A framework for expected capability sets

Nicolas Fayard[1*], David Rios Insua[2] and Alexis Tsoukiàs[1]

[1*]LAMSADE – CNRS, Paris-Dauphine PSL-University, Pl. du Maréchal de Lattre de Tassigny, Paris, 75016, France.

[2]Institute of Mathematical Sciences (ICMAT), Campus Cantoblanco UAM, C/ Nicolás Cabrera 13-15, Madrid, 28049, Spain.

*Corresponding author(s). E-mail(s): nicolas.fayard@dauphine.psl.eu;
Contributing authors: david.rios@icmat.es;
tsoukias@lamsade.dauphine.fr;

Abstract

This paper addresses decision-aiding problems involving multiple objectives and uncertain states of the world. Inspired by the capability approach, we focus on cases where a policy maker chooses an act that, combined with a state of the world, leads to a set of choices for citizens. While no preferential information is available to construct importance parameters for the criteria, we can obtain probabilities for the different states. To effectively support decision-aiding in this context, we propose two procedures that merge the potential set of choices for each state of the world taking into account their respective probabilities. Our procedures satisfy several fundamental and desirable propositions that characterize the outcomes.

Keywords: Decision analysis, Uncertainty, Multiobjective Decision Making, Capability sets

1 Introduction

Consider the case of a Policy Maker (PM) deciding how to allocate resources for climate change mitigation projects for a given territory. A simplified version of a decision-analytic approach to this case would consist of assessing the current welfare of a territory based on the distribution of private assets and the accessibility to common resources. This would involve identifying states of potential damage to private property and common resources, along with their likelihood of occurrence, and then solving an expected utility optimisation problem. From a decision support perspective, this implies that the welfare, the consequences, and the impacts of policies need to be measured using a single utility function. Among other things, this entails that the PM considers all citizens as homogeneous rational consumers all behaving in the same way with respect to welfare and risks. From a normative standpoint, a policy aligned with what a generic citizen would rationally decide is deemed the appropriate decision to make.

As suggested in Fayard, Mazri, and Tsoukiàs (2022); Sen (1985) and Sen (1993), while this approach aids in rationalizing the decision-making process of PMs, it fails to address, at least, two important issues:

- the impact of common resources on the welfare of citizens, extending beyond their mere material wealth;
- the fact that the citizens do not behave as indistinguishable consumers, which entails that considering the welfare as just a utility maximisation problem is misleading.

In the aforementioned paper (Fayard et al., 2022), the authors introduce the idea of using Sen's Capability Theory (Sen, 1980, 1985, 1993, 1999, 2009) to overcome both issues. Their approach assumes that the welfare of a single citizen is the Pareto frontier obtained by "solving" a mixed integer multi-objective linear programme (MI-MOLP) whose constraints are related to the citizen's private assets as well as their accessibility to the common resources and the objectives correspond to the maximisation of several welfare value functions. These functions reflect the individual's preferences across different welfare dimensions, such as being safe, being well-nourished, being in good health, or being happy. The rationale for adopting this perspective is grounded on the meaning of "freedom of choice" for citizens. Essentially, welfare is determined not only by private possessions but also by the opportunities and capabilities that these possessions bring along. When all significant welfare dimensions are taken into accoun, the capability set can be perceived as a Pareto frontier, where, with no loss of generality, we may assume that all relevant dimensions are to be maximized.

Once we know each citizen's capability set, we can cluster the population of a given community according to their capability set similarity, which relates to how similar two Pareto frontiers are. The consequence of this approach is that determining the welfare of a community is not merely the sum of the utilities of its citizens. Instead, it involves a Pareto frontier that represents multidimensional welfare for each cluster of citizens with similar values. As a result, the decision problem of the PM consists of establishing and choosing a policy that maximises the capability of clusters of citizens, that is, maximising both the level of the outcomes for each citizen and their freedom of choice.

It is crucial for our purposes to emphasize at this point how the notion of capability relates to freedom. Consider the following example from Sen (1985), if a citizen has access to a capability set containing multiple solutions among which option b is considered to be the best, according to his or her preferences, a second capability set comprising only option b would be deemed less desirable than the first set, even if the citizen would end up choosing such option in both cases. This is because the second set provides less freedom of choice, in the sense of allowing fewer choices. The result of these assumptions is that the PM will choose a policy which is not the behaviour of an undistinguishable consumer, but the one which maximises the value and the diversity of the options offered through the Pareto frontier of one or more clusters of citizens. However, the PM ignores, and actually does not care to know it, how each single citizen will take profit from the policy.

The PM's problem further complicates if several possible states of the world need to be considered. Accounting for such states of the world, potentially affecting private resources or the commons, implies acknowledging the likelihood of such events. Assuming such likelihoods are measurable through probabilities, the standard approach consists of computing expected utilities. However, for our PM aiming at maximising welfare for the citizens in her territory, assuming our approach, there is no single utility function to consider for each citizen, but a Pareto frontier representing the welfare supposed to be achieved through the policies. It is therefore necessary to extend the traditional public policy approach considering public policies as impacting Pareto frontiers. Note though that these Pareto frontiers would result from different potential events, meaning that Pareto frontiers corresponding to individual states of the world must be somehow integrated, taking into account the likelihood of those states. In other words, instead of computing an expected utility, the PM has to compute an expected Pareto frontier. The open technical question is therefore how to compute such "Expected Pareto Frontiers"? To our knowledge this is an open question despite the importance it has for policy design purposes.

Thus, our normative position stands around the following policy design question: "*How can a PM design policies that expand citizens' freedom of choice by focusing on actions that consistently improve aspects of their lives?*". When multiple states are plausible, and their probabilities are available, even if based on subjective estimates, we aim to develop policies that maximize freedom of choice across all potential states of the world factoring in their relative chance of occurrence.

This capability theory based perspective differs from more classical approaches. For example, traditional decision-making methods like multi-attribute utility theory (Fishburn, 1977; Keeney & Raiffa, 1993) aim to identify the *best solution* by quantifying and aggregating multiple attributes of each alternative based on the maximum expected utility principle. Similarly, approaches at the juncture of scenario analysis and multi-criteria decision making (Durbach & Stewart, 2012; Ríos Insua & French, 1991) focus on the optimization of outcomes, but they do so under the assumption that each state and decision lead to a single output, whereas we consider a set of possible outcomes. They aim to balance the overall risk and return across a "portfolio" of choices, taking into account potential correlations among them for optimal diversification and risk management. In our approach, we assume we have available probabilities for the states, differing from the traditional assumption in scenario planning (Stewart, French, & Rios, 2013), and have capability sets rather than single consequences, even if multi-attribute, differing from standard scenario-based portfolio approaches (Liesiö & Salo, 2012; Ríos Insua & French, 1991; Vilkkumaa, Liesiö, Salo, & Ilmola-Sheppard, 2018). Our final discussion points out additional issues concerning partial information about the states' probabilities.

To summarize, the PM allocates resources within society, with the actual distribution ultimately shaped by external factors and uncertainties. As an example, consider the case where the PM considers introducing a policy concerning air quality improvement in a town. A major industrial accident could occur, leading to medium to long-term impacts over air quality for residents in certain districts. Citizens will need to adapt their behavior to comply with the policy, while also responding to the environmental or external factors that may influence the outcomes such as the accident if occurs. Therefore, the PM needs to anticipate how the citizens will adapt to a policy, but also how they will adapt if an accident occurs and the citizens behave differently.

Following standard jargon from capability theory (Fayard et al., 2022), we define *capability sets* as the range of opportunities individuals have to achieve various *functionings*. Functionings refer to concrete aspects of life, representing what people are effectively able to do and to be, such as being healthy, feeling safe, maintaining social relationships, engaging in meaningful work, or participating in community life. Within their capability set, citizens choose a *being-vector*, which corresponds to a specific combination of functionings that reflects their personal conception of a good life. Such choices will depend on the states holding, in case multiple states do exist. How citizens employ their resources and which being-vectors they pursue are matters of personal agency and should not be normatively dictated by the PM. The objective of the PM is to ensure that citizens have access to a diverse and appreciable capability set. The PM should consider the resources they allocate, as well as those determined by nature. Should state probabilities be available, the PM might then adopt a policy that optimizes the aggregated capability set, as represented by the Pareto Frontier.

Our primary focus is on the exploration of mixed capability sets within the previously outlined framework. By that we understand the combination or aggregation of capability sets corresponding to different states of the world, each weighted by its respective probability. To achieve this, we begin by formulating the problem precisely in Section 2, where we introduce the required terminology from capability theory. Section 3.1 introduces a natural approach for mixing capability sets, termed *average capability sets*; in turn, Section 3.2 proposes a more sophisticated approach called *expected capability sets*, whose major properties are explored in Section 4. Section 5 further explores the distinctions between both approaches, expanding our understanding of the possibilities and considerations involved in decision-making within the capability approach context.

2 Problem formulation

A PM faces the task of selecting an act in an environment of risk. To represent this risk, a set S of possible states of the world is employed. S is assumed to be finite, with $S = \{s_1, s_2, ..., s_{l^*}\}$, where states are assumed to be mutually exclusive and collectively exhaustive. The PM's beliefs about the likelihood of each state being the actual one are modeled using probabilities $p(s_l)$ for $l \in \{1, 2, \cdots, l^*\}$, with $\sum_{l=1}^{l^*} p(s_l) = 1$ and $p(s_l) > 0$ for all s_l, in accordance with well-established results from the literature as in DeGroot (2000); French and Ríos Insua (2000) or Scott (1964).

The PM has to select an act from a set $F = \{f_1, f_2, ..., f_{m^*}\}$ which maps states from S into consequences. To evaluate these consequences, we employ a function U^i resulting in a capability set $U^i(f_m(s_l)) \subseteq \mathbb{R}^{h^*}_{\geq 0}$ for each $m \in \{1, 2, \ldots, m^*\}$ and $l \in \{1, 2, \ldots, l^*\}$, corresponding to a (group of) citizen(s) i. Here, $\mathbb{R}^{h^*}_{\geq 0}$ denotes the set of h^*-dimensional real vectors with non-negative components. For notational simplicity, we will usually omit the superscript i and refer directly to U, except when individual-specific capability functions need to be explicitly distinguished.

The elements within these sets, designated *being-vectors* in the capability approach jargon, are denoted as $\vec{b} \in \{U^i(f_m(s_l)) \,|\, \forall (f_m, s_l) \in F \times S\}$, and are h^*-dimensional vectors encapsulating various welfare dimensions such as health, security, or pleasure. Note that capability sets are generally not singletons and are guaranteed to be non-empty ($U(f_m(s_l)) \neq \emptyset$), potentially encompassing an infinite number of elements. We will represent sure acts $U(f_m(S)), U(f_{m'}(S)), U(f_{m''}(S)), \ldots$ as $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ respectively; similarly, $U(f_m(s_l)), U(f_{m'}(s_l)), U(f_{m''}(s_l))$ will be denoted as $\mathbf{A}_l, \mathbf{B}_l, \mathbf{C}_l$. This notation will be utilized to discuss below capability sets in general terms. When a capability set $\mathbf{A}_l$ comprises a finite number n^*_l of solutions, we represent its elements by $\vec{b_{l,1}}, \ldots, \vec{b_{l,n^*_l}}$, where $b_{l,n,h}$ specifies the value of the n^{th} being-vector for the l^{th} state in the h^{th} dimension.

