Representing preferences using intervals

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Abstract

In this paper we present a general framework for the comparison of intervals when preference relations have to established. The use of intervals in order to take into account imprecision and/or uncertainty in handling preferences is well known in the literature, but a general theory on how such models behave is lacking. In the paper we generalise the concept of interval (allowing the presence of more than two points). We then introduce the structure of the framework based on the concept of relative position and component set. We provide an exhaustive study of 2-point and 3-point intervals comparison and show the way to generalise such results to n-points intervals.

Key words: preference modelling, interval representation, intransitivity, thresholds

1 Introduction

Dealing with preferences is an important issue in many fields including Computer Science and Artificial Intelligence (see [6], [8], [12]). In general, preferences are represented by binary relations defined on a set A (finite or infinite) of alternatives to be compared or evaluated. The classical theory of preference modelling considers two relations, strict preference P and indifference I (for a more general presentation on preference modelling see [21], [24]). Such a representation admits the existence of a complete preference structure, *i.e.* the decision maker is supposed to be able to compare any pair alternatives (for all object a and b in A, aPb or bPa or aIb holds). Other types of preference structures have been studied in the literature, either partial ones [9], [10], [32] and/or admitting more relations [7], [23], [25], [36], [31], [33], [34].

In this paper we focus on complete preference structures defined on a finite set A admitting two binary relations P and I. P is assumed to be an asymmetric relation and I is defined as the symmetric complement of P. The union of P and

I is denoted by R (by construction R is complete and reflexive and the relation $P \cap I$ is empty) and the affirmation aRb holds if and only if "a is at least as good as b". Among others, completeness is a crucial property in order to obtain a numerical representation of the preference structure. In fact, exploiting preferences requires naturally a model and a majority of existing models are quantitative ones, the quantification of preferences rendering easier the search for optimal or near-optimal decisions. In this perspective, a number of contributions in decision theory are based on the representational theory of measurement, formalized by Scott and Suppes ([26]) and presented in details in the three-volume set by Krantz et al. [14], Suppes et al. [29] and Luce et al. [16]. Generally speaking representation theorems represent a crucial aspect in handling preferences. Consider a recommender system trying to understand the preference structure of a user through a number of preferential statements. If the user claims that a is indifferent to b and this indifferent to c, but a is better than c, then we know that we need to use a numerical representation using intervals instead of single numbers in order to handle such preferences. On the other hand consider an agent who is trying to compare objects whose values (on some attribute) are expressed imprecisely: a is between 10 and 12, b is between 11 and 14, c is between 13 and 15. How do we compare such objects? There are preference structures (in this case interval orders) who allow to establish a preference among a, b and c.

Linear orders and weak orders are well known complete structures. A linear order consists of an arrangement of objects from the best one to the worst one without any ex aequo while a weak order defines the indifference relation as an equivalence relation (reflexive, symmetric and transitive). A weak order is indeed a total order of the equivalence (indifference) classes of A. Such preference structures have a limited representation capacity. In particular, a well known problem with linear orders or weak orders is that the associated indifference relation is necessarily transitive and such a property may be violated in the presence of thresholds as in the famous example given by Luce [15] on a cup of coffee. Different structures have been introduced for handling such cases. Indeed, in contrast to the strict preference relation, the indifference relation induced by such structures is not necessarily transitive. Semiorders may form the simplest class of such structures and they appear as a special case of interval orders. The axiomatic analysis of what we call now interval orders has been given by Wiener [37], then the term "semiorders" has been introduced by Luce [15] and many results about their representations are available in the literature (for more details see [10], [22]). Fishburn ([11]) has distinguished nine nonequivalent ordered sets defined as a generalisation of semiorders (using preference structures allowing only strict preference and indifference). These are interval orders, split semiorders, split interval orders, tolerance orders, bitolerance orders, unit tolerance orders, bisemiorders, semitransitive orders and subsemitransitive orders.

The use of simple numbers appears insufficient for the representation of ordered sets having a non transitive indifference relation. For instance, the numerical representation of an interval order makes use of intervals in a way that each alternative is represented by an interval (with a uniform length in the case of semiorders) and is said preferred to another alternative if and only if its associated interval is completely to the right of the other's interval. It is known that a majority of the structures belonging to the classification given by Fishburn ([11]) has a numerical representation using intervals.

However, the literature lacks a systematic study of such structures. Indeed as soon as we allow to compare "intervals" we can accept several different ways to do so. Just consider the case of the well known model of interval order where strict preference corresponds to the case where an interval is "completely to the right" (in the sense of the reals) of the other one. We could also consider as strict preference the case where an interval is just to the right of the other one despite having a non empty intersection. The problem becomes more complicated if we admit the existence of "intermediate points" within an interval. The number of possible comparisons increases dramatically and we would like to have a general framework within which studying them. In this paper we propose such a general framework for the study of preference structures to be used when we compare intervals. Our objective is to propose a systematic analysis of such structures and their numerical representations. We generalise the concept of interval allowing, besides the two extreme points of an interval, the existence of a certain number of intermediate points. We call such intervals n-point intervals. The comparison rules on these intervals are supposed to satisfy some hypotheses that we define at the beginning of our study.

The paper is organised as follows: Section 2 introduces basic notions, Section 3 presents hypotheses on the comparison rules and numerical representations that we can create in our framework. Section 4 shows some general results related to our study. Section 5 makes an exhaustive study of 2-point intervals, while Section 6 does the same for 3-point intervals. Section 7 concludes the paper.

2 Relative positions

Consider a finite set of alternatives A where each alternative x of A is associated a n-tuple of points of the real line $\mathbb{R}$; these n points are distinct and ranked in increasing order w.r.t. the natural order on the reals. Such a representation can also be seen as an interval with $n-2$ interior points. Therefore we call these objects "*n-point intervals*". If not otherwise mentioned, we use the same notation, typically x or y, for designating an alternative or its associated interval. A n-point interval x is specified by a vector of n elements: $\langle f_1(x), \cdots, f_n(x)\rangle$, with $f_i(x) < f_{i+1}(x)$, for all x in A and i in $\{1, \ldots, n-1\}$. Note that numbers $f_i(x)$ are not necessarily equally spaced. Figure 1 shows the graphical representation of an n-point interval.

Since our interest focuses on the possible preference structures arising from the

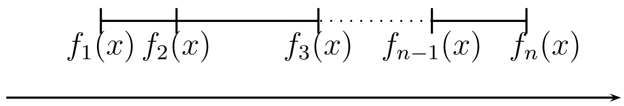


Fig. 1. n-point interval representation

comparison of n-point intervals, the position of one interval with respect to another is especially important. In case two n-point intervals x and y have no point in common, their relative position can be described by a total order on $2n$ points (n points for x + n points for y) as in the following example.

