A new axiomatic foundation of the partial comparability theory [1]

Alexis Tsoukiàs(+), Philippe Vincke(*)

(+) LAMSADE, Université Paris Dauphine
Place du Maréchal de Lattre de Tassigny
75775 Paris Cédex 16, France
(*) Université Libre de Bruxelles
CP 210/01 Bd. du Triomphe
1050 Bruxelles, Belgium

Abstract

The paper presents some results obtained in searching a new axiomatic foundation for the partial comparability theory (PCT) in the frame of non conventional preference modeling. The basic idea is to define an extended preference structure able to represent lack of information, uncertainty, ambiguity, multidimensional and conflicting preferences, using formal logic as the basic formalism.

A four valued paraconsistent logic is therefore described in the paper as a more suitable language for the purposes of the research. Then the concepts of partition, general binary relations properties, foundamental relational system of preferences (f.r.s.p.), maximal f.r.s.p. and well founded f.r.s.p. are introduced and some theorems are demonstrated in order to provide the axiomatic foundation of the PCT. The main result obtained is a preference structure that is a maximal well founded f.r.s.p.. This preference structure enables a more flexible, reliable and robust preference modeling. Moreover it can be viewed as a generalization of the conventional approach so that all the results obtained until now under it can be used.

Two examples are provided at the end of the paper in order to give an account of the operational potentialities of the new theory, mainly in the area of multicriteria decision aid and social choice theory. Further research directions conclude the paper.

Introduction

Preference modeling has been a basic subject in many theoretical and applicative domains as economic theory, operational research, psychology etc.. The principal scope of this discipline is to give a solid theoretical foundation to models of preferences, mainly human preferences, expressed on a set of objects (actions, alternatives) henceforth called A. Conventional preference modeling (see Roubens and Vincke, 1985) is based on binary relations representing statements of the kind "a is preferred to b", "a is indifferent to b" etc. characterized by different properties (reflexivity, asymmetry, transitivity etc.). A particular interest has been dedicated in developing preference models that allow an easy operational use in order to obtain a prescription towards possible

[1] This research has been done while the first author was in the Université Libre de Bruxelles under the "Research in Brussels actions"

decisions. That is the construction of "orders" (weak orders, semi-orders, interval orders etc.). from which is possible to extract a final choice or classification.

Since the fifties the conventional approach has been argued, on the basis of empirical, experimental and theoretical research (see Luce, 1958; Tversky, 1969; Roy, 1977; Fishburn, 1991), of limited representational power. Problems arise when particular situations have to be faced as discrimination problems, uncertainty, ambiguity, conflicting and multidimensional preferences. Such situations are very common when operational uses of the preference models are requested. On this basis non-conventional preference modeling theories have been developed (see Kacprzyk and Roubens, 1988; Tsoukiàs and Vincke, 1992) in order to improve the representational and operational power of the theory. In this direction two non exclusive paths have been explored, that is the identification of new preference relations that could represent, at least partially, particular situations and the definition of new "languages" under which more flexible models could be available. In this second direction the theory about "fuzzy preference relations" has been developed (see Perny and Roy, 1992; Ovchinnikov and Roubens 1991 and 1992), based on the concepts defined in fuzzy logic. In this frame (see also Kacprzyk and Roubens, 1988), with the exception of Roy (1977), the conventional preference relations are considered as fuzzy sets and the relative theoretical problems are faced.

In our paper we try to combine the idea of extending the set of possible preference relations and of defining a new appropriate language, but not in the frame of fuzzy logic. The language we are looking for should fulfill the following characteristics:

- it should provide a truth calculus so that unambiguous sentences can be written and a specific truth value can be always defined for them;
- it should be able to capture and distinguish both the situations of incomplete information (uncertainty, imprecision, missing data etc.) and of conflicting information (ambiguity, incoherence, multidimensional preferences etc.) as they are the main problems where conventional preference models fail to provide satisfactory representations;
- it should provide easy operational extensions, as an objective of the research is to define a new family of decision aid methods to be used in real world problems.

The natural choice is to use formal logic. Problems of this kind has already been faced in multi-valued logics. We will consider a four-valued paraconsistent logic studied in the sixties by Dubarle (1963) and presented by Belnap (1976 and 1977) as a basis for a questioning-answering system capable of making inferences from a database. An extension of this logic has been recently developed by Doherty et al. (1992) and with some further variations it will be used as the basis for the development of our theory.

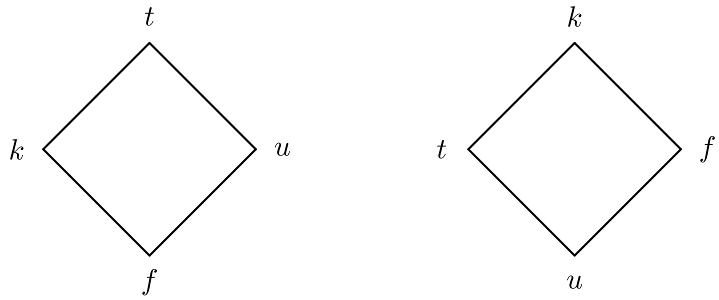
The reason for this choice will be more clear in the body of the paper. However, the only realistic alternative is fuzzy logic which fails to give an explicit difference between contradictory information and incomplete information. Modal logics of knowledge and belief have been also initially tested, but since their first order extensions are not immediate (sometimes not even available;) we decided not to continue.

The main objectives of the paper can be summarized as follows:
- provide a general framework for the axiomatization of preference modeling theories based on finite (discrete) valued logics;

- introduce a new paraconsistent four valued logic as a basis for preference modeling;
- demonstrate the limited representational power of classic logic in this domain;
- give an axiomatization of the partial comparability theory using the four valued logic.

The paper is organized as follows. In the first section the four-valued paraconsistent logic is presented as a new language for preference modeling reasons. In the second section the partial comparability theory is enhanced and axiomatized using the new language. The principal definitions and theorems are contained in this section. In the third section an example is presented in order to verify the operational potentialities of the new theory. The results obtained and further research are discussed in the conclusive section.

1 A four valued paraconsistent logic



L4 **A4**

Figure 1. The **L4** and **A4** lattices.

The principal idea (see Doherty et al., 1993), introduced by Dubarle (1963) and Belnap (1976 and 1977) was to define a logic where the truth values are partially ordered on a lattice. The four values; t, f, u, and k, may be read as *true, false, none* and *both*, respectively. The work done by Scott (see Scott, 1982) on the approximation lattices provided the formal basis for such a development. The intuition is that the truth values could be partially ordered on the basis of a "more information" relation (the correspondent lattice called **A4**) and by a "more truth" relation (the correspondent lattice called **L4**). The two lattices are shown in figure 1. Under this perspective we can introduce the concept of a "bilattice" because the set $\{t, k, u, f\}$ is ordered under the two partial orders in a way that present interesting functional correspondences that will be exploited in the definitions of the logical operators. The use of such lattices has been recently studied in Artificial Intelligence research (see Ginsberg, 1988; and Fitting, 1990).

1.1 Definitions

An alphabet of the first order language $\mathcal{L}$, henceforth called DDT, consists of a denumerable set of *individual variables*, the *logical connectives* "$\vee$" (or), "$\wedge$" (and), "$\sim$" (complementation), "$\not\sim$" (semi-negation) and "$\neg$' (negation), the *quantifiers* "$\forall$ (for all) and "$\exists$" (exists), a countable set of *predicate constants* including "=" for identity, the *monadic operators* "**T**" (true), "**F**" (false), “**U**” (unknown), and “**K**” (both), and the symbols “(” and “)” serving as punctuation marks. We use special letters (possibly with subscripts and/or primes): x, y and z for individual variables; and i, p, q and r for predicate constants of positive arity. We use the letters I, P, Q and R for the corresponding sets. We also use greek letters $\alpha, \beta, \gamma, \cdots$ to represent general formula of the language. For the truth definitions of the basic logical operators see Appendix A. In the following tables 1 and 2 we show the truth tables of the basic operators defined.

α	$\not\sim\alpha$	$\sim\not\sim\alpha$	$\sim\alpha$	$\neg\alpha$	$\neg\not\sim\alpha$	$\neg\sim\not\sim\alpha$	$\neg\sim\alpha$
t	k	u	f	f	k	u	t
k	t	f	u	k	f	t	u
u	f	t	k	u	t	f	k
f	u	k	t	t	u	k	f

Table 1. The truth tables of $\neg$, $\not\sim$ and $\sim$ and their combinations.

$\wedge$	u	f	t	k
u	u	f	u	f
f	f	f	f	f
t	u	f	t	k
k	f	f	k	k

$\vee$	u	f	t	k
u	u	u	t	t
f	u	f	t	k
t	t	t	t	t
k	t	k	t	k

$\rightarrow$	t	k	u	f
t	t	k	u	f
k	t	t	u	u
u	t	k	t	k
f	t	t	t	t

Table 2. The truth tables of $\wedge$, $\vee$ and $\rightarrow$.

What we actually use in the definitions of the negations are functional transformations of the bilattice previously defined. We have a strong negation ($\neg$: defined as usual) preserving the order of the ”truth dimension” and a weak negation ($\not\sim$) necessary to preserve the order of the ”information dimension”. The weak negation alone operates the transformation among the two orders. We call the weak negation a ”semi-negation”. These two operators do not introduce the complement, necessary to define a complete Boolean algebra on the bilattice. We call this operator ($\sim$) ”complementation”. By this operator is possible to define mutually inconsistent sentences (which are not a-priori possible with the strong negation). From an intuitive point of view the strong negation has the usual sense, the complementation introduces inconsistency and the semi-negation introduces the concept of ”perhaps” or ”it is possible”.