How citizens use their resources and which being-vector they pursue are matters of personal agency and should not be normatively dictated by the PM, who intentionally avoids making assumptions about which being-vector will actually be chosen for two primary reasons: first, the PM may not have access to the individual's preferences across various welfare dimensions; second, the PM aims to enhance freedom without dictating a specific lifestyle. Indeed, the *sole* assumption the PM makes about the citizen is that all aspects relevant to the citizens' choices are captured through the h^* welfare dimensions, and that, assuming increasing monotonic preferences over welfare, citizens are rational. Thus, we should choose a $\vec{b}$ in $\mathbf{A}_l$ that is in $PF(\mathbf{A}_l)$, where PF represents the Pareto frontier, as in Fayard et al. (2022).

Let us define by $\mathbf{A} - \mathbb{R}^{h^*}_{\geq 0}$ the set encompassing all *solutions* weakly Pareto dominated by $\mathbf{A}$: it includes all vectors $\vec{b}$ for which there exists a $\vec{b'} \in \mathbf{A}$ such that $b_h \leq b'_h$ for all dimensions h, denoted $\vec{b} \leq \vec{b'}$.[1] One method to determine whether a capability set $\mathbf{B}$ is preferred to a capability set $\mathbf{A}$, when the state of nature is known, is to verify that for each $\vec{b} \in \mathbf{A}$, there is a $\vec{b'} \in \mathbf{B}$ such that $\vec{b} \leq \vec{b'}$; additionally, there should be at least one $\vec{b'} \in \mathbf{B}$ for which there is no $\vec{b} \in \mathbf{A}$ that is at least as good as $\vec{b'}$. In other words, if $\mathbf{B}$ is "above" $\mathbf{A}$, or $\mathbf{A} \subsetneq \mathbf{B} - \mathbb{R}^{h^*}_{\geq 0}$, then $\mathbf{B}$ is preferred to $\mathbf{A}$, and if $\mathbf{A} \subseteq \mathbf{B} - \mathbb{R}^{h^*}_{\geq 0}$ then $\mathbf{B}$ is at least as good as $\mathbf{A}$. This ensures that any rational individual, assuming they aim to maximize their welfare across all dimensions, would choose $\mathbf{B}$ over $\mathbf{A}$, regardless of their specific preferences.

Example 1. *Figure 1 depicts three capability sets with $h^* = 2$. Two of the capability sets, respectively denoted $\mathbf{A}$ and $\mathbf{B}$, are finite, while the third one, denoted $\mathbf{C}$, contains an infinite number of solutions. The red, resp. blue dotted, areas represent the spaces dominated by $\mathbf{A}$, resp. $\mathbf{B}$, that is, $\mathbf{A} - \mathbb{R}^2_{\geq 0}$, resp. $\mathbf{B} - \mathbb{R}^2_{\geq 0}$.*[2]

[1] Note that $\vec{b} = \vec{b'} \implies \vec{b} \leq \vec{b'}$ and $\vec{b'} \leq \vec{b}$

[2] For ease of representation, the figures only display the non-negative region.

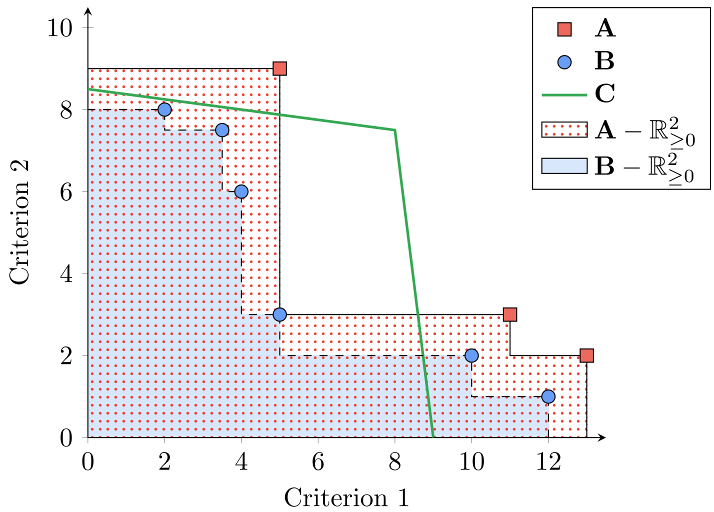

Fig. 1 Three capability sets $\mathbf{A}, \mathbf{B}, \mathbf{C}$

In this example, $\mathbf{A}$ *should be preferred to* $\mathbf{B}$ *because* $\mathbf{B}$ *is entirely contained within* $\mathbf{A} - \mathbb{R}^2_{\geq 0}$*. On the other hand, when comparing* $\mathbf{C}$ *to* $\mathbf{A}$ *using the above method, no comparison can be made. This is because neither* $\mathbf{C}$ *is a subset of* $\mathbf{A} - \mathbb{R}^2_{\geq 0}$*, nor is* $\mathbf{A}$ *a subset of* $\mathbf{C} - \mathbb{R}^2_{\geq 0}$*.* △

Under classical assumptions (Savage, 1972), decision support within F is carried out as follows. For a given act f_m and state s_l, there exists a unique consequence $f_m(s_l)$. There is also a utility function u so that $u(f_m(s_l)) \in \mathbb{R}$ for all $f_m \in F$ and $s_l \in S$, such that $f_{m'}$ is at least as preferred as f_m if and only if the expected utility of $f_{m'}$ is greater than or equal than that of f_m, i.e.,

$$E(u(f_{m'})) = \sum_{l=1}^{l^*} p(s_l) \cdot u(f_{m'}(s_l)) \geq \sum_{l=1}^{l^*} p(s_l) \cdot u(f_m(s_l)) = E(u(f_m)).$$

Consequently, the PM is recommended to select the act f_m^* maximizing expected utility, namely

$$\max_{f_m \in F} (E(u(f_m)).$$

However, in our framework we use capability sets $U(f_m(s_l)) \subseteq \mathbb{R}^{h^*}_{\geq 0}$, instead of utilities $u(f_m(s_l)) \in \mathbb{R}$. The consequence is that, in order to include probabilities and compare expectations, we need to “mix” the capability sets. Subsequent sections elucidate the manner in which a mixed capability set can be established.

3 Average and expected capability sets

As mentioned, in classical decision theory, uncertainty is typically handled through expected utility theory, in which outcomes are evaluated using a single evaluation function, for instance, money or utility. However, in our context, as stated in Section 2, citizens' outcomes are evaluated through multiple welfare (value) functions. As a result, aggregating these multidimensional outcomes under uncertainty requires careful consideration, since a "multidimensional expected utility" can be constructed in several ways.

This section discusses two such ways. The first one is an "obvious extension": *averaging capabilities* using the probabilities as weights; the second consists of computing an *"expected capability set,"* which takes into account both the averaging of the values and the maximisation of freedom of choice, as we shall see, an aspect neglected by simple averaging.

3.1 Average capability sets

A natural approach to address the problem involves considering every combination of vectors from various capability sets. These combinations are then aggregated based on the probability of their corresponding states of the world. This method of aggregation is referred to as the *average capability set*, denoted $\overline{\mathbf{A}}$ for an act f_m:

$$\overline{\mathbf{A}} = \left\{ \sum_{l=1}^{l^*} p(s_l) \cdot X_l, \text{for all } X \text{ such that } X = (\vec{b_1}, \ldots, \vec{b_{l^*}}) \right. \\ \left. \text{with } \vec{b_l} \in \mathbf{A}_l \text{ for every } l \in \{1, \ldots, l^*\} \right\}.$$

This is a natural extension of the expected utility concept and is consistent with it. Indeed, if the capability sets are singletons and assessed using only one dimension, then expected capability sets are equivalent to expected utilities, see proof in Appendix A (Prop 1 bis). If our goal is to identify and retain only the efficient Pareto set of the set $\overline{\mathbf{A}}$, denoted $PF(\overline{\mathbf{A}})$, we calculate it by solving Problem (1)

$$\begin{aligned} & PF(\overline{\mathbf{A}}) = PF\ \left(\textstyle\sum_{l=1}^{l^*} b_{l,1} \cdot p(s_l), \cdots, \sum_{l=1}^{l^*} b_{l,h^*} \cdot p(s_l) \right) \\ & \text{s.t.} \\ & \qquad \vec{b_l} \in \mathbf{A}_l \qquad\qquad \forall l \in \{1, \cdots, l^*\}. \end{aligned} \tag{1}$$

In general terms, the reduction to the Pareto frontier is not mandatory. However, it will serve to elucidate the concept of expected capability set and facilitate the comparison of both approaches. The decision of whether or not to use the Pareto frontier will be discussed in Section 5.

Example 2. *Consider a case where* $U(f_m(s_1)) = \mathbf{A}_1 = \{(2,7),(3,4)\}$ *and* $U(f_m(s_2)) = \mathbf{A}_2 = \{(4,3),(7,2)\}$. *Should* s_1 *be the actual state, a citizen would choose between solutions* $(2,7)$ *and* $(3,4)$, *whereas if* s_2 *was the actual one, such citizen would choose between* $(4,3)$ *and* $(7,2)$. *If* $p(s_1) = p(s_2) = 0.5$, *the average capability set* $\overline{\mathbf{A}}$ *is* $\{(3,5),(3.5,3.5),(4.5,4.5),(5,3)\}$ *and its Pareto frontier* $PF(\overline{\mathbf{A}})$ *is*

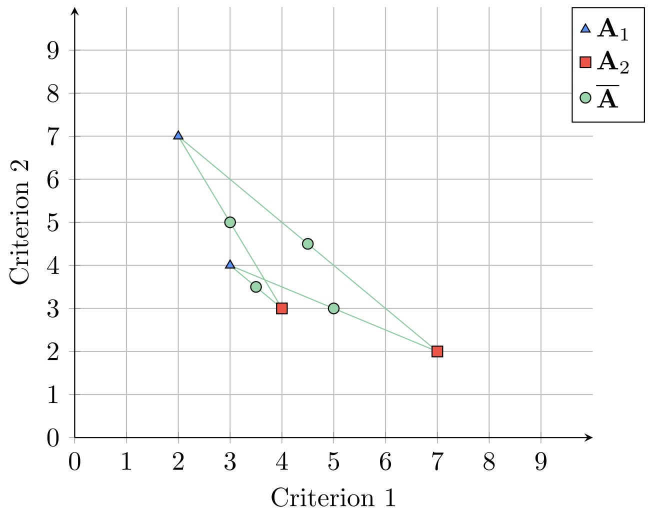

Fig. 2 Example 2: Average capability set

$\{(3,5),(4.5,4.5),(5,3)\}$. *Refer to Table 1 for the steps used to construct this set and Figure 2 for a visual depiction.*

X_i	$\vec{b} \in \overline{\mathbf{A}}$
$\{(2,7),(4,3)\}$	$(\mathbf{3},\mathbf{5})$
$\{(2,7),(7,2)\}$	$(\mathbf{4.5},\mathbf{4.5})$
$\{(3,4),(4,3)\}$	$(3.5,3.5)$
$\{(3,4),(7,2)\}$	$(\mathbf{5},\mathbf{3})$

Table 1 Solutions of $\overline{\mathbf{A}}$

Note that this mixed solution includes the possibility of achieving $(3.5,3.5)$*, which is dominated by* $(4.5,4.5)$*. Additionally, in this particular case, the average capability set is not dominated by any solutions from* $\mathbf{A}_1$ *or* $\mathbf{A}_2$*. In fact, no solution within the average capability set is dominated by solutions from either* $\mathbf{A}_1$ *or* $\mathbf{A}_2$*.* △

Although the average capability set provides a straightforward method for aggregating sets of capabilities, it may not satisfy all the desirable requirements that a solution to our problem must meet. To wit, the main issue with average capability sets is that they may not be dominated by the union of all being-vectors from all states of the world. This idea will be further developed in Sections 3.2 and 4, but an illustration of why this is a problem can be better grasped by considering a one-dimensional setting.