Example 1 *Let x and y be two 3-point intervals such that $x = \langle f_1(x), f_2(x), f_3(x)\rangle$, $y = \langle f_1(y), f_2(y), f_3(y)\rangle$ with their relative position represented schematically in Figure 2. The relative position of x and y is described by the total order: $f_1(y) < f_2(y) < f_1(x) < f_3(y) < f_2(x) < f_3(x)$.*

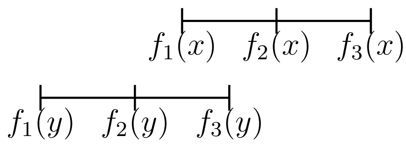


Fig. 2. Relative position of x and y

A convenient manner of representing the relative position of two n-point intervals is obtained using the n-tuple of numbers $\varphi(x, y)$ defined below.

Definition 1 (Relative position) *The relative position $\varphi(x, y)$ is an n-tuple $(\varphi_1(x, y),$-$\cdots, \varphi_i(x, y), \cdots, \varphi_n(x, y))$ where $\varphi_i(x, y)$ encodes the number of values of index j such that $f_i(x) \leq f_j(y)$.*

Intuitively, $\varphi(x, y)$ can be seen as representing to what extent the relative position of x and y is close to the case of two disjoint intervals. Indeed, in case $\varphi(x, y)$ is the null vector, x lies entirely to the right of y: no point of y is to the right of any point of x. The latter case is of particular interest as will become clear by the end of this section. Number $\varphi_i(x, y)$ represents the number of points of interval y that $f_i(x)$ must become greater than in order to reach the disjoint case.

For instance, the relative positions of the n-point intervals shown in figure 2, are:

$$\begin{aligned} \varphi(x, y) &= (1, 0, 0) \\ \varphi(y, x) &= (3, 3, 2). \end{aligned} \tag{1}$$

Clearly, if we assume that x and y have no points in common (i.e. $f_i(x) \neq f_j(y)$ for all i, j), giving either $\varphi(x, y)$ or $\varphi(x, y)$ allows us to reconstruct the weak order on the $2n$ points representing x and y. Having $\varphi(x, y) = (1, 0, 0)$ means that only

$f_1(x)$ lies to the left of some point representing y, the other two points of x being greater than all the points representing y.

It is readily seen that any vector $\varphi(x,y) = (\varphi_1(x,y), \cdots, \varphi_i(x,y), \cdots, \varphi_n(x,y))$ with $0 \leq \varphi_i(x,y) \leq n$ and $\varphi_i(x,y) \geq \varphi_{i+1}(x,y)$ corresponds to the relative position of feasible n-point intervals on the real line. Indeed we have that: for all $i = 1, \ldots, n$,

$$\begin{cases} f_i(x) \leq f_1(y) & \text{if } \varphi_i(x,y) = n \\ f_n(y) < f_i(x) & \text{if } \varphi_i(x,y) = 0 \\ f_{n-\varphi_i(x,y)}(y) < f_i(x) \leq f_{n+1-\varphi_i(x,y)}(y) & \text{otherwise} \end{cases} \tag{2}$$

These simple remarks allow us to derive the following result, which we state without further proof. In this result we limit ourselves to the case where the compared n-point intervals have no point in common.

Proposition 1 *For any vector $\varphi(x,y) = (\varphi_1(x,y), \cdots, \varphi_i(x,y), \cdots, \varphi_n(x,y))$ with $0 \leq \varphi_i(x,y) \leq n$ for all $i = 1, \ldots, n$ and $\varphi_i(x,y) \geq \varphi_{i+1}(x,y)$ for all $i = 1, \ldots, n-1$, there is a pair x, y of n-point intervals of the real line, with no points in common, such that the order on the $2n$ points representing x and y is uniquely determined. These two sets of n points are unique up to an increasing transformation of the real line.*

Given the relative position $\varphi(x,y)$ of x with respect to y, the relative position $\varphi(y,x)$ of y with respect to x can be easily computed.

Proposition 2 *Let $\varphi(x,y)$ be the relative position of the n-point interval x with respect to the n-point interval y, then, for all $i = 1, \ldots, n$,*

$$\begin{cases} \varphi_i(y,x) = n + 1 - |\{j, \varphi_j(x,y) \geq (n+1-i)\}| & \textit{if } \exists k, f_i(y) = f_k(x) \\ \varphi_i(y,x) = n - |\{j, \varphi_j(x,y) \geq (n+1-i)\}| & \textit{otherwise} \end{cases} \tag{3}$$

Proof.
We start with the proof of the second case. Using definition 1, we have $\forall i,\ \varphi_i(y,x) = |\{, f_j(x) \geq f_i(y)\}|$, hence $\forall i,\ \varphi_i(y,x) = n - |\{, f_j(x) < f_i(y)\}|$. On the other hand, $f_j(x) < f_i(y) \iff (n + 1 - \varphi_j(x,y)) \leq i$ (inequality 2). Replacing $f_j(x) < f_i(y)$ by $(n+1-i) \leq \varphi_j(x,y)$ in the above expression of $\varphi_i(y,x)$ we get $\forall i, \varphi_i(y,x) = n - |\{j, \varphi_j(x,y) \geq (n+1-i)\}|$.

In case $f_i(y)$ coincides with some point of the n-point interval$x$, we have to add 1 to the previously computed value of $\varphi_i(y,x)$. ■

The reader can check formula 3 against Example 1 (see equation (1)).

The number of possible relative positions of n-point intervals grows with n as stated in the next proposition.

Proposition 3 *Let x and y be two n-point intervals. The number of possible relative positions $\varphi(x, y)$ is*

$$m = \frac{(2n)!}{(n!)^2} = \begin{pmatrix} 2n \\ n \end{pmatrix}.$$

Proof . This number is the number of linear arrangements of $2n$ distinct points of the real line, n of which belonging to x and the other n to y, hence the formula. This is also the number of nondecreasing functions from $\{1, \ldots, n\}$ to $\{0, \ldots, n\}$. This sequence of integers is known as the sequence of *of central binomial coefficients* $A000984$ [27]. ■

For instance, the six relative positions of 2-point intervals can be described as follows: interval x completely lies to the right of interval y; intervals x and y have non empty intersection, without one being included in the other and x lying to the right of y; interval x is included in interval y; and the symmetric cases of these three situations (see Figure 8).

Table 1 shows the number of possible relative positions depending on number n, for $n = 2, 3, 4$.

$n =$	2	3	4	n
Relative positions	6	20	70	$\frac{(2n)!}{(n!)^2}$

Table 1
Number of relative positions depending on n

When alternatives are represented by n-point intervals of the real line, it is natural to assume that some relative positions of two intervals are more representative of a clear preference than others (from a cognitive and/or intuitive point of view). For instance, in the case of two disjoint intervals, it is more likely that we acknowledge a strict preference than in a case where one interval is included in the other. If the orientation of the real axis, say from left to right, is related to growing preference, we will be all the more ready to say that x is preferred to y that the interval representing x lies more to the right of the interval representing y. If x lies at least as much to the right of y then x' lies to the right of y', we say that the relative position $\varphi(x, y)$ is at least as strong as $\varphi(x', y')$ and we denote this by $\varphi(x, y) \trianglerighteq \varphi(x', y')$. A formal definition of $\trianglerighteq$ is as follows.

