Decisions with multiple attributes

A brief introduction to conjoint measurement models

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Doctoral School COST IC0602
Catania, Italy, April 11 – 16, 2008

Introduction

Aims

mainly pedagogical

- present elements of the classical theory
- position some extensions w.r.t. this classical theory

Introduction

Comparing holiday packages

	cost	# of days	travel time	category of hotel	distance to beach	Wifi	cultural interest
A	200 €	15	12 h	***	45 km	Y	++
B	425 €	18	15 h	****	0 km	N	−−
C	150 €	4	7 h	**	250 km	N	+
D	300 €	5	10 h	***	5 km	Y	−

Central problems

- helping a DM choose between these packages
- helping a DM structure his preferences

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Two different contexts

1. decision aiding
   - careful analysis of objectives
   - careful analysis of attributes
   - careful selection of alternatives
   - availability of the DM
2. recommendation systems
   - no analysis of objectives
   - attributes as available
   - alternatives as available
   - limited access to the user

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Introduction

Basic model

- additive value function model

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

$$x, y: \text{ alternatives}$$

$$x_i: \text{ evaluation of alternative } x \text{ on attribute } i$$

$$v_i(x_i): \text{ number}$$

- underlies most existing MCDM techniques

Underlying theory: conjoint measurement

- Economics (Debreu, 1960)
- Psychology (Luce & Tukey, 1964)
- tools to help structure preferences

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Outline: Classical theory

1. An aside (second desert): measurement in Physics
2. An example (Primo piatto): even swaps
3. Notation
4. Additive value functions (Main dish): outline of theory
5. Additive value functions: implementation

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Outline: Extensions

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6. Models with interactions
7. Ordinal models

Part I

Classical theory: conjoint measurement

Outline

Aside: measurement of physical quantities

Lonely individual on a desert island

- no tools, no books, no knowledge of Physics
- wants to rebuild a system of physical measures

A collection of rigid straight rods

- problem: measuring the length of these rods
  - pre-theoretical intuition
    - length
    - softness, beauty

3 main steps

- comparing objects
- creating and comparing new objects
- creating standard sequences

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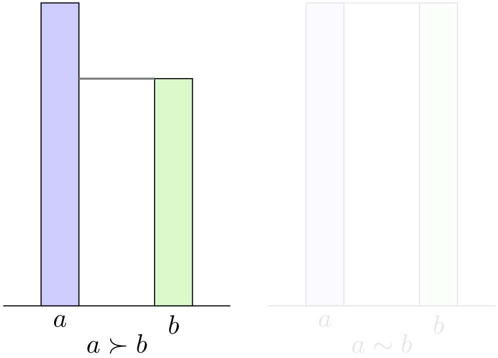

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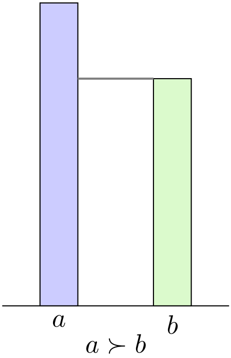

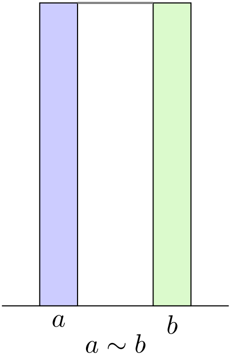

Comparing objects

Results

- $a \succ b$: extremity of rod a is higher than extremity of rod b
- $a \sim b$: extremity of rod a is as high as extremity of rod b

Expected properties

- $a \succ b$, $a \sim b$ or $b \succ a$
- $\succ$ is asymmetric
- $\sim$ is symmetric
- $\succ$ is transitive
- $\sim$ is transitive
- $\succ$ and $\sim$ combine "nicely"
  - $a \succ b$ and $b \sim c \Rightarrow a \succ c$
  - $a \sim b$ and $b \succ c \Rightarrow a \succ c$

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Comparing objects

Summary of experiments

- binary relation $\succsim = \succ \cup \sim$ that is a weak order
  - complete ($a \succsim b$ or $b \succsim a$)
  - transitive ($a \succsim b$ and $b \succsim c \Rightarrow a \succsim c$)

Consequences

- associate a real number $\Phi(a)$ to each object a
- the comparison of numbers faithfully reflects the results of experiments

$$a \succ b \Leftrightarrow \Phi(a) > \Phi(b) \qquad a \sim b \Leftrightarrow \Phi(a) = \Phi(b)$$

- the function Φ defines an ordinal scale
  - applying an increasing transformation to Φ leads to a scale that has the same properties
  - any two scales having the same properties are related by an increasing transformation