Example 3. *Suppose we have a one-dimensional setting with a single act and four possible states of the world so that* $u(f_m(s_1)) = 8$*,* $u(f_m(s_2)) = 3$*,* $u(f_m(s_3)) = 4$*, and* $u(f_m(s_4)) = 5.5$*. Figure 3 illustrates the space dominated in each state. For example, under state* s_2*, a citizen would be able to choose an outcome that is at least as good as* *the one represented by the dashed green region, that is, any solution not worse than 3. The space below 3 is also dominated under states s_1, s_3, and s_4, meaning that this portion of the outcome space is dominated under all states of the world.*

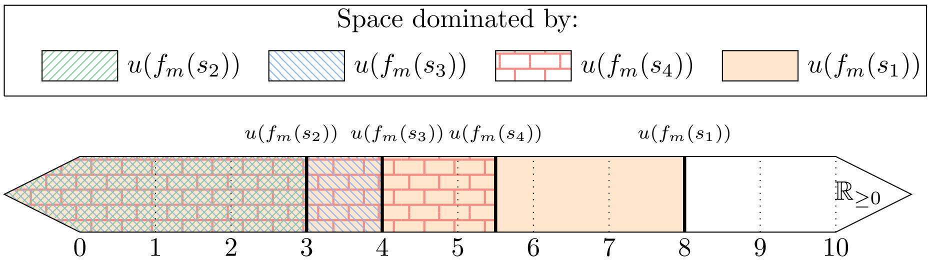

Fig. 3 Representation of the dominated space in one dimension.

It is therefore reasonable for the PM to expect an outcome at least as good as 3*, since this value is dominated in every state:* 3 *corresponds to the value that lies within the intersection of all dominated regions. On the other hand, the PM should not expect to achieve more than* 8*, as this value is not dominated in any state of the world. That is, it lies outside the union of the dominated spaces across all states.* △

The following subsection demonstrates the need to impose stronger properties on how to mix capability sets, leading to what we shall refer to as the *expected capability set*. Section 5 compares both types of sets and analyses how the average capability set can be useful in a problem that slightly differs from the one we explore.

3.2 Expected capability sets

Motivated by Example 3, let us introduce two properties that naturally capture any citizens' preferences in relation to capability sets that will inform our discussion. The first one, called *greater choice* property, reflects the fundamental issue within the capability approach that citizens prefer to have access to more and better being-vectors.

Property 1 (Greater choice)**.** *For all capability sets* $\mathbf{A}$ *and* $\mathbf{B}$*, it holds that* $\mathbf{A} \cup \mathbf{B}$ *is at least as preferred as* $\mathbf{A}$*.*

By iteratively applying this property, we find that the union of capability sets resulting from an act f_m across all states, that is $\cup_{l=1}^{l^*} \mathbf{A}_l$, should be not less preferred to the capability set $\mathbf{A}_l$ corresponding to each individual state s_l. Accordingly, we should also expect that such union will also be not less preferred to the expected capability set, denoted $E(\mathbf{A})$, i.e., $E(\mathbf{A}) \preceq \cup_{l=1}^{l^*} \mathbf{A}_l$.

The second one, designated *fewer choice* property, captures another aspect of citizens' preferences: for any capability set, citizens do not prefer having less choice.

Property 2 (Fewer choice)**.** *For all capability sets* $\mathbf{A}$ *and* $\mathbf{B} \subseteq \mathbb{R}^{h^*}_{\geq 0}$*, it holds that* $\mathbf{A}$ *is at least as preferred as* $(\mathbf{A} - \mathbb{R}^{h^*}_{\geq 0}) \cap (\mathbf{B} - \mathbb{R}^{h^*}_{\geq 0})$*.*

Based on this property, the capability set associated with each individual state should not be less preferred to the intersection of the solutions dominated by the capability sets resulting from an act f_m across all states, that is, $\cap_{l=1}^{l^*}(\mathbf{A}_l - \mathbb{R}^{h^*}_{\geq 0}) \preceq \mathbf{A}_l$. Accordingly, the expected capability set $E(\mathbf{A})$ should not be less preferred than such a set.

A simple example illustrates these properties.

Example 2 (Cont)**.** *Figure 4 depicts the capability sets $\mathbf{A}_1$ and $\mathbf{A}_2$, as well as the space dominated by the solutions in the union of the capability sets under both states, denoted $(\mathbf{A}_1 - \mathbb{R}^2_{\geq 0}) \cup (\mathbf{A}_2 - \mathbb{R}^2_{\geq 0})$, orange dotted area, and the space that can be dominated in any state, denoted by $(\mathbf{A}_1 - \mathbb{R}^2_{\geq 0}) \cap (\mathbf{A}_2 - \mathbb{R}^2_{\geq 0})$, gray. The average capability set $\overline{\mathbf{A}}$ is not contained within the orange dotted area and does not fulfill Property 1.*

Following this property, the expected capability set $E(\mathbf{A})$ should be contained within the dotted orange area, which is not the case for the average capability set. The space above the dotted orange area represents the being-vectors that we are sure will not be able to dominate, regardless of the actual state. On the other hand, following Property 2, the dark gray area should be dominated by $E(\mathbf{A})$, since in any state, we should find a solution dominating such area and, therefore, are sure to be able to dominate it. △

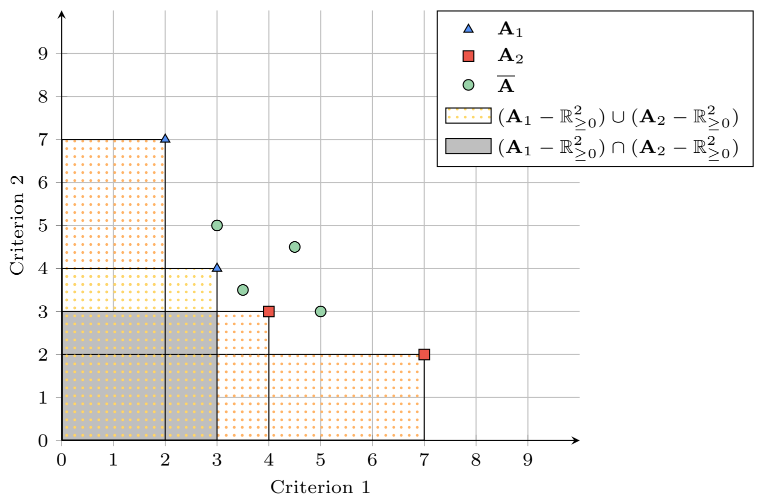

Fig. 4 Example 2 used to illustrate Properties 1 and 2.

Our objective is thus to establish a mixing procedure that guarantees fulfilling the conditions

$$\bigcap_{l=1}^{l^*}(\mathbf{A}_l - \mathbb{R}^{h^*}_{\geq 0}) \subseteq E(\mathbf{A}) - \mathbb{R}^{h^*}_{\geq 0} \subseteq \bigcup_{l=1}^{l^*} \mathbf{A}_l - \mathbb{R}^{h^*}_{\geq 0}.$$

Note that this is not per se a definition of the expected capability set $E(\mathbf{A})$, but rather a crucial requirement that such a set must fulfill to be considered interesting in our context. To operationalize it, we propose a model ensuring that for any $\vec{b} \in E(\mathbf{A})$, there exists $\vec{b}' \in \bigcup_{l=1}^{l^*} \mathbf{A}_l$ such that $\vec{b} \leq \vec{b}'$, i.e., $b_h \leq b'_h$ for all h. Additionally, for any $\vec{b}''$ in $\bigcap_{l=1}^{l^*} \mathbf{A}_l - \mathbb{R}^{h^*}_{\geq 0}$, there exists $\vec{b} \in E(\mathbf{A})$ such that $\vec{b}'' \leq \vec{b}$.

For this, let us provide several mathematical programming formulations to define and compute $E(\mathbf{A})$. The first one, referred to as Problem 2, is designed to define an expected capability set. Within this framework, solutions $\vec{b_l}$ are not contained in $\mathbf{A}_l$ as in the averaging Problem 1, but are just weakly Pareto dominated by $\mathbf{A}_l$ and are combined using the probabilities associated with their specific states, as in Problem 1. Additionally, a necessary condition for aggregation is imposed: preserving weak Pareto dominance across solutions in different states. This process ensures that the resulting expected capability set fulfills both Properties 1 and 2. Specifically, the expected capability set construction is formulated through a Pareto frontier problem as:

$$
\begin{aligned}
E(\mathbf{A}) = PF\Bigg(\sum_{l=1}^{l^*} b_{l,1} \cdot p(s_l), \ldots \qquad & \ldots, \sum_{l=1}^{l^*} b_{l,h^*} \cdot p(s_l)\Bigg) \\
s.t. \quad \vec{b_l} \in \mathbf{A}_l - \mathbb{R}^{h^*}_{\geq 0} \qquad & \forall l \in \{1, \cdots, l^*\}, \qquad (2a) \\
\vec{b_l} \geq \vec{b_{l'}} \text{ or } \vec{b_l} \leq \vec{b_{l'}} \qquad & \forall l, l' \in \{1, \ldots, l^*\}, \;\; l \neq l'. \qquad (2b)
\end{aligned}
$$

Condition (2a) directs our focus towards mixing all solutions *dominated* by $\mathbf{A}_l$ for all $s_l \in S$, rather than just those of $\mathbf{A}_l$. On the other hand, condition (2b) stipulates the existence of a total order among all $\vec{b_l}$. This implies that any two solutions $\vec{b_l} \in \mathbf{A}_l, \vec{b_{l'}} \in \mathbf{A}_{l'}$ which are incomparable (i.e., neither $\vec{b_l} \geq \vec{b_{l'}}$ nor $\vec{b_l} \leq \vec{b_{l'}}$) are inappropriate to lead to an expected solution of the type $\vec{b} = p(s_l) \cdot \vec{b_l} + p(s_{l'}) \cdot \vec{b_{l'}}$. Note that in the classical setting (Savage, 1972) with one-dimensional utilities ($h^* = 1$) and capability sets that are singletons, all solution are comparable as $\geq$ is a total order in $\mathbb{R}$.