It is easy to verify the following identities:

$\sim\alpha \equiv \neg\not\sim\neg\not\sim\alpha \equiv \not\sim\neg\not\sim\neg\alpha$

$\neg\neg\alpha \equiv \alpha$

$\sim\sim\alpha \equiv \alpha$

$\not\sim\not\sim\alpha \equiv \alpha$

$\neg\sim\alpha \equiv \sim\neg\alpha$

$\sim\not\sim\alpha \equiv \not\sim\sim\alpha$

$\neg\not\sim\alpha \equiv \sim\not\sim\neg\alpha \equiv \not\sim\sim\neg\alpha \equiv \not\sim\neg\sim\alpha$

$\not\sim\neg\alpha \equiv \sim\neg\not\sim\alpha \equiv \neg\sim\not\sim\alpha \equiv \neg\not\sim\sim\alpha$

$\neg(\alpha\wedge\beta) \equiv \neg\alpha\vee\neg\beta$

$\sim(\alpha\wedge\beta) \equiv \sim\alpha\vee\sim\beta$
$\nsim(\alpha\wedge\beta) \equiv (\sim\alpha\vee\nsim\beta)\wedge(\alpha\vee\nsim\alpha)$

The logic defined fulfills some nice properties:

- all the negations observe the duality condition;
- the De Morgan laws are satisfied for the strong negations;
- the lattice properties are satisfied for the binary operators defined; if $v(\alpha)$ is an evaluation function from $\mathcal{L}$ to $\{t, f, u, k\}$ ordered by the logical lattice **L4** (by the relation $\preceq$) then:
  - $v(\alpha\wedge\beta) = \inf(v(\alpha), v(\beta))$
  - $v(\alpha\vee\beta) = \sup(v(\alpha), v(\beta))$
  - $v(\alpha\rightarrow\beta) = t$ **iff** $v(\alpha) \preceq v(\beta)$
  - $v(\alpha \equiv \beta) = t$ **iff** $v(\alpha) = v(\beta)$
  - the formula $\alpha\rightarrow\alpha$ is a tautology;
- the logic is functionally complete (see Dubarle, 1963);
- the logic is sound and complete (see Dubarle, 1963).

We can give now the definitions of four strong monadic operators enabling to obtain "non-contradictory" (only true or false) statements from a proposition. We have:

1. $\mathbf{T}\alpha =_{def} \alpha\wedge\sim\neg\alpha$.
2. $\mathbf{F}\alpha =_{def} \sim\alpha\wedge\neg\alpha$.
3. $\mathbf{U}\alpha =_{def} \sim\nsim\alpha\wedge\neg\nsim\alpha$.
4. $\mathbf{K}\alpha =_{def} \nsim\alpha\wedge\nsim\neg\alpha$.

The truth tables for the defined operators are in table 3.

α	$\mathbf{T}\alpha$	$\mathbf{K}\alpha$	$\mathbf{U}\alpha$	$\mathbf{F}\alpha$
t	t	f	f	f
k	f	t	f	f
u	f	f	t	f
f	f	f	f	t

Table 3. Truth Tables for the strong monadic operators.

1.2 A Set Theoretical Perspective

It is useful to interpret DDT from a set theoretic perspective, especially in the context of the proposed application of preference modeling where one normally works with a set of candidates and a binary relation applied to these candidates.

Given a model structure M with domain $|M|$ (see appendix A) and a predicate constant p where $n_P = 2$, we observe that

- $P^+ = \{\langle x, y\rangle \mid M \models_t p(x, y)\}$
- $P^- = \{\langle x, y\rangle \mid M \models_f p(x, y)\}$

We can give also the following definitions:

- $P^{*+} = \{\langle x, y\rangle \mid M \models_t \sim p(x, y)\}$
- $P^{*-} = \{\langle x, y\rangle \mid M \models_f \sim p(x, y)\}$

We have that $P^{*+} = |M| \setminus P^+$ and that $P^{*-} = |M| \setminus P^-$. Observe that an object $(x, y) \in |M|$ may be in both P^+ and P^- ($\langle x, y\rangle \in P^+ \cap P^-$) or in none of P^+ and P^- ($\langle x, y\rangle \in P^{*+} \cap P^{*-}$).

The set theoretic extensions of the four strong monadic operators are defined as follows:

- $Ext(\mathbf{T}p(x, y)) \equiv \{\langle x, y\rangle \mid \langle x, y\rangle \in (P^+ \cap P^{*-})\}$.
- $Ext(\mathbf{K}p(x, y)) \equiv \{\langle x, y\rangle \mid \langle x, y\rangle \in (P^+ \cap P^-)\}$.
- $Ext(\mathbf{U}p(x, y)) \equiv \{\langle x, y\rangle \mid \langle x, y\rangle \in (P^{*+} \cap P^{*-})\}$.
- $Ext(\mathbf{F}p(x, y)) \equiv \{\langle x, y\rangle \mid \langle x, y\rangle \in (P^- \cap P^{+*})\}$.

In the following sections, we will be describing binary preference relations. We will use the following notation for the extensions of a relation R: R^t, R^k, R^u, R^f respectively. Notice also that the usual correspondence between set operations and logical operators hold: inclusion ($\subseteq$) corresponds to implication ($\rightarrow$), intersection ($\cap$) corresponds to "and" ($\wedge$) and union ($\cup$) corresponds to "or" ($\vee$).

Concluding this section we have defined a powerful representation language achieving all the objectives previously given.

- We have a strong truth calculus that enables us to obtain unambiguous and meaningful sentences.
- The language is able to represent both incompleteness and inconsistency.
- The language has an immediate first order extension and therefore is suitable for preference modeling propositions.

2 A new axiomatic foundation of the PCT

Roy and Vincke (1984), when introducing their partial comparability axiom, claim that the two relations of strict preference and indifference fail to represent the whole set of situations that may occur when preferences have to be modelled. They introduce therefore four foundamental relations (strict preference, weak preference, indifference and incomparability) that define a "foundamental relational system of preferences" in the sense that define a set of exhaustive and mutual exclusive binary relations. However the definitions of such relations while offer some strong operational directions for their use, suffer from a theoretical point of view because it is impossible to give a complete axiomatization of them, unless valued binary relations are used. This theoretical lack could be naturally compensated by the operational validity of the approach, but it leaves space to ambiguities on the use of such relations and some arbitrariness on their use in decision aid methods. We will thus use the DDT logic in order to build a new theory about binary relations and verify its application in the definition of extended preference structures.

2.1 Binary relations and Preference Structures

In the following we give some general definitions and theorems about binary relations, some of them independently of the truth values of the language used. As stated in the introduction we work with languages having a finite (discrete) number of truth values, thus excluding fuzzy logic.

Definition 2.1 *Given a set A and a language $\mathcal{L}$, a set of n-ary relations $< R_1, \cdots R_m >$ defines a partition of A^n* **iff**

$$\forall (x_1, \cdots x_n) \in A^n, \ \ \exists R_i : \ \ v(R_i(x_1, \cdots x_n)) = t$$

and

$$\nexists (x_1, \cdots x_n) \in A^n, \ \ \exists R_i, R_j (i \neq j) : \ v(R_i(x_1, \cdots x_n)) = v(R_j(x_1, \cdots x_n)) = t$$

where $v(\alpha)$ is the evaluation function of $\mathcal{L}$.

♣

In other words, for any element of A^n at least one relation has the truth value t (true)", while is impossible to have more than one relation having contemporaneously the truth value t (true) on the same element of A^n. This definition applies whatever is the set of possible truth values of the language used. Let us bound ourselves now to binary relations.

Let A be a set of objects and let R be a binary relation on A. R will be defined as a subset of the cartesian product of A with itself ($R \subseteq A \times A$). Such a binary relation can be viewed as a binary predicate r with extensions on the domain $A \times A$ defined by the entailment relations of the language used. We define the *inverse* relation R^-1 of R as follows:

$$\forall x, y \in A \quad r^{-1}(x, y) \equiv r(y, x)$$

Preference relations are binary relations. Any collection of binary relations $< P, Q, \cdots >$ that model preference situations can be considered as a preference structure. In order to build sound theories about preference structures we wish to identify the ones fulfilling some properties and principles. A preference structure may be characterized by a binary relation (let us call it S) in the sense that the collection of binary relations should be defined by combinations of the epistemic states of this characteristic relation expressed on the same couple of elements of A. More formally, given a language $\mathcal{L}$, a set A and a set $\mathcal{R}$ of binary relations $r_1, \cdots, r_n$ we desire to find a set $\mathcal{A}$ of formula $\alpha_1, \cdots \alpha_n$ such that for all $r_i \in \mathcal{R}$

$$\forall x, y \;\; r_i(x, y) \equiv \alpha_i(s(x, y), s^{-1}(x, y))$$

This is a principle known also as "independence of irrelevant alternatives". In other words we wish to establish the truth value of a relation for a given couple (x, y) based uniquely on the truth state of the characteristic relation. Such a relation summarizes some way the information about the comparison of the elements of the couple. We consider such a principle as sound basis for the definition of preference relations mainly in the case of preference aggregation. We give now two definitions in order to identify two relevant features of a preference structure.

Definition 2.2 *Given a set A, a preference structure $< P, Q, \cdots >$, applied on A, is a "fundamental relational system of preferences" (f.r.s.p.)* **iff** *the set of relations $\mathcal{F} = \{P, Q, \cdots\}$ defines a partition of $A \times A$.*

♣

Given a language $\mathcal{L}$, a set $\mathcal{P}$ of binary relations properties defined in $\mathcal{L}$ we have the set $2^{\mathcal{P}}$ of all subsets of $\mathcal{P}$. For each element $\mathcal{P}_i \in 2^{\mathcal{P}}$ we can associate a logical formula α_i derived as the logical conjunction of the properties contained in $\mathcal{P}_i$. We can define now the set $2^{*\mathcal{P}}$ as any set of subsets of $\mathcal{P}$ such that the formulas associated to its elements are not logically equivalent.

$$2^{*\mathcal{P}} \;\; = \;\; \{\mathcal{P}_i, \mathcal{P}_j : \;\; \mathcal{P}_i, \mathcal{P}_j \in 2^{\mathcal{P}} \text{ and } \alpha_i \not\equiv \alpha_j\}$$

Definition 2.3 *Given a language $\mathcal{L}$, a set $\mathcal{P}$ of binary relations properties defined in $\mathcal{L}$, a set A, a preference structure $< P, Q, \cdots >$, applied on A, is a "well founded f.r.s.p."* **iff** *$< P, Q, \cdots >$ is a f.r.s.p. and it exists an injective application from $\mathcal{F}$ to $2^{*\mathcal{P}}$.*

♣

These two kind of preference structures are of particular interest because provide sets of preference relations that are exhaustive and mutually exclusive in the first case and well characterized in the second case as each binary relation is defined by its properties and there is no other binary relation having the same properties or their equivalent.

