<table>
<thead>
<tr>
<th></th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$\cdots$</th>
<th>$R_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>1</td>
<td>3</td>
<td>$\cdots$</td>
<td>$2n - 1$</td>
</tr>
<tr>
<td>-</td>
<td>2</td>
<td>4</td>
<td>$\cdots$</td>
<td>$2n$</td>
</tr>
</tbody>
</table>
The associated revision operator \( *_P \) is then given by
\[
x \leq \alpha y \text{ iff either } x < y \text{ or } (x \sim y \text{ and } x \leq \alpha y).
\]

This is a “reverse” lexicographic method, studied in the context of iterated belief revision [19]. This time it corresponds to \( \leq \) being refined by \( \leq \alpha \). In this case new evidence is not always believed.

**Proposition 10** If \( * \) is generated from some \( \leq \)-faithful tpo over \( W^+ \) satisfying \( (\leq P) \) then \( * \) satisfies
\[
(P) \quad \text{If } x \in [-\alpha], \ y \in [\alpha] \text{ and } x < y \text{ then } x <^*_\alpha y.
\]

Furthermore if \( * \) is any revision operator for \( \leq \) which satisfies \((P)\) then the \( \leq \)-faithful tpo \( \leq_* \) defined right after Theorem 1 satisfies \((P)\).

**Proof:** Suppose \( * = *_* \) for some \( \leq \) satisfying \((P)\). To show \( * \) satisfies \((P)\) suppose \( x \in [-\alpha], \ y \in [\alpha] \) and \( x < y \). Then \( r_\alpha(x) = x^- \) and \( r_\alpha(y) = y^+ \). Since \( x < y \) we may apply \((P)\) to deduce \( r_\alpha(x) < r_\alpha(y) \), i.e., \( x <^*_\alpha y \) as required.

For the second part let \( * \) be a revision operator which satisfies \((P)\) and suppose \( x < y \). We want to show \( x^- \prec_* y^+ \), i.e., both \( x^- \leq_* y^+ \) and \( y^+ \nprec_* x^- \). If \( x < y \) then clearly \( x \neq y \), hence by construction this is equivalent to showing \( x \leq^*_y y \) and \( y \nprec_* x \), i.e., \( x <^*_y y \). But from \((P)\) we know \( z <^*_y y \) for all \( z \neq y \) such that \( z < y \). Hence \( x <^*_y y \) as required. \( \blacksquare \)

This result implies \( *_P \) may be characterised axiomatically by \((1)\)–\((7)\) plus \((P)\). However we may significantly simplify this list by observing the following:

**Proposition 11** Let \( * \) be any revision operator for \( \leq \) satisfying \((3), (4) \) and \((5)\). Then \( * \) together satisfies \((6), (7) \) and \((P)\) iff \( * \) satisfies:
\[
(p) \quad \leq \subseteq <^*_\alpha.
\]

**Proof:** \((6), (7), (P) \Rightarrow (p)\)
In fact we show that, in the presence of the other rules, \((P)\) is enough to prove \((p)\) on its own. Suppose \( x < y \). To show \((p)\) we must show \( x <^*_\alpha y \). We look at each of the cases \( y <^\alpha x \), \( x <^\alpha y \) and \( x \sim^\alpha y \). If \( y <^\alpha x \) then the required conclusion follows immediately from \((P)\). If \( x <^\alpha y \) then the conclusion follows from \((5)\). Finally if \( x \sim^\alpha y \) then the conclusion follows from \((3)\) or \((4)\).

\((p) \Rightarrow (6), (7), (P)\)
\((p) \Rightarrow (P)\) is immediate. To show \((p)\) implies the other two rules we show in fact \((p)\) implies the following property, which is easily seen to be stronger than both \((6)\) and \((7)\):

If \( x <^\alpha y \) and \( y \leq^*_\alpha x \) then \( y <^*_\alpha x \).

This property holds since \( x <^\alpha y \) and \( y \leq^*_\alpha x \) then \( y < x \) by \((5)\). Hence \( y <^*_\alpha x \) follows by \((p)\).

Again \((1)\) becomes redundant, and so we arrive at the following characterisation of \( *_P \).
**Proposition 12** $\ast_P$ is the unique revision operator for $\leq$ which satisfies ($\ast 3$)–($\ast 5$) plus ($\ast p$).

It is easy to see the sentential counterpart of ($\ast p$) is the following rule:

$$(\circ p) \leq \circ \beta \subseteq \leq_{\alpha} \circ \beta.$$ 

($\circ p$) states that *all* conditional beliefs in $\leq$ are preserved after revision.

As the following example shows (partly based on one by [7]), rigid use of either of these limiting cases $\ast_L$ and $\ast_P$ can lead to counter-intuitive results.

**Example 2** Suppose we have a murder trial with two main suspects, John and Mary. Let $p$ represent “John is the murderer” and $q$ represent “Mary is the murderer”. Furthermore let $r$ represent “The victim is an alien from outer-space”.