While Problem 2 has been employed to elucidate the concept of expected capability set, we still need to provide computational schemes to construct it. To this end, we rely on the assumption that each capability set $\mathbf{A}_l$ is a *polytope*, i.e., a non-empty, bounded convex set in $\mathbb{R}^{h^*}_{\geq 0}$ that can be fully characterized either by a finite number of linear inequalities or by its extreme points. Such sets arise, for example, as the feasible regions of MI-MOLPs, as in Fayard et al. (2022). Under this assumption, we propose Problem 3 as an operational formulation to compute $E(\mathbf{A})$, based on a MI-MOLP, where M denotes a sufficiently large number.

$$
\begin{aligned}
E(\mathbf{A}) = PF\Bigg(\sum_{l=1}^{l^*} b_{l,1} \cdot p(s_l), \ldots \qquad & \ldots, \sum_{l=1}^{l^*} b_{l,h^*} \cdot p(s_l)\Bigg) \\
\text{s.t.} \qquad & \\
b_{l,h} \leq z_{l,h} \qquad & \forall l \in \{1, \cdots, l^*\} \qquad (3a) \\
& \forall h \in \{1, \ldots, h^*\} \\
\vec{z_l} \in \mathbf{A}_l \qquad & \forall l \in \{1, \ldots, l^*\} \qquad (3b)
\end{aligned}
$$

$$
\begin{aligned}
& d_{l,l'} \in \{0,1\} && l \neq l' \quad l,l' \in \{1,\cdots,l^*\} && \text{(3c)}\\
& b_{l,h} \leq b_{l',h} + d_{l,l'} \cdot M && l \neq l' \quad l,l' \in \{1,\cdots,l^*\} && \text{(3d)}\\
& d_{l,l'} + d_{l',l} \leq 1 && l < l' \quad l,l' \in \{1,\cdots,l^*\} && \text{(3e)}\\
& b_{l,h} \in \mathbb{R} && \forall l \in \{1,\cdots,l^*\} && \text{(3f)}\\
& && \forall h \in \{1,\cdots,h^*\} &&
\end{aligned}
$$

In this formulation, inequality (3a) guarantees that we aggregate vectors $\vec{b_l}$ that are dominated by a solution $\vec{z_l}$ in $\mathbf{A}_l$, as expressed by (3b) which compiles the constraints to find the capability set of $\mathbf{A}_l$, as in Fayard et al. (2022). To establish a total order among the solutions $\vec{b_l} = (b_{l,1}, \ldots, b_{l,h^*})$, we introduce binary decision variables $d_{l,l'}$ using (3c), which ensure that if $d_{l,l'} = 0$, then $b_{l,h} \leq b_{l',h}$ for all h, as in inequality (3d). We guarantee that this order exists between all solutions $\vec{b_l}$ by using inequality (3e).

Two key factors influence the computational complexity of Problem 3. First, the number l^* of states plays a significant role. Indeed, $(2 \times \sum_{l=1}^{l^*-1} l)$ decision variables $d_{l,l'}$ are introduced to establish a total order between the weakly Pareto dominated solutions at each state. Thus, the bigger the number of states, the more decision variables and constraints need to be considered. Second, the number h^* of welfare dimensions affects the complexity of finding the expected capability set, a crucial aspect in our framework. When the dimensionality of the criteria is large, computational demands become significant, scaling typically exponentially with the number h^* of criteria (Ehrgott, 2005).

When the capability sets $\mathbf{A}_l$ are finite $\{\vec{z_{l,1}}, \cdots, \vec{z_{l,n_l^*}}\}$, the expected capability set $E(\mathbf{A})$ can be found through the following MI-MOLP (Problem 4), where, again, M is a sufficiently large number

$$
E(\mathbf{A}) = PF\left(\sum_{\forall l=1}^{l^*} b_{l,1} \cdot p(s_l), \qquad \cdots, \sum_{\forall l=1}^{l^*} b_{l,h^*} \cdot p(s_l)\right)
$$

s.t.

$$
\begin{aligned}
& \sum_{n=1}^{n_l^*} \delta_{l,n} \leq n_l^* - 1 && \forall l \in \{1,\cdots,l^*\} && \text{(4a)}\\
& b_{l,h} \leq z_{l,n,h} + \delta_{l,n} \cdot M && \forall l \in \{1,\cdots,l^*\} && \text{(4b)}\\
& && \forall n \in \{1,\cdots,n_l^*\} &&\\
& && \forall h \in \{1,\cdots,h^*\} &&\\
& d_{l,l'} \in \{0,1\} && l \neq l' \quad l,l' \in \{1,\cdots,l^*\} && \text{(4c)}\\
& b_{l,h} \leq b_{l',h} + d_{l,l'} \cdot M && l \neq l' \quad l,l' \in \{1,\cdots,l^*\} && \text{(4d)}\\
& d_{l,l'} + d_{l',l} \leq 1 && l < l' \quad l,l' \in \{1,\cdots,l^*\} && \text{(4e)}\\
& \delta_{l,n} \in \{0,1\} && \forall l \in \{1,\cdots,l^*\} && \text{(4f)}\\
& && \forall n \in \{1,\cdots,n_l^*\} &&\\
& b_{l,h} \in \mathbb{R} && \forall l \in \{1,\cdots,l^*\} && \text{(4g)}
\end{aligned}
$$

$\forall h \in \{1, \cdots, h^*\}$

Constraints (4c), (4d) and (4e) establish an order among the aggregated solutions as with constraints (3c), (3d), and (3e). To determine dominated solutions, we no longer use constraints (3a) and (3b) but rather introduce the new constraints (4a), (4b) and (4f). We introduce decision variables $\delta_{l,n}$ using constraint (4f). Constraint (3b) is replaced by (4b). If $\delta_{l,n}$ equals 0, then $\vec{b_l} = (b_{l,1}, \ldots, b_{l,h^*})$ is considered weakly dominated by the n^{th} solution in $\mathbf{A}_l$. Finally, constraint (4a) ensures that at least one of the decision variables $\delta_{l,n}$ equals 0, indicating that $\vec{b_l}$ is dominated by at least one solution in $\mathbf{A}_l$.

In addition to the number of states and the number of dimensions, a third factor influences the computational complexity of Problem 4: the *size of the capability sets*. Observe that we introduce $\sum_{l=1}^{l^*} n_l^*$ decision variables $\delta_{l,n}$, where n_l^* represents the number of solutions in the capability set $\mathbf{A}_l$. Thus, larger capability sets involve more decision variables and constraints, increasing computational demands.

Example 4. *Consider an act f_m with two states: $U(f_m(s_1)) = \mathbf{A}_1 = \{(3,10),(4,5),(7,3),(8,1)\}$ and $U(f_m(s_2)) = \mathbf{A}_2 = \{(2,5),(5,4),(10,2)\}$. Figure 5 displays $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_1 - \mathbb{R}^2_{\geq 0}$ and $\mathbf{A}_2 - \mathbb{R}^2_{\geq 0}$ as all the points on and under the dotted blue (resp. dashed red) lines. It also shows $E(f_m)$ according to the formulation given in Problem 4, when $p(s_1) = 0.8$ (and $p(s_2) = 0.2$). We obtain $E(f_m) = \{(2.8,9), (3,8.8), (4,4.8), (4.2,4), (5,3.2), (6.6,3), (7,2.8), (7.6,2), (8.4,1.2)\}$. As an example, $(2.8,9)$ is obtained by aggregating $\vec{b}_1 = (3,10)$ and $\vec{b}_2 = (2,5)$. We have $\vec{b}_1 \leq \vec{z}_{1,1} = (3,10)$ and $\vec{b}_2 \leq \vec{z}_{2,1} = (2,5)$; since $\vec{b}_1 \geq \vec{b}_2$, we have $d_1, d_2 = (1,0)$. The other solutions are displayed in Table 2.* △

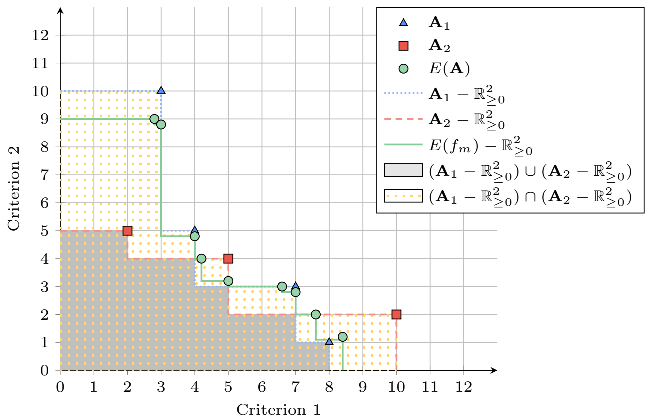

Fig. 5 Example 4: The expected capability set $E(\mathbf{A})$ with $\mathbf{p(s_1) = 0.8}$ and $\mathbf{p(s_2) = 0.2}$

Example 2 (Cont). *Figure 6 depicts the expected capability set $E(f_m)$ when $p(s_1) = p(s_2) = 0.5$. Details for the MI-MOLP used and the solutions obtained are in Appendix B. As required, $E(\mathbf{A})$ is located between the set of solutions dominated across all states,*

$\vec{b} \in E(\mathbf{A})$	$\vec{z_1}$	$\vec{b_1}$	$\vec{z_2}$	$\vec{b_2}$	(d_1, d_2)
$(2.8, 9)$	$(3, 10)$	$(3, 10)$	$(2, 5)$	$(2, 5)$	$(1, 0)$
$(3, 8.8)$	$(3, 10)$	$(3, 10)$	$(5, 4)$	$(3, 4)$	$(1, 0)$
$(4, 4.8)$	$(4, 5)$	$(4, 5)$	$(5, 4)$	$(4, 4)$	$(1, 0)$
$(4.2, 4)$	$(4, 5)$	$(4, 4)$	$(5, 4)$	$(5, 4)$	$(0, 1)$
$(5, 3.2)$	$(7, 3)$	$(5, 3)$	$(5, 4)$	$(5, 4)$	$(0, 1)$
$(6.6, 3)$	$(7, 3)$	$(7, 3)$	$(5, 4)$	$(5, 3)$	$(1, 0)$
$(7, 2.8)$	$(7, 3)$	$(7, 3)$	$(10, 2)$	$(7, 2)$	$(1, 0)$
$(7.6, 2)$	$(7, 3)$	$(7, 2)$	$(10, 2)$	$(10, 2)$	$(0, 1)$
$(8.4, 1.2)$	$(8, 1)$	$(8, 1)$	$(10, 2)$	$(10, 2)$	$(0, 1)$

Table 2 Example 4: Calculation of $\mathbf{E}(\mathbf{A})$

i.e., $(\mathbf{A}_1 - \mathbb{R}^2_{\geq 0}) \cap (\mathbf{A}_2 - \mathbb{R}^2_{\geq 0}) \subseteq E(\mathbf{A})$, *and the set of solutions dominated by at least one state of the world, i.e.,* $E(\mathbf{A}) \subseteq (\mathbf{A}_1 \cup \mathbf{A}_2) - \mathbb{R}^2_{\geq 0}$. $\triangle$

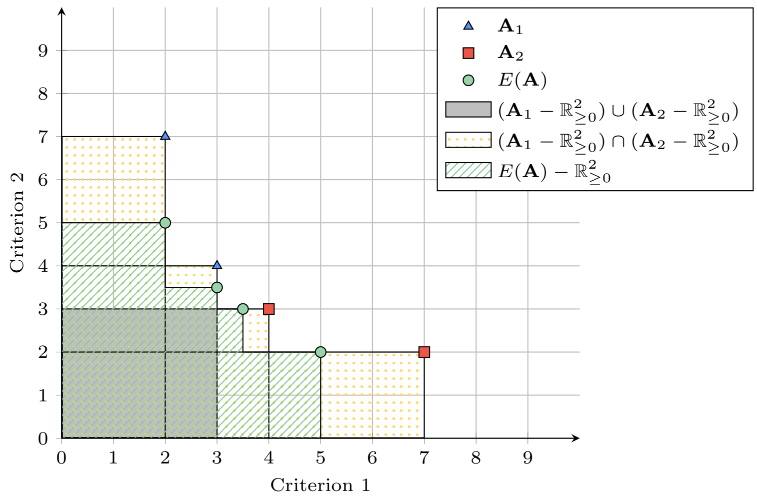

Fig. 6 Example 2: The expected capability set $\mathbf{E}(\mathbf{A})$

Our proposal provides valuable insights into decision-aiding under uncertainty within the context of the capability approach. However, it is important to consider carefully the computational limitations we already introduced when applying the proposed framework. Addressing them presents a compelling line of research that could significantly improve the tractability of our model. As a last resort, we would need to draw on multiobjective heuristics to generate Pareto solutions as in Branke (2016); Chugh, Sindhya, Hakanen, and Miettinen (2019); Martín, Bielza, and Ríos Insua (2005), or Deb, Pratap, Agarwal, and Meyarivan (2002).

