



Methods and Models for Decision Making

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God in 7 steps:

- Four classes of decision problems
- The main two (in this context) → ranking, rating
- Binary relations ($A > B$, $A \geq B$, $A \sim B$, $A ? B$)
- Ranking-1 → the risk analysis
- Non-deterministic environment (random outcomes)
- Lotteries to measure the risk attitude of the DM
- Utility function (one for each indicator) of this DM

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- **(7) Ranking-2, multicriteria**
- **(8) A tentative case**
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- (11) Group decision
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- (14) Case results (if any ...)
- (15) Conclusions

Ranking-2: multicriteria analysis

MultiObjective / MultiCriteria

- ❑ Decision problem with one DM and full information
- ❑ Different points of view (objectives or criteria)
- ❑ Final solution = a good trade-off between the criteria
- ❑ Various phases →
 - Ph1: from indicators to utilities
 - Ph2: subset of efficient solutions
 - Ph3: preference and final solution
- ❑ Two cases (one continuous, one discrete) for understanding

The various phases

- ❑ A decision problem with different (conflicting) objectives/criteria
- ❑ Objectives = continuous case // Criteria = discrete case
- ❑ The need of a synthesis (considering different points of view)

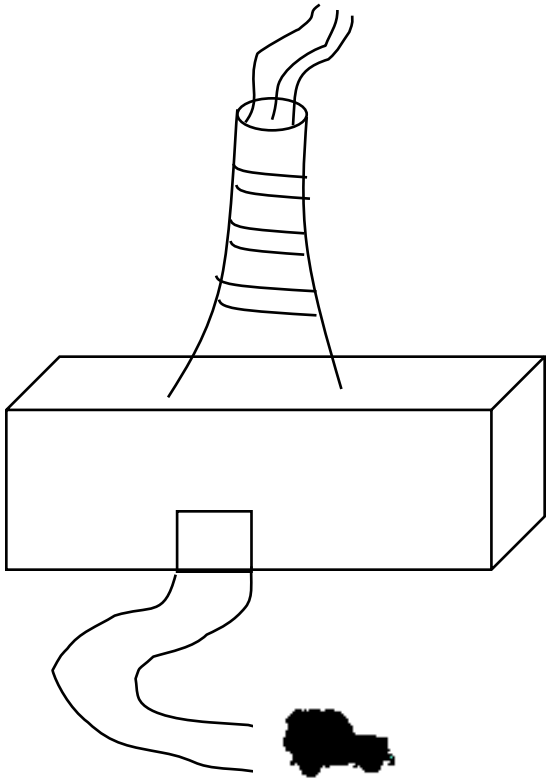
Ph1 – The treatment of different data (from indicators to utilities)

Ph2 – The search of efficient (or non-dominated or Pareto) solutions

Ph3 – The final (best trade-off solution) choice and the sensitivity

- ❑ The procedure is not “objective”, but the analysis can point out the crucial aspects of subjectivity (what influence, where, ...)

Example – The incinerator project



*There is a air
standard quality Q^**

- **Variables of decision:**
 - D = plant dimension,
 - H = smokestack height,
 - P = % of pollutant eliminated
- **Sectors of attention:**
 - economics,
 - waste service,
 - fly safety (the smokestack),
 - local viability (congestion),
 - environment.

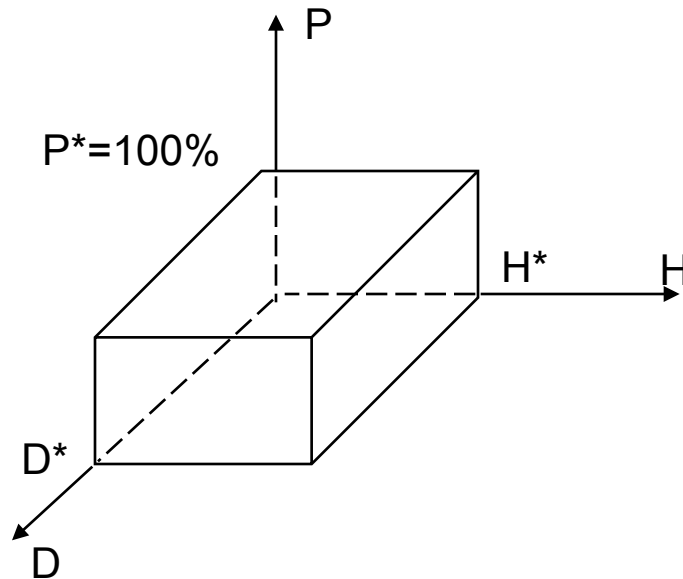
Indicators (measures of the effects)

Sector	Indicator	Constraint	Objective
Economics	R (benefits-costs)	-	max R
Service	D (smaltiti wastes)	$D \leq D^*$	-
Security	H (air trafic)	$H \leq H^*$	-
Viability	D (number of vehicles)	$D \leq D^*$	-
Environm.	P (% removed particules)	$P \leq 100\%$	max Q/Q*

- Indicators →
 - directly in the constraints
 - directly in the obj. functions
 - undirectly in the o.f. (i.e. particules)
- Sector models (to supply the measures)

The variable space (decisions)

- Three (continuous) variables



$$D \leq D^*$$

$$H \leq H^*$$

$$P \leq P^* = 100\%$$

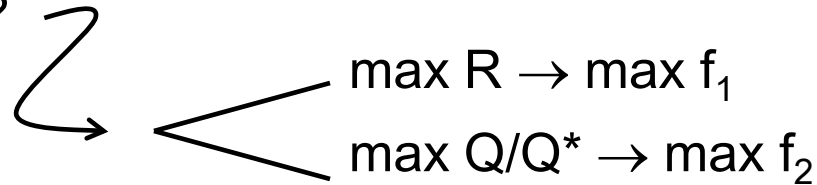


The feasible region X
is a parallelepiped

- Each point $\mathbf{x} \in X$ is a feasible solution (∞ solutions)
- For each point \mathbf{x} it is possible to compute the values of R & Q
(sector models)

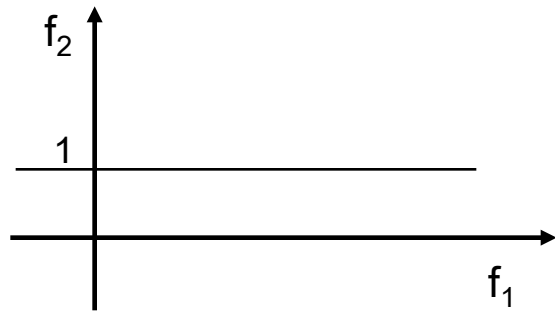
The objective space (results)

- How many objectives ?



[attention: it must be $f_2 = Q/Q^* \geq 1 \rightarrow$ why ?]

- Two dimensions



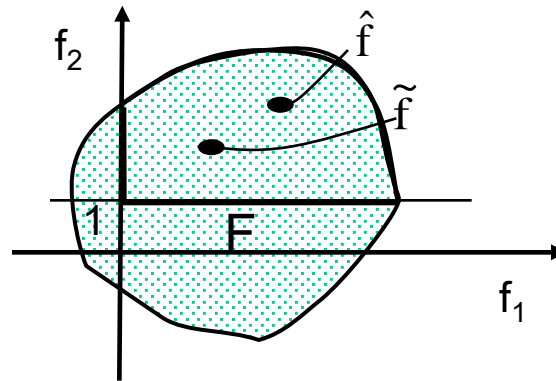
$$f_1 = R$$
$$f_2 = Q/Q^*$$

Because
it must be
 $Q \geq Q^*$

- Each vector \mathbf{x} (a tern of decision variables)
corresponds to a vector \mathbf{f} (a couple of results)

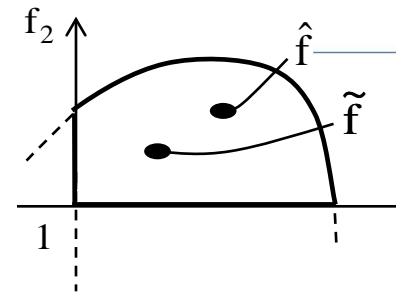
How obtaining F from X

- Region X is known (you can explore it)
- For each $\bar{x} \rightarrow$ the corresponding \bar{f}
- In general: $x \in X \rightarrow f \in F$
- So, you have F



- Question: given two vectors of results, is it better \hat{f} or \tilde{f} ?