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Nature of the scale

- Φ is quite far from a full-blown measure of length...
- useful though since it allows the experiments to be done only once

Hypotheses are stringent

- highly precise comparisons
- several practical problems
  - any two objects can be compared
  - connections between experiments
  - comparisons may vary in time
- idealization of the measurement process

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Step 2: creating and comparing new objects

- use the available objects to create new ones
- concatenate objects by placing two or more rods "in a row"

$$a \circ b \succ c \circ d$$

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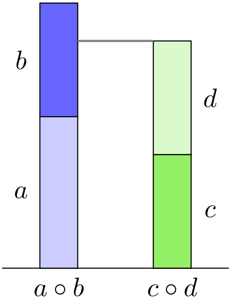

$$a \circ b \succ c \circ d$$

Concatenation

- we want to be able to deduce $\Phi(a \circ b)$ from $\Phi(a)$ and $\Phi(b)$
- simplest requirement

$$\Phi(a \circ b) = \Phi(a) + \Phi(b)$$

- monotonicity constraints

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go faster

Example

- five rods: $r_1, r_2, \ldots, r_5$
- we may only concatenate two rods (space reasons)
- we may only experiment with different rods
- data:

$$r_1 \circ r_5 \succ r_3 \circ r_4 \succ r_1 \circ r_2 \succ r_5 \succ r_4 \succ r_3 \succ r_2 \succ r_1$$

- all constraints are satisfied: weak ordering and monotonicity

Example

$$r_1 \circ r_5 \succ r_3 \circ r_4 \succ r_1 \circ r_2 \succ r_5 \succ r_4 \succ r_3 \succ r_2 \succ r_1$$

	Φ	Φ'	Φ''
r_1	14	10	14
r_2	15	91	16
r_3	20	92	17
r_4	21	93	18
r_5	28	100	29

- Φ, Φ' and Φ'' are equally good to compare simple rods
- only Φ and Φ'' capture the comparison of concatenated rods
- going from Φ to Φ'' does not involve a "change of units"

- it is tempting to use Φ or Φ'' to infer comparisons that have not been performed…
- disappointing

$$\Phi : r_2 \circ r_3 \sim r_1 \circ r_4 \quad \Phi'' : r_2 \circ r_3 \succ r_1 \circ r_4$$

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Step 3: creating and using standard sequences

- choose a standard rod
- be able to build perfect copies of the standard
- concatenate the standard rod with its perfects copies

$$S(8) \succ a \succ S(7)$$
$$\Phi(s) = 1 \Rightarrow 7 < \Phi(a) < 8$$

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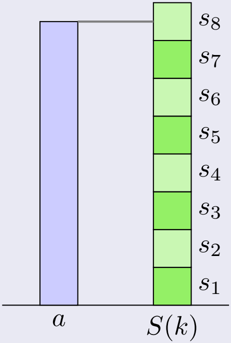

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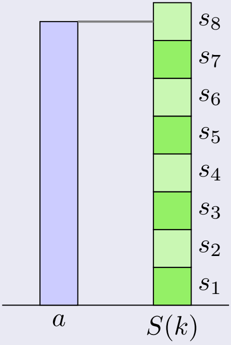

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Convergence

First method

- choose a smaller standard rod
- repeat the process

Second method

- prepare a perfect copy of the object
- concatenate the object with its perfect copy
- compare the “doubled” object to the original standard sequence
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Extensive measurement

- Krantz, Luce, Suppes & Tversky (1971, chap. 3)

4 Ingredients

1. well-behaved relations $\succ$ and $\sim$
2. concatenation operation $\circ$
3. consistency requirements linking $\succ$, $\sim$ and $\circ$
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Question

Can this be applied outside Physics?

- no concatenation operation (intelligence!)

What is conjoint measurement?

Conjoint measurement

- mimicking the operations of extensive measurement
  - when there are no concatenation operation readily available
  - when several dimensions are involved

Seems overly ambitious

- let us start with a simple example

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Example: Hammond, Keeney & Raiffa

Choice of an office to rent

- five locations have been identified
- five attributes are being considered
  - *Commute* time (minutes)
  - *Clients*: percentage of clients living close to the office
  - *Services*: ad hoc scale
    - A (all facilities), B (telephone and fax), C (no facility)
  - *Size*: square feet ($\simeq 0.1$ m^2)
  - *Cost*: $ per month

Attributes

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- *Clients* is a proxy attribute
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Data

	a	b	c	d	e
Commute	45	25	20	25	30
Clients	50	80	70	85	75
Services	A	B	C	A	C
Size	800	700	500	950	700
Cost	1850	1700	1500	1900	1750