4 Properties of expected capability sets

Inspired by Savage (1972) axioms, this section presents core properties of expected capability sets. For comparison, properties for average capability sets are provided in Appendix A.

The first proposition expresses consistency with the classical maximum expected utility principle under the original conditions in Savage's setup (univariate utilities and single consequences).

Proposition 1 (Consistency with expected utility)**.** *If the capability sets are singletons and are assessed using only one dimension, then expected capability sets are equivalent to expected utilities.*

Proof. Assume that the capability sets are singletons for each state ($|\mathbf{A}_l| = 1$ for all s_l) and that there is only one assessment dimension ($h^* = 1$). In this case, we represent each $\mathbf{A}_l$ as a single real number, denoted b_l. Since there is a total order among all b_l, and each state has a unique corresponding solution, we derive that

$$b = \max \left\{ \sum_{l=1}^{l^*} p(s_l) \cdot (b_l - \mathbb{R}_{\geq 0}) \mid b_l \in \mathbf{A}_l \text{ for all } l \right\} \quad \text{if and only if} \quad b = \sum_{l=1}^{l^*} p(s_l) \cdot b_l.$$

■

The second proposition, derived from Property 1, states that the expected capability set should be weakly Pareto dominated by the union of the capability sets across all states. This proposition is considered desirable because if PMs are certain that citizens cannot achieve a solution $\vec{b}$ that dominates others in any possible state, then $\vec{b}$ should not be expected to be dominant overall. In other words, $\vec{b} \notin E(\mathbf{A})$. By adhering to this principle, PMs maintain a realistic understanding of the citizens' feasible choices and avoid incorporating solutions that citizens are certain not to dominate, regardless of the actual state of nature.

As an illustration, consider the case of a one-dimensional expected capability set, where each capability set $\mathbf{A}_l$ is one-dimensional and consists of a single element, i.e., $\mathbf{A}_l = \{b_l\}$ with $b_l \in \mathbb{R}_{\geq 0}$. In this setting, the expected capability set is necessarily dominated by the best outcome among all individual capability sets across the different states. Returning to Example 3, the expected capability must be less than or equal to 8, since 8 is the highest achievable value across all states. Formally, the expected capability set is dominated by the union of all capability sets, expressed as: $E(\mathbf{A}) \leq \max_{s_l \in S} U(f_m(s_l))$ or equivalently $E(\mathbf{A}) \subseteq \bigcup_{l=1}^{l^*} (\mathbf{A}_l - \mathbb{R}_{\geq 0})$.

Proposition 2 (Sure domination of the Expected Capability)**.** *The expected capability set is dominated by the union of capability sets over all possible states, that is,*

$$E(\mathbf{A}) \subseteq \bigcup_{l=1}^{l^*} (\mathbf{A}_l - \mathbb{R}^{h^*}_{\geq 0})$$

Proof. Let $\vec{b} \in E(\mathbf{A})$ be a solution obtained by aggregating $\sum_{l=1}^{l^*} p(s_l) \cdot \vec{b_l}$. Since the $\vec{b_l}$ are ordered, $\vec{b}$ is weakly Pareto dominated by the best among them, which itself is weakly Pareto dominated by a solution in $\mathbf{A}_l$. ■

Proposition 3 follows from Property 2 and states that the expected capability set should dominate all outcomes that are dominated in every individual capability set across all states. This is desirable because, if $\vec{b} \in \bigcap_{l=1}^{l^*}(\mathbf{A}_l - \mathbb{R}_{\geq 0}^{h^*})$, then there exists a solution that dominates $\vec{b}$ in every state. To illustrate this, consider the setting where capability sets are one-dimensional singletons. In this case, the expected capability set necessarily dominates the worst outcome among all states. In Example 3, it is clear that the expected utility must be at least 3, since this value is dominated in all states. In other words, the expected capability set dominates the intersection of the spaces dominated by the capability sets in each state. Formally, this is written as:

$$\min_{s_l \in S} U(f_m(s_l)) \leq E(\mathbf{A}) \quad \text{or equivalently} \quad \bigcap_{l=1}^{l^*}(\mathbf{A}_l - \mathbb{R}_{\geq 0}) \subseteq E(\mathbf{A}) - \mathbb{R}_{\geq 0}.$$

Proposition 3 (Sure domination by the Expected Capability)**.** *Expected capability sets dominate the intersection of all solutions dominated by all states' capability sets, that is*

$$\bigcap_{l=1}^{l^*}(\mathbf{A}_l - \mathbb{R}_{\geq 0}^{h^*}) \subseteq E(\mathbf{A}) - \mathbb{R}_{\geq 0}^{h^*}$$

Proof Consider any $\vec{b}$ that is dominated by every capability set, i.e. $\vec{b} \in \bigcap_{l=1}^{l^*}(\mathbf{A}_l - \mathbb{R}_{\geq 0}^{h^*})$. For each $l \in \{1, \dots, l^*\}$, there exists $\vec{b_l} \in (\mathbf{A}_l - \mathbb{R}_{\geq 0}^{h^*})$ such that $\vec{b_l} = \vec{b}$. Therefore, the solution $\sum_{i=l}^{l^*} p(s_l) \cdot \vec{b_l} \in E(\mathbf{A})$ is equal to $\vec{b}$. ■

The fourth proposition shows that the expected procedures are preserved by positive affine transformations. This relates to the positive affine uniqueness property of utility functions (French & Ríos Insua, 2000).

Proposition 4 (Linearity)**.** (*a*) *Preservation of addition:*

$$E(\mathbf{A} + \vec{c}\,) = E(\mathbf{A}) + \vec{c} \text{ with } \vec{c} \in \mathbb{R}^{h^*}$$

(*b*) *Preservation of positive multiplication*

$$E(\mathbf{A} \cdot \vec{c}\,) = E(\mathbf{A}) \cdot \vec{c} \text{ with } \vec{c} \in \mathbb{R}_{\geq 0}^{h^*}$$

Proof To prove (*a*), consider any $\vec{b}$ in $E(\mathbf{A} + \vec{c}\,)$. This means $\vec{b} = \sum_{l=1}^{l^*}(p(s_l) \cdot \vec{b_l})$, with an order between all $\vec{b_l}$, and $\vec{b_l} \in \mathbf{A}_l + \vec{c} - \mathbb{R}_{\geq 0}^{h^*}$. Similarly, for any $\vec{b}'$ in $E(\mathbf{A}) + \vec{c}$, we have $\vec{b}' \in \sum_{l=1}^{l^*}(p(s_l) \cdot \vec{b}_l') + \vec{c}$, with an order between all $\vec{b}_l'$, and $\vec{b}_l' \in \mathbf{A}_l - \mathbb{R}_{\geq 0}^{h^*}$.

Now, for each $\vec{b}$ in $E(\mathbf{A}+\vec{c}\,)$, we can find a $\vec{b}'$ in $E(\mathbf{A})+\vec{c}$ such that $\vec{b}'=\vec{b}$, by setting $\vec{b}'_l=\vec{b}_l-\vec{c}$. Conversely, for each $\vec{b}'$ in $E(f_m)+\vec{c}$, we can find a $\vec{b}\in E(\mathbf{A}+\vec{c}\,)$ such that $\vec{b}=\vec{b}'$, by setting $\vec{b}_l=\vec{b}'_l+\vec{c}$.

The proof for (b) follows a similar approach. ■

Proposition 5 stipulates that extending capability sets on states cannot lead to a reduction of the expected capability set.

Proposition 5 (Monotonicity over capability domination)**.** *If for all $s_l\in S$ we have $\mathbf{A}_l\subseteq[\mathbf{B}_l-\mathbb{R}^{h^*}_{\geq 0}]$, then $E(\mathbf{A})\subseteq E(\mathbf{B})-\mathbb{R}^{h^*}_{\geq 0}$.*

Proof For every $\vec{b}$ in $E(\mathbf{A})$, where $\vec{b}=\sum_{l=1}^{l^*}p(s_l)\cdot\vec{b}_l$, there exists a $\vec{b}'=\sum_{l=1}^{l^*}p(s_l)\cdot\vec{b}'_l$ in $E(\mathbf{B})-\mathbb{R}^{h^*}_{\geq 0}$ such that $\vec{b}'=\vec{b}$ by setting $\vec{b}_l=\vec{b}'_l$ for all $l\in\{1,\ldots,l^*\}$. ■

Our final proposition shows that increasing the probability of one state of the world dominating another one should lead to the expected capability set of the first one to dominate the expected capability set of the second one.[3]

Proposition 6 (Monotonicity over probabilities)**.** *Suppose we have $\mathbf{A}_z\subseteq\mathbf{A}_{z'}-\mathbb{R}^{h^*}_{\geq 0}$ with $(s_z,s_{z'})\in S^2$ and*

$$p'(s_l)=\begin{cases}p(s_l)+c & ,\ for\quad l=z'\\ p(s_l)-c & ,\ for\quad l=z\\ p(s_l) & ,\ otherwise\end{cases}$$

with $c\in(0;p(s_z)]$. Then, $E(\mathbf{A},p)\subseteq E(\mathbf{A},p'))-\mathbb{R}^{h^*}_{\geq 0}$.