Dominance



- Comparison between \hat{f} and \tilde{f}
- \hat{f} dominates \tilde{f} (and the solution \hat{x} dominates the solution \tilde{x}): why ?
- Definition (1), **dominance** \rightarrow in a decision problem with m objectives (to be maximized) $\max f_1(x), \dots, \max f_m(x)$, a solution x dominates a solution y if $f_1(x) \geq f_1(y), \dots, f_m(x) \geq f_m(y)$, that is the solution x obtains better (or equivalent) results with respect to the solution y , for all the objectives.
- Definition (2), **efficient solution** \rightarrow a solution x non dominated by any other solution is called efficient (or parentian).

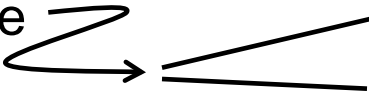
Example2 – A sabbatical year

- Professor C. has to decide where going for a sabbatical year
- Data are the following:

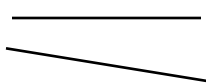
	<i>Rome</i>	<i>Berlin</i>	<i>Geneva</i>	<i>Moscow</i>	<i>Tokio</i>
Reward	5	7	10	2	7
University prestige	3	9	4	6	5
Quality of life	10	4	5	3	3

- Qualitative scales, converted in numerical [0, 10] ones
- Search for the best choice, between the 5 alternatives
- A multi-criteria (discrete set of options) decision problem

More about dominance

- In this context it is still valid the concept of dominance ?
- There are  2 dominated solutions
3 efficient (non dominated) solutions
- If the data are correct and if the teacher is rational, he must choose only between → **Rome – Berlin – Geneva**
- So he has reduced the options, but he doesn't already chosen the final solution
- What option ? It depends on the importance that the teacher acknowledges to the various criteria: economics, working place, environment
- The preference structure of the DM could be very complex; but in the simpler case it is a vector with dimension equal to the number of criteria (3 in this case)

Common & different features

- Common elements:
 - deterministic problems (all the data are known)
 - multi objective/criteria  2 in the case of incinerator
3 in the case of sabbatical
 - only one DM
- Decision problems → 1/m/d (1 dec. maker / m criteria / det. info)
- Different elements:
 - continuous problema with ∞ solutions (MODM),
discrete problem with only 5 alternatives (MCDM)
 - in one case (incinerator) we have done only definitions,
in the other (sabbatical) we obtained the efficient solutions
- MODM (or MCDM) → trade-off → **subjectivity**

The reference frame

- Three axis

- The 1/m/d case →

Decision with
m objectives

- Formulation →

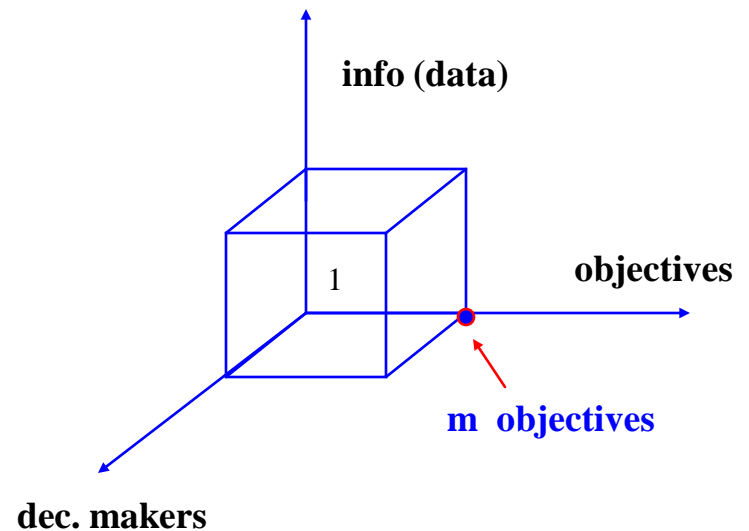
Min or max
with $x \in X$

(a vector of obj. functions) ←

$$\begin{array}{|l} f_1(x) \\ f_2(x) \\ : \\ f_m(x) \end{array}$$

- Problems

continuous case → multi-objective analys
discrete case → multi-criteria analys



Three phases of the choice

- **Phase 1 → Data analysis**
 - the objectives of the decision maker are measured by functions
 - each function shows the value of an indicator
 - each indicator has his own unit
 - to compare a common scale is needed
 - the scale is the measure of the utilities perceived by the decision maker
- **Phase 2 → Efficient solutions**
 - are there some dominated solutions among the others (infinite or prearranged)?
 - elimination of the dominated solutions
 - not dominated or efficient or Pareto solutions (synonyms) remain
- **Phase 3 → Final choice**
 - Analysis of the preferences structure of the decision maker
 - vector of weights (pair comparison)
 - weighted sum of the utility of each alternative
 - ranking, final choice, sensitivity

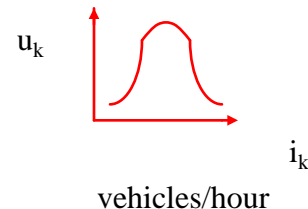
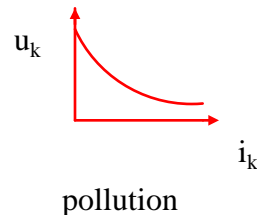
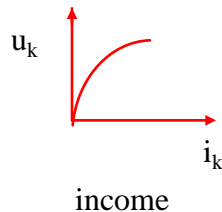
Phase 1 – Indicators (and their units of measure)

- **Example of the incinerator :**

$\max f_1$ (profit) → millions €/year

$\max f_2$ (air quality) → fraction between 2 values in mg/m^3

- **What:** to analyze the link between a certain indicator and **utility** perceived by the decision maker → a function $u_k(i_k)$, where i_k represent the value of the indicator related to the objective-function $f_k(x)$
- **Why:** the **utility function** u_k allows to affirm that the solution \hat{x} is better than the solution \bar{x} (following that objective or criteria) if $u_k(\hat{x}) > u_k(\bar{x})$ while there is no preference if $u_k(\hat{x}) = u_k(\bar{x})$
- **Examples of utility functions**