Hypotheses and context

- a single cooperative DM
- choice of a single office
- ceteris paribus reasoning seems possible
  Commute: decreasing *Clients*: increasing
  Services: increasing *Size*: increasing
  Cost: decreasing
- dominance has meaning

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- all alternatives except c have a common evaluation on *Commute*
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  - starting with c, what is the gain on *Clients* that would exactly compensate a loss of 5 min on *Commute*?
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Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

- no more dominance
- assessing tradeoffs
- all alternatives except c have a common evaluation on *Commute*
- modify c in order to bring it to this level
  - starting with c, what is the gain on *Clients* that would exactly compensate a loss of 5 min on *Commute*?
  - difficult but central question

	c	c'
Commute	20	**25**
Clients	70	$\mathbf{70} + \delta$
Services	C	C
Size	500	500
Cost	1500	1500

find δ such that $c' \sim c$

Answer

- for $\delta = 8$, I am indifferent between c and c'
- replace c with c'

	c	c'
Commute	20	**25**
Clients	70	$\mathbf{70} + \delta$
Services	C	C
Size	500	500
Cost	1500	1500

find δ such that $c' \sim c$

Answer

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- all alternatives have a common evaluation on *Commute*
- divide and conquer: dropping attributes
  - drop attribute *Commute*

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- check again for dominance
- unfruitful
- assess new tradeoffs
  - neutralize Service using *Cost* as reference

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Questions

- what maximal increase in monthly cost would you be prepared to pay to go from C to B on service for c'?
  - answer: 250 $
- what minimal decrease in monthly cost would you ask if we go from A to B on service for d?
  - answer: 100 $

	b	c'	c''	d	d'
Clients	80	78	78	85	85
Services	B	C	**B**	A	**B**
Size	700	500	500	950	950
Cost	1700	1500	1500 + **250**	1900	1900 − **100**

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Size	700	500	500	950	950
Cost	1700	1500	$1500 + \mathbf{250}$	1900	$1900 - \mathbf{100}$

- replacing c' with c''
- replacing d with d'
- dropping Service

	b	c''	d'
Clients	80	78	85
Size	700	500	950
Cost	1700	1750	1800

- checking for dominance: c'' is dominated by b
- c'' can be dropped

- replacing c' with c''
- replacing d with d'
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	b	c''	d'
Clients	80	78	85
Size	700	500	950
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- checking for dominance: c'' is dominated by b
- c'' can be dropped

- dropping c''

	b	d'
Clients	80	85
Size	700	950
Cost	1700	1800

- no dominance
- question: starting with b what is the additional cost that you would be prepared to pay to increase size by 250?
  - answer: 250 $

	b	b'	d'
Clients	80	80	85
Size	700	**950**	950
Cost	1700	1700 + **250**	1800

- dropping c''

	b	d'
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Size	700	**950**	950
Cost	1700	1700 + **250**	1800

- replace b with b'
- drop *Size*

	b'	d'
Clients	80	85
Size	950	950
Cost	1950	1800

	b'	d'
Clients	80	85
Cost	1950	1800

- check for dominance
- d' dominates b'

Conclusion

- Recommend d as the final choice

- replace b with b'
- drop *Size*

	b'	d'
Clients	80	85
Size	950	950
Cost	1950	1800

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Clients	80	85
Cost	1950	1800

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Size	950	950
Cost	1950	1800

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Summary

Remarks

- very simple process
- no question on "intensity of preference"
- notice that importance plays absolutely no rôle
- why be interested in something more complex?

Problems

- set of alternative is small
  - many questions otherwise
- output is not a preference model
  - if new alternatives appear, the process should be restarted
- what are the underlying hypotheses?

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Monsieur Jourdain doing conjoint measurement

Similarity with extensive measurement

- $\succ$: preference, $\sim$: indifference
- we have implicitly supposed that they combine nicely

Monsieur Jourdain doing conjoint measurement

OK… but where are the standard sequences?

- hidden… but really there!
- standard sequence for length: objects that have exactly the same length
- tradeoffs: preference intervals on distinct attributes that have the same length
  - $c \sim c'$
  - [25, 20] on *Commute* has the same length as [70, 78] on *Client*

	c	c'	f	f'
Commute	20	**25**	*20*	**25**
Clients	70	**78**	*78*	**82**
Services	C	C	C	C
Size	500	500	500	500
Cost	1500	1500	1500	1500