Proof For every $\vec{b}$ in $E(\mathbf{A},p)$, defined as $\vec{b}=\sum_{l=1}^{l^*}p(s_l)\cdot\vec{b}_l$, there always exists a corresponding $\vec{b}'$ in $E(\mathbf{A},p')$, defined as $\vec{b}'=\sum_{l=1}^{l^*}p'(s_l)\cdot\vec{b}'_l$, such that $\vec{b}'\geq\vec{b}$:

- If $\vec{b}_z\leq\vec{b}_{z'}$, let $\vec{b}_l=\vec{b}'_l$ for all $l\in\{1,\ldots,l^*\}$. It then follows that $\sum_{l=1}^{l^*}p'(s_l)\cdot\vec{b}'_l\geq\sum_{l=1}^{l^*}p(s_l)\cdot\vec{b}_l$.
- If $\vec{b}_z>\vec{b}_{z'}$, let $\vec{b}'_l$ be defined as:

$$\vec{b}'_l=\begin{cases}\vec{b}_z & \text{for }\ l=z'\\ \vec{b}_{z'} & \text{for }\ l=z\\ \vec{b}_l & \text{Otherwise}\end{cases}$$

Then, we deduce that $\sum_{l=1}^{l^*}p'(s_l)\cdot\vec{b}'_l\geq\sum_{l=1}^{l^*}p(s_l)\cdot\vec{b}_l$. ■

5 Expected vs average capability sets

Average capability sets respect the same properties than expected capability sets except for the sure domination of the expected capability (Proposition 2), and the monotonicity over probabilities (Proposition 6), see Appendix A for proofs.

[3] In this proposition, we use $E(\mathbf{A},p)$ to denote the set of expected capability vectors formed by weighting elements from each $\mathbf{A}_{s_l}$ using the probability distribution p. This allows us to compare how the expected set changes when using p versus p'.

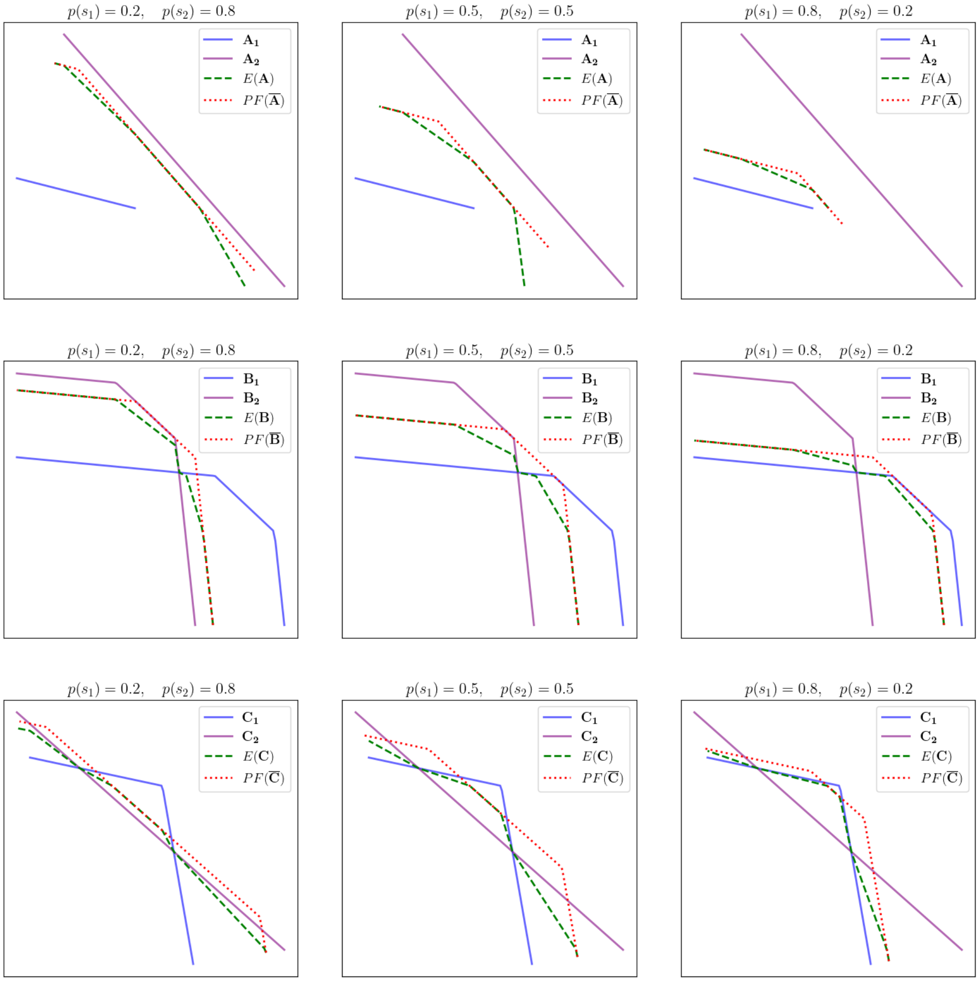

Fig. 7 Comparison of the expected and average capability sets for three sets and three probability distributions.

An important relation between both concepts is that for all solutions in the expected capability set, it is possible to find a solution in the average capability set that is at least as preferred as it, i.e. the expected capability set is in the space dominated by the average capability set.

Proposition 7. *We have*

$$E(\mathbf{A}) \subseteq \overline{\mathbf{A}} - \mathbb{R}^{h^*}_{\geq 0}$$

Proof. For every $\vec{b}$ in $E(\mathbf{A})$ such that $\vec{b} = \sum_{l=1}^{l^*} p(s_l) \cdot \vec{b}_l$, we have that for each $\vec{b}_l$ there exists a corresponding $\vec{b}'_l$ in $\mathbf{A}_l$ such that $\vec{b}'_l \geq \vec{b}_l$. Consequently, $\vec{b}' = \sum_{l=1}^{l^*} p(s_l) \cdot \vec{b}'_l$ is in $\overline{\mathbf{A}}$ and satisfies $\vec{b}' \geq \vec{b}$. ■

To illustrate the distinction between both sets, consider the example of mixing two capability sets as in Figure 7. For both concepts, the mixed capability set tends to align more closely with $\mathbf{A}_1$ (resp. $\mathbf{B}_1$, $\mathbf{C}_1$) as the probability $p(s_1)$ increases, and, conversely, with $\mathbf{A}_2$ (resp. $\mathbf{B}_2$, $\mathbf{C}_2$) as the probability $p(s_2)$ increases. Furthermore, observe that the expected capability sets are always "below" the union of the capabilities in both states of the world, as well as "below" the average capability set.

Since expected and average capability sets fulfill different properties, a careful scrutiny seems necessary to determine which concept to employ under given circumstances. Broadly speaking, the expected capability approach is best suited when calculating the anticipated expected being-vectors of citizens. In this context, Proposition 2 becomes very relevant. Indeed, it is crucial that PMs should not expect citizens to achieve solutions that cannot be weakly Pareto dominated in any possible state of the world. Moreover, when increasing the probabilities of a state of the world leading to a capability set that is preferred over another one generated in a different state (Proposition 6), it is crucial that the resulting expected capability set is improved for the citizen.

Conversely, the average capability approach is better suited in situations requiring social aggregation. In this approach, the average capability set of a group reflects a compound of all possible solutions that any citizen within the group could choose. Therefore, it is not surprising that average capability sets do not fulfill Proposition 2. This is because *social being-vectors* in an average capability set represent a combination of *individual being*-vectors. This is also why we recommend compiling $\overline{\mathbf{A}}$ and not just $PF(\overline{\mathbf{A}})$ because $\overline{\mathbf{A}}$ embodies actual social aggregation. This aggregation may not be efficient in the sense that individuals maximize their own solution regardless of the aggregated solutions at the social level, while $PF(\overline{\mathbf{A}})$ would represent the social aggregation where all individuals choose in accordance with an efficient solution at the social level. Non-fulfillment of Proposition 6 is to be expected since increasing the probability of a better capability set will make the average capability set "closer" to it, but not necessarily dominate the previous average capability set.

We illustrate such distinctions with a final example.

Example 5. *A PM is interested in managing food production in a given region with two objectives: (1) guarantee that local food producers can sustain their livelihoods and continue producing food; and, (2) ensure that citizens have access to a sufficient and diverse food supply. For this the PM considers two possible acts within the set* $F = \{f_1 = \textit{do nothing},\ f_2 = \textit{change land field allocation}\}$.

The region produces two types of food: rice and wine. Agricultural productivity in the region is highly sensitive to both terrain and annual weather conditions. The land consists of two types of terrain:

- *a rain-abundant valley, ideal for rice cultivation; and,*
- *sun-soaked hillside terraces, well-suited for growing wine grapes.*

While each crop has a preferred terrain, it is still possible—though with reduced yields—to grow rice on the terraces and wine in the valley. On the other hand, the set of possible states of the world (weather states) is $S = \{s_{rainy},\ s_{sunny},\ s_{normal}\}$, *with associated the following probabilities and consequences:*

- $p(s_{rainy}) = 0.25$. *Excess rainfall reduces productivity in the valley due to flooding, but it benefits rice cultivation on the hillside terraces, as rice thrives with additional moisture. Wine yields on terraces slightly decrease due to excessive humidity.*
- $p(s_{sunny}) = 0.25$. *Sun boosts wine yields in the valley but decreases rice yields. On the terraces, both rice and wine yields suffer due to heat stress.*
- $p(s_{normal}) = 0.5$. *Standard productivity across all terrains and crops.*

There are two producers in the region, Alice and Bob. Currently, Alice is allocated all the hillside terraces, and Bob, all the valley land. If the PM selects f_1 (i.e., do nothing), let us denote the evaluation function of the consequence $f_1(s)$ for Alice as U and for Bob as U'. The capability sets for Alice and Bob at each state $s \in S$ are, respectively, denoted $U(f_1(s)) = \mathbf{A}_s$ and $U'(f_1(s)) = \mathbf{A}'_s$. Their capability sets for each state and corresponding expected capability sets are reflected in Figures 8 and 9.

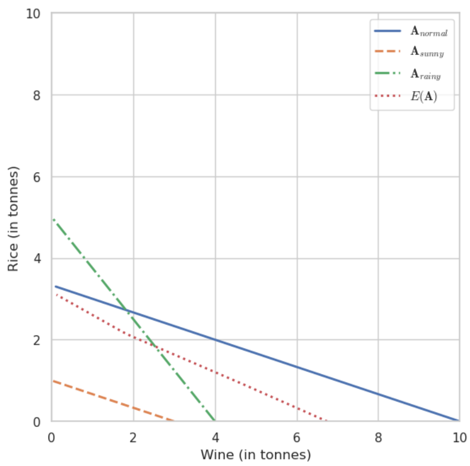

Fig. 8 Capability sets for Alice under f_1.

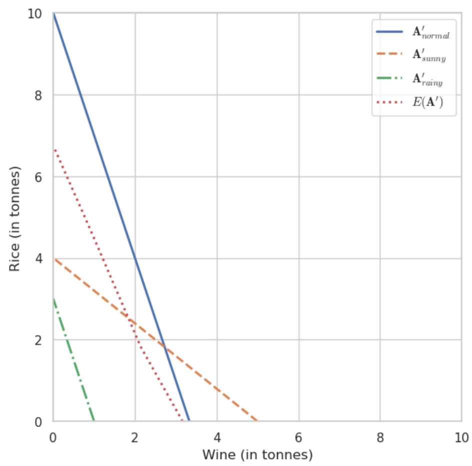

Fig. 9 Capability sets for Bob under f_1.