The following theorem is a preliminar result for the development of the rest of the paper.

Theorem 2.1 *Given a functionally complete language $\mathcal{L}$ with n (finite) truth values, a set A and a f.r.s.p. $< R_1, \cdots R_m >$ on A, characterized by a binary relation S, the number of possible R_i is $m \leq \frac{n(n+1)}{2}$*

Proof. By the independence principle each $r_i \equiv \alpha_i(s, s^{-1})$ where $\alpha_i(s, s^{-1})$ is a formula of $\mathcal{L}$ containing only s, s^{-1} and logical operators. By functionally completeness of $\mathcal{L}$ we can reduce $\alpha(s, s^{-1})$ in a normal conjuctive form $\bigwedge(N(s), N(s^{-1}))$, therefore using only conjunctions $(\wedge)$

and sequences of negations (N). From the lattice properties of the evaluation function (v) of $\mathcal{L}$ we have that: $v(r_i) = v(\bigwedge(N(s), N(s^{-1}))) = \inf(v(N(s)), v(N(s^{-1})))$. If the preference structure is a partition then it exists always a couple $(x, y) \in A \times A$ such that $v(r_i) = t$ and $v(r_{j \neq i}) \prec v(r_i)$ where $\prec$ (read: less) is the ordering relation of the truth values lattice. Therefore should be $v(N(s)) = t$ and $v(N(s^{-1})) = t$. Now if there exist n truth values there exist n sequences of negations giving $v(N(s)) = t$ and n sequences of negations giving $v(N(s^{-1})) = t$ depending on the epistemic states of s and $s^{-}1$. Then there exist n^2 combinations of s and s^{-1} forming an $n \times n$ matrix. This matrix has a symmetric form because for symmetric combinations of s and s^{-1} we obtain r and r^{-1}. Therefore the maximum number of relations that can be defined in this way corresponds to the upper or lower triangular elements of the matrix and these are exactly $\frac{n(n+1)}{2}$.

We can now give a third definition that completes the set of basic properties of a preference structure.

Definition 2.4 *Given a functionally complete language $\mathcal{L}$ with n truth values, a set A, a f.r.s.p. $< R_1, \cdots R_m >$ is a "maximal f.r.s.p."* **iff** $m = \frac{n(n+1)}{2}$.

♣

2.2 The PCT Preference Structure

In the following we will use the DDT language and we will define a new preference structure: the partial comparability theory (PCT) preference structure. We will then try to identify the properties of the PCT preference structure.

Using the DDT language we can write the properties of a generic binary relation as sentences in the logic itself (this is only a partial list of properties as others may be yet defined):

Definition 2.5 *Given a set A and a binary relation R on it, then R is:*

1. *reflexive* **iff**: $\forall x \in A \quad r(x, x)$
2. *semi-reflexive* **iff**: $\forall x \in A \quad \not\sim r(x, x)$
3. *quasi-reflexive* **iff**: $\forall x \in A \quad \neg{\sim} r(x, x)$
4. *semi-quasi-reflexive* **iff**: $\forall x \in A \quad \neg\not\sim r(x, x)$
5. *irreflexive* **iff**: $\forall x \in A \quad \sim r(x, x)$
6. *semi-irreflexive* **iff**: $\forall x \in A \quad \sim\not\sim r(x, x)$
7. *quasi-irreflexive* **iff**: $\forall x \in A \quad \neg r(x, x)$
8. *semi-quasi-irreflexive* **iff**: $\forall x \in A \quad \not\sim\neg r(x, x)$
9. *non-contradictory* **iff**: $\forall x, y \in A \quad \neg \mathbf{K} r(x, y) \wedge \neg \mathbf{U} r(x, y)$.
10. *symmetric* **iff**: $\forall x, y \in A \quad r(x, y) \rightarrow r^{-1}(x, y)$
11. *weakly-symmetric* **iff**: $\forall x, y \in A \quad r(x, y) \rightarrow \not\sim r^{-1}(x, y)$

12. *quasi-symmetric* **iff**: $\forall x, y \in A \quad r(x,y) \rightarrow \neg\sim r^{-1}(x,y)$

13. *quasi-weakly-symmetric* **iff**: $\forall x, y \in A \quad r(x,y) \rightarrow \nsim\neg r^{-1}(x,y)$

14. *asymmetric* **iff**: $\forall x, y \in A \quad r(x,y) \rightarrow \sim r^{-1}(x,y)$

15. *quasi-asymmetric* **iff**: $\forall x, y \in A \quad r(x,y) \rightarrow \neg r^{-1}(x,y)$

16. *weakly-asymmetric* **iff**: $\forall x, y \in A \quad r(x,y) \rightarrow \sim\nsim r^{-1}(x,y)$

17. *quasi-weakly-asymmetric* **iff**: $\forall x, y \in A \quad r(x,y) \rightarrow \neg\nsim r^{-1}(x,y)$

18. *complete* **iff**: $\forall x, y \in A \quad r(x,y) \vee r^{-1}(x,y)$

19. *transitive* **iff**: $\forall x, y, z \in A \quad r(x,y) \wedge r(y,z) \rightarrow r(x,z)$

20. *weakly-transitive* **iff**: $\forall x, y, z \in A \quad r(x,y) \wedge r(y,z) \rightarrow \nsim r(x,z)$

We call this set of properties $\mathcal{D}$.

$\clubsuit$

We may introduce now a preference structure characterized by a relation s and we will read $s(a,b)$ as "a is at least as good as b". We assume that $\forall x \in A \ s(x,x)$ has the truth value t ($v(s(x,x)) = t$; that is the relation s is reflexive). We will then define the PCT preference structure on the basis of the independence principle.

Definition 2.6 *Given a set A, the language DDT, and the characteristic relation s, a "PCT preference structure" $< p, q, k, l, i, j, h, r, u, v >$ is an n-uple of ten relations defined as follows:*

- $i(x,y) \equiv s(x,y) \wedge s^{-1}(x,y)$ *(indifference);*
- $j(x,y) \equiv \nsim s(x,y) \wedge \nsim s^{-1}(x,y)$ *(weak indifference);*
- $h(x,y) \equiv s(x,y) \wedge \nsim s^{-1}(x,y)$ *(semi-indifference);*
- $p(x,y) \equiv s(x,y) \wedge \sim s^{-1}(x,y)$ *(strict preference);*
- $q(x,y) \equiv \nsim s(x,y) \wedge \sim s^{-1}(x,y)$ *(weak preference);*
- $k(x,y) \equiv s(x,y) \wedge \sim\nsim s^{-1}(x,y)$ *(semi-preference);*
- $l(x,y) \equiv \nsim s(x,y) \wedge \neg\nsim s^{-1}(x,y)$ *(semi-weak-preference);*
- $r(x,y) \equiv \sim s(x,y) \wedge \sim s^{-1}(x,y)$ *(incomparability);*
- $u(x,y) \equiv \sim\nsim s(x,y) \wedge \sim\nsim s^{-1}(x,y)$ *(weak incomparability);*
- $v(x,y) \equiv \sim s(x,y) \wedge \sim\nsim s^{-1}(x,y)$ *(semi-incomparability).*

$\clubsuit$

We give now the following theorem.

Theorem 2.2 *Between the PCT preference structure and the set of properties* $2^{*\mathcal{D}}$ *(see definition 2.2.) it exists an injective application defined as follows:*

- *strict preference is irreflexive and asymmetric;*
- *weak preference is irreflexive and weakly asymmetric;*
- *semi-preference is semi-irreflexive and weakly asymmetric;*
- *semi-weak-preference is semi-quasi-irreflexive and quasi-asymmetric;*
- *indifference is reflexive and symmetric;*
- *weak indifference is semi-reflexive and symmetric;*
- *semi-indifference is semi-reflexive and weakly symmetric;*
- *incomparability is irreflexive and symmetric;*
- *weak incomparability is semi-irreflexive and symmetric;*
- *semi-incomparability is irreflexive and weakly symmetric.*

Proof. See Appendix B.

We introduce now the following lemma.

Lemma 2.1 *The PCT preference structure is a f.r.s.p.*

Proof. See Appendix C.

We can give now the principal result of this work in the following theorem.

Theorem 2.3 *The PCT preference structure is a well-founded, maximal f.r.s.p..*

Proof. From lemma 2.1. and theorem 2.2. we have that the PCT preference structure is a well founded f.r.s.p. and from cardinality of PCT we have that PCT is a well founded maximal f.r.s.p..

We can give now the following theorem.

Theorem 2.4 *Given the preference relations* $< P, Q, I, ... >$ *as defined in definition 2.6., the relation* S *is defined as follows:*
$\mathbf{T}s \equiv \mathbf{T}p \vee \mathbf{T}k \vee \mathbf{T}i \vee \mathbf{T}h$
$\mathbf{K}s \equiv \mathbf{T}q \vee \mathbf{T}l \vee \mathbf{T}j \vee \mathbf{T}h^{-1}$
$\mathbf{U}s \equiv \mathbf{T}u \vee \mathbf{T}l^{-1} \vee \mathbf{T}v^{-1} \vee \mathbf{T}k^{-1}$
$\mathbf{F}s \equiv \mathbf{T}r \vee \mathbf{T}v \vee \mathbf{T}q^{-1} \vee \mathbf{T}p^{-1}$.

Proof. See Appendix D.

The PCT preference structure therefore provides a solid formal structure for preference modeling purposes under incomplete and/or conflictual information. However it may appear as a rather complicated structure to use towards a real problem situation. We can always define "easier" structures and f.r.s.p. by combining the relations of the PCT preference structure or adding more specific properties. The $< P, Q, I, R >$ preference structure proposed by Roy (1985) can be obtained for instance as follows:
- $P_{Roy} = P \cup K$
- $Q_{Roy} = Q \cup L$
- $R_{Roy} = R \cup U \cup V$
- $I_{Roy} = I \cup J \cup H$

This is of course only a tentative definition, not necessarily respecting the exact semantics of the relations defined by B. Roy. We consider however that this is the more realistic definition at least when a formal language, as DDT is, is adopted.

The conventional preference structure can be also obtained imposing on the characteristic relation S the property of "non contradictory" and obtaining only the relations P, I and R. Under this point of view the PCT preference structure and the DDT logic appear as a generalization of the theories developed until now.