[70, 78] has the same length [78, 82] on *Client*

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Outline

Setting

- $N = \{1, 2, \ldots, n\}$ set of attributes
- X_i: set of possible levels on the ith attribute
- $X = \prod_{i=1}^{n} X_i$: set of all conceivable alternatives
  - X include the alternatives under study... and many others
- $\succsim$: binary relation on X: "at least as good as"
- $x \succ y \Leftrightarrow x \succsim y$ and $Not[y \succsim x]$
- $x \sim y \Leftrightarrow x \succsim y$ and $y \succsim x$

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Preference relations on Cartesian products

Applications

- Economics: consumers comparing bundles of goods
- Decision under uncertainty: consequences in several states
- Inter-temporal decision making: consequences at several moments in time
- Inequality measurement: distribution of wealth across individuals

- Decision making with multiple attributes
  - in all other cases, the Cartesian product is homogeneous

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Independence

Main idea

- Crucial hypothesis in the additive value model
- Main idea: it makes sense to talk about preference on a single attribute (or a subset of attributes) without referring to the values on the others
- $\succsim$ is weakly independent if one can talk of preference on each single attribute
- $\succsim$ is independent if one can talk of independence on each subset of attributes
- a weaker form: separability

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Is this a reasonable hypothesis ?

- it is easy to imagine examples in which independence is violated
    - Main course and Wine example
- it is nearly hopeless to try to work if weak independence (at least weak separability) is not satisfied

Dominance

- as soon as I have a weakly independent weak order
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Outline of theory: 2 attributes

Question

- suppose I can "observe" $\succsim$ on $X = X_1 \times X_2$
- what must be supposed to guarantee that I can represent $\succsim$ in the additive value function model

$$v_1 : X_1 \to \mathbb{R}$$
$$v_2 : X_2 \to \mathbb{R}$$
$$(x_1, x_2) \succsim (y_1, y_2) \Leftrightarrow v_1(x_1) + v_2(x_2) \geq v_1(y_1) + v_2(y_2)$$

- $\succsim$ must be an independent weak order

Method

- try building standard sequences and see if it works!

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Uniqueness

Important observation

Suppose that there are v_1 and v_2 such that

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If $\alpha > 0$

$$w_1 = \alpha v_1 + \beta_1 \quad w_2 = \alpha v_2 + \beta_2$$

is also a valid representation

Consequences

- fixing $v_1(x_1) = v_2(x_2) = 0$ is harmless
- fixing $v_1(y_1) = 1$ is harmless if $y_1 \succ_1 x_1$

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Standard sequences

Preliminaries

- choose arbitrarily two levels $x_1^0, x_1^1 \in X_1$
- make sure that $x_1^1 \succ_1 x_1^0$
- choose arbitrarily one level $x_2^0 \in X_2$
- $(x_1^0, x_2^0) \in X$ is the reference point (origin)
- the preference interval $[x_1^0, x_1^1]$ is the unit

Building a standard sequence on X_2

- find a "preference interval" on X_2 that has the same "length" as the reference interval $[x_1^0, x_1^1]$
- find x_2^1 such that

$$(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$$

$$v_1(x_1^0) + v_2(x_2^1) = v_1(x_1^1) + v_2(x_2^0) \text{ so that}$$
$$v_2(x_2^1) - v_2(x_2^0) = v_1(x_1^1) - v_1(x_1^0)$$

- the structure of X_2 has to be "rich enough"

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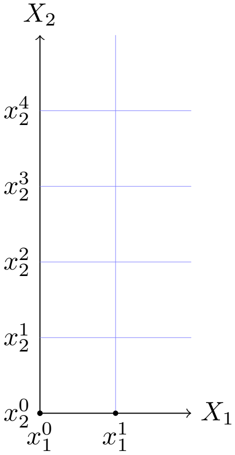
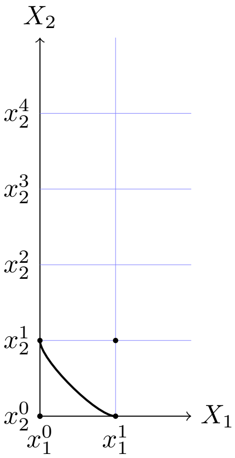


X_2
x_2^4
x_2^3
x_2^2
x_2^1
x_2^0
X_1
x_1^0
x_1^1



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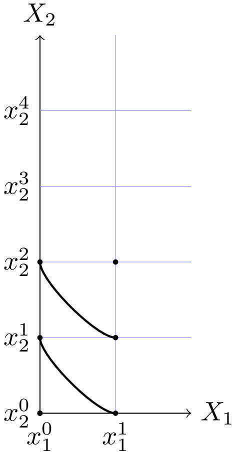
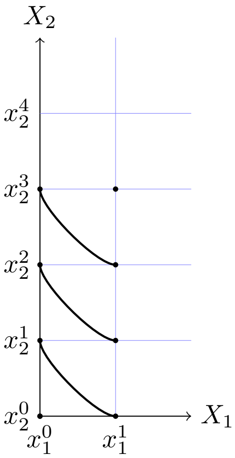
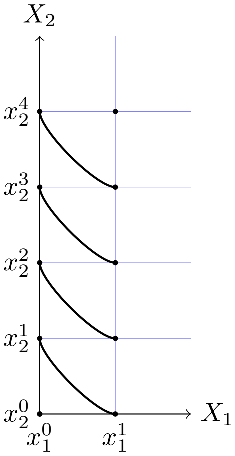
X_1