Definition 2 ("Stronger than" relation) *Let $\varphi(x, y)$ and $\varphi(x', y')$ denote the relative positions of two pairs of alternatives, respectively (x, y) and (x', y'). We say that $\varphi(x, y)$ is "at least as strong as" $\varphi(x', y')$ and note $\varphi(x, y) \trianglerighteq \varphi(x', y')$ if and only if $\forall i \in \{1, \ldots, n\},\ \varphi_i(x, y) \leq \varphi_i(x', y')$. We denote by $\triangleright$ the asymmetric* *part of $\trianglerighteq$. We say that $\varphi(x,y)$ is "stronger than" $\varphi(x',y')$ if and only if $\varphi(x,y) \trianglerighteq \varphi(x',y')$ and not $(\varphi(x',y') \trianglerighteq \varphi(x,y))$, which is denoted by $\varphi(x,y) \triangleright \varphi(x',y')$.*

This definition is consistent with intuition. Indeed, $\varphi_i(x,y) = 0$ for all i means that x lies totally to the right of y, which is the strongest possible position; if $\varphi_i(x,y) \neq 0$, the smaller the value of $\varphi_i(x,y)$, the stronger the position of x w.r.t. y. The following example illustrates this further.

Example 2 *Let $\varphi(x,y)$ and $\varphi(x,t)$ be the relative positions of the 3-points intervals represented in Figure 3. We have $\varphi(x,y) = (1,1,0)$, $\varphi(x,t) = (2,1,0)$. We get "$\varphi(x,y)$ is stronger than $\varphi(x,t)$" since $1 \leq 2,\ 1 \leq 1$ and $0 \leq 0$.*

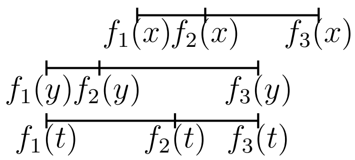


Fig. 3. Example: $(1,1,0) \triangleright (2,1,0)$

The "*at least as strong as*" relation $\trianglerighteq$ is a partial order (reflexive, antisymmetric and transitive relation). It is not a complete relation since there may always exist two relative positions φ and φ' for which $\exists i, j \in \{1, \ldots, n\}$ such that $\varphi_i < \varphi'_i$ and $\varphi'_j < \varphi_j$.

It is quite natural to represent relation $\trianglerighteq$ as a directed graph. We denote by G^n, the graph of all the possible[1] relative positions of n-point intervals. In G^n, the nodes represent the relative positions φ and the arcs, the relation $\trianglerighteq$. We denote by SG^n a subgraph of G^n, N_{G^n} the set of nodes of G^n and N_{SG^n} the set of nodes of SG^n. For the sake of getting readable graphical representations of partial orders, one often represents the cover relation associated with a partial order. The cover relation is a relation on the same set of objects N_{G^n}, but not all arcs of the graph are drawn. There is an arc from a to b if and only if there is no c such that $a \triangleright c \triangleright b$. This relation contains all the information needed to reconstruct the partial order $\trianglerighteq$ (add the loops and the arcs joining the initial vertex to the final vertex of all directed paths of the graph of the cover relation). Figure 4 represents the graph of the cover relation of $\trianglerighteq$ for 3-point intervals (G^3).

If x and y are 3-point intervals without common points, the correspondence between $\varphi(x,y)$ and $\varphi(y,x)$ defines a symmetry of the graph in Figure 4. Using proposition 2 we see e.g.that $\varphi(x,y) = (2,0,0)$ corresponds to $\varphi(y,x) = (3,2,2)$, $\varphi(x,y) = (2,1,0)$ to $\varphi(y,x) = (3,2,1)$ (assuming that x and y have no points in common). In general, for n-point intervals this symmetry is a transformation on the

[1] By "possible" relative positions, we understand the relative positions appearing in all possible sets A of n-point intervals.

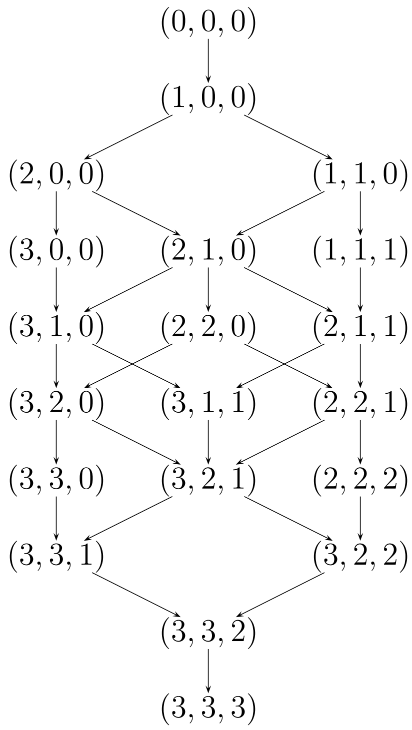


Fig. 4. Graph of the cover relation of the “at least as strong as” relation for 3-point intervals

set of relative positions, which we call *inversion*, and define by adapting formula (3):

Definition 3 *For any relative position φ in the set N_{G^n}, the inverse of φ is denoted by $(\varphi)^{-1}$ and is defined as follows:*

$$(\varphi)_i^{-1} = n - |\{j : \varphi_j \geq n + 1 - i\}| \tag{4}$$

Proposition 4 *The transformation of N_{G^n} that maps any relative position φ onto its inverse $(\varphi)^{-1}$ has the following properties:*

- *it is involutive, i.e. $\varphi = ((\varphi)^{-1})^{-1}$,*
- *and antitone with respect to the partial order $\trianglerighteq$, i.e. $\varphi \trianglerighteq \varphi'$ implies $(\varphi')^{-1} \trianglerighteq (\varphi)^{-1}$.*

Proof. The involutive character of the transformation results directly from the fact that φ and $(\varphi)^{-1}$ are respectively the relative positions $\varphi(x, y)$ and $\varphi(y, x)$ for some concrete n-point intervals x and y having no points in common. Hence $((\varphi)^{-1})^{-1}$ is just $\varphi(x, y)$. Verifying that the transformation is antitone can be done directly by using formula (4). ■

Partial order $\trianglerighteq$ defines a lattice on the set of possible relative positions N_{G^n}. A partially ordered (finite) set is a lattice if every pair of elements has a unique smallest upper bound (*join*) and a unique greatest lower bound (*meet*). Upper and lower bounds of a subset of relative positions are defined as follows. Let φ_* be a relative position. We say that:

- φ_* is a *lower bound* of the graph G^n (resp. of the subgraph SG^n) if $\varphi_* \in N_{G^n}$ (resp. $\varphi_* \in N_{SG^n}$) and $\neg\exists\varphi \in N_{G^n}$ (resp. $\neg\exists\varphi \in N_{SG^n}$) such that $\varphi_* \rhd \varphi$;
- φ^* is an *upper bound* of the graph G^n (resp. of the subgraph SG^n) if $\varphi^* \in N_{G^n}$ (resp. $\varphi^* \in N_{SG^n}$) and $\neg\exists\varphi \in N_{G^n}$ (resp. $\neg\exists\varphi \in N_{SG^n}$) such that $\varphi \rhd \varphi^*$.