3 Two examples

In the following we will give two small examples about the use of the new formalism. We are conscious that the two examples do not present all the potentialities of the new theory as they treat with conventional problems already known in literature. On the other hand we preferred such examples because better known.

3.1 The pseudo order

In 1984 Roy and Vincke introduced the concept of the "pseudo order" as a double threshold preference structure characterized by complete comparability. Three basic preference relations have been introduced p, q, i and defined as follows.

Given a real valued function $g(.)$, two thresholds γ and δ ($\gamma < \delta$) and a set of alternatives A then:
- $\forall x, y \in A \;\; p(x, y) \Leftrightarrow g(x) - g(y) \geq \delta$
- $\forall x, y \in A \;\; q(x, y) \Leftrightarrow \delta > g(x) - g(y) > \gamma$
- $\forall x, y \in A \;\; i(x, y) \Leftrightarrow \gamma \geq g(x) - g(y) \geq -\gamma$

Such a preference structure has a large field of applications and has been used in a lot of real world decision aid situations. However it presents some theoretical problems. In our point of view this is not by itself a problem, but standing such theoretical ambiguities the pseudo order lacks a solid foundation and it is often misused by practitioners not aware and/or not honest. What are such problems?

Suppose you want to define p, q, i via a characteristic relation s of the pseudo order. Then (using classic logic notation) $\forall x, y \in A \;\; p(x,y) \equiv s(x,y) \wedge \neg s^{-1}(x,y)$ and if $\forall x, y \in A \;\; i(x,y) \equiv s(x,y) \wedge s^{-1}(x,y)$ what is $q(x,y)$? On the other hand if you want to establish s (which is a complete relation) from p, q, i following the definition given in Roy and Vincke (1984) $S = P \cup Q \cup I$ you have that $\forall x, y \in A \;\; s(x,y) \Leftrightarrow g(y) \leq g(x) + \gamma$. But in this case the definitions of $p(x,y)$ and of $p^{-1}(x,y)$ include the ones of $q(x,y)$ and of $q^{-1}(x,y)$. If $\forall x, y \in A \;\; s(x,y) \Leftrightarrow g(y) \leq g(x) + \delta$ then it does not hold $S = P \cup Q \cup I$.

The usual way by which these problems have been faced has been the use of fuzzy sets. Actually when $\gamma <\mid g(x) - g(y) \mid< \delta$ the membership function of $s(x,y)$ linearly drops from 1 to 0; $s(x,y)$ is a "valued" binary relation.

In our example we will give a representation using the DDT logic and the PCT preference structure. For two reasons: 1) we think it appears a natural modelization of the "conflicting" situation represented by the double threshold; 2) the fuzzy modelization does not give an explicit account of such "conflicting" dimension (the membership function could be between 0 and 1 because of incomplete information).

We give the following definitions:
- $\forall x, y \in A \;\; s(x,y) \Leftrightarrow g(y) \leq g(x) + \delta$
- $\forall x, y \in A \;\; \neg s(x,y) \Leftrightarrow g(y) > g(x) + \gamma$

Under these definitions the critical interval defined by the double threshold (that is the space beteen the two thresholds) is interpreted as a contradictory interval in which both the relation and its negation may hold. This corresponds with the truth definitions of the DDT logic. We can give now also the following definitions:
- $\forall x, y \in A \;\; \not\sim s(x,y) \Leftrightarrow g(y) > g(x) + \gamma$ and $g(y) \leq g(x) + \delta$
- $\forall x, y \in A \;\; \sim s(x,y) \Leftrightarrow g(y) > g(x) + \delta$
- $\forall x, y \in A \;\; \neg\sim s(x,y) \Leftrightarrow g(y) \leq g(x) + \gamma$

It is easy to verify that the "true extension" of the relation s ($\mathbf{T}s$) holds when $g(y) \leq g(x) + \gamma$ that is when $\forall x, y \in A \;\; s(x,y) \wedge \neg\sim s(x,y)$. It is also easy to verify that we have five distinguished intervals where we can apply the definitions of the PCT preference structure.

$g(y) < g(x) - \delta$	$\equiv$	$s(x,y) \wedge \sim s^{-1}(x,y)$	$\equiv p(x,y)$
$g(y) \geq g(x) - \delta$ and $g(y) < g(x) - \gamma$	$\equiv$	$s(x,y) \wedge \not\sim s^{-1}(x,y)$	$\equiv h(x,y)$
$g(y) \geq g(x) - \gamma$ and $g(y) \leq g(x) + \gamma$	$\equiv$	$s(x,y) \wedge s^{-1}(x,y)$	$\equiv i(x,y)$
$g(y) > g(x) + \gamma$ and $g(y) \leq g(x) + \delta$	$\equiv$	$\not\sim s(x,y) \wedge s^{-1}(x,y)$	$\equiv h^{-1}(x,y)$
$g(y) > g(x) + \delta$ and	$\equiv$	$\sim s(x,y) \wedge s^{-1}(x,y)$	$\equiv p^{-1}(x,y)$

We therefore have obtained the same definitions of the pseudo order replacing the relation of weak preference with the one of weak indifference as defined in the PCT preference structure. This is only a linguistic difference. We think that our formal definition is semantically equivalent to the definition given by Roy and Vincke (1984) under which the weak preference is a situation where ".... it is not possible to distinguish among preference and indifference while it is sure that there is no strict preference in the other sense."

There are two advantages in this definition. The first one is that we gave an explicit account of the contradictory information contained in the critical interval defined by the double threshold. The second one is the concrete possibility to give a characterization of the pseudo order.

3.2 Aggregation of partial orders

Suppose you have five examinators and five candidates (a, b, c, d, e). Each examinator used a multicriteria model to evaluate the candidates. The results are presented in figure 2 where each arc should be read as "strict preference", the sign of equality as "indifference" and the absence of arcs as "incomparability". These are well known partial orders that can be easily obtained when preferences are aggregated on more criteria as in our case.

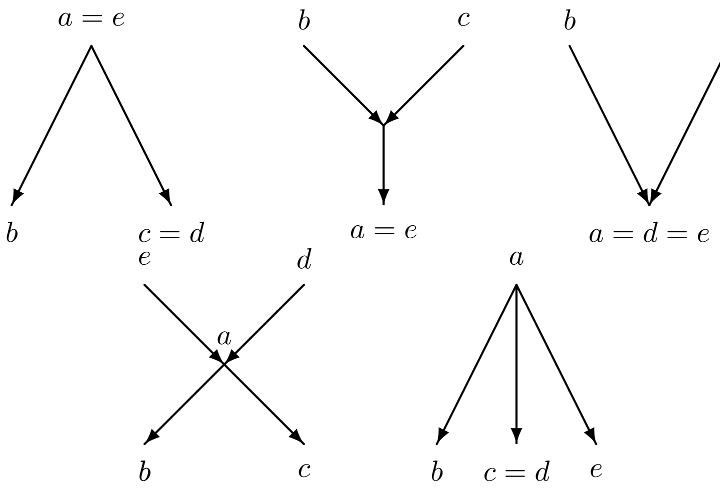


Figure 2: five partial orders that have to be aggregated

We are interested in obtaining a final ranking. There are two ways. The first one is to reduce these partial orders to one of their underlying complete orders and then aggregate the obtained rankings. This is the usual solution in such cases, but presents the inconvenient to eliminate useful information as the incomparabilities and to impose technical transformations that may be obscure to the decision maker. The second way (our choice) is to try to directly aggregate the five partial orders using the decision rule: "x is at least as good as y iff it exists a majority in this sense and there is no strong opposition against it".

Using a classic logic representation the decision rule could be formalized as follows:

$$\forall x, y \;\; s(x, y) \Leftrightarrow \mid s_j(x, y) \mid \geq \alpha \text{ and } \mid p_j^{-1}(x, y) \mid < \beta$$

where
$s(x, y)$ means that x is globally at least as good as y
$\mid s_j(x, y) \mid$ is the number of preorders where x is at least as good as y (preferred or indifferent)
$\mid p_j^{-1}(x, y) \mid$ is the number of preorders where y is strictly preferred to x

Let us take for example $\alpha = 3$ and $\beta = 2$ (these are the values which give the richest global relation). We obtain the following result

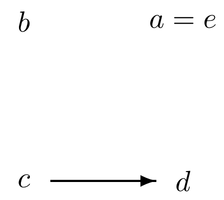


Figure 3: the result of a direct conventional aggregation of the five partial orders

The information contained in this result is rather poor: c is better than d, a and e are indifferent and every other pairwise comparison leads to incomparability. However, it is clear from the data that all these incomparabilities do not have the same causes.

We may try to apply the DDT logic and the PCT preference structure. As the original data is "non contradictory" it is useful to introduce the appropriate notation in order to handle this situation. We write for any formula α in DDT:
- $\triangle\alpha =_{def} \mathbf{T}\alpha \wedge \mathbf{K}\alpha$
and we obtain:
- $\triangle\neg\alpha =_{def} \mathbf{F}\alpha \wedge \mathbf{K}\alpha$
- $\neg\triangle\alpha =_{def} \mathbf{F}\alpha \wedge \mathbf{U}\alpha$
- $\neg\triangle\neg\alpha =_{def} \mathbf{T}\alpha \wedge \mathbf{U}\alpha$

The decision rule can be formalized now as follows. Let

$$\begin{cases} \triangle s(x,y) \Leftrightarrow \mid s_j(x,y) \mid \geq \alpha, \\ \triangle\neg s(x,y) \Leftrightarrow \mid p_j^{-1}(x,y) \mid \geq \beta. \end{cases}$$

We obtain

$$\begin{array}{l} \mathbf{T}s(x,y) \Leftrightarrow \mid s_j(x,y) \mid \geq \alpha \wedge \mid p_j^{-1}(x,y) \mid < \beta, \\ \mathbf{K}s(x,y) \Leftrightarrow \mid s_j(x,y) \mid \geq \alpha \wedge \mid p_j^{-1}(x,y) \mid \geq \beta, \\ \mathbf{U}s(x,y) \Leftrightarrow \mid s_j(x,y) \mid \geq \alpha \wedge \mid p_j^{-1}(x,y) \mid < \beta, \\ \mathbf{F}s(x,y) \Leftrightarrow \mid s_j(x,y) \mid \geq \alpha \wedge \mid p_j^{-1}(x,y) \mid \geq \beta, \end{array}$$

Applying these rules to the previous example leads to the following extended preference structure:

	a	b	c	d	e
a	I	Q	Q	J	I
b	Q^{-1}	I	U	V^{-1}	R
c	Q^{-1}	U	I	H	R
d	J	V	H^{-1}	I	U
e	I	R	R	U	I

Please notice that we used the non contradictory ("true") extensions of the preference relations obtained by simple transformations of the definitions of the PCT preference structure of the kind:

$\forall x, y \;\; \mathbf{T}p(x, y) \equiv \mathbf{T}s(x, y) \wedge \mathbf{F}s^{-1}(x, y)$
For the demonstrations it can be used the theorem 2.4.