X_2
x_2^4
x_2^3
x_2^2
x_2^1
x_2^0
X_1
x_1^0
x_1^1



X_2
x_2^4
x_2^3
x_2^2
x_2^1
x_2^0
x_1^0
x_1^1
X_1


X_2
x_2^4
x_2^3
x_2^2
x_2^1
x_2^0
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x_1^1
X_1

Standard sequence

Archimedean

- we have to make an hypothesis (that was left implicit for length)
  - the standard sequence can reach the length of any object

$$\forall x, y \in \mathbb{R}, \exists n \in \mathbb{N} : ny > x$$

- rough interpretation : there are not "infinitely" liked or disliked consequences

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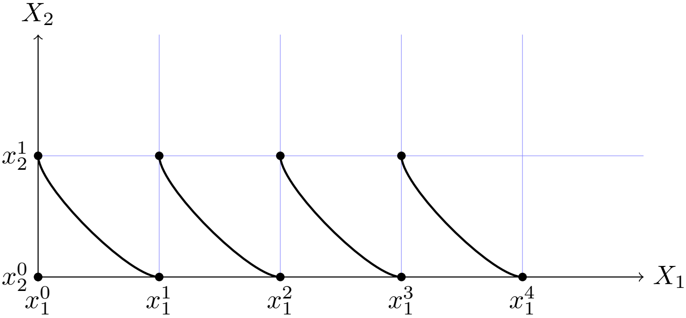
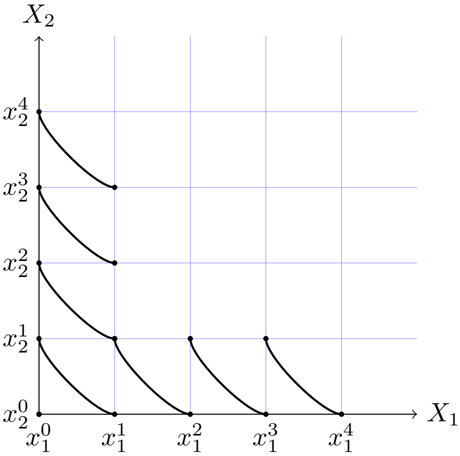


X_2
x_2^1
x_2^0
X_1
x_1^0
x_1^1
x_1^2
x_1^3
x_1^4



X_2
x_2^4
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x_2^2
x_2^1
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x_1^2
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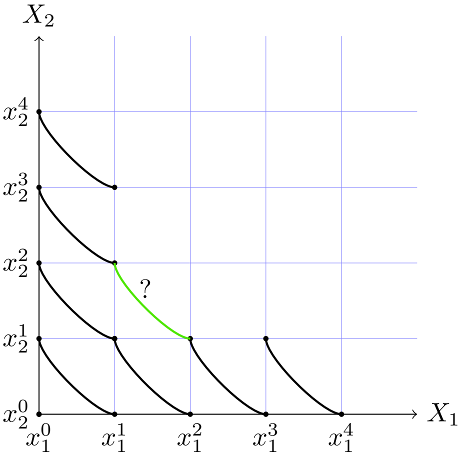
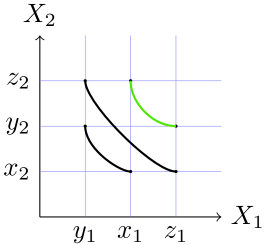
x_1^4
X_1



X_2
x_2^4
x_2^3
x_2^2
x_2^1
x_2^0
?
X_1
x_1^0
x_1^1
x_1^2
x_1^3
x_1^4

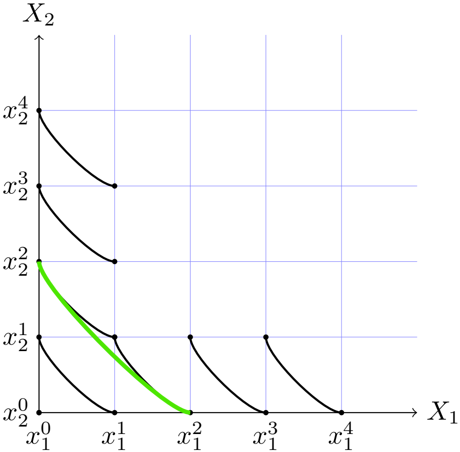
Thomsen condition

$$\begin{array}{c}(x_1, x_2) \sim (y_1, y_2) \\ \text{and} \\ (y_1, z_2) \sim (z_1, x_2)\end{array} \Rightarrow (x_1, z_2) \sim (z_1, y_2)$$