Initially we believe the murder was committed by one person, either John or Mary. However we wouldn’t be surprised to discover that either both or neither were involved in the crime. What would be surprising – indeed highly shocking – would be if we found out the victim was an alien. However we are still capable of imagining a hypothetical situation in which this turns out to be the case, and we think this would not alter our belief that either John or Mary acted alone. If we were to represent all this using a tpo $\leq$, it seems the following is the best candidate:

$$\begin{array}{cccc}
100 & 110 & 101 & 111 \\
010 & 000 & 011 & 001 \\
\end{array}$$

Now during the trial we receive testimony that John is the murderer, leading us to revise $\leq$ by $p$. Supposing we then receive testimony that Mary is the murderer, the most reasonable conclusion would be that both John and Mary were involved in the murder. But using the operator $\ast_P$ gives

$$\leq_{\ast p} \circ q = Cn(\neg p \land q \land \neg r)$$

We are forced to drop our belief that John is the murderer.

Now consider the situation where we receive testimony that John is the murderer, followed by the supposition that if John is the murderer, then the victim is an alien. In this case it seems the reasonable thing to do is drop the acquired belief that John is the murderer. However, using the operator $\ast_L$ gives

$$\leq_{\ast L} \circ (p \rightarrow r) = Cn(p \land \neg q \land r)$$

That is, we end up believing John murdered an alien!

The move to our more general family of tpo-revision operators enables a correct treatment of both these scenarios simultaneously. Consider the $\leq$-faithful tpo $\leq$ represented by:

$$\begin{array}{cccc}
R_1 & R_2 & R_3 & R_4 \\
+ & 1 & 2 & 5 & 6 \\
- & 3 & 4 & 7 & 8 \\
\end{array}$$
In the first case where we receive evidence pointing towards John’s guilt followed by the supposi-
tion Mary did it, we have
\[ \leq_p^* \circ q = Cn(p \land q \land \neg r) \]
which is the intuitive result. In the case where we receive evidence for John being the murderer,
followed by supposing that if John is the murderer then the victim is an alien, we have
\[ \leq_p^* \circ (p \to r) = Cn(\neg p \land q \land \neg r) \]
which is what we would expect.

7 A further sub-class

Close inspection reveals that both the limiting cases mentioned above share something in common – in both cases we have \( \propto \leq \propto \). Writing out this condition in full, the unique \( \propto \) defined in each
case satisfies:
\[(\leq 5) \quad x^- \leq y^+ \iff x^- \prec y^+.\]
This condition states that no \( x^- \) appears in the same \( \leq \)-rank as a \( y^+ \). In this section we take a look
at the subclass of our family of revision operators defined by enforcing this condition.

Firstly, in terms of the graphical representation of \( \propto \) the effect of \( (\leq 5) \) is simple: it just means
that no number is allowed to appear twice in the array of numbers. Another thing to notice is
that if \( \propto \propto \propto \) then the distinction between the overrules relation and the strictly
overrules relation relative to \( \propto \)-disappears – they collapse into the same binary relation. As for an axiomatic
characterisation of this subfamily, the next result points the way:

**Proposition 13** If \( * \) is generated from some \( \leq \)-faithful tpo over \( W^\pm \) satisfying \( (\leq 5) \) then \( * \) satisfies
\[(*8) \quad \text{For } x \in [\alpha] \text{ and } y \in [\neg \alpha], \text{ either } x \prec_\alpha y \text{ or } y \prec_\alpha x.\]
Furthermore if \( * \) is any revision operator for \( \leq \) which satisfies \((*8)\) then the \( \leq \)-faithful tpo \( \leq_s \)
defined right after Theorem 1 satisfies \((\leq 5)\).

**Proof:** For the first part let \( * = *_\leq \) for some \( \leq \)-faithful tpo satisfying \((\leq 5)\). Let \( x \in [\alpha] \) and
\( y \in [-\alpha] \). Then to show the consequent of \((*8)\) we need to show that either \( x^+ \prec y^- \) or \( y^- \prec x^+ \).
By \((\leq 5)\) we can replace the second disjunct here by \( y^- \leq x^+ \). But since \( \leq \) is a tpo (by \((\leq 1)\)) we
always have either \( x^+ \prec y^- \) or \( y^- \leq x^+ \). Hence the consequent of \((*8)\) holds.

For the second part let \( * \) be a revision operator satisfying \((*8)\). We want to show \( x^- \leq_s y^+ \) iff \( x^- \prec_s y^+ \). If \( x = y \) we know \( x^+ \prec_s x^- \) so neither of these conditions can hold, making the
biconditional true in this case. So suppose \( x \neq y \). In this case the first condition is equivalent to
\( x \leq_y y \) while the second is equivalent to \( x \prec_y y \). But from \((*8)\) (since \( x \neq y \)) we know either
\( x \prec_y y \) or \( y \prec_x x \), i.e., \( x \not\prec_y y \). This means \( x \leq_s y \) can hold iff \( x \prec_y y \), as required.