We do not yet obtained a final ranking for which an appropriate exploiting procedure should be used, but nevertheless the advantage of this result is that it provides a very richer interpretation than the result obtained with the classic logic, allowing for example to distinguish the different kinds of incomparabilities which can occur in an aggregation problem.

Two main sources of incomparability are ignorance (lack of information) and conflict (contradictory preferences). These situations are illustrated by pairs (b, c) (complete ignorance) and (b, e) (strong conflict). It is also interesting to have a distinction between a pair like (b, e) (incomparability due to a conflict and reinforced by an individual incomparability) and a pair like (a, d) (incomparability due to a conflict but with an indication that they could be considered as indifferent). In the case of (a, e), the conflict is less strong than for (a, d) and there is a presumption of indifference.

Some other pairs lead to a presumption of preference. It is the case of (a, b), (b, d) and (c, d), but with a conflict for (a, b), an indication of ignorance for (b, d) and an indication of indifference for (c, d).

It must be pointed out here that our purpose is not to introduce a different relation (a different symbol) in each situation which can occur in an aggregation problem. However, we have the feeling that classic logic is not rich enough to represent the diversity of situations which can be obtained. The logic introduced in this paper has three advantages: it is richer than classic logic, it is well-founded and it is the less complex possibility to build a well-founded logic other than the classic one.

4 Conclusions

In this paper some key results on the axiomatic foundation of the "partial comparability theory" (PCT) are presented. When lack of information, uncertainty, ambiguity, multidimensional and conflicting preferences have to be modeled then the non conventional preference modeling theories have to be used. In order to build coherent and complete theories however is often necessary to adopt appropriate languages overcoming the classic formal logic generally used.

Under this perspective the paper introduces a new logic, called DDT, that is a four-valued paraconsistent logic, with very interesting properties (sound and complete, functionally complete, nice algebraic properties), enabling to represent and distinguish both the situations of incomplete and conflicting information. The first order extension of this logic provides thus a powerful language to be used in non conventional preference modeling theories.

Independently of the language used we define in the paper the concepts of partition, fundamental relational system of preferences (f.r.s.p.), well founded f.r.s.p. and maximal f.r.s.p.. Under the DDT language we redefined the properties of binary relations as formulas of the language and provide the definition of the PCT preference structure. We therefore demonstrated that such preference structure is a well founded, maximal f.r.s.p.. It is possible to observe that under incomplete and/or conflicting information it is possible to define a preference structure with *"higher granularity"* than the conventional one. On the other hand all the preference structures defined in literature until now can be viewed as particular situations of the PCT preference structure.

Some questions arise immediately after these results giving the stimulus for further research in the field opened by this paper.

1. Uniqueness of the *PCT* preference structure. Is *PCT* the only well founded maximal f.r.s.p. that can be defined using the DDT language and under which eventual supplementary conditions?

2. Characterization of particular preference structures under the DDT language. Semi-orders, pseudo-orders, different kinds of partial orders, how can be characterized under the DDT language?

3. Operational applications of the *PCT* preference structure and the DDT language. A small example is already included in the paper, but we think that a more thorough investigation of the potentialities of the new theory should be pursued.

References

1. Belnap N.D., (1976), "How a computer should think", *Proceedings of the Oxford International Symposium on Contemporary Aspects of Philosophy*, Oxford, England, 30 - 56.

2. Belnap N.D., (1977), "A useful four-valued logic", in G. Epstein, J. Dumm, (eds.), *Modern uses of multiple valued logics*, D. Reidel, Dordrecht, 8 -37.

3. Doherty P., Driankov D., Tsoukiàs A., (1993), "Partiality, Para-consistency and Preference Modeling", submitted, (preliminar version in *IDA research report Lith-IDA-R-92-18*, Linköping University).

4. Dubarle D., (1963), "Essai sur la généralisation naturelle de la logique usuelle", manuscript published posthumously in *Mathématique, Informatique, Sciences Humaines*, N^o 107, 1989, 17 - 73.

5. Fishburn P.C., (1991), "Nontransitive Preferences in Decision Theory", *Journal of Risk and Uncertainty*, vol., 4, 113 - 134.

6. Fitting M.C., (1990), "Bilattices in Logic Programming", in G. Epstein (ed.), *The Twentieth International Symposium on Multiple - Valued Logic*, IEEE, 238 - 246.

7. Ginsberg M., (1988), "Multivalued logics: A uniform approach to reasoning in artificial intelligence", *Computational Intelligence*, vol. 4, 265 - 316.

8. Kacprzyk J., Roubens M., (eds.), (1988), *Non Conventional Preference Relations in Decision Making*, Springer Verlag, LNMES n. 301, Berlin.

9. Luce R.D., (1956), "Semiorders and a theory of utility discrimination", *Econometrica*, vol. 24, 178 - 191.

10. Ovchinnikov S., Roubens M., (1991), "On strict preference relations", *Fuzzy Sets and Systems*, vol. 43, 319 - 326.

11. Ovchinnikov S., Roubens M., (1992), "On fuzzy strict preference, indifference and incomparability relations", *Fuzzy Sets and Systems*, vol. 47, 313 - 318.

12. Perny P., Roy B., (1991), "The use of fuzzy outranking relations in preference modeling", *Fuzzy Sets and Systems*, vol. 49, 33 - 53.

13. Roubens M., Vincke Ph., (1985), *Preference Modelling*, Springer Verlag, Berlin.

14. Roy B., (1977), "Partial Preference Analysis and Decision Aid: The fuzzy outranking relation concept, in D.E. Bell, R.L. Keeney, H. Raiffa, (eds.), *Conflicting objectives in Decisions*, J. Wiley, New York, 40 - 75.

15. Roy B., (1985), *Mèthodologie multicritère d'aide à la décision*, Economica, Paris.

16. Roy B., Vincke Ph., (1984), "Relational systems of preferences with one or more pseudo-criteria: some new concepts and results", *Management Science*, vol. 30, 1323 - 1335.

17. Scott D., (1982), "Some ordered sets in computer science", in I. Rival, (ed.), *Ordered Sets*, D. Reidel, Dordrecht, 677 - 718.

18. Tsoukiàs A., Vincke Ph., (1992), "A survey on non conventional preference modelling", *Ricerca Operativa*, n. 61, 5 - 49.

19. Tversky A., (1969), "Intransitivity of preferences", *Psychological Review*, vol. 76, 31 - 48.

Appendix A: truth definitions for DDT

A *similarity type* ρ is a finite set of predicate constants R, where each R has an arity $n_R \leq \omega$. Every alphabet uniquely determines a class of *formulas*. Relative to a given similarity type ρ, $R(x_1, \ldots, x_m)$ is an atomic formula iff $x_1, \ldots, x_m$ are individual variables, $R \in \rho$, and $n_R = m$. Similarly, , $(x = y)$ is an atomic formula iff x and y are variables. The definitions of $\mathcal{L}[\rho]$ formulas, free variables, etc., are defined in the usual way. In this paper, formulas are denoted by the letters α, β, γ, $\cdots$, possibly subscripted.

A *structure* or *model* M for similarity type ρ consists of a non-empty domain $|M|$ and, for each predicate symbol $R \in \rho$, an ordered pair $R^M = \langle R^{M^+}, R^{M^-} \rangle$ of sets (not necessarily disjoint) of n_R-tuples from $|M|$.

A *variable assignment* is a mapping from the set of variables to objects in the domain of the model. Capital letters from the beginning of the alphabet are used to range over variable assignments.

The truth definition for DDT is defined via two semantic relations, $\models_t$ (true entailment) and $\models_f$ (false entailment), by simultaneous recursion as follows:

Definition 4.1 *Let M be a model structure and A a variable assignment.*

1. $M \models_t R(x_1, \ldots, x_n)[A]$ *iff* $\langle A(x_1), \ldots, A(x_n) \rangle \in R^{M^+}$.

2. $M \models_f R(x_1, \ldots, x_n)[A]$ *iff* $\langle A(x_1), \ldots, A(x_n) \rangle \in R^{M^-}$.

3. $M \models_t \sim R(x_1, \ldots, x_n)[A]$ *iff* $\langle A(x_1), \ldots, A(x_n) \rangle \in M \setminus R^{M^+}$.

4. $M \models_f \sim R(x_1, \ldots, x_n)[A]$ *iff* $\langle A(x_1), \ldots, A(x_n) \rangle \in M \setminus R^{M^-}$.

5. $M \models_t (x = y)[A]$ *iff* $A(x) = A(y)$.

6. $M \models_f (x = y)[A]$ *iff* $A(x) \neq A(y)$.

7. $M \models_t \neg\alpha[A]$ *iff* $M \models_f \alpha[A]$.

8. $M \models_f \neg\alpha[A]$ *iff* $M \models_t \alpha[A]$.

9. $M \models_t \not\sim\alpha[A]$ *iff* $M \models_t \alpha[A]$.

10. $M \models_f \not\sim\alpha[A]$ *iff* $M \models_f \sim\alpha[A]$.

11. $M \models_t (\alpha \vee \beta)[A]$ *iff* $M \models_t \alpha[A]$ *or* $M \models_t \beta[A]$.

12. $M \models_f (\alpha \vee \beta)[A]$ *iff* $M \models_f \alpha[A]$ *and* $M \models_f \beta[A]$.

13. $M \models_t (\alpha \wedge \beta)[A]$ *iff* $M \models_t \alpha[A]$ *and* $M \models_t \beta[A]$.