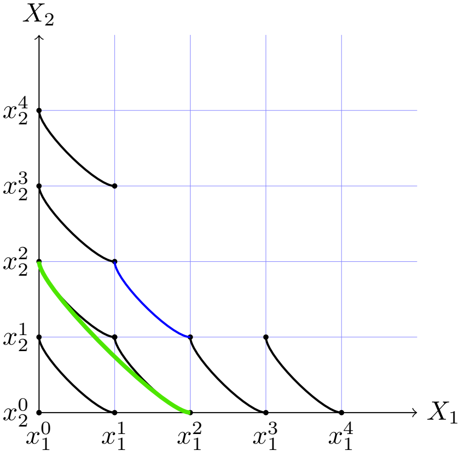


Consequence

- there is an additive value function on the grid



X_2
x_2^4
x_2^3
x_2^2
x_2^1
x_2^0
X_1
x_1^0
x_1^1
x_1^2
x_1^3
x_1^4



X_2
x_2^4
x_2^3
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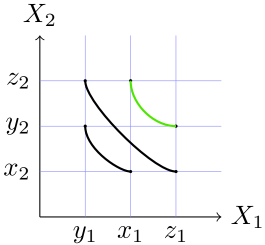

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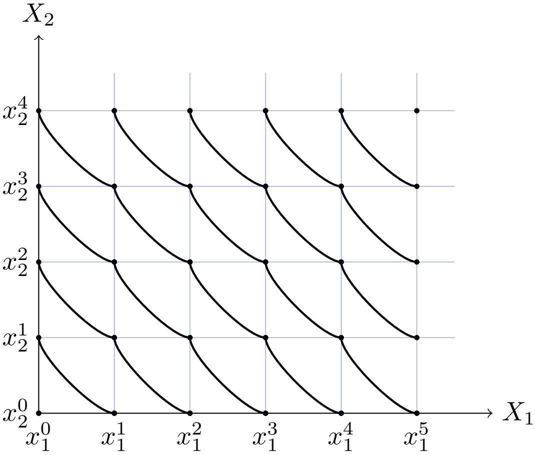
X_2
x_2^4
x_2^3
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x_2^1
x_2^0
X_1
x_1^0
x_1^1
x_1^2
x_1^3
x_1^4
x_1^5

Summary

- we have defined a “grid”
- there is an additive value function on the grid

- iterate the whole process with a “denser grid”

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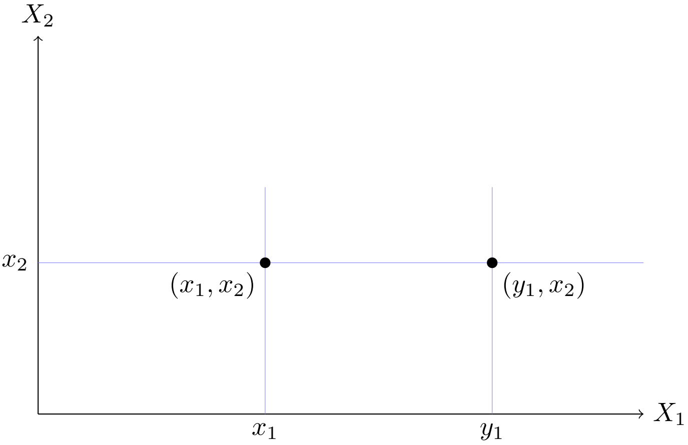
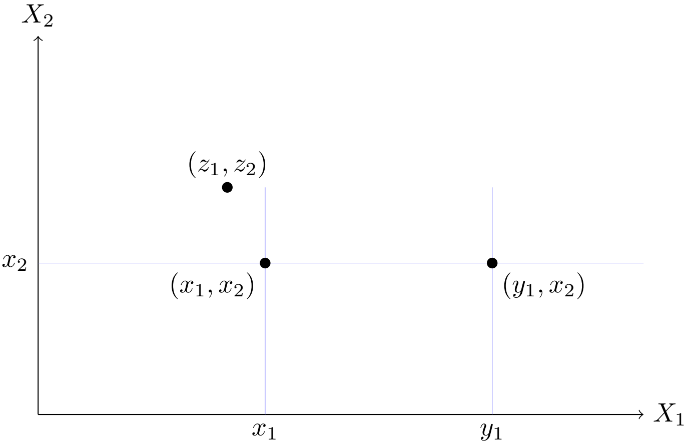
- iterate the whole process with a “denser grid”