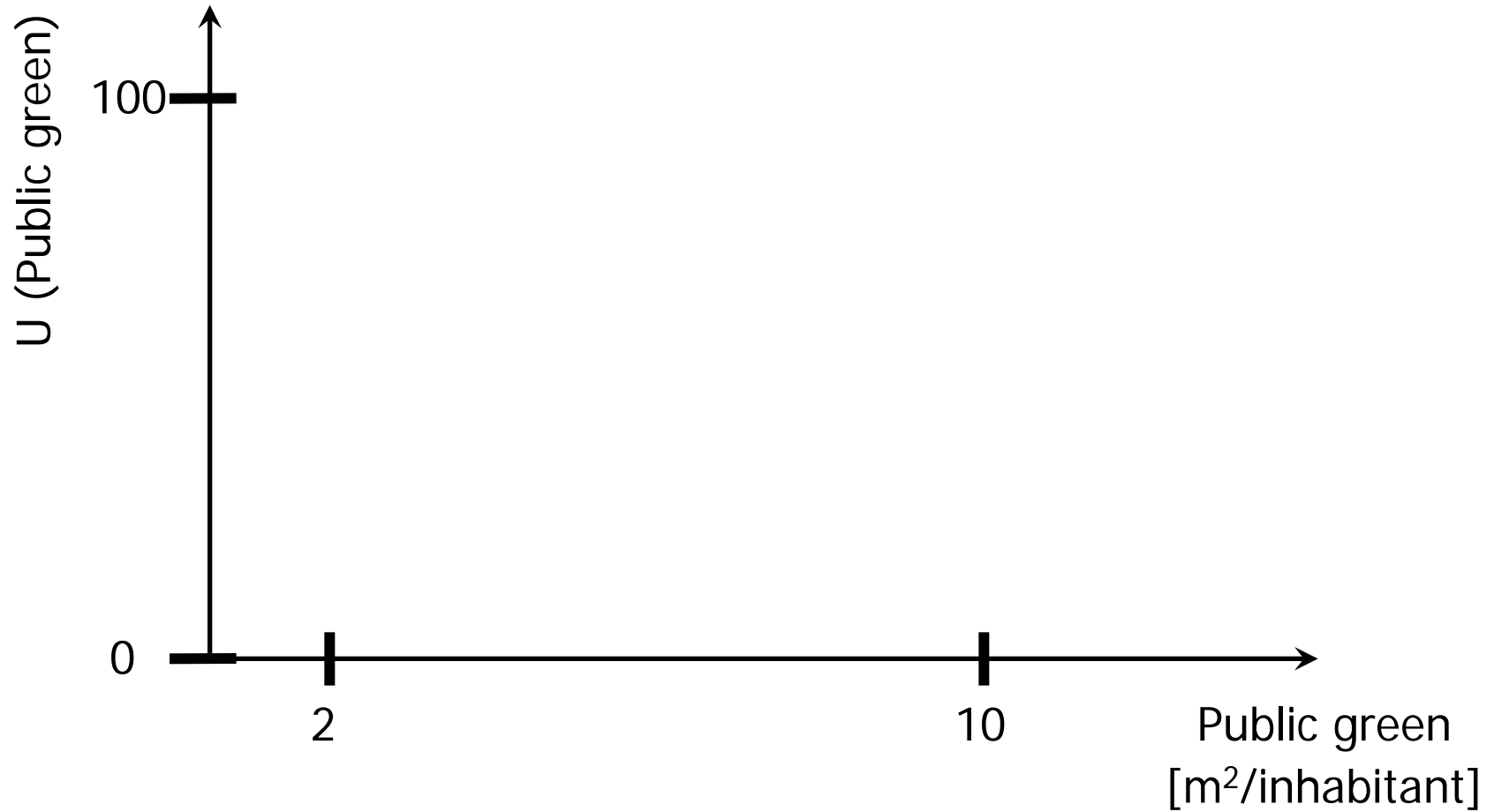
Estimation of the utility functions

- By the literature
- By an empirical procedure (points):

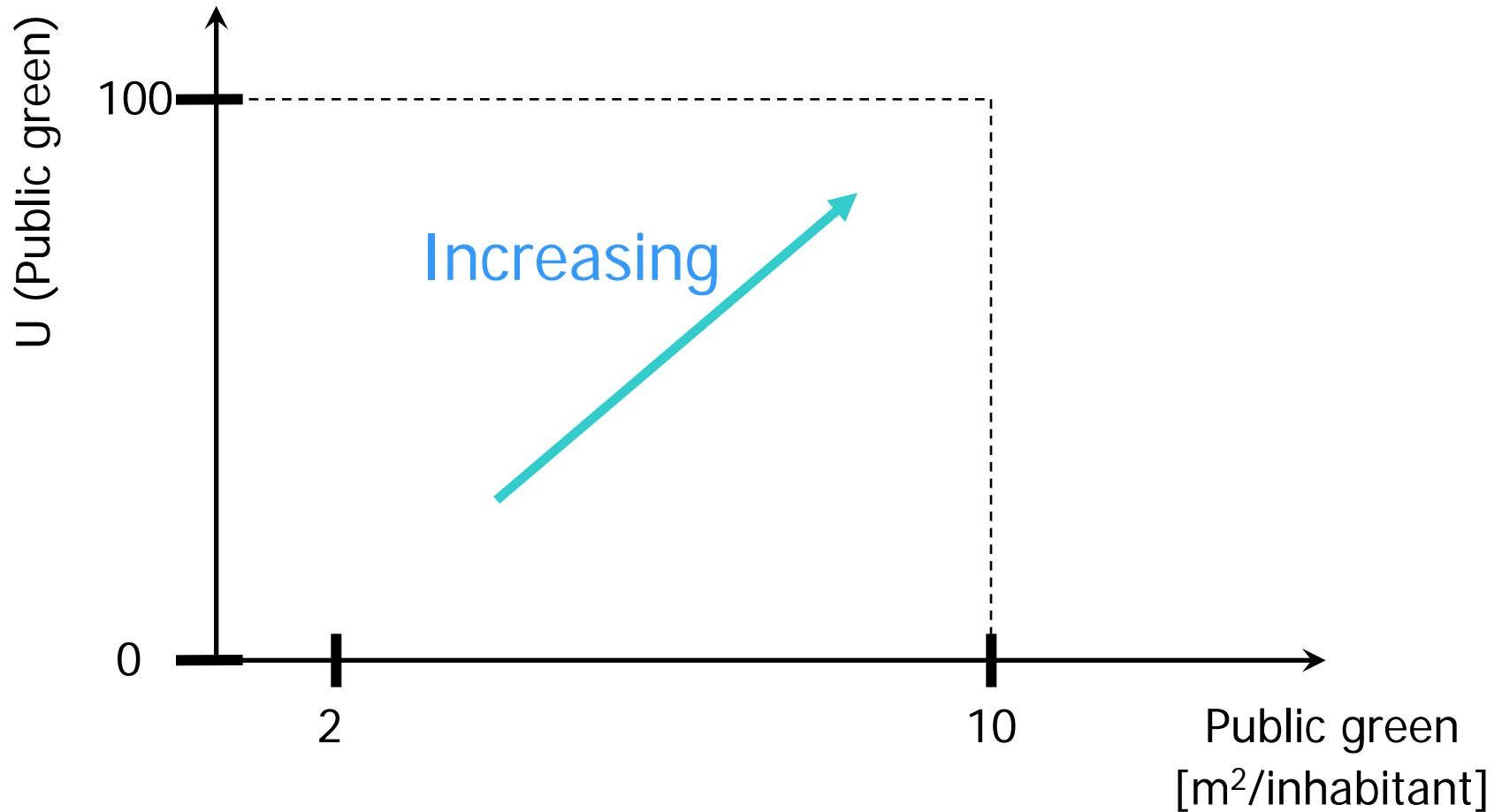


1. To define the range of admissible value for the considered attribute (wide? narrow?)
2. To state the shape of the utility function (increasing? decreasing? Non-monotonic?)
3. To estimate the function