Notice that for every n, G^n has a unique lower bound (φ, with $\forall i, \varphi_i = n$) and a unique upper bound (φ, with $\forall i, \varphi_i = 0$). But a subgraph may have more than one lower or upper bound because of the existence of incomparable nodes (consider e.g. the subgraph containing nodes $(0,0,0)$, $(1,0,0)$, $(2,0,0)$, $(1,1,0)$; there are two lower bounds: $(2,0,0)$ and $(1,1,0)$ and one upper bound: $(0,0,0)$.

Considering a relative position φ, we respectively denote by $D^+(\varphi)$, $D^-(\varphi)$ and $J(\varphi)$ the set of relative positions φ' such that φ is at least as strong as φ', which are at least as strong as φ, and which are incomparable to φ. We have:

$$\begin{aligned} D^+(\varphi) &= \{\varphi', \varphi \unrhd \varphi'\}, \\ D^-(\varphi) &= \{\varphi', \varphi' \unrhd \varphi\}, \\ J(\varphi) \quad &= \{\varphi', \varphi \ntrianglerighteq \varphi' \wedge \varphi' \ntrianglerighteq \varphi\}. \end{aligned}$$

3 Preference rules for comparing n-point intervals

The main goal of this paper is to explore *preference rules* used to interpret the relative positions of n-point intervals in terms of preference. Let A be any finite set of n-point intervals. A *preference rule* π assigns any pair (x, y) of $A^2 = A \times A$ to one in four exclusive categories that are denoted by P, P^{-1}, I or J. P, P^{-1}, I and J are just labels but we want to interpret P as *preference*, i.e. $\pi(x,y) = P$ if x is preferred to y; P^{-1} is *inverse preference*, i.e. $\pi(x,y) = P^{-1}$ if y is preferred to x; I denotes *indifference* and J, *incomparability*. For a given set A of n-point intervals, we denote by $P^A, (P^{-1})^A, I^A, J^A$ the following relations on A (i.e. the following subsets of A^2):

$$\begin{aligned} P^A \quad &= \{(x,y) \in A, \pi(x,y) = P\} \\ (P^{-1})^A &= \{(x,y) \in A, \pi(x,y) = P^{-1}\} \\ I^A \quad &= \{(x,y) \in A, \pi(x,y) = I\} \\ J^A \quad &= \{(x,y) \in A, \pi(x,y) = J\} \end{aligned} \tag{5}$$

Whenever there is no ambiguity, we shall abuse notation and drop superscript A, writing P (resp. P^{-1}, I, J) instead of P^A (resp. $(P^{-1})^A, I^A, J^A$), hence designating the relations defined on A by generic labels.

Following [24], the triple P^A, I^A, J^A of relations on A is a preference structure if P^A is an asymmetric relation, I^A a reflexive and symmetric one, J^A an irreflexive and symmetric relation and $P^A \bigcup (P^{-1})^A \bigcup I^A \bigcup J^A = A^2$, this union being a union of disjoint sets.

Obviously, not any rule that determines a partition of A^2 (whenever A is a set of n-point intervals) can be said a *preference* rule. In this paper, we are interested in preference rules that assign pairs of n-point intervals taking only into account their relative positions. Moreover, we shall restrict ourselves to *complete* preference rules π, for which there is no incomparability ($J = \emptyset$). Hence the resulting preference structure (P, I) is complete, i.e. $P^A \bigcup (P^{-1})^A \bigcup I^A = A^2$. We emphasise that this implies that the whole (P, I) structure is determined as soon as we know the sole strict preference relation P; indeed, $I^A = A^2 \setminus P^A \bigcup (P^{-1})^A$. The next definition lists the properties that we shall impose to preference rules in the rest of this study.

Definition 4 *A (complete) preference rule for n-point intervals, π, is a function defined on any Cartesian product A^2, where A is a finite set of n-point intervals, which assigns a label from the set $\{P, P^{-1}, I\}$ to any pair $(x, y) \in A^2$, respecting the following requirements:*

Axiom 1 *For all finite sets of n-point intervals A and B, and for all $x, y \in A$ and $z, t \in B$, if $\varphi(x, y) = \varphi(z, t)$, then $\pi(x, y) = \pi(z, t)$.*

Axiom 2 *For all $x, y, z, t \in A$, if $\varphi(x', y') \trianglerighteq \varphi(x, y)$ and $\pi(x, y) = P$, then $\pi(x', y') = P$.*

Axiom 1 tells that the assignment of a pair (x, y) to one of the relations $P, (P^{-1}), I$ only depends on the relative position $\varphi(x, y)$ of x w.r.t. y. This is *a fortiori* true when $A = B$. Axiom 1 allows us to talk about relative positions without referring to any particular set of n-point intervals A. The second axiom clearly interprets as a monotonicity condition w.r.t. relation "at least as strong as" on relative positions.

In view of axioms 1 and 2, a *complete* preference rule is entirely determined if we know the set of relative positions that lead to the assignment of label P to a pair (x, y) (independently of the set A which x and y are elements of). Indeed, letting $\Phi(P)$ be the set of such positions, we have $\pi(x, y) = P^{-1}$ if and only if $\pi(y, x) = P$, i.e. $\varphi(y, x) \in \Phi(P)$. We may thus define the set of relative positions $\Phi(P^{-1})$ leading to $\pi(x, y) = P^{-1}$ as the set of positions $\varphi(x, y)$ such that their inverse $\varphi(y, x)$ belongs to $\Phi(P)$. Since, by definition, π assigns a label to all pairs (x, y), we have $\pi(x, y) = I$ if and only if $\varphi(x, y) \in \Phi(I)$, which is the complement of $\Phi(P) \bigcup \Phi(P^{-1})$ in the set N_{G^n} of all relative positions.

The set of relative positions $\Phi(P)$ associated with a complete preference rule π has the following properties; reciprocally, these properties characterize those sets of relative positions that are associated with strict preference by some complete preference rule.

Proposition 5 *Let $\Phi(P)$ be the set of relative positions corresponding to preference for a given complete preference rule π. For all φ in $\Phi(P)$, we have:*

(1) φ' in N_{G^n} and $\varphi' \trianglerighteq \varphi$ imply $\varphi' \in \Phi(P)$;
(2) $(\varphi)^{-1} \notin \Phi(P)$.