Suppose now that the PM selects f_2, redistributing the valley and terrace equally between Alice and Bob. The new capability sets for Alice and Bob at each state are, respectively, denoted $U(f_2(s)) = \mathbf{B}_s$ and $U'(f_2(s)) = \mathbf{B}'_s$. The resulting capability and expected capability sets are in Figure 10.
Figure 11 compares the expected capability sets before and after redistribution. We observe that both individuals benefit overall from the more balanced land distribution, as most of the redistributed expected capabilities lie above *those of $E(\mathbf{A})$ and $E(\mathbf{A}')$, indicating an overall improvement. However, from a societal perspective, the total food production capability remains unchanged across acts. This is because the same land is cultivated—only its allocation changes. Figure 12 shows the average capability sets under each weather state, illustrating that the aggregate productive potential of the region is invariant.*

Clearly, the average capability set and the expected capability set aggregate different information. The average capability set *aggregates capability sets across citizens for*

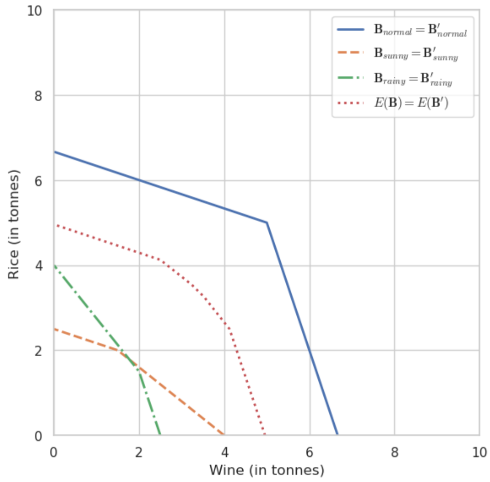

Fig. 10 Expected capability sets for Alice and Bob under equal land sharing (f_2).

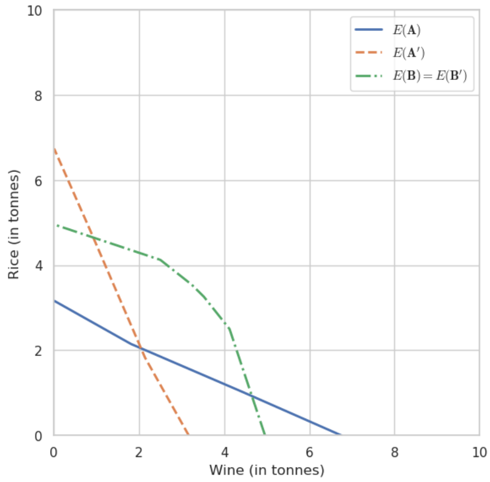

Fig. 11 Comparison of expected capability sets before and after equal sharing.

the same state, helping the PM verify that their policy (f_2) does not affect the aggregate amount of food production capabilities in society. The expected capability set, *however, aggregates capability sets across states for the same citizen, providing insights into how each individual's situation is improved by the policy. These two perspectives are complementary and jointly support policy decisions by addressing the PM's dual objectives (1) and (2).* △

This example highlights the critical distinction between average and expected capability sets. While average capability sets reflect the combined possible outcomes in a population, expected capability sets provide a probabilistic forecast of the individual's

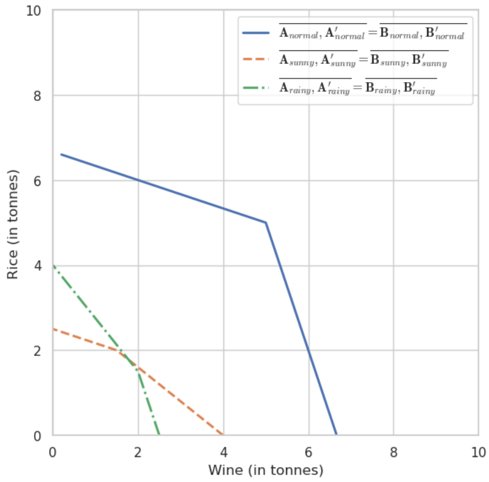

Fig. 12 Average capability sets under different weather states (invariant across allocations).

potential outcomes. The choice between both measures would depend on whether it requires a societal or an individual perspective.

6 Conclusion

We have provided two mixing concepts, expected and average capability sets, that can be used to summarise capability sets in a multiobjective decision-making setup under risk within the capability approach. Both have similar properties but are crucially different concerning sure domination of the expected capability (Proposition 2). The choice of which concept to apply depends on the PM's ultimate goal. If the PM is dealing with capability sets linked to states of the world, each having definite probabilities, and only one of these states of the world will materialize, then using expected capability sets is recommended. Conversely, if the situation involves the combination of multiple capability sets, and all being-vectors can be achieved by different individuals, average capability sets seem a more suitable choice.

This general framework of representing sets of choices in a multidimensional space extends beyond the capability approach within welfare economics, potentially finding relevance and application in other fields and contexts where decision-making involves multiple choice and criteria:

- In an uncertain environment, capability sets could refer to the potential results associated with risk treatments before we are able to undertake a full probabilistic assessment of impacts (Bossert, Pattanaik, & Xu, 2000; Cohen & Jaffray, 1980).
- In research and development (R&D) management, capability sets can refer to the potential outcomes through projects. R&D decisions are often made over multiple periods, where choices made today impact options available in the future. The question then arises as to which development option should be chosen today, knowing that it will influence future development choices. Furthermore, in many cases,

the PM may not have precise knowledge of their future possible choices due to uncertainties related to factors such as the economy, societal values, environmental conditions, future laws, etc. The sets of possible future choices would also be uncertain due to the inherent uncertainties associated with R&D projects, such as performance, development time, and costs. Representing these choices as capability sets would enable flexibility and resilience (Evans, 1991; Koopmans, 1962; Kreps, 1979), allowing for better adaptation to unforeseen events.
- Lastly, in a multi-objective setting (Ehrgott, 2005), capability sets would refer to the multiple objective levels achievable under various states, when we have not yet undertaken a process to aggregate results.

From a framework perspective, the proposed setting extends naturally to two important cases. Firstly, we assumed a multi-objective setting essentially entailing using the whole set of monotonic utility functions (rather than a single utility function as in Savage's framework). In between, we could conceive cases in which a smaller class of utility functions is used; as an example, we could consider the set of all monotonic risk-averse utility functions, as in second order stochastic dominance. Secondly, we assumed a single probability distribution over the states, but again, there could be cases in which we have a class of probability distributions as in robust Bayesian settings (Ríos Insua & Ruggeri, 2010).

Finally, another exciting direction for future research lies in developing new methodologies to assess expected capability sets. This exploration should go beyond our proposed approach, exploring more nuanced strategies as in Barberà, Bossert, and Pattanaik (2004); Foster (2011); Gaertner (2012); Gaertner and Xu (2006, 2008); Pattanaik and Xu (1990). The prevailing challenge, which remains unaddressed, lies in finding a way to account for the diversity of being-vectors and how these being-vectors are assessed by citizens. Advances in this direction will undoubtedly expand the potential use of capability sets within a decision-making framework, offering further enrichment to this field of study.

7 Acknowledgment

The second author acknowledges the support of: the AXA-ICMAT Chair in Adversarial RIsk Analysis, the Spanish Ministry of Science program PID2021-124662OB-I00, the Severo Ochoa Excellence Programme CEX-2023-001347-S, and a grant from the FBBVA (Amalfi). The first and third author acknowledge the support of the project CARE: Capabilities for risk Acceptability and REsilience, funded by the Franch Research Agency (ANR) and the Regions Normandy and North France. Three anonymous referees helped to improve the paper with their comments and we are grateful to them for their work.

References

Barberà, S., Bossert, W., & Pattanaik, P. K. (2004). Ranking sets of objects. In *Handbook of utility theory* (pp. 893–977). Springer.

Bossert, W., Pattanaik, P. K., & Xu, Y. (2000). Choice under complete uncertainty: axiomatic characterizations of some decision rules. *Economic Theory*, *16*(2), 295–312.

Branke, J. (2016). Mcda and multiobjective evolutionary algorithms. *Multiple criteria decision analysis: state of the art surveys*, 977–1008.

Chugh, T., Sindhya, K., Hakanen, J., & Miettinen, K. (2019). A survey on handling computationally expensive multiobjective optimization problems with evolutionary algorithms. *Soft Computing*, *23*, 3137–3166.

Cohen, M., & Jaffray, J.-Y. (1980). Rational behavior under complete ignorance. *Econometrica*, 1281–1299.

Deb, K., Pratap, A., Agarwal, S., & Meyarivan, T. (2002). A fast and elitist multiobjective genetic algorithm: Nsga-ii. *IEEE transactions on evolutionary computation*, *6*(2), 182–197.

DeGroot, M. H. (2000). *Optimal statistical decisions.* McGraw Hill, NY. (Original edition published in 1970.)

Durbach, I. N., & Stewart, T. J. (2012). Modeling uncertainty in multi-criteria decision analysis. *European journal of operational research*, *223*(1), 1–14.

Ehrgott, M. (2005). *Multicriteria optimization* (2nd ed., Vol. 491). Springer Science & Business Media. (Original edition published in 2000.)

Evans, J. S. (1991). Strategic flexibility for high technology manoeuvres: a conceptual framework. *Journal of management studies*, *28*(1), 69–89.

Fayard, N., Mazri, C., & Tsoukiàs, A. (2022). Capability theory inspired tools for aiding policy design. *EURO Journal on Decision Processes*, *10*, 100024.

Fishburn, P. C. (1977). Multiattribute utilities in expected utility theory. *Conflicting objectives in decisions*, 172–196.

Foster, J. E. (2011). Freedom, opportunity, and well-being. In *Handbook of social choice and welfare* (Vol. 2, pp. 687–728). Elsevier.

French, S., & Ríos Insua, D. (2000). *Statistical decision theory.* Wiley.

Gaertner, W. (2012). Evaluating sets of objects in characteristics space. *Social Choice and Welfare*, *39*(2), 303–321.

Gaertner, W., & Xu, Y. (2006). Capability sets as the basis of a new measure of human development. *Journal of Human Development*, *7*(3), 311–321.

Gaertner, W., & Xu, Y. (2008). A new class of measures of the standard of living based on functionings. *Economic Theory*, *35*(2), 201–215.

Keeney, R. L., & Raiffa, H. (1993). *Decisions with multiple objectives: preferences and value trade-offs.* Cambridge university press.

Koopmans, T. C. (1962). On flexibility of future preference. *Cowles Foundation Discussion Papers*, *379*.

Kreps, D. M. (1979). A representation theorem for "preference for flexibility". *Econometrica*, *47*(3), 565–577.

Liesiö, J., & Salo, A. (2012). Scenario-based portfolio selection of investment projects with incomplete probability and utility information. *European Journal of Operational Research*, *217*(1), 162–172.