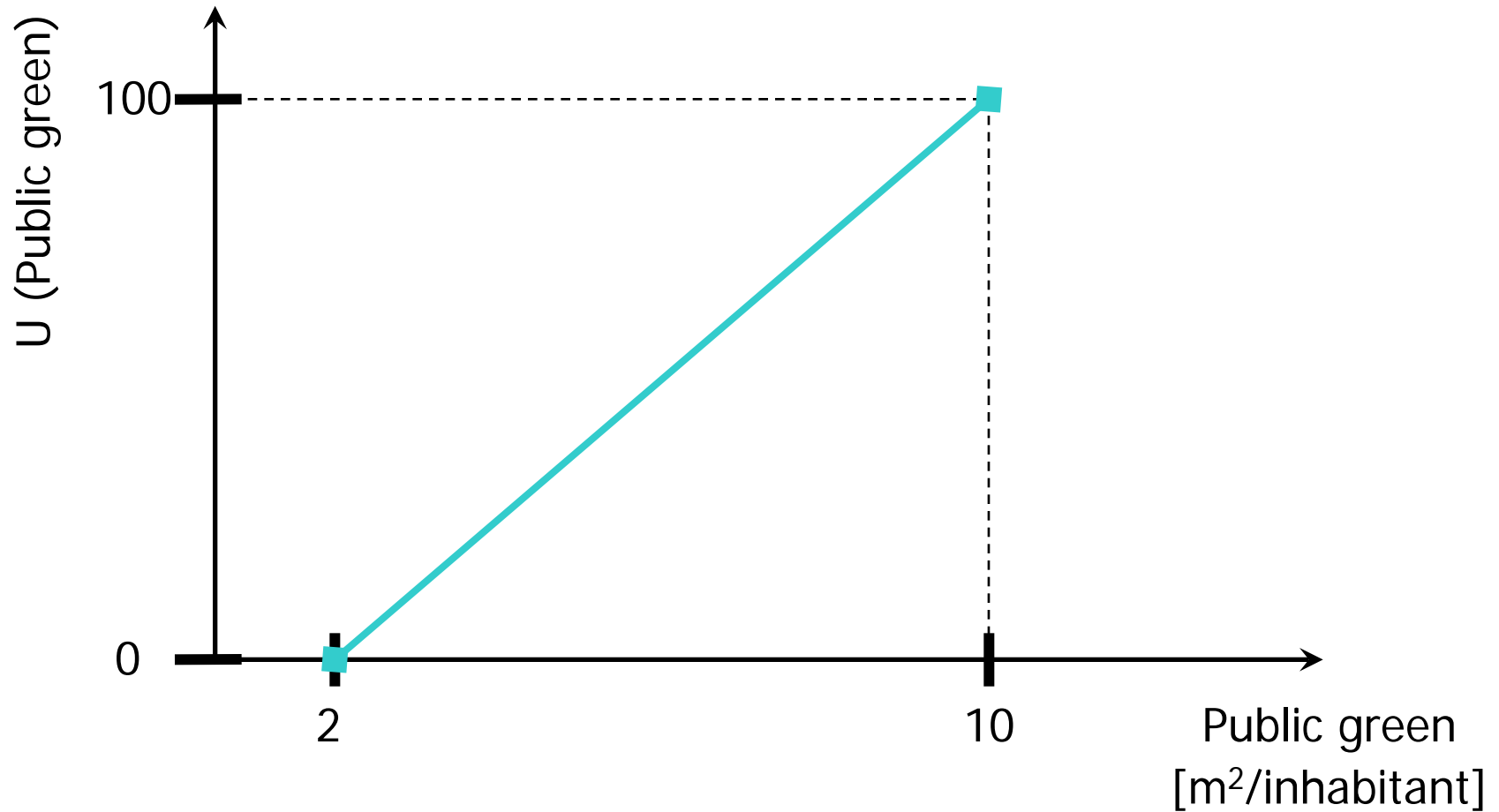
The mean fraction: step 1



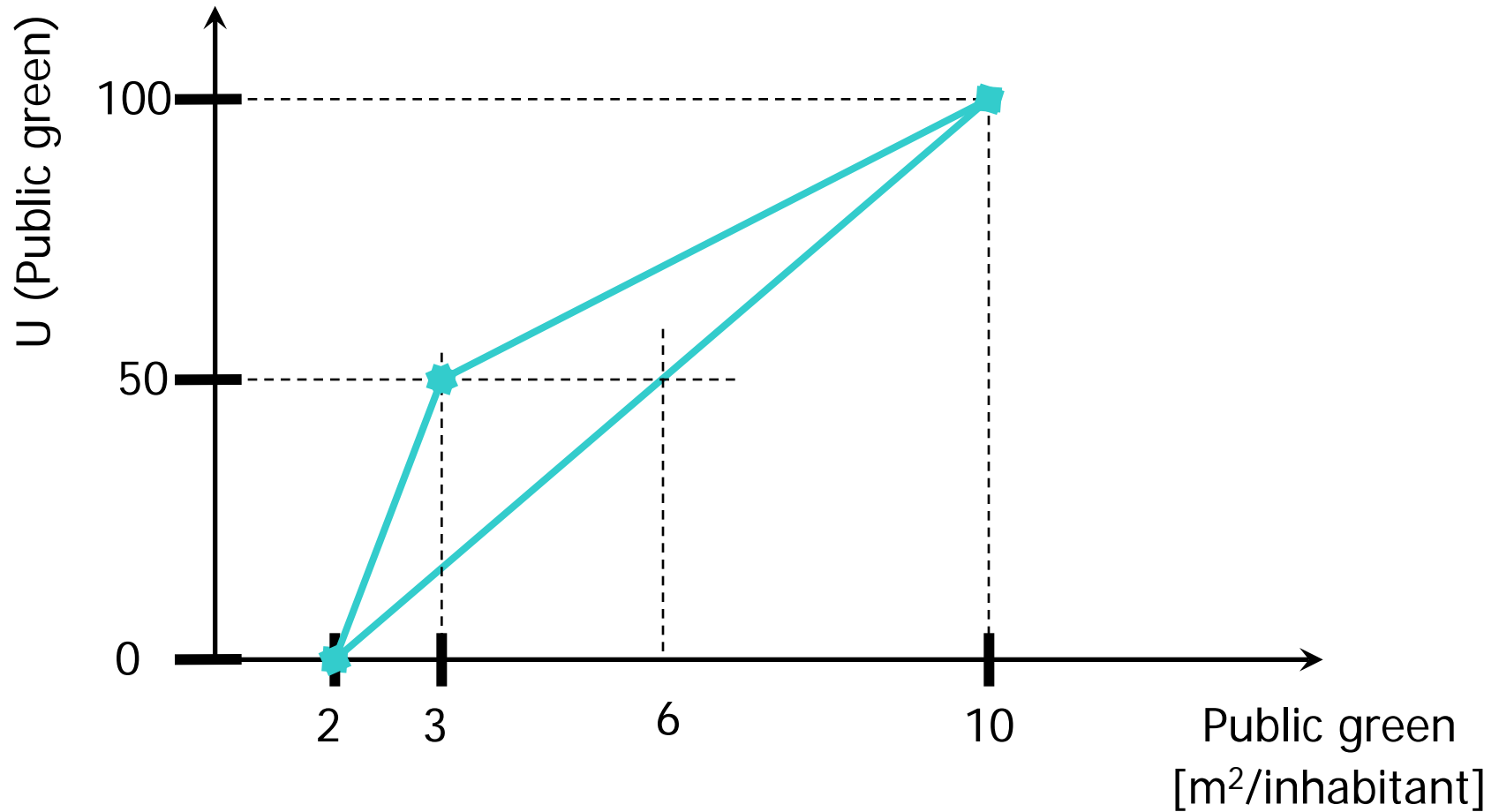
The mean fraction : step 2



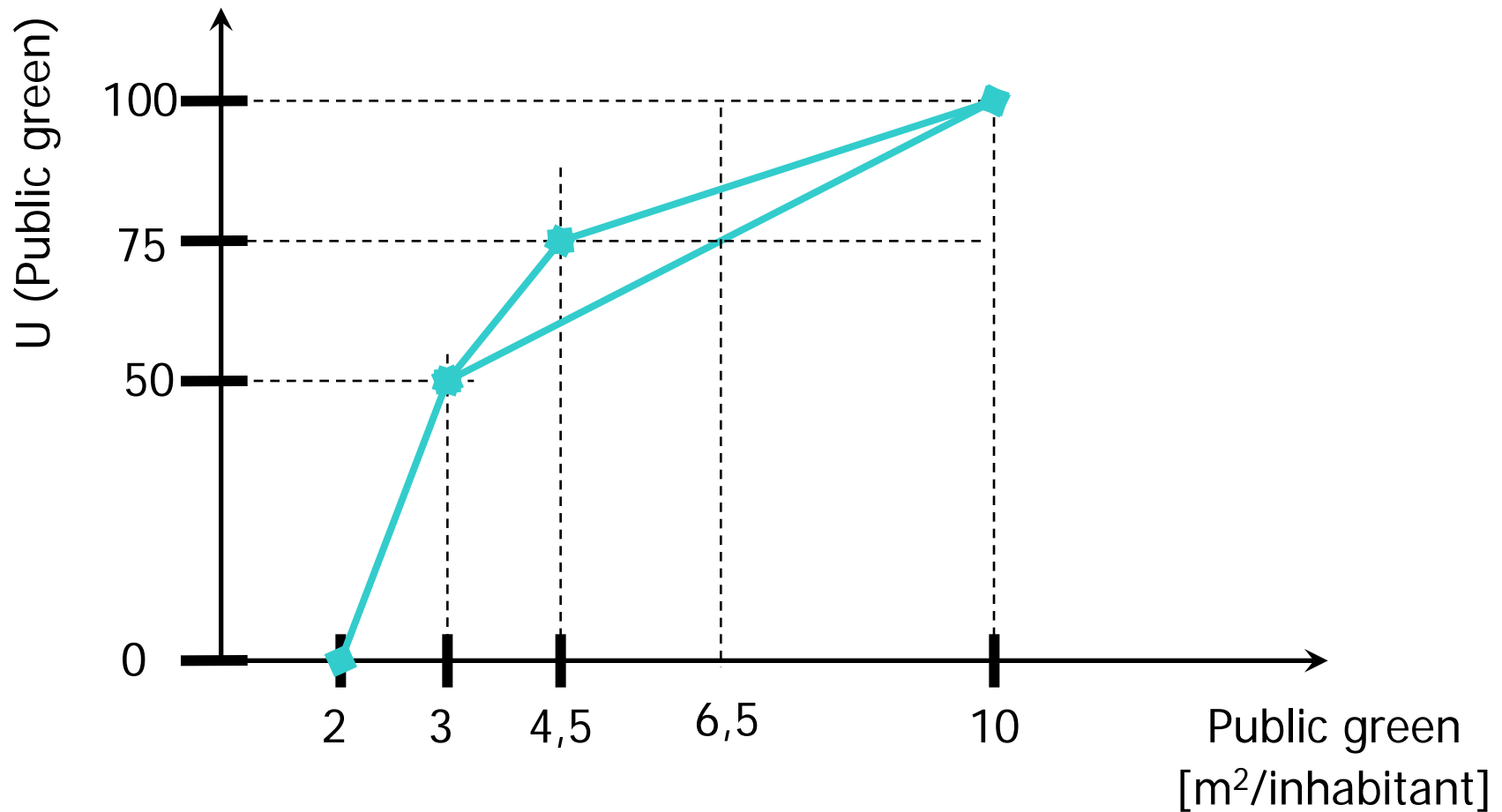
The mean fraction : step 3



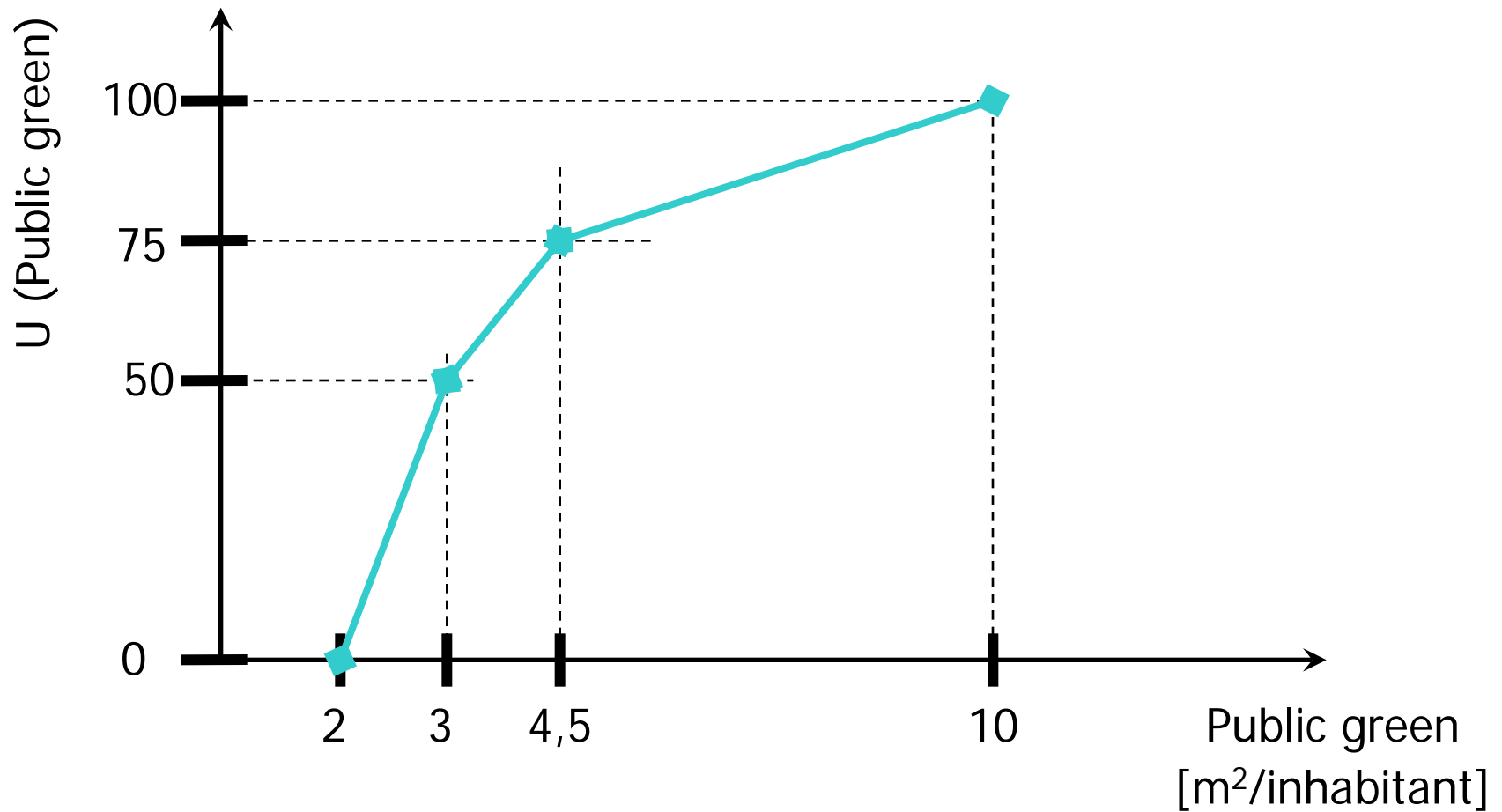
The mean fraction : step 3



The mean fraction : step 3



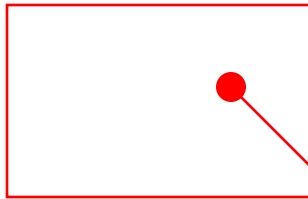
The mean fraction : step 3



Phase 2 – Evaluation matrix

- **Discrete case: Multi Criteria Analysis (MCA)**
 - a finite number (usually small) of alternatives
 - a finite number of criteria (m)