Conversely, if a set $\Phi \subset N_{G^n}$ enjoys the two above properties it is the set $\Phi(P)$ associated with the complete preference rule π defined as follows: for all n-point intervals x, y,

$$\begin{aligned}
\pi(x,y) = P \quad & \Leftrightarrow \varphi(x,y) \in \Phi \\
\pi(x,y) = (P)^{-1} & \Leftrightarrow \varphi(y,x) \in \Phi \\
\pi(x,y) = I \quad & \Leftrightarrow \varphi(x,y) \notin \Phi \text{ and } \varphi(y,x) \notin \Phi.
\end{aligned} \tag{6}$$

Proof. The first property is a direct consequence of the definition of π and of axioms 1 and 2. The second results from the asymmetry of relation P and the fact that any pair $\varphi, (\varphi)^{-1} \in N_{G^n}$ describes the relative positions of a pair x, y of n-point intervals. For proving the converse statement, it is easy to see that π as defined by (6) unambiguously assigns one label in the set $\{P, (P)^{-1}, I\}$ to any pair of n-point intervals x, y. In particular, property 2 guarantees that no pair (x, y) will receive both labels P and $(P)^{-1}$. Indeed, if $\varphi(x, y) = \varphi$ and x and y have no points in common–which can be assumed without loss of generality–then $\varphi(y, x) = (\varphi)^{-1}$). By definition, π satisfies axiom 1. Property 1 ensures that it also fulfills axiom 2.■

The asymmetry of relation P can also be put in relation with the description of n-point intervals as n-tuples of real numbers.

Proposition 6 *Let π be a complete preference rule. If for some n-point intervals x, y we have $f_i(x) \leq f_i(y)$ for all $i = 1, \ldots, n$, then we may not have $\pi(x, y) = P$.*

Proof. If $f_i(x) \leq f_i(y)$ for all $i = 1, \ldots, n$, then $\varphi(y, x) \trianglerighteq \varphi(x, y)$. Using axiom 2, $\pi(x, y) = P$ implies $\pi(y, x) = P$, which means that (x, y) both belongs to P and P^{-1}. This contradicts the definition of π. ■

The conclusion of proposition 6 gives credit to a natural interpretation of n-point intervals w.r.t. preference: if none of the n points of x is better placed than the corresponding point of y, we cannot reasonably say that x is preferred to y.

3.1 Preference rules with a single weakest relative position

In view of proposition 5, any complete preference rule π on n-point intervals is determined by a set of relative positions $\Phi(P)$ that contains all relative positions stronger than any of its elements. As a consequence the weakest elements of such a set play an important role since all the other elements of the set can be determined from these ones. Let us consider two examples for 3-point intervals (the set of all relative positions for 3-point intervals is represented on figure 4). They differ by the number of lower bounds in $\Phi(P)$.

Example 3 *Let $\Phi(P)$ be the set of all relative positions at least as strong as $\varphi = (2, 1, 0)$. Then $\Phi(P) = \{(2, 1, 0), (2, 0, 0), (1, 1, 0), (1, 0, 0), (0, 0, 0)\}$ because of axiom 2. It is easy to see that the corresponding preference rule assigns a pair (x, y) of 3-point intervals to P if and only if $f_1(x) \geq f_1(y)$, $f_2(x) \geq f_2(y)$ and $f_3(x) \geq f_3(y)$.*

Example 4 *Define $\Phi(P)$ as the set of all relative positions at least as strong as $\varphi = (2, 0, 0)$ or $\varphi = (1, 1, 0)$. Note that these relative positions cannot be compared using relation $\trianglerighteq$. Then $\Phi(P) = \{(2, 0, 0), (1, 1, 0), (1, 0, 0), (0, 0, 0)\}$ because of axiom 2. The corresponding preference rule assigns a pair (x, y) of 3-point intervals to P if and only if at least one of the following conjunctions of conditions is fulfilled:*

$$\begin{cases} f_1(x) \geq f_1(y) \\ \text{and} \\ f_2(x) \geq f_3(y) \end{cases} \quad \text{or} \quad \begin{cases} f_1(x) \geq f_2(y) \\ \text{and} \\ f_3(x) \geq f_3(y). \end{cases} \tag{7}$$

These examples illustrate two typical cases. In the first case, $\Phi(P)$ has a single lower bound as in the former example (the unique lower bound is $(2, 1, 0)$); we call the corresponding decision rules *simple*. The second situation occurs when $\Phi(P)$ has more than one lower bound, as in the latter example (two lower bounds: $(2, 0, 0)$ and $(1, 1, 0)$); the corresponding preference rules are called *compound*. With simple preference rules, as in example 3, the conditions on $f_i(x)$ and $f_j(y)$ ensuring that $\pi(x, y) = P$ can be expressed as a single system of inequality constraints; for compound rules, as in example 4, the conditions will be a disjunction of systems of inequality constraints (such as (7)). For the reader's convenience, we state below the definition of a simple rule.

Definition 5 *A (complete) preference rule π as defined in definition 4 is* simple *if there is a unique relative position φ such that for all n-point intervals x and y, we have $\pi(x, y) = P$ if and only if their relative position $\varphi(x, y)$ is at least as strong as φ.*

In the sequel, we concentrate on *simple* preference rules for the following reason. In sections 5 and 6, we shall study systematically the preference structures (P, I) that are obtained when using simple preference rules in the cases of 2 and 3-point intervals. Compound preference rules will just yield disjunctions of the types of preferences structures obtained with simple rules. For instance in example 4, the preference structure P, I associated with the rule is such that P is the union of the following two strict preference relations:

- the strict preference relation $P_{(2,0,0)}$ associated with the simple preference rule $\pi_{(2,0,0)}$ defined by $\pi_{\trianglerighteq(2,0,0)}(x,y) = P$ if and only if $\varphi(x,y) \ \trianglerighteq (2,0,0)$
- the strict preference relation $P_{(1,1,0)}$ associated with the simple preference rule $\pi_{(1,1,0)}$ defined by $\pi_{\trianglerighteq(1,1,0)}(x,y) = P$ if and only if $\varphi(x,y) \ \trianglerighteq (1,1,0)$;

the indifference relation I is the symmetric complement of P, i.e. x and y are indifferent is and only if neither x is preferred to y nor y is preferred to x.

Which relative positions can be considered the weakest position of a set $\Phi(P)$ associated with a simple preference rule? A necessary and sufficient condition is established in the following lemma.

Lemma 1 *The set of relative positions that are not weaker than a given relative position φ is the set $\Phi(P)$ associated with some simple decision rule π if and only if*

$$\mathrm{Not}[(n, n-1, n-2, \ldots, 1) \trianglerighteq \varphi]. \tag{8}$$

Proof. Assume on the contrary that $[(n, n-1, n-2, \ldots, 1) \trianglerighteq \varphi]$. Using the definition of the inverse transformation of the set of relative positions and its antitone character (proposition 4), we obtain:

$$(\varphi)^{-1} \trianglerighteq (n, n-1, n-2, \ldots, 1)^{-1} = (n-1, n-2, n-3, \ldots, 0).$$

Since $(n-1, n-2, n-3, \ldots, 0) \triangleright (n, n-1, n-2, \ldots, 1)$ and using the transitivity of $\trianglerighteq$, we get $(\varphi)^{-1} \triangleright \varphi$ which contradicts proposition 5.2. The condition is thus necessary.