Martín, J., Bielza, C., & Ríos Insua, D. (2005). Approximating nondominated sets in continuous multiobjective optimization problems. *Naval Research Logistics*,

52(5), 469–480.
Pattanaik, P. K., & Xu, Y. (1990). On ranking opportunity sets in terms of freedom of choice. *Recherches Economiques de Louvain/Louvain Economic Review*, *56*(3-4), 383–390.
Ríos Insua, D., & French, S. (1991). A framework for sensitivity analysis in discrete multi-objective decision-making. *European Journal of Operational Research*, *54*(2), 176-190.
Ríos Insua, D., & Ruggeri, F. (2010). *Robust bayesian analysis.* Boston: Springer Verlag.
Savage, L. J. (1972). *The foundations of statistics* (2nd ed.). Dover Publications. (Original edition published in 1954.)
Scott, D. (1964). Measurement structures and linear inequalities. *Journal of mathematical psychology*, *1*(2), 233–247.
Sen, A. (1980). Equality of what? In S. McMurrin (Ed.), *Tanner lectures on human values, volume 1.* Cambridge: Cambridge University Press.
Sen, A. (1985). *Commodities and capabilities.* Elsevier Science Publishing Company.
Sen, A. (1993). Capability and well-being. In M. Nussbaum & A. Sen (Eds.), *The quality of life* (p. 30 - 53). Oxford: Clarendon Press.
Sen, A. (1999). *Development as freedom.* Alfred Knopf.
Sen, A. (2009). *The idea of justice.* Harvard University Press.
Stewart, T. J., French, S., & Rios, J. (2013). Integrating multicriteria decision analysis and scenario planning—review and extension. *Omega*, *41*(4), 679–688.
Vilkkumaa, E., Liesiö, J., Salo, A., & Ilmola-Sheppard, L. (2018). Scenario-based portfolio model for building robust and proactive strategies. *European Journal of Operational Research*, *266*(1), 205–220.

Appendix A Proofs for average capability sets

This Appendix provides proofs of the propositions satisfied by average capability sets.

Proposition 1bis: Consistency with average utility *If the capability sets are singletons and are only assessed using one dimension, then expected capability sets are equivalent to expected utilities.*
Proof Suppose we are dealing with only one solution ($|\mathbf{A}| = 1$) and one assessment dimension ($h^* = 1$). In such case, we can express $u(f_m(s_l))$ as a single real number b_l. Then $\overline{\mathbf{A}} = \left\{\sum_{l=1}^{l^*} p(s_l) \cdot X_{i,l}, \text{ for all } X_i \text{ such that } X_i = (\vec{b_1}, \ldots, \vec{b_{l^*}}) | \vec{b_l} \in \mathbf{A}_l \text{ for every } l \in \{1, \ldots, l^*\}\right\}$ is equivalent to $\sum_{l=1}^{l^*} p(s_l) \cdot b_l$ which is equivalent to $\sum_{l=1}^{l^*} p(s_l) \cdot u(f_m(s_l))$. ■

Proposition 3bis: Sure domination by the average capability *The average capability sets include the intersection of all vectors dominated by states' capabilities,* that is

$$\bigcap_{l=1}^{l^*}(\mathbf{A}_l - \mathbb{R}^{h^*}_{\geq 0}) \subseteq \overline{\mathbf{A}} - \mathbb{R}^{h^*}_{\geq 0}$$

Proof Consider any solution $\vec{b}$ dominated by every capability set, i.e. $\vec{b} \in \bigcap_{l=1}^{l^*}(\mathbf{A}_l - \mathbb{R}^{h^*}_{\geq 0})$. For each $l \in \{1, \ldots, l^*\}$, there exists $\vec{b}'_l \in \mathbf{A}_l$ such that $\vec{b}'_l \geq \vec{b}$. Therefore, the solution $\sum_{i=l}^{l^*} p(s_l) \cdot X_{i,l}$ with $X_i = (\vec{b}'_1, \ldots, \vec{b}'_{l^*})$ is in $\overline{\mathbf{A}}$ and is at least weakly dominating $\vec{b}$. ■

Proposition 4bis: Linearity
(*a*) *Preservation of addition:*

$$\forall \vec{c} \in \mathbb{R}^{h^*},\ \overline{\mathbf{A} + \vec{c}} = \overline{\mathbf{A}} + \vec{c}$$

(*b*) *Preservation of positive multiplication*

$$\forall \vec{c} \in \mathbb{R}^{h^*}_{\geq 0},\ \overline{\mathbf{A} \cdot \vec{c}} = \overline{\mathbf{A}} \cdot \vec{c}$$

Proof To prove (*a*), we simply remark that

$$\overline{\mathbf{A}} + \vec{c} = \Bigg\{ \sum_{l=1}^{l^*} \big(p(s_l) \cdot X_{i,l}\big) + \vec{c}, \text{for all } X_i \text{ such that } X_i = (\vec{b_1}, \ldots, \vec{b_{l^*}}) \ \mid \vec{b}_l \in \mathbf{A}_l \text{ for every } l \in \{1, \ldots, l^*\} \Bigg\}$$

and

$$\overline{\mathbf{A} + \vec{c}} = \Bigg\{ \sum_{l=1}^{l^*} p(s_l) \cdot X_{i,l}, \text{for all } X_i \text{ such that } X_i = (\vec{b_1}, \ldots, \vec{b_{l^*}}) \ \mid \vec{b}_l \in \mathbf{A}_l + \vec{c} \text{ for every } l \in \{1, \ldots, l^*\} \Bigg\}$$

The proof for (*b*) follows a similar approach. ■

Proposition 5bis: Monotonicity over capability domination *If for all $s_l \in S$ we have $\mathbf{A}_l \subseteq [\mathbf{B}_l - \mathbb{R}^{h^*}_{\geq 0}]$, then $\overline{\mathbf{A}} \subseteq \overline{\mathbf{B}} - \mathbb{R}^{h^*}_{\geq 0}$.*

Proof For every $\vec{b}$ in $\overline{\mathbf{A}}$, where $\vec{b} = \sum_{l=1}^{l^*} p(s_l) \cdot X_{i,l}$ and $X_i = (\vec{b_1}, \ldots, \vec{b_{l^*}})$, there is a set of being-vectors $X'_i = (\vec{b'_1}, \ldots, \vec{b'_{l^*}})$ such that $b_l \leq b'_l$ for all l (since for all $s_l \in S$ we have $\mathbf{A}_l \subseteq [\mathbf{B}_l - \mathbb{R}^{h^*}_{\geq 0}]$). Thus $\vec{b}$ is an element of $\vec{b'} - \mathbb{R}^{h^*}_{\geq 0}$ with $\vec{b'} = \sum_{l=1}^{l^*} p(s_l) \cdot X'_{i,l}$ and $\vec{b'}$ is in $\overline{\mathbf{A}'}$. ■

We provide now counterexamples to the propositions not fulfilled by average capability sets.

Proposition 2bis: Possible domination of the average capability *The average capability set is not always dominated by the set of all possible states, that is, we can have $\overline{\mathbf{A}} \nsubseteq \bigcup_{l=1}^{l^*}(\mathbf{A}_l - \mathbb{R}^{l^*}_{\geq 0})$.*

Proof See sets **B** and **C** in Figure 7. ■

Proposition 6bis: Non-monotonicity over probabilities *If we have* $\mathbf{A}_z \subseteq \mathbf{A}_{z'} - \mathbb{R}^{h^*}_{\geq 0}$ *with* $(s_z, s_{z'}) \in S^2$ *and*

$$p'(s_l) = \begin{cases} p(s_l) + c & for \quad l = z' \\ p(s_l) - c & for \quad l = z \\ p(s_l) & Otherwise \end{cases}$$

with $c \in (0; p(s_z)]$. *Then, we do not necessarily have* $\overline{\mathbf{A}} \subseteq \overline{\mathbf{A'}} - \mathbb{R}^{h^*}_{\geq 0}$.

Proof Consider a decision problem with two states, s_1 and s_2, and their associated capability set $\mathbf{A}_1 = U(f_m(s_1)) = \{(0,1)\}$ and $\mathbf{A}_2 = U(f_m(s_2)) = \{(0,1),(1,0)\}$. We have $\mathbf{A}_1 \subseteq \mathbf{A}_2 - \mathbb{R}^2_{\geq 0}$.

Initially, the PMs base their decision on $\overline{\mathbf{A}}$ with equal probabilities, $p(s_1) = p(s_2) = 0.5$, resulting in $\overline{\mathbf{A}} = \{(0,1),(0.5,0.5)\}$.

Now, with increased importance on the "best" set, i.e. $\mathbf{A}_2$, the new choice is $\overline{\mathbf{A'}} = \{(0,1),(0.75,0.25)\}$ when using probabilities $p'(s_1) = 0.25$ and $p'(s_2) = 0.75$, resulting in $\overline{\mathbf{A}} \nsubseteq \overline{\mathbf{A'}}$. ■

Appendix B Solving the Expected capability set of Example 2 using Problem 4

With $\mathbf{A}_1 = \{\vec{z_{1,1}} = (2,7), \vec{z_{1,2}} = (3,4)\}$ and $\mathbf{A}_2 = \{\vec{z_{2,1}} = (4,3), \vec{z_{2,2}} = (7,2)\}$, $M = 10$, the expected capability set is obtained through

$$E(\mathbf{A}) = PF\left(0.5 \cdot b_{1,1} + 0.5 \cdot b_{2,1}, 0.5 \cdot b_{1,2} + 0.5 \cdot b_{2,2}\right)$$

s.t.

$$\begin{aligned}
&b_{1,1} \leq 2 + \delta_{1,1} \cdot 10 && \delta_{1,1} + \delta_{1,2} \leq 1\\
&b_{1,1} \leq 3 + \delta_{1,2} \cdot 10 && \delta_{2,1} + \delta_{2,2} \leq 1\\
&b_{1,2} \leq 7 + \delta_{1,1} \cdot 10 && b_{1,1} \leq b_{2,1} + d_{1,2} \cdot 10\\
&b_{1,2} \leq 4 + \delta_{1,2} \cdot 10 && b_{1,2} \leq b_{2,2} + d_{1,2} \cdot 10\\
&b_{2,1} \leq 4 + \delta_{2,1} \cdot 10 && b_{2,1} \leq b_{1,1} + d_{2,1} \cdot 10\\
&b_{2,1} \leq 7 + \delta_{2,2} \cdot 10 && b_{2,2} \leq b_{1,2} + d_{2,1} \cdot 10\\
&b_{2,2} \leq 3 + \delta_{2,1} \cdot 10 && d_{1,2} + d_{2,1} \leq 1\\
&b_{2,2} \leq 2 + \delta_{2,2} \cdot 10 && b_{1,1}, b_{1,2}, b_{2,1}, b_{2,2} \in \mathbb{R}\\
& && \delta_{1,1}, \delta_{1,2}, \delta_{2,1}, \delta_{2,2} \in \{0; 1\}\\
& && d_{1,2}, d_{2,1} \in \{0; 1\}
\end{aligned}$$

The solution and the associated variable values are displayed in Table B1.

$\vec{b} \in E(\mathbf{A})$	$b_{1,1}$	$b_{1,2}$	$b_{2,1}$	$b_{2,2}$	$\delta_{1,1}$	$\delta_{1,2}$	$\delta_{2,1}$	$\delta_{2,2}$	$d_{1,2}$	$d_{2,1}$
(2, 5)	2	7	2	3	0	1	1	0	1	0
(3,3.5)	3	4	3	3	1	0	0	1	1	0
(3.5,3)	3	3	4	3	1	0	0	1	0	1
(5, 2)	3	2	7	2	0	1	1	0	0	1

Table B1 Being-vectors of the expected capability set and the associated variable values