- **Evaluation matrix**



rows (m) \rightarrow criteria

columns (n) \rightarrow alternatives

u_{kj} = **utility** with respect to criterion k of the alternative j

- **Example (sabbatical):**

	R	B	G	M	T
Reward	5	7	10	2	7
University prestige	3	9	4	6	5
Quality of life	10	4	5	3	3

Values are in the conventional scale

[0, 10]

Phase 2 – Efficient solutions

- Are there cities in which the teacher will not (...) in the future ?

Search of the dominated alternatives
(and then of the efficient alternatives)



Phase 2

- **Dominance** → alternative A dominates alternative B if:

$$u_{1A} \geq u_{1B}, u_{2A} \geq u_{2B}, \dots, u_{mA} \geq u_{mB}$$

(and if for at least an attribute there is >)



- **Search of efficient solutions**

comparison between r columns
(how many comparisons?)

- **Example** R dominates B, or vice versa ?
R dominates G, or vice versa ?
M dominates T, or vice versa ?

NO

B dominates M
B dominates T

Efficient solutions are → Z, R, B


Phase 3 – The final choice

- One more element → the preferences structure

- Matrix

	Rome	Berlin	Geneve	Moscow *	Tokyo *
Reward	5	7	10	2	7
University prestige	3	9	4	6	5
Quality of life	10	4	5	3	3

Evaluation matrix



0.3
0.6
0.1

weights

(*) dominated alternative

- The vector of the weights measures the importance that the decision maker gives to the criteria (objectives)

- Weighted sum:

	Rome	Berlin	Geneva	Moscow	Tokyo
	4.3	7.9	5.9	4.5	5.4
	(5°)	(1°)	(2°)	(4°)	(3°)

These values (total utility) are calculated as sum of the products of the rows and the weights

- What does it mean ? What is his use?



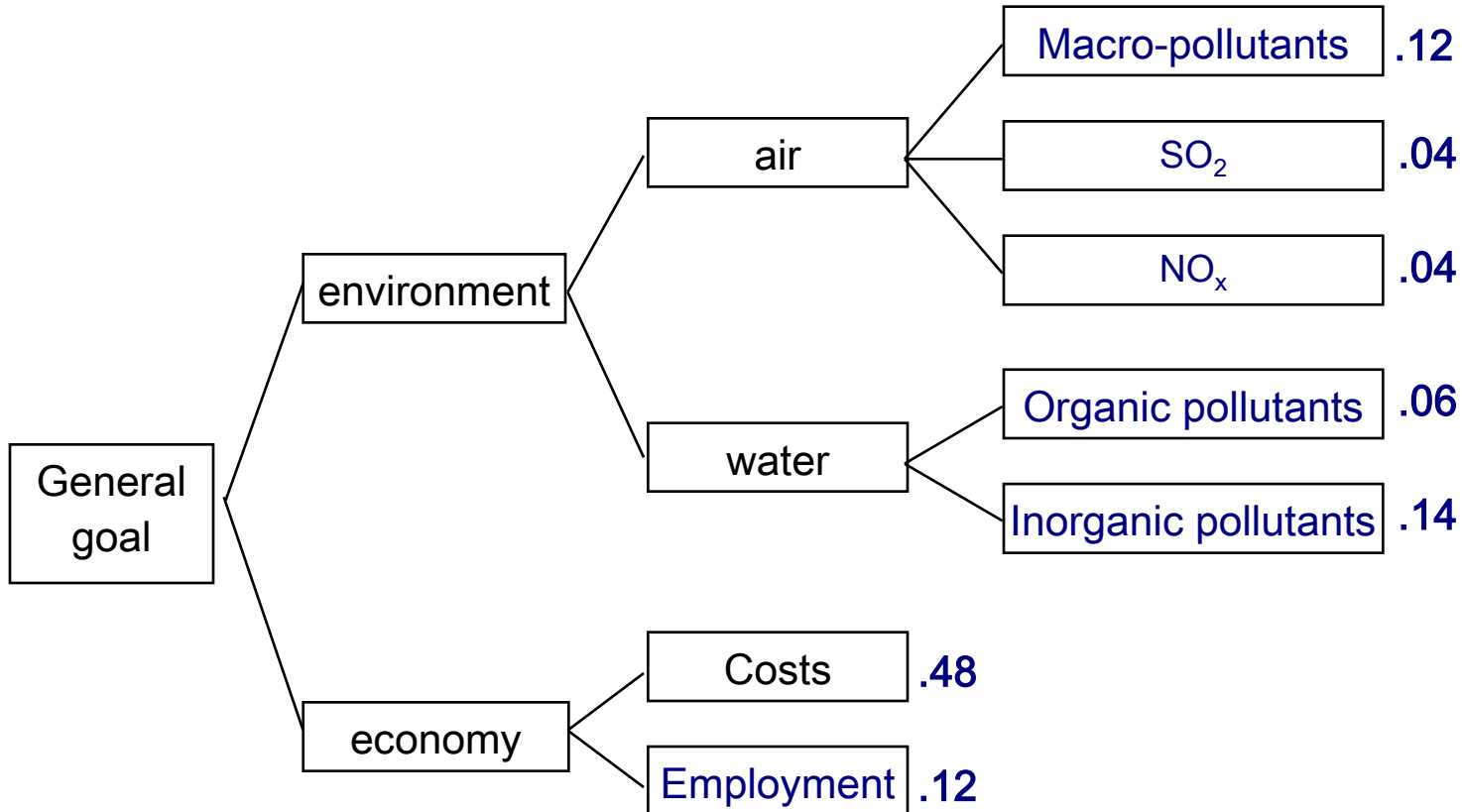
The satisfaction related to each alternative