14. $M \models_f (\alpha \wedge \beta)[A]$ *iff* $M \models_f \alpha[A]$ *or* $M \models_f \beta[A]$.

15. $M \models_t (\alpha \rightarrow \beta)[A]$ *iff* $M \models_t (\sim\alpha \vee \beta)$.

16. $M \models_f (\alpha \rightarrow \beta)[A]$ *iff* $M \models_f (\sim\alpha \vee \beta)$.

17. $M \models_t (\alpha \equiv \beta)[A]$ *iff* $M \models_t ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$.

18. $M \models_f (\alpha \equiv \beta)[A]$ *iff* $M \models_f ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$.

19. $M \models_t \exists x\alpha[A]$ *iff* $M \models_t \alpha[A']$ *for an* A' *differing with* A *at most at* x.

20. $M \models_f \exists x\alpha[A]$ *iff* $M \models_t \alpha[A']$ *for all* A' *differing with* A *at most at* x.

21. $M \models_t \forall x\alpha[A]$ *iff* $M \models_t \alpha[A']$ *for all* A' *differing with* A *at most at* x.

22. $M \models_f \forall x\alpha[A]$ *iff* $M \models_t \alpha[A']$ *for an* A' *differing with* A *at most at* x.

♣

Appendix B: Proof of theorem 2.2.

In order to prove the ten propositions of the theorem we will use the evaluation function of the DDT logic and the assumption that $\forall x \in A \;\; v(s(x, x)) = t$ where S is the usual global preference relation. Remind that:
$v(\alpha \wedge \beta) = \inf(v(\alpha), v(\beta))$
$v(\alpha \vee \beta) = \sup(v(\alpha), v(\beta))$
$v(\alpha \rightarrow \beta) = t$ **iff** $v(\alpha) \preceq v(\beta)$
$v(\alpha \equiv \beta) = t$ **iff** $v(\alpha) = v(\beta)$
We give also the following identities that are very easy to prove:
$\alpha \wedge \beta \rightarrow \alpha \vee \beta$

$\not\sim\alpha\wedge\not\sim\beta\rightarrow\not\sim(\alpha\wedge\beta)$
$\sim\not\sim\alpha\wedge\sim\not\sim\beta\rightarrow\sim\not\sim(\alpha\wedge\beta)$

We take now the ten propositions one by one (in the following is omitted $\forall x$ and $\forall x, y$ as extremely repetitive; the reader should associated them to all relations as appropriate).

1. indifference is reflexive and symmetric

   $i(x,x) \;\equiv\; s(x,x)\wedge s^{-1}(x,x)$
   $v(s(x,x)) = t$ and $v(s^{-1}(x,x)) = t$ therefore $v(s(x,x)\wedge s^{-1}(x,x)) = t$
   therefore $v(i(x,x)) = t$. This proves reflexivity.

   $i(x,y) \;\equiv\; s(x,y)\wedge s^{-1}(x,y) \;\equiv\; i^{-1}(x,y)$
   This proves symmetry.

2. weak indifference is semi-reflexive and symmetric

   $j(x,x) \;\equiv\; \not\sim s(x,x)\wedge\not\sim s^{-1}(x,x)$
   $v(\not\sim s(x,x)) = k$ and $v(\not\sim s^{-1}(x,x)) = k$ therefore $v(\not\sim s(x,x)\wedge\not\sim s^{-1}(x,x)) = k$
   therefore $v(j(x,x)) = k$ and $v(\not\sim j(x,x)) = t$. This proves semi-reflexivity.

   $j(x,y) \;\equiv\; \not\sim s(x,y)\wedge\not\sim s^{-1}(x,y) \;\equiv\; j^{-1}(x,y)$
   This proves symmetry.

3. semi-indifference is semi-reflexive and weakly symmetric

   $h(x,x) \;\equiv\; s(x,x)\wedge\not\sim s^{-1}(x,x)$
   $v(s(x,x)) = t$ and $v(\not\sim s^{-1}(x,x)) = k$ therefore $v(s(x,x)\wedge\not\sim s^{-1}(x,x)) = k$
   therefore $v(h(x,x)) = k$ and $v(\not\sim h(x,x)) = t$. This proves semi-reflexivity.

   $h(x,y) \;\equiv\; s(x,y)\wedge\not\sim s^{-1}(x,y) \;\equiv$
   $\equiv\; \not\sim\not\sim s(x,y)\wedge\not\sim s^{-1}(x,y) \;\rightarrow$
   $\rightarrow\; \not\sim(\not\sim s(x,y)\wedge s^{-1}(x,y)) \;\equiv$
   $\equiv \not\sim h^{-1}(x,y)$. This proves weak symmetry.

4. incomparability is irreflexive and symmetric

   $r(x,x) \;\equiv\; \sim s(x,x)\wedge\sim s^{-1}(x,x)$
   $v(\sim s(x,x)) = f$ and $v(\sim s^{-1}(x,x)) = f$ therefore $v(\sim s(x,x)\wedge\sim s^{-1}(x,x)) = f$
   therefore $v(r(x,x)) = f$ and $v(\sim r(x,x)) = t$. This proves irreflexivity.

   $r(x,y) \;\equiv\; \sim s(x,y)\wedge\sim s^{-1}(x,y) \;\equiv\; r^{-1}(x,y)$
   This proves symmetry.

5. weak incomparability is semi-irreflexive and symmetric

$u(x,x) \;\equiv\; \sim\nsim s(x,x) \wedge \sim\nsim s^{-1}(x,x)$
$v(\sim\nsim s(x,x)) = u$ and $v(\sim\nsim s^{-1}(x,x)) = u$ therefore $v(\sim\nsim s(x,x) \wedge \sim\nsim s^{-1}(x,x)) = u$
therefore $v(u(x,x)) = u$ and $v(\sim\nsim u(x,x)) = t$. This proves semi-irreflexivity.

$u(x,y) \;\equiv\; \sim\nsim s(x,y) \wedge \sim\nsim s^{-1}(x,y) \;\equiv\; u^{-1}(x,y)$
This proves symmetry.

6. semi-incomparability is irreflexive and weakly symmetric

$v(x,x) \;\equiv\; \sim s(x,x) \wedge \sim\nsim s^{-1}(x,x)$
$v(\sim s(x,x)) = f$ and $v(\sim\nsim s^{-1}(x,x)) = u$ therefore $v(\sim s(x,x) \wedge \sim\nsim s^{-1}(x,x)) = f$
therefore $v(v(x,x)) = f$ and $v(\sim v(x,x)) = t$. This proves irreflexivity.

$v(x,y) \;\equiv\; \sim s(x,y) \wedge \sim\nsim s^{-1}(x,y) \;\equiv$
$\equiv\; \nsim\nsim\sim s(x,y) \wedge \sim\nsim s^{-1}(x,y) \;\rightarrow$
$\rightarrow\; \nsim(\sim\nsim s(x,y) \wedge \sim s^{-1}(x,y)) \;\equiv$
$\equiv \nsim v^{-1}(x,y)$. This proves weak symmetry.

7. strict preference is irreflexive and asymmetric

$p(x,x) \;\equiv\; s(x,x) \wedge \sim s^{-1}(x,x)$
$v(s(x,x)) = t$ and $v(\sim s^{-1}(x,x)) = f$ therefore $v(s(x,x) \wedge \sim s^{-1}(x,x)) = f$
therefore $v(p(x,x)) = f$ and $v(\sim v(x,x)) = t$. This proves irreflexivity.

$p(x,y) \;\equiv\; s(x,y) \wedge \sim s^{-1}(x,y) \;\rightarrow$
$\rightarrow\; s(x,y) \vee \sim s^{-1}(x,y) \;\equiv$
$\equiv\; \sim(\sim s(x,y) \wedge s^{-1}(x,y)) \;\equiv$
$\equiv \sim p^{-1}(x,y)$. This proves asymmetry.

8. weak preference is irreflexive and weakly asymmetric

$q(x,x) \;\equiv\; \nsim s(x,x) \wedge \sim s^{-1}(x,x)$
$v(\nsim s(x,x)) = k$ and $v(\sim s^{-1}(x,x)) = f$ therefore $v(\nsim s(x,x) \wedge \sim s^{-1}(x,x)) = f$
therefore $v(q(x,x)) = f$ and $v(\sim q(x,x)) = t$. This proves irreflexivity.

$q(x,y) \;\equiv\; \nsim s(x,y) \wedge \sim s^{-1}(x,y) \;\equiv$
$\equiv\; \sim\sim\nsim s(x,y) \wedge \nsim\nsim\sim s^{-1}(x,y) \;\rightarrow$
$\rightarrow\; \sim\nsim(\sim s(x,y) \wedge \nsim s^{-1}(x,y)) \;\equiv$
$\equiv \sim\nsim q^{-1}(x,y)$. This proves weak asymmetry.

9. semi-preference is semi-irreflexive and weakly asymmetric

$k(x,x) \;\; \equiv \;\; s(x,x) \wedge \sim\nsim s^{-1}(x,x)$
$v(s(x,x)) = t$ and $v(\sim\nsim s^{-1}(x,x)) = u$ therefore $v(s(x,x) \wedge \sim\nsim s^{-1}(x,x)) = u$
therefore $v(k(x,x)) = u$ and $v(\sim\nsim v(x,x)) = t$. This proves semi-irreflexivity.

$k(x,y) \;\; \equiv \;\; s(x,y) \wedge \sim\nsim s^{-1}(x,y) \;\; \equiv$
$\equiv \;\; \nsim\nsim\sim\sim s(x,y) \wedge \sim\nsim s^{-1}(x,y) \;\; \rightarrow$
$\rightarrow \;\; \sim\nsim(\sim\nsim s(x,y) \wedge s^{-1}(x,y)) \;\; \equiv$
$\equiv \sim\nsim k^{-1}(x,y)$. This proves weak asymmetry.

10. semi-weak-preference is semi-quasi-irreflexive and quasi asymmetric

$l(x,x) \;\; \equiv \;\; \nsim s(x,x) \wedge \neg\nsim s^{-1}(x,x)$
$v(\nsim s(x,x)) = k$ and $v(\neg\nsim s^{-1}(x,x)) = k$ therefore $v(\nsim s(x,x) \wedge \neg\nsim s^{-1}(x,x)) = k$
therefore $v(l(x,x)) = k$ and $v(\nsim\neg l(x,x)) = t$. This proves semi-quasi-irreflexivity.