Hypotheses

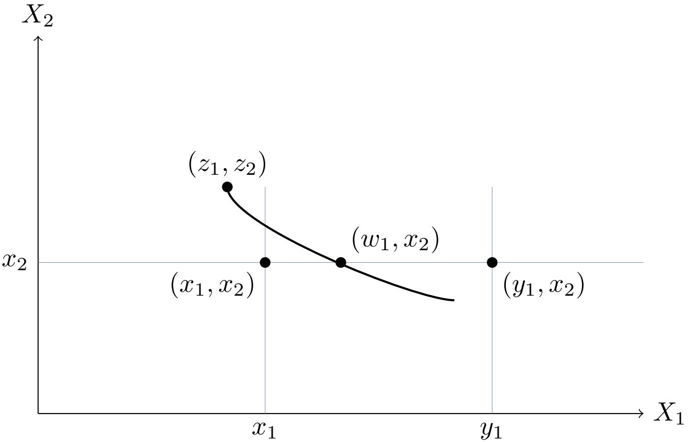
- Archimedean: every strictly bounded standard sequence is finite
- essentiality: both $\succ_1$ and $\succ_2$ are nontrivial
- restricted solvability



X_2
x_2
(x_1, x_2)
(y_1, x_2)
X_1
x_1
y_1



$$\left.\begin{array}{l}(y_1, x_2) \succ (z_1, z_2) \\ (z_1, z_2) \succ (x_1, x_2)\end{array}\right\}$$



$$\left.\begin{array}{l}(y_1, x_2) \succ (z_1, z_2)\\(z_1, z_2) \succ (x_1, x_2)\end{array}\right\} \Rightarrow \exists w_1 \text{ such that } (z_1, z_2) \sim (w_1, x_2)$$

Basic result

Theorem (2 attributes)

If

- restricted solvability holds
- each attribute is essential

then
the additive value function model holds
if and only if
$\succsim$ is an independent weak order satisfying the Thomsen and the Archimedean conditions

The representation is unique up to scale and location

General case

Good news

- entirely similar. . .
- with a very nice surprise: Thomsen can be forgotten
  - if $n = 2$, independence is identical with weak independence
  - if $n > 3$, independence is much stronger than weak independence

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Basic result

Theorem (more than 2 attributes)

If

- restricted solvability holds
- at least three attributes are essential

then
the additive value function model holds
if and only if
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The representation is unique up to scale and location

Outline

Assessing value functions

Standard technique

- check independence
- build standard sequences

Problems

- many difficult questions
- discrete attributes
- propagation of "errors"

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In practice

Methods to assess v_i independently on each attribute

- direct rating
- bisection techniques
- ...

UTA: outline

Principle

- select a number of reference alternatives that the DM knows well
- rank order these alternatives
- test, using LP, if this information is compatible with an additive value function
  - if yes, present a central one
    - interact with the DM
    - apply the resulting function to the whole set of alternatives
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Other indirect methods

UTA-GMS

Greco, Mousseau, Slowinski

- works with families of additive value functions

MACBETH

Conventions

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} \lambda_i u_i(x_i) \geq \sum_{i=1}^{n} \lambda_i u_i(y_i)$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$u_1(x_{1*}) = u_2(x_{2*}) = \ldots = u_n(x_{n*}) = 0$$

$$u_1(x_1^*) = u_2(x_2^*) = \ldots = u_n(x_n^*) = 1$$

Principles

- assess the u_i independently on each attribute using "preference differences"
- assess the λ_i to fit together these functions

Summary

Conjoint measurement

- highly consistent theory
- together with practical assessment techniques

Why consider extensions?