Condition \((*8)\) means that after revising by \( \alpha \), there is a separation between \( \alpha \)-worlds and \( -\alpha \)-worlds, in the sense that each \( \leq^*_\alpha \)-rank contains either only \( \alpha \)-worlds or only \( -\alpha \)-worlds. This
property is called (UR) by [5], where it is shown that its sentential counterpart is:

\((\circ 8)\) If \(\neg\alpha \not\leq \circ_{\alpha} \circ \beta\) then \(\alpha \leq \circ_{\alpha} \circ \beta\).

The postulate \((\circ 8)\) does have a certain amount of intuitive appeal. It says that after receiving \(\alpha\) as evidence and then making the supposition \(\beta\), \(\alpha\) should be believed as long as it is consistent to do so.

\((\ast 8)\), alias \((\circ 8)\), is quite a strong rule, and adding it to the list \((\ast 1)\)–\((\ast 7)\) causes some redundancies. Since \((\ast 8)\) implies the equivalence of \(x \leq_{\alpha} y\) with \(x <_{\alpha} y\) for \(x \not=_{\alpha} y\), we see \((\ast 6)\) now follows from \((\ast 7)\). Meanwhile \((\ast 5)\) becomes equivalent to “if \(x <_{\alpha} y\) and \(x \leq y\) then \(x \leq_{\alpha} y\)” (i.e., (C4R) proposed by [7]). But using the fact that \(\leq = \leq_{\top}\) (which follows from \((\ast 3)\)), this is seen as just the instance of \((\ast 7)\) in which \(\gamma = \top\). Hence \((\ast 5)\) also disappears. Thus the class of tpo-revision operators generated by those \(\leq\)-faithful tpos over \(W_{\pm}\) satisfying \((\leq 5)\) may be characterised as follows:

**Theorem 2** Let \(*\) be a revision operator for \(\leq\). Then \(*\) is generated from some \(\leq\)-faithful tpo over \(W_{\pm}\) satisfying \((\leq 5)\) iff \(*\) satisfies \((\ast 1)\), \((\ast 3)\), \((\ast 4)\), \((\ast 7)\) and \((\ast 8)\).

Of course we can if we wish replace the last four rules above with their sentential equivalents.

## 8 Conclusion and future work

We have introduced a new family of operators for revising total preorders by sentences based on the simple intuitive idea that when we compare possibilities, we are often able to imagine these possibilities with regard to “best case” and “worst case” scenarios. We have placed this family firmly in the context of the problem of iterated belief revision, and have shown that our results significantly extend current work on this topic.

On the level of belief sets, our operators fall within the realm of non-prioritised revision, in that revision inputs are not necessarily elements of the belief set associated to the revised preorder. This is in contrast to most works on iterated belief change, which are usually given in the “prioritised” setting (with the work of [4] being an exception). We envisage prioritised revision by \(\alpha\) as a two-stage process, with the first stage being carried out by one of the operators in this paper, and then the second stage consisting of an application of Boutilier’s natural revision [6] of the resulting tpo by \(\alpha\), i.e., the most preferred \(\alpha\)-worlds are simply brought if necessary to the front of the new tpo. For the special case of the operator \(*_{p}\), this was already done by [5], leading to the restrained revision operator (see Section 5 of that paper). For future work we plan to apply this to the more general family.

Another direction for future research is the investigation of larger families of revision operators, such as those obtained by weakening one, or both, of \((\leq 2)\) and \((\leq 3)\). Observe that this is equivalent to weakening \((\ast 3)\) and \((\ast 4)\), or \((\circ 3)\) and \((\circ 4)\). The weakening of \((\circ 4)\) will be of particular interest, since it is essentially equivalent to the much-criticised postulate (C2) proposed by [7].

Conversely, it would be interesting to consider special subclasses of our general family. We considered one in the last section. Another example could be the family obtained by taking \(\ll\) or \(\ll\) to be modular orderings. Finally note that our operators do not conform to the principle of
categorical matching – from an initial tpo $\leq$ together with a $\preceq$-faithful tpo $\preceq$ over $W^\pm$ they return a new tpo $\leq_{\alpha^*}$, but give no help on defining a new $\leq_{\alpha^*}$-faithful tpo over $W^\pm$ which can then be used to further revise $\leq_{\alpha^*}$. One way of rectifying this might be to preserve as much of $\ll$ and $\ll$ as possible.

9 Acknowledgements

Most of the paper was written during a visit by Richard Booth to the KRR group at NICTA in Sydney. Some valuable feedback was received from audience members during a presentation of this material to the KR group at Leipzig University. Thanks are also due to the reviewers for some helpful suggestions. National ICT Australia is funded by the Australia Government’s Department of Communications, Information and Technology and the Arts and the Australian Research Council through Backing Australia’s Ability and the ICT Centre of Excellence program. It is supported by its members the Australian National University, University of NSW, ACT Government, NSW Government and affiliate partner University of Sydney.

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