For proving sufficiency, we assume that φ is such that $\mathrm{Not}[(n, n-1, n-2, \ldots, 1) \trianglerighteq \varphi]$ and we prove that $\Phi = \{\varphi' \text{ such that } \varphi' \trianglerighteq \varphi\}$ is the set of relative positions leading to strict preference for some simple preference rule. This amounts to proving that Φ enjoys properties 1 and 2 in proposition 5. The former property is obvious by construction. Let us prove that for all $\varphi' \in \Phi$, $(\varphi')^{-1} \notin \Phi$. We start by proving that $(\varphi)^{-1} \notin \Phi$. By hypothesis (8), there is $i \leq n$ such that $\varphi_i < n - i + 1$. Due to the fact that $\varphi_j \geq \varphi_{j+1}$ for all j, we have $|\{j : \varphi_j \geq n - i + 1\}| \leq i - 1$. Hence, $(\varphi')^{-1}_i = n - |\{j : \varphi_j \geq n + 1 - i\}| \geq n - i + 1 > \varphi_i$, which implies $\mathrm{Not}[(\varphi)^{-1} \trianglerighteq \varphi]$. Let us finally consider any $\varphi' \in \Phi$. By proposition 4, we know that $\varphi' \trianglerighteq \varphi$ implies $(\varphi)^{-1} \trianglerighteq (\varphi')^{-1}$. Assuming $(\varphi')^{-1} \in \Phi$ would imply $(\varphi)^{-1} \in \Phi$, which has just been shown to be untrue. ■

We now introduce notation specific to simple rules, for which the strict preference relation is determined by their unique weakest relative position. Let φ be a relative position such that $\mathrm{Not}[(n, n-1, n-2, \ldots, 1) \trianglerighteq \varphi]$. We denote by $\pi_{\trianglerighteq\varphi}$ the corresponding simple preference rule, and by P_φ the set of relative positions that are at least as strong as φ. For ease of further reference, we give a direct formal definition of the preference structure arising from a simple preference rule, without referring explicitly to this rule; we emphasise here the relations that are defined on the set of n-point intervals as a result of using the decision rule. From this point, we shall use the notation $P_\varphi(x, y)$ (resp. $I_\varphi(x, y)$) as an alias for $\pi_{\trianglerighteq\varphi}(x, y) = P$ (resp. $\pi_{\trianglerighteq\varphi}(x, y) = I$).

Definition 6 *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a vector of relative positions in N_{G^n} such that Not$[(n, n-1, n-2, \ldots, 1) \trianglerighteq \varphi]$. Let x and y be any pair of n-point intervals. Relations P_φ and I_φ associated with φ (i.e. φ represents the weakest relative position such that P holds) are defined as follows:*

$$
\begin{aligned}
&P_\varphi(x, y) \Longleftrightarrow \varphi(x, y) \trianglerighteq \varphi, \\
&I_\varphi(x, y) \Longleftrightarrow \neg P_\varphi(x, y) \wedge \neg P_\varphi(y, x).
\end{aligned}
$$

3.2 Compact description of a preference structure

In this section, we come back to the construction of systems of inequalities expressing that $P_\varphi(x, y)$ according to a simple preference rule with weakest position φ. We have already obtained such descriptions for examples 3 and 4.

Let us consider the strict preference relation, represented in Figure 5, having (2,0,0) as its weakest relative position. Applying formula (2), we express the conditions for having $P_{(2,0,0)}(x, y)$ by means of the following inequalities: $f_1(y) < f_1(x)$, $f_3(y) < f_2(x)$ and $f_3(y) < f_3(x)$. Note that the third inequality is redundant. In order to avoid such redundancies and hence dispose of a compact coding of such inequalities, we introduce a new object that we call the *"component set"* of an n-tuple φ and that we denote by Cp_φ.

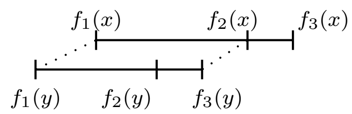


Fig. 5. $P_{(2,0,0)}(x, y) \Longleftrightarrow \varphi(x, y) \in \{(0, 0, 0) \cup (1, 0, 0) \cup (2, 0, 0)\}$

For the example in figure 5, we have $Cp_{(2,0,0)} = \{(1, 1)(3, 2)\}$. The pair $(1, 1)$ corresponds to inequality $f_1(y) < f_1(x)$, while $(3, 2)$ corresponds to $f_3(y) < f_2(x)$. Hence the representation convention is as follows: a pair (j, k) in Cp_φ represents inequality $f_j(y) < f_k(x)$. In the example, we do not need to include pair $(3, 3)$ corresponding to the redundant equation $f_3(y) < f_3(x)$.

In general, starting with a vector φ of relative positions, we have that $\varphi(x, y) \trianglerighteq \varphi$ if and only if for all i, $f_{n-\varphi_i}(y) < f_i(x)$; each such inequality is coded $(n - \varphi_i, i)$. From all these pairs we may remove those for which there exists $i' < i$ with $\varphi_{i'} \leq \varphi_i$. Indeed, the inequality corresponding to $n - \varphi_{i'}, i'$ yields $f_{n-\varphi_{i'}}(y) \leq f_{i'}(x)$ and we have $f_{i'}(x) < f_i(x)$ and $f_{n-\varphi_i}(y) < f_{n-\varphi_{i'}}(y)$.

The definition of Cp_φ below guarantees that the encoded systems of constraints are non redundant.

Definition 7 *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a relative position in N_{G^n} such that condition* (8) *is fulfilled. The* component set Cp_φ *associated with φ is the set of pairs $(n - \varphi_j, j)$ such that there is no $j' < j$ with $\varphi_{j'} \leq \varphi_j$.*

The component set Cp_φ encodes the minimal information needed to determine the preference structure (P_φ, I_φ). In particular, the strict preference relation P_φ is determined as follows:

$$\forall x, y,\ P_\varphi(x, y) \Longleftrightarrow \forall (i, j) \in Cp_\varphi,\ f_i(y) < f_j(x). \tag{9}$$

The indifference relation I_φ is obtained by expressing that $I_\varphi(x, y)$ if and only if $\neg P_\varphi(x, y)$ and $\neg P_\varphi(y, x)$, i.e.

$$\begin{aligned} \forall x, y,\ I_\varphi(x, y) \Longleftrightarrow & \exists (i, j) \in Cp_\varphi,\ f_i(y) \geq f_j(x), and \\ & \exists (k, l) \in Cp_\varphi,\ f_k(x) \geq f_l(y). \end{aligned} \tag{10}$$

Condition (8) determines the relative positions that generate simple preference rules. This condition translates into the following property of Cp_φ.