- hypotheses may be violated
- assessment is demanding
  - time
  - cognitive effort

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Part II

A glimpse at possible extensions (desert)

Summary

Additive value function model

- requires independence
- requires a finely grained analysis of preferences

Two main types of extensions

1. models with interactions
2. more ordinal models

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Outline

6 Models with interactions
- Rough sets
- GAI networks
- Fuzzy integrals

7 Ordinal models

Interactions

Two extreme models

- additive value function model
    - independence
- decomposable model
    - only weak independence

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

$$x \succsim y \Leftrightarrow F[v_1(x_1), \ldots v_n(x_n)] \geq F[v_1(y_1), \ldots v_n(y_n)]$$

Decomposable models

$$x \succsim y \Leftrightarrow F[v_1(x_1), \ldots v_n(x_n)] \geq F[v_1(y_1), \ldots v_n(y_n)]$$
$$F \text{ increasing in all arguments}$$

Result

Under mild conditions, any weakly independent weak order may be represented in the decomposable model

Problem

- all possible types of interactions are admitted
- assessment is a very challenging task

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Two main directions

Extensions

1. work with the decomposable model
   - rough sets
2. find models "in between" additive and decomposable
   - CP-nets, GAI
   - fuzzy integrals

Rough sets

Basic ideas

- work within the general decomposable model
- use the same principle as in UTA
- replacing the numerical model by a symbolic one
- infer decision rules

IF

$x_1 \geq a_1, \ldots, x_i \geq a_i, \ldots, x_n \geq a_n$ and

$y_1 \leq b_1, \ldots, y_i \leq b_i, \ldots, y_n \leq b_n$

THEN

$x \succsim y$

- many possible variants
- Greco, Matarazzo, Słowiński

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GAI: Example

Choice of a meal: 3 attributes

$X_1 = \{\text{Steak}, \text{Fish}\}$
$X_2 = \{\text{Red}, \text{White}\}$
$X_3 = \{\text{Cake}, \text{sherBet}\}$

Preferences

$$x^1 = (S, R, C) \quad x^2 = (S, R, B) \quad x^3 = (S, W, C) \quad x^4 = (S, W, B)$$
$$x^5 = (F, R, C) \quad x^6 = (F, R, B) \quad x^7 = (F, W, C) \quad x^8 = (F, W, B)$$

$$x^2 \succ x^1 \succ x^7 \succ x^8 \succ x^4 \succ x^3 \succ x^5 \succ x^6$$

- the important is to match main course and wine
- I prefer Steak to Fish
- I prefer Cake to sherBet if Fish
- I prefer sherBet to Cake if Steak

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Independence

$$x^1 \succ x^5 \Rightarrow v_1(S) > v_1(F)$$
$$x^7 \succ x^3 \Rightarrow v_1(F) > v_1(S)$$

Grouping main course and wine?

$$x^7 \succ x^8 \Rightarrow v_3(C) > v_3(B)$$
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Model

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$$u_{12}(S, R) = 6 \quad u_{12}(F, W) = 4 \quad u_{12}(S, W) = 2 \quad u_{12}(F, R) = 0$$
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Generalized Additive Independence

GAI (Gonzales & Perny)

- axiomatic analysis
- if interdependences are known
  - assessment techniques
  - efficient algorithms (compactness of representation)

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Fuzzy integrals

Choquet and Sugeno

- Other types of interactions can be modelled by fuzzy integrals
  - Choquet integral
  - Sugeno integral
- Encompassed in the framework of the decomposable model
- Difficult to analyze due to the commensurability hypothesis

Outline

Observations

Classical model

- deep analysis of preference that may not be possible
  - preference are not well structured
  - several or no DM
  - prudence

Idea

- it is not very restrictive to suppose that levels on each X_i can be ordered
- aggregate these orders
- possibly taking importance into account

Social choice

- aggregate the preference orders of the voters to build a collective preference

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Outranking methods

ELECTRE

$x \succsim y$ if

Concordance a "majority" of attributes support the assertion

Discordance the opposition of the minority is not "too strong"

$$x \succsim y \Leftrightarrow \begin{cases} \sum_{i: x_i \succsim_i y_i} w_i \geq s \\ Not[y_i \; V_i \; x_i], \forall i \in N \end{cases}$$

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Such relations can be analyzed within a conjoint measurement model

Drawbacks

- Condorcet paradox: $\succsim$ may have cycles
- Arrow's theorem

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$$x \succsim y \Leftrightarrow \begin{cases} \sum_{i:x_i \succsim_i y_i} w_i \geq s \\ \\ Not[y_i \; V_i \; x_i], \forall i \in N \end{cases}$$

Model

Such relations can be analyzed within a conjoint measurement model

Drawbacks

- Condorcet paradox: $\succsim$ may have cycles
- Arrow's theorem

Ways out

Accepting intransitivity

- find way to extract information in spite of intransitivity
  - ELECTRE I, II, III, IS
  - PROMETHEE I, II

Do not use paired comparisons

- only compare x with carefully selected alternatives
  - ELECTRE TRI
  - methods using reference points

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Explicit models

Advantage of considering explicit models:

- analysis of their expressivity (through axioms)
- their analysis may provide hints for elicitation

General frameworks

- allows to compare models within the framework (common axioms)
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