$l(x,y) \;\; \equiv \;\; \nsim s(x,y) \wedge \neg\nsim s^{-1}(x,y) \;\; \rightarrow$
$\rightarrow \;\; \nsim s(x,y) \vee \neg\nsim s^{-1}(x,y) \;\; \equiv$
$\equiv \;\; \neg(\neg\nsim s(x,y) \wedge \nsim s^{-1}(x,y)) \;\; \equiv$
$\equiv \neg l^{-1}(x,y)$. This proves quasi asymmetry.

It remains to demonstrate that there is no logical equivalence among the subsets of properties associated to each binary relation of the PCT preference structure.

Let us begin with formulas containing a different property about reflexivity. For instance the formula *"reflexive and symmetric"* and the formula *"irreflexive and symmetric"*. Suppose we accept that are equivalent. We have therefore (from definition of equivalence):
$\forall x, y \in A \;\; v(h(x,x) \wedge h(x,y) \rightarrow h^{-1}(x,y)) \;\; = \;\; v(\sim h(x,x) \wedge h(x,y) \rightarrow h^{-1}(x,y))$
where $v(\alpha)$ is the evaluation function of DDT and $h(x,y)$ is a generic binary relation. We can write now that:
$\exists x, y \in A \;\; v(h(x,x) \wedge h(x,y) \rightarrow h^{-1}(x,y)) \;\; = \;\; v(\sim h(x,x) \wedge h(x,y) \rightarrow h^{-1}(x,y)) \;\; = \;\; t$
and from definition of $\wedge$ we have:
$\exists x, y \in A \;\; v(h(x,x)) \;\; = \;\; v(h(x,y) \rightarrow h^{-1}(x,y)) \;\; = \;\; v(\sim h(x,x)) \;\; = \;\; t$
and this is impossible because $h(x,x)$ and $\sim h(x,x)$ have contemporaneously the truth value t. Therefore the two formulas are not equivalent. The same reasoning applies among:
- reflexivity and semi-quasi-irreflexivity
(it results $v(h(x,x)) \;\; = \;\; v(\nsim\neg h(x,x)) \;\; = \;\; t$; impossible);
- reflexivity and semi-reflexivity
(it results $v(h(x,x)) \;\; = \;\; v(\nsim h(x,x)) \;\; = \;\; t$; impossible);
- reflexivity and semi-irreflexivity
(it results $v(h(x,x)) \;\; = \;\; v(\sim\nsim h(x,x)) \;\; = \;\; t$; impossible);
- irreflexivity and semi-reflexivity
(it results $v(\sim h(x,x)) \;\; = \;\; v(\nsim h(x,x)) \;\; = \;\; t$; impossible);
- irreflexivity and semi-quasi-irreflexivity
(it results $v(h(x,x)) \;\; = \;\; v(\nsim\neg h(x,x)) \;\; = \;\; t$; impossible);

- irreflexivity and semi-irreflexivity
(it results $v(\sim h(x,x)) = v(\sim\not\sim h(x,x)) = t$; impossible);
- semi-reflexivity and semi-irreflexivity
(it results $v(\not\sim h(x,x)) = v(\sim\not\sim h(x,x)) = t$; impossible);
- semi-irreflexivity and semi-quasi-irreflexivity
(it results $v(\sim\not\sim h(x,x)) = v(\not\sim\neg h(x,x)) = t$; impossible);
On the other hand we have that among semi-quasi-irreflexivity and semi-reflexivity it could be possible an equivalence. We associate to each of these properties the binary relations that can be characterized by them. We have therefore that not equivalence is demonstrated for the following couples (the inverse couple is omitted because not equivalence is symmetric):
$(p,k),(p,l),(p,i),(p,j),(p,h),(p,u),(q,k),(q,l),(q,i),(q,j),(q,h),(q,u),$
$(k,l),(k,i),(k,j),(k,h),(k,r),(k,v),(l,i),(l,r),(l,v),(l,u),(i,j),(i,h),$
$(i,r),(i,u),(i,v),(j,r),(j,u),(j,v),(h,r),(h,u),(h,v),(r,u),(u,v)$

Let us work now with formulas containing different properties about symmetry. For instance the formula *"irreflexive and asymmetric"* and the formula *"irreflexive and weakly asymmetric"*. As in the previous paragraph we have:
$\forall x,y \in A\ \ v(\sim h(x,x)\wedge h(x,y)\rightarrow\sim h^{-1}(x,y)) = v(\sim h(x,x)\wedge h(x,y)\rightarrow\sim\not\sim h^{-1}(x,y))$
We then have:
$\exists x,y \in A\ \ v(\sim h(x,x)\wedge h(x,y)\rightarrow\sim h^{-1}(x,y)) = v(\sim h(x,x)\wedge h(x,y)\rightarrow\sim\not\sim h^{-1}(x,y)) = t$
and therefore:
$\exists x,y \in A\ \ v(\sim h(x,x)) = v(h(x,y)\rightarrow\sim h^{-1}(x,y)) = v(h(x,y)\rightarrow\sim\not\sim h^{-1}(x,y)) = t$
It is compatible with this case the following:
$\exists x,y \in A\ \ v(h(x,y)) = t$
The conjunction of the last two sentences gives (from definition of implication):
$\exists x,y \in A\ \ v(h(x,y)) = v(\sim h^{-1}(x,y)) = v(\sim\not\sim h^{-1}(x,y)) = t$
and this is impossible because $\sim h^{-1}(x,y)$ and $\sim\not\sim h^{-1}(x,y)$ cannot have contemporaneously the truth value t. Therefore the two sentences are not equivalent. The same reasoning applies among:
- asymmetry and symmetry
(it results $v(\sim h^{-1}(x,y)) = v(h^{-1}(x,y)) = t$; impossible);
- asymmetry and weak symmetry
(it results $v(\sim h^{-1}(x,y)) = v(\not\sim h^{-1}(x,y)) = t$; impossible);
- weak asymmetry and quasi asymmetry
(it results $v(\sim\not\sim h^{-1}(x,y)) = v(\neg h^{-1}(x,y)) = t$; impossible);
- weak asymmetry and symmetry
(it results $v(\sim\not\sim h^{-1}(x,y)) = v(h^{-1}(x,y)) = t$; impossible);
- weak asymmetry and weak symmetry
(it results $v(\sim\not\sim h^{-1}(x,y)) = v(\not\sim h^{-1}(x,y)) = t$; impossible);
- quasi asymmetry and symmetry
(it results $v(\neg h^{-1}(x,y)) = v(h^{-1}(x,y)) = t$; impossible);
- quasi asymmetry and weak symmetry
(it results $v(\neg h^{-1}(x,y)) = v(\not\sim h^{-1}(x,y)) = t$; impossible);
- symmetry and weak symmetry
(it results $v(h^{-1}(x,y)) = v(\not\sim h^{-1}(x,y)) = t$; impossible);

On the other hand among asymmetry and quasi asymmetry it could be possible a situation of equivalence. As previously not equivalence is demonstrated now for the following couples (again the inverse couples are omitted):
$(p,q),(p,k),(p,i),(p,j),(p,h),(p,r),(p,u),(p,v),(q,l),(q,i),(q,j),(q,h),$
$(q,r),(q,u),(q,v),(k,l),(k,i),(k,j),(k,h),(k,r),(k,u),(k,v),(l,i),(l,j),$
$(l,h),(l,r),(l,u),(l,v),(i,h),(i,v),(j,h),(j,v),(h,r),(h,u),(r,v),(u,v)$

And this concludes the proof as the union of the two sets of couples for which not equivalence has been demonstrated is exhaustive of all the possible combinations.

Appendix C: proof of lemma 2.1.

In order to demonstrate that the PCT preference structure forma a partition of $A \times A$ we introduce the truth tables of the ten preference relations and their inverse as they result combining $s(x,y)$ and $s^{-1}(x,y)$ by their definitions. In the following tables the truth values of s are on the top of the rows and the truth values of s^{-1} are on the top of the columns.

P	t	f	u	k
t	f	t	k	u
f	f	f	f	f
u	f	u	f	u
k	f	k	k	f

P^{-1}	t	f	u	k
t	f	f	f	f
f	t	f	u	k
u	k	f	f	k
k	u	f	u	f

Q	t	f	u	k
t	f	k	k	f
f	f	u	f	u
u	f	f	f	f
k	f	t	k	u

Q^{-1}	t	f	u	k
t	f	f	f	f
f	k	u	f	t
u	k	f	f	k
k	f	u	f	u

K	t	f	u	k
t	u	k	t	f
f	f	f	f	f
u	u	f	u	f
k	f	k	k	f

K^{-1}	t	f	u	k
t	u	f	u	f
f	k	f	f	k
u	t	f	u	k
k	f	f	f	f

L	t	f	u	k
t	k	f	k	f
f	f	u	u	f
u	f	f	f	f
k	k	u	t	f

L^{-1}	t	f	u	k
t	k	f	f	k
f	f	u	f	u
u	k	u	f	t
k	f	f	f	f

I	t	f	u	k
t	t	f	u	k
f	f	f	f	f
u	u	f	u	f
k	k	f	f	k

$$\forall x, y \;\; i(x,y) \equiv i^{-1}(x,y)$$

J	t	f	u	k
t	k	f	f	k
f	f	u	f	u
u	f	f	f	f
k	k	u	f	t

$$\forall x, y \;\; j(x,y) \equiv j^{-1}(x,y)$$

H	t	f	u	k
t	k	u	f	t
f	f	f	f	f
u	f	u	f	u
k	k	f	f	k

H^{-1}	t	f	u	k
t	k	f	f	k
f	u	f	u	f
u	f	f	f	f
k	t	f	u	k

R	t	f	u	k
t	f	f	f	f
f	f	t	k	u
u	f	k	k	f
k	f	u	f	u

$$\forall x, y \;\; r(x,y) \equiv r^{-1}(x,y)$$

U	t	f	u	k
t	u	f	u	f
f	f	k	k	f
u	u	k	t	f
k	f	f	f	f

$$\forall x, y \;\; u(x,y) \equiv u^{-1}(x,y)$$

V	t	f	u	k
t	f	f	f	f
f	u	k	t	f
u	f	k	k	f
k	u	f	u	f

V^{-1}	t	f	u	k
t	f	u	f	u
f	f	k	k	f
u	f	t	k	u
k	f	f	f	f

To demonstrate that these ten relations form a partition of $A \times A$ we have to make the following two demonstrations:

1. at least one of these relations has to be true. Under the DDT language means to demonstrate that the following is an identity.