Proposition 7 *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a relative position in N_{G^n} such that condition* (8) *is fulfilled. In the component set Cp_φ associated with φ, there is at least one pair (i, j) with $(i \geq j)$.*

Proof. On the contrary assume that for all pairs (i, j) in Cp_φ we have $i < j$. Consider a pair (x,y) of n-point intervals such that:

$$\begin{aligned} f_1(x) < f_1(y) < f_2(x) < f_2(y) < \ldots f_k(x) < f_k(y) < f_{k+1}(x) \\ < f_{k+1}(y) < \ldots < f_n(x) < f_n(y). \end{aligned}$$

For all $k = 2, \ldots n$ we have $f_k(y) < f_{k+1}(x)$ which implies $f_i(y) < f_j(x)$ for all (i, j) in Cp_φ, hence $P_\varphi(x, y)$. The relative position of x w.r.t. y is characterized by $\varphi' = (n, n-1, \ldots, 1)$. Since $P_\varphi(x, y)$, we have $\varphi' = (n, n-1, \ldots, 1) \trianglerighteq \varphi$ violating (8). ■

3.3 Constructing all simple preference rules

In this section, we present an algorithm yielding all possible sets of relative positions which may determine a strict preference relation P associated with a simple preference rule (definition 5). For this purpose we consider each relative position φ in turn; if φ can be the weakest relative position leading to strict preference (i.e. if it satisfies condition (8)), we build a set of nodes N_{SG^n}, which consists of all relative positions at least as strong as φ.

Algorithm Unique Cuts:

$L := \emptyset$;

For all nodes φ in the graph G^n do

if $\exists i, \varphi_i < n - i + 1$ then

$N_{SG^n} := D^- = \{(\varphi) = \varphi' : \varphi' \trianglerighteq \varphi\}$;

$L := L \cup \{N_{SG^n}\}$;

end if;
od;
Return L;

Each iteration of this algorithm provides a subgraph SG^n of the graph G^n with just one upper bound ($\forall i, \varphi_i = 0$) and just one lower bound. As a consequence each relative position becomes a lower bound of a SG^n once and only once except those that do not satisfy (8). In Figure 6 we show the result of the algorithm when the lower bound is $P_{(3,1,0)}$.

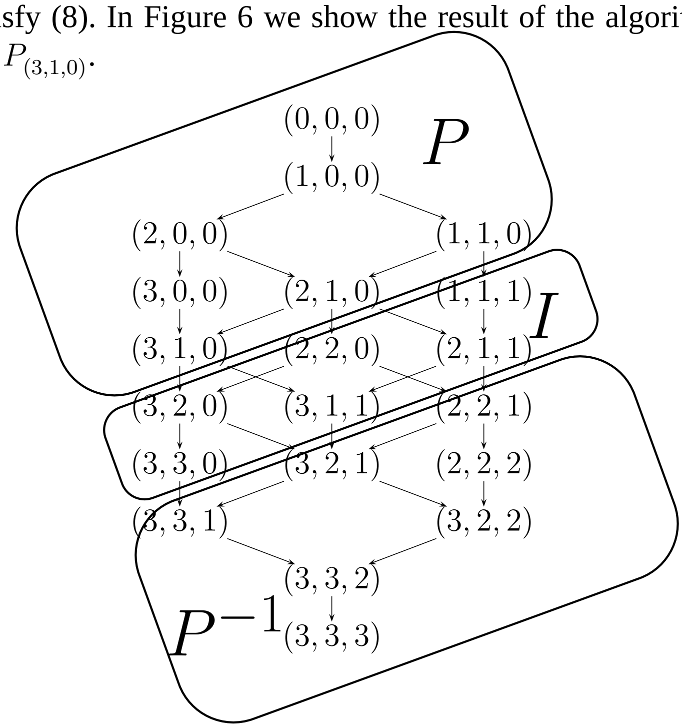


Fig. 6. The preference structure resulting when the lower bound is (3,1,0)

It is easy to compute the number of different sets of relative positions (equal to the number of possible SG^n) that our algorithm calculates when n is known.

Proposition 8 *Let sm be the number of sets of relative positions having a single* *weakest element, containing all positions at least as strong as any of their elements and never containing a position and its inverse. We have:*

$$sm = \binom{2n}{n+1}$$

Proof. Number sm is equal to the number all relative positions of n-point intervals as computed in proposition 3 minus the number of relative positions that cannot be the weakest element of a set P_φ, i.e. φ's such that $(n, n-1, \ldots, 1) \trianglerighteq \varphi$, i.e. $n - i + 1 \leq \varphi_i$ for all $i = \{1, \ldots, n\}$. Rephrasing these conditions in terms of inequalities involving $f_j(x)$ and $f_k(y)$, we get, using (2), $f_i(x) \leq f_i(y)$ for all i. Hence, we have to compute the number of relative positions of n-point intervals x and y such that $f_i(x) \leq f_i(y)$. Since there is no loss of generality in assuming strict inequalities in the latter, this is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Indeed C_n is known to be, among many other characterizations (see [28]), the number of Dyck words of length $2n$, i.e. the number of sequences of n X's and n Y's such that no initial segment of the sequence has more Y's than X's. The correspondence with our case is clear. Consequently, we have:

$$sm = \frac{(2n)!}{(n!)^2} - \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n+1}.$$

■

This number is also the number of simple preference rules on n-point intervals.

3.4 The case where n-point intervals have points in common

At this point let us make a comment on the reason why we have assumed that the n-point intervals under consideration have no points in common. The reason is not that the latter case is not interesting. In the framework of temporal reasoning, for instance, Allen [3,4] has investigated relations between time intervals, which distinguish the cases where intervals start at the same time, finish at the same time or both. His work has generated a large literature (see e.g. [35]).

In contrast, in preference modeling, the case where $f_i(x) = f_j(y)$ for some i, j is not dealt with separately. It leads either to preference (P) or non preference in a systematic way. In view of our definition of relative positions (definition 1), we assimilate the case $f_i(x) = f_j(y)$ to the case $f_i(x) < f_j(y)$. Hence, while assigning a pair of n-point intervals to P or to $\mathrm{Not}P$, we make no distinction between a pair (x, y) such that $f_i(x) = f_j(y)$ and a pair (x', y) in which $f_k(x') = f_k(x)$ for all $k \neq i$ and $f_i(x') = f_i(x) - \varepsilon$, provided ε is positive but small enough to guarantee that $f_k(x) > f_l(y)$ implies $f_k(x') > f_l(y)$ for all k, l. In other words, we may always break equalities in such a way that we get a pair of n-point intervals with no points in common, which receives the same assignment P or $\mathrm{Not}P$ as the original pair. Figure 7 shows an example for $n = 3$ in which $f_2(x) = f_3(y)$ and $f_2(x') < f_3(y)$. For all preference rule π, we have $\pi(x, y) = P$ if and only if $\pi(x', y) = P$.