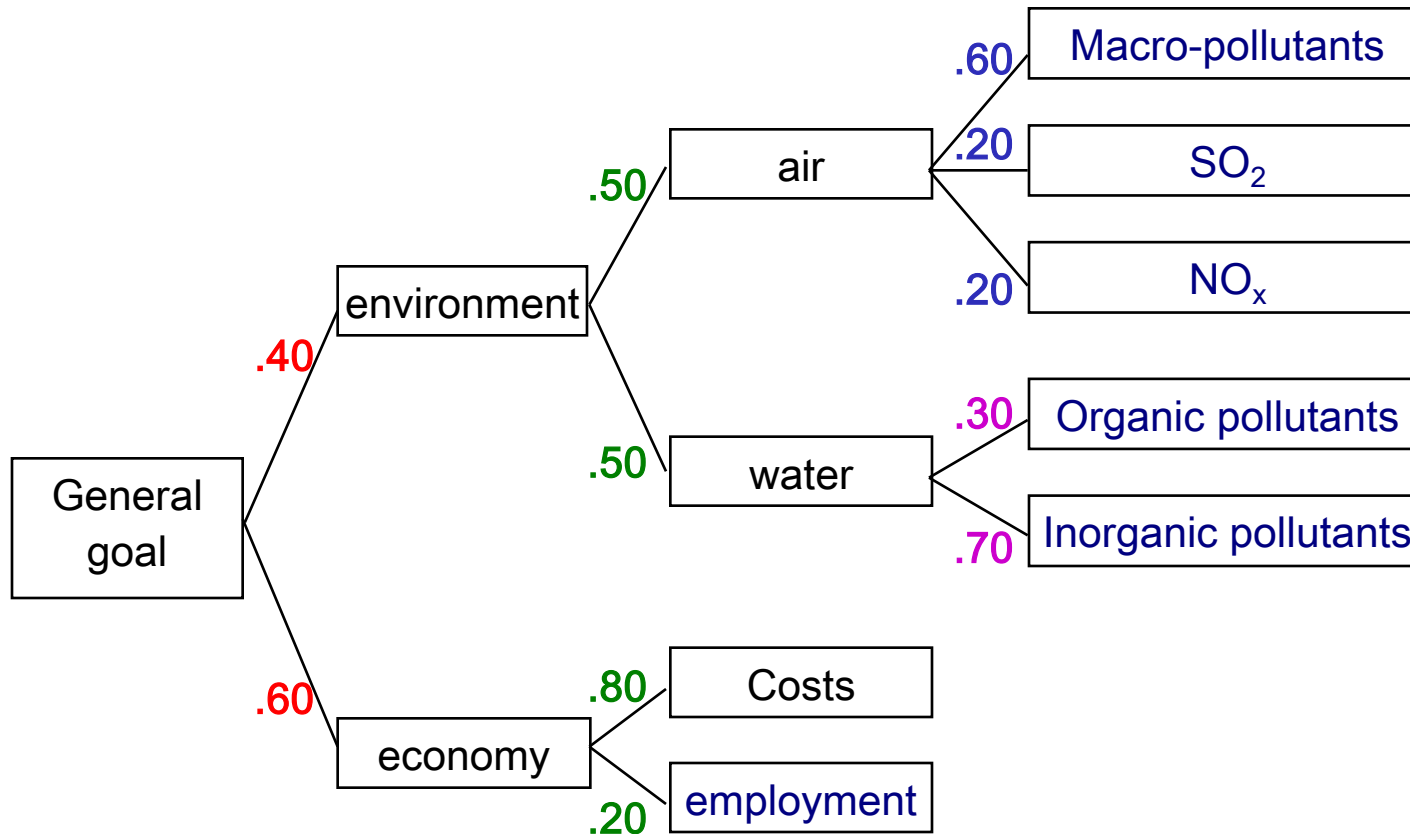
To rank the alternatives giving the choice →

Berlin

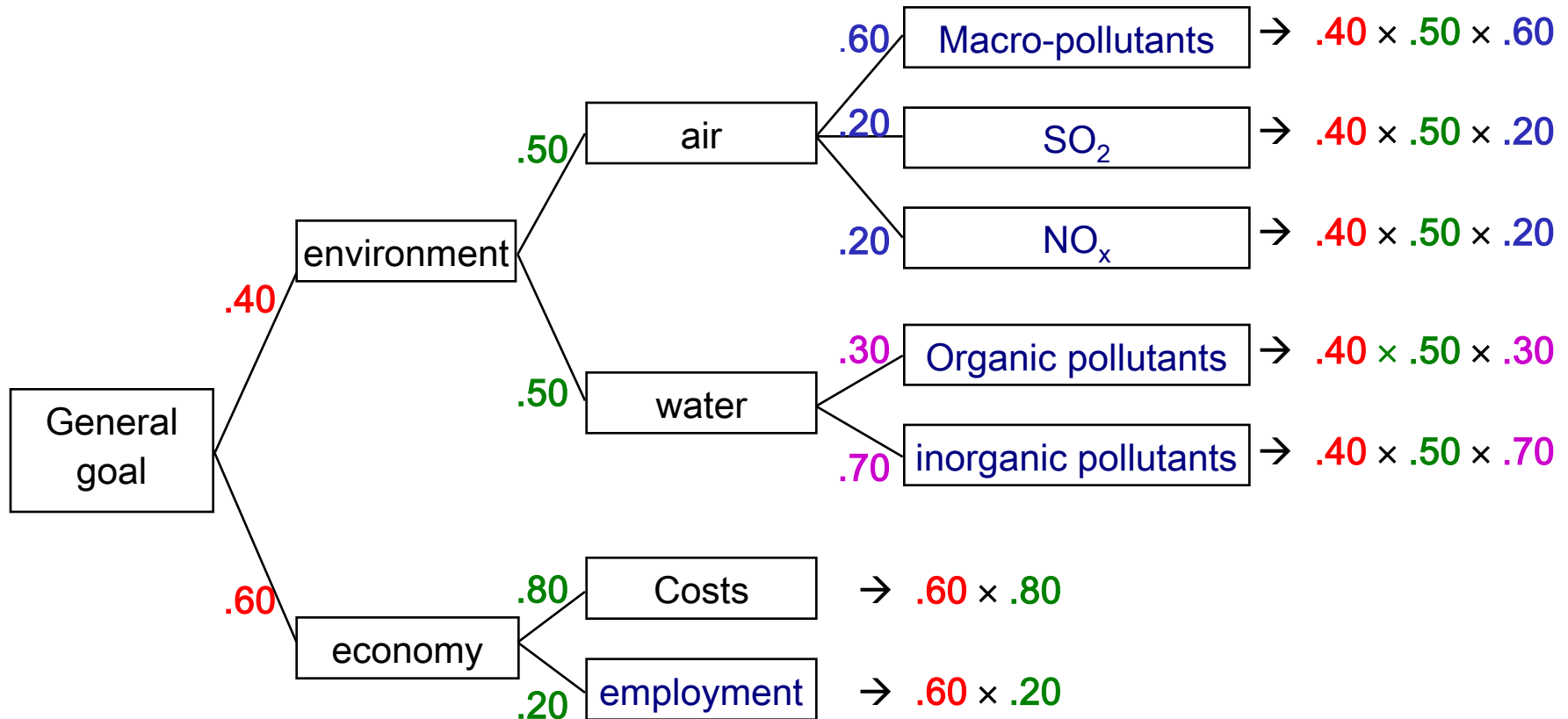
Weight assignment: list



Weight assignment: hierarchy



Weights on the hierarchy



Pair comparison

- **How to obtain the vector of the weights?**

- Thanks to many pair comparison between criteria

- **Example:**

$$\begin{array}{ccc} c_1 & c_2 & c_3 \\ \left(\begin{array}{ccc} 1 & 1/2 & 3 \\ 2 & 1 & 6 \\ 1/3 & 1/6 & 1 \end{array} \right) & \begin{array}{l} c_1 \\ c_2 \\ c_3 \end{array} \end{array}$$

$$\begin{pmatrix} * & * & \bullet \\ * & * & * \\ * & * & * \end{pmatrix}$$

a_{ij} = how criterion c_i is more important than criterion c_j

- Responses of the decision maker:

- c_2 is 2 times more important than c_1
- c_1 is 3 times more important than c_3
- c_2 is 6 times more important than c_3

- **Substitution rate**

- To a worsening of 1 unit as regards c_2 must correspond an improvement of 2 units as regard c_1 so that the DM considers equally (indifferently) the two alternatives
- The same for the other pair comparisons: c_1 in comparison with c_3 , c_2 in comparison with c_3

Consistency (internal coherence of judgements)