$\forall x, y\ \mathbf{T}p(x,y) \vee \mathbf{T}q(x,y) \vee \mathbf{T}k(x,y) \vee \mathbf{T}l(x,y) \vee \mathbf{T}i(x,y) \vee \mathbf{T}j(x,y) \vee \mathbf{T}h(x,y) \vee \mathbf{T}r(x,y) \vee \mathbf{T}u(x,y)$
$\vee \mathbf{T}v(x,y) \vee \mathbf{T}p^{-1}(x,y) \vee \mathbf{T}q^{-1}(x,y) \vee \mathbf{T}k^{-1}(x,y) \vee \mathbf{T}l^{-1}(x,y) \vee \mathbf{T}h^{-1}(x,y) \vee \mathbf{T}v^{-1}(x,y)$.

Using the truth tables previously defined we can build progressively the truth table of this formula as follows:

$\mathbf{T}p \vee \mathbf{T}q$	t	f	u	k
t	f	t	f	f
f	f	f	f	f
u	f	f	f	f
k	f	t	f	f

$\mathbf{T}p \vee \mathbf{T}q \vee \mathbf{T}k$	t	f	u	k
t	f	t	t	f
f	f	f	f	f
u	f	f	f	f
k	f	t	f	f

$\dots$

It is easy to observe that following this direction, when the last relation will be added to the truth table, we obtain the truth table of an identity (all truth values are t). This concludes the first demonstration.

2. It is not possible to have two relations having the truth value t on the same couple (x, y). Under the DDT language we have to demonstrate a list of implications of the kind $\forall x, y\ \ \mathbf{T}p(x,y) \rightarrow \neg \mathbf{T}q(x,y)$ for all the combinations (90) of the ten binary relations. In the following we demonstrate the truthness of the previous formula. All the other cases can be demonstrated in exactly the same way and are therefore omitted.

$\mathbf{T}p$	t	f	u	k
t	f	t	f	f
f	f	f	f	f
u	f	f	f	f
k	f	f	f	f

$\neg \mathbf{T}q$	t	f	u	k
t	t	t	t	t
f	t	t	t	t
u	t	t	t	t
k	t	f	t	t

From the definition of the value function in the case of implication we can observe that the formula concerned is always true. And this concludes the demonstration.

From these two demonstration we can conclude that the PCT preference structure forms a partition of $A \times A$ and therefore is a f.r.s.p..

Appendix D.

We represent by $\top$ the logical constant representing "truth" ($v(\top) = t$). We remind also that $\mathbf{T}\alpha \equiv \alpha \wedge \neg \sim \alpha$. We therefore have:

1. $\mathbf{T}p \vee \mathbf{T}k \vee \mathbf{T}i \vee \mathbf{T}h \equiv$
$((s \wedge \sim s^{-1}) \wedge \sim \neg (s \wedge \sim s^{-1})) \vee ((s \wedge \sim \not\succ s^{-1}) \wedge \sim \neg (s \wedge \sim \not\succ s^{-1})) \vee$
$((s \wedge s^{-1}) \wedge \sim \neg (s \wedge s^{-1})) \vee ((s \wedge \not\succ s^{-1}) \wedge \sim \neg (s \wedge \not\succ s^{-1})) \equiv$
$(s \wedge \sim s^{-1} \wedge \sim \neg s \wedge \neg s^{-1}) \vee (s \wedge \sim \not\succ s^{-1} \wedge \sim \neg s \wedge \neg \not\succ s^{-1}) \vee$
$(s \wedge s^{-1} \wedge \sim \neg s \wedge \sim \neg s^{-1}) \vee (s \wedge \not\succ s^{-1} \wedge \sim \neg s \wedge \neg \sim \not\succ s^{-1}) \equiv$

$(s\wedge\sim\neg s)\wedge((\sim s^{-1}\wedge\neg s^{-1})\vee(\sim\nsim s^{-1}\wedge\neg\nsim s^{-1})\vee(s^{-1}\wedge\sim\neg s^{-1})\vee(\nsim s^{-1}\wedge\nsim\neg s^{-1}))\equiv$
$\mathbf{T}s\wedge(\mathbf{F}s^{-1}\vee\mathbf{U}s^{-1}\mathbf{T}s^{-1}\mathbf{K}s^{-1})\equiv$
$\mathbf{T}s\wedge\top\equiv\mathbf{T}s$

2. $\mathbf{T}q\vee\mathbf{T}l\vee\mathbf{T}j\vee\mathbf{T}h^{-1}\equiv$
$((\nsim s\wedge\sim s^{-1})\wedge\sim\neg(\nsim s\wedge\sim s^{-1}))\vee((\nsim s\wedge\neg\nsim s^{-1})\wedge\sim\neg(\nsim s\wedge\sim\nsim s^{-1}))\vee$
$((\nsim s\wedge\nsim s^{-1})\wedge\sim\neg(\nsim s\wedge\nsim s^{-1}))\vee((\nsim s\wedge s^{-1})\wedge\sim\neg(\nsim s\wedge s^{-1}))\equiv$
$(\nsim s\wedge\sim s^{-1}\wedge\nsim\neg s\wedge\neg s^{-1})\vee(\nsim s\wedge\sim\nsim s^{-1}\wedge\nsim\neg s\wedge\neg\nsim s^{-1})\vee$
$(\nsim s\wedge\nsim s^{-1}\wedge\nsim\neg s\wedge\nsim\neg s^{-1})\vee(\nsim s\wedge s^{-1}\wedge\nsim\neg s\wedge\sim\neg s^{-1})\equiv$
$(\nsim s\wedge\nsim\neg s)\wedge((\sim s^{-1}\wedge\neg s^{-1})\vee(\sim\nsim s^{-1}\wedge\neg\nsim s^{-1})\vee(s^{-1}\wedge\sim\neg s^{-1})\vee(\nsim s^{-1}\wedge\nsim\neg s^{-1}))\equiv$
$\mathbf{K}s\wedge(\mathbf{F}s^{-1}\vee\mathbf{U}s^{-1}\mathbf{T}s^{-1}\mathbf{K}s^{-1})\equiv$
$\mathbf{K}s\wedge\top\equiv\mathbf{K}s$

3. $\mathbf{T}u\vee\mathbf{T}l^{-1}\vee\mathbf{T}v^{-1}\vee\mathbf{T}k^{-1}\equiv$
$((\sim\nsim s\wedge\sim\nsim s^{-1})\wedge\sim\neg(\sim\nsim s\wedge\sim\nsim s^{-1}))\vee((\nsim s\wedge\neg\nsim s^{-1})\wedge\sim\neg(\nsim s\wedge\neg\nsim s^{-1}))\vee$
$((\sim\nsim s\wedge\sim s^{-1})\wedge\sim\neg(\sim\nsim s\wedge\sim s^{-1}))\vee((\sim\nsim s\wedge s^{-1})\wedge\sim\neg(\sim\nsim s\wedge s^{-1}))\equiv$
$(\sim\nsim s\wedge\sim\nsim s^{-1}\wedge\neg\nsim s\wedge\neg\nsim s^{-1})\vee(\sim\nsim s\wedge\nsim s^{-1}\wedge\neg\nsim s\wedge\nsim\neg s^{-1})\vee$
$(\sim\nsim s\wedge\sim s^{-1}\wedge\neg\nsim s\wedge\neg s^{-1})\vee(\sim\nsim s\wedge s^{-1}\wedge\neg\nsim s\wedge\neg\sim s^{-1})\equiv$
$(\sim\nsim s\wedge\neg\nsim s)\wedge((\sim s^{-1}\wedge\neg s^{-1})\vee(\sim\nsim s^{-1}\wedge\neg\nsim s^{-1})\vee(s^{-1}\wedge\sim\neg s^{-1})\vee(\nsim s^{-1}\wedge\nsim\neg s^{-1}))\equiv$
$\mathbf{U}s\wedge(\mathbf{F}s^{-1}\vee\mathbf{U}s^{-1}\mathbf{T}s^{-1}\mathbf{K}s^{-1})\equiv$
$\mathbf{U}s\wedge\top\equiv\mathbf{U}s$

4. $\mathbf{T}r\vee\mathbf{T}v\vee\mathbf{T}q^{-1}\vee\mathbf{T}p^{-1}\equiv$
$((\sim s\wedge\sim s^{-1})\wedge\sim\neg(\sim s\wedge\sim s^{-1}))\vee((\sim s\wedge\sim\nsim s^{-1})\wedge\sim\neg(\sim s\wedge\sim\nsim s^{-1}))\vee$
$((\sim s\wedge\nsim s^{-1})\wedge\sim\neg(\sim s\wedge\nsim s^{-1}))\vee((\sim s\wedge\ s^{-1})\wedge\sim\neg(\sim s\wedge\ s^{-1}))\equiv$
$(\sim s\wedge\sim s^{-1}\wedge\neg s\wedge\neg s^{-1})\vee(\sim s\wedge\sim\nsim s^{-1}\wedge\neg s\wedge\neg\nsim s^{-1})\vee$
$(\sim s\wedge\nsim s^{-1}\wedge\neg s\wedge\nsim\neg s^{-1})\vee(\sim s\wedge s^{-1}\wedge\neg s\wedge\neg\sim s^{-1})\equiv$
$(\sim s\wedge\neg s)\wedge((\sim s^{-1}\wedge\neg s^{-1})\vee(\sim\nsim s^{-1}\wedge\neg\nsim s^{-1})\vee(s^{-1}\wedge\sim\neg s^{-1})\vee(\nsim s^{-1}\wedge\nsim\neg s^{-1}))\equiv$
$\mathbf{F}s\wedge(\mathbf{F}s^{-1}\vee\mathbf{U}s^{-1}\mathbf{T}s^{-1}\mathbf{K}s^{-1})\equiv$
$\mathbf{F}s\wedge\top\equiv\mathbf{F}s$