- **Consistent DM:**

$$a_{ij} = a_{ik} \cdot a_{kj}$$

- In this case each column of the matrix, after normalization, (dividing by the sum of the values of the column), gives the vector of the weights

$$\begin{pmatrix} 1 & 1/2 & 3 \\ 2 & 1 & 6 \\ 1/3 & 1/6 & 1 \end{pmatrix} \Rightarrow \begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix} = w$$

- **Non-consistent DM:**

- An ad hoc procedure of the matrix calculation is needed (calculation of eigenvalue-eigenvector) to obtain the w vector of the weights

- **From the vector of the weights:**

- i) weighted sum of the columns of the evaluation matrix
- ii) calculation of utility u_j ($j=1,2,\dots,r$) and ranking of alternatives

Phase 3 – Subjectivity (the wife decision)

- Another possible Decision Maker → the wife

- Her structure of preferences

↳ The wife gives much more importance to the life quality (and much less importance to the university prestige)

0.4
0.1
0.5

- Wife weighted sum and ranking

<i>Roma</i>	<i>Berlin</i>	<i>Geneva</i>	<i>Moscow</i>	<i>Tokyo</i>
7.3	5.7	6.9	2.9	4.8
(1°)	(3°)	(2°)	(5°)	(4°)

the choice of the wife would be for Roma

- Conclusion:

↓
Subjectivity

Though the use of the same data (evaluation matrix) different DM can make different choice → it depends on the structure of preference (vectors of weight)

Note: the dominated alternatives cannot win given any preference structure

Dependance by weights

Overall Function: weighted sum of the utilities:

$$\max_j f_i = \sum_{i=1}^p w_i z_{ij}$$

$z_{ij} = u_i(x_j)$

Example 4

	A1	A2	A3	w
Ob. 1	90	100	80	.20
Ob. 2	100	70	40	.20
Ob. 3	60	80	100	.50
Ob. 4	80	100	90	.10

← w_3

76	84	83	(u_j)
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Alt. 2

The last row (overall utility of each alternative) determine the ranking: the best alternative is A2 (utility = 84), followed first by A3 and then by A1.

How the final choice depend on the weights? (i.e. if w_3 changes...)

Sensitivity

The result depend on the weights w_i (and on something else ...) →

$$\max_j f_j = \sum_{i=1}^p w_i z_{ij}$$

	A1	A2	A3
Ob. 1	90	100	80
Ob. 2	100	70	40
Ob. 3	60	80	100
Ob. 4	80	100	90

w
.20
.20
.50
.10

→ .51 → .52 → ...
 ↘ .49 → .48 → ...

76	84	83	(f_j)
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(best)

Changing the w_3 value:

.50	→ .51 ... → .55	79	88	88
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if $w_3 > 0.55$ the best is A3

.50	→ .49 ... → .10	52	52	43
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if $w_3 < 0.10$ the best is A1

Sensitivity and RR (Rank Reversal)

- **Goal:**

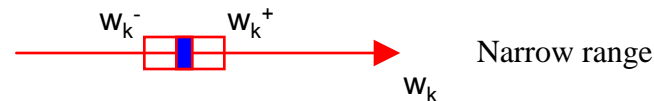
- To find the variations w_k^+ (increasing) e w_k^- (decreasing) of the weight of the k^{th} criteria w_k within which the choice doesn't change (cioè l'alternativa in 1^a posizione)

- **Method:**

- keep all the weights w_i ($i=1, \dots, m; i \neq k$) except w_k with the values given by the DM and calculate the overall utilities of the alternatives as functions of w_k
- calculate the values of w_k given which the alternative ranked first keep having the higher utility

- **Result:**

- “narrow” range, little changes in the weight w_k
would cause a different choice of the alternative



- “wide” range, big changes in the weight w_k
wouldn't cause a different choice of the alternative



An example of sensitivity

- Does the choice of the professor change, if the weight w_1 change ?

- Vector of weights (non-normalized) $\begin{bmatrix} w_1 \\ 0.6 \\ 0.1 \end{bmatrix}$

- Comparison of the utility when w_1 changes

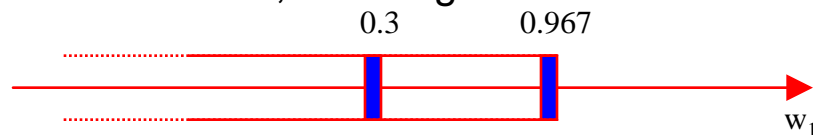
$$\left. \begin{array}{l} u_R = 5w_1 + 2.8 \\ u_B = 7w_1 + 5.8 \\ u_G = 10w_1 + 2.9 \\ u_M = 2w_1 + 3.9 \\ u_T = 7w_1 + 3.3 \end{array} \right\} \text{looking for the values } u \text{ so that}$$

$$\begin{array}{l} u_B \geq u_R ? \rightarrow \text{ALWAYS} \\ u_B \geq u_G ? \rightarrow \\ u_B \geq u_M ? \rightarrow \text{USELESS} \\ u_B \geq u_T ? \rightarrow \text{USELESS} \end{array}$$

- The choice (B) doesn't change for $\rightarrow w_1 \leq 0.967$

- Result

- To modify the final choice, the weight of the reward should be bigger then triple



Summary

- ❑ We discussed the decisional problem in a more **general frame**
- ❑ We saw the **three phases** needed to solve a multi-objectives or a multi-criteria problem, analyzing their own aspects
- ❑ We obtained a different result depending on the DM (the professor or his wife) → **subjective evaluation**

MultiCriteria Decision Making (MCDM)

Relevant characters of a MCDM problem

- Analyze the model of the specific application as a multi criteria analysis problem
- Build utility functions (asking to the DM)
- Build the vector of the weights (asking to the DM)
- Document the subjectivity in the choice (it can not be removed, only documented);
- Be supported by specific software

Test-1

In a multi objectives (criteria) problem:

- the 2 phase is the only one not dependant on the DM **true / false**

What does it mean to pass from indicators to objectives?

- to correct the results of the measurements **true / false**
- to modify the values of the indicators so that the maximum value become 1 and the minimum become 0 **true / false**
- to modify the indicators in utility value, in a conventional scale, ie from 0 (worst case) to 1 (best case) **true / false**

Test-2: sabbatical year

A inequality shows that the utility of Berlin is higher than the one of Genève:
which one ?

- 1 $7w_1 + 5.8 \geq 2w_1 + 3.9$
- 2 $7w_1 + 5.8 \geq 10w_1 + 2.9$
- 3 $7w_1 + 5.8 \geq 7w_1 + 3.3$

Test-3: Pair comparison

The following matrix of pair comparisons is consistent.

$$\begin{pmatrix} 1 & 2 & 5 \\ 1/2 & 1 & 4 \\ 1/5 & 1/4 & 1 \end{pmatrix}$$

true / false

Test-4: Sensitivity

The sensitivity analysis consists in changing simultaneously all the weights in a multi criteria problem to check if some dominated solutions become efficient.

true / false

**A tentative case:
the “Colorni award”
(the best italian newspaper on the web)**

The best italian newspaper on the web

(you are in the jury ...)

- What are the alternatives ?
- What are the attributes ?
- What are the utility functions ?
- What are the weights ?
- What is the ranking ? (the personal one and the collective one)

Alternatives

- Repubblica → www.repubblica.it
- Corriere Sera → www.corriere.it
- Sole24Ore → www.sole24.it
- Ansa online → www.ansa.it
- RaiNews24 → www.rainews24.it
- Foglio → www.ilfoglio.it

- Gazzetta d. Sport ? **NO**, because it is too specific (only sports)
- Novella 2000 ? **NO**, because it is a magazine (weekly)

The alternatives must be “similar” (but the concept of similarity is subjective)

What are the rules ? (if there are rules ...)