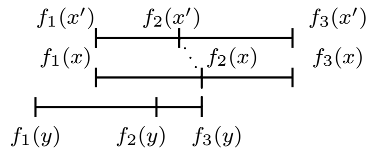


Fig. 7. $\pi(x, y) = P \;\Leftrightarrow\; \pi(x', y) = P$

Note that it is perfectly possible to adopt the reverse convention in the definition of relative positions, hence assimilating $f_i(x) = f_j(y)$ to the case $f_i(x) > f_j(y)$. In such a case we would make no distinction between a pair (x, y) such that $f_i(x) = f_j(y)$ and a pair (x', y) in which $f_k(x') = f_k(x)$ for all $k \neq i$ and $f_i(x') = f_i(x+\varepsilon)$, provided ε is positive but small enough to guarantee that $f_k(x) < f_l(y)$ implies $f_k(x') < f_l(y)$ for all k, l. The important thing is that the rule is systematically applied.

4 General results

In this section, we characterize the simple preference rules inducing preference structures (P_φ, I_φ) that enjoy some classical properties such as transitivity of preference and indifference, Ferrers property, etc. Note that we shall not refer to any specific set A of n-point intervals in the sequel. When we say that P_φ is transitive for some simple preference rule, we mean that the relation P_φ induced by this rule on any set of n-point intervals is systematically transitive. Clearly, for a simple preference rule that does not guarantee that P_φ is transitive, it may happen that it is for some specific sets of n-point intervals but not for all (consider e.g. the case in which A contains only one n-point interval; in this case, P_φ is trivially transitive). We emphasize that the properties of P_φ and I_φ listed below are valid for all sets of n-point intervals. Our first result is concerned with the transitivity of the preference relation. We start with a lemma.

Lemma 2 *Let φ be the relative position associated with a simple preference rule. If Cp_φ contains the pair (i, j), then*

(1) $\varphi_j = n - i$
(2) if $j > 1$, $\varphi_{j-1} \geq n - i + 1$

(3) the relative position φ' defined by:

$$\begin{cases} \varphi'_j = n - i \\ \varphi'_k = n - i + 1 \quad \forall k < j \\ \varphi'_l = 0 \qquad\qquad \forall l > j \end{cases} \tag{11}$$

is such that $\varphi' \trianglerighteq \varphi$.

Proof. 1. The first assertion is a direct consequence of definition 7.

2. We have $\varphi_{j-1} \geq n - i$. Assume that $\varphi_{j-1} = n - i$. This would contradict the definition of Cp_φ since there would exist $j' = j - 1$ with $\varphi_{j'} = \varphi_j$.

3. In view of 1 and 2, we have $\varphi'_j = \varphi_j$, $\varphi'_k \leq \varphi_k$ for all $k < j$ and, obviously, $\varphi'_l \leq \varphi_l$ for all $l > j$, hence $\varphi' \trianglerighteq \varphi$. ■

Proposition 9 *Let P_φ be the preference relation obtained by applying a simple decision rule as described in definition 6 and Cp_φ be the corresponding component set as described in definition 7. P_φ is guaranteed to be transitive (on all sets of n-point intervals) if and only if $\forall (i,j) \in Cp_\varphi,\ i \geq j$,*

Proof.
$\Leftarrow$ Suppose that $P_\varphi(x,y)$ and $P_\varphi(y,z)$ hold, then we get $\forall (i,j) \in Cp_\varphi,\ f_i(y) < f_j(x)$ and $f_i(z) < f_j(y)$. Since $i \geq j$, we have $f_j(y) \leq f_i(y)$ hence, $\forall (i,j) \in Cp_\varphi,\ f_i(z) < f_j(y) \leq f_i(y) < f_j(x)$.This implies $P_\varphi(x,z)$.

$\Rightarrow$ We will prove that:
$\exists (i,j) \in Cp_\varphi\ i < j \Longrightarrow \exists x,y,z,\ P_\varphi(x,y) \wedge P_\varphi(y,z)$ and$\neg P_\varphi(x,z)$.

Assume first that $1 < i$ and $j < n$. Consider n-point intervals x, y, z satisfying the following constraints:

$$\begin{aligned} f_1(z) < \dots f_{i-1}(z) < f_1(y) < \dots < f_i(y) < f_1(x) < \dots < f_i(x) < \dots f_j(x) \\ < f_i(z) < \dots f_n(z) < f_{i+1}(y) < \dots < f_n(y) < \dots < f_{j+1}(x) < \dots < f_n(x). \end{aligned} \tag{12}$$

We have $P_\varphi(x,y)$. Indeed $\varphi_k(x,y) = n - i$ for all $k \leq j$ and $\varphi_l(x,y) = 0$ for all $l > j$. Using φ' in lemma 2, yields $\varphi(x,y) \trianglerighteq \varphi' \trianglerighteq \varphi$, hence $P_\varphi(x,y)$. We show similarly that $P_\varphi(y,z)$ since $\varphi(y,z) = \varphi'$. However, xPz does not hold since $f_i(z) > f_j(x)$.

We now examine the cases in which conditions $1 < i$ and $j < n$ may fail to be fulfilled. The positions of x, y, z as described in (12) can easily be adapted:

$(i = 1)$: there is no $f_k(z)$ before $f_1(y)$, which is the only one before $f_1(x)$;

$(j = n)$: all $f_k(x)$ lie between $f_i(y)$ and $f_i(z)$.

In both these cases, the same conclusions as in the general case can be drawn. ■

Most preference structures induced by simple decision rules have a transitive preference relation. However, we do not exclude rules that violate this property as in the case of $P_{\leq(3,2,0)}$ (for more details see Section 6). It is indeed possible to consider preferences in which the asymmetric part would not be transitive. The *tangent circle "order"* is an example of such a structure. It describes the order and the intersection structure of circles of different diameters all tangent to an horizontal line of the plane (see [2]).

We now present a characterization of decision rules that guarantee the transitivity of the indifference relation I_φ.

Proposition 10 *Let I_φ be the indifference relation obtained by applying a simple decision rule as described in definition 6 and Cp_φ be the corresponding component set. I_φ is guaranteed to be transitive on all sets of n-point intervals if and only if*

$$\exists i \in \{1, \dots, n\},\ Cp_\varphi = \{(i, i)\} \tag{13}$$

Proof. $\Leftarrow$ Suppose that $Cp_\varphi = \{(i, i)\}$. Then $\forall x, y,\ I_\varphi(x, y) \iff f_i(y) \geq f_i(x) \wedge f_i(x) \geq f_i(y)$, which is equivalent to $I_\varphi(x, y) \iff f_i(y) = f_i(x)$. Since equality is transitive, I_φ is transitive.

$\Rightarrow$ We prove this result by contradiction. We suppose that $Cp_\varphi \neq \{(i, i)\}$ and we analyze two different cases.

1. $\exists (i, j) \in Cp_\varphi, i \neq j$**.** In this case, using (10), we have $f_i(y) \geq f_j(x) \wedge f_i(x) \geq f_j(y) \Longrightarrow I_\varphi(x, y)$. Let x, y, z be three n-point intervals such that

$$f_j(z) < f_j(y) < f_i(z) < f_n(z) < f_1(x) < f_j(x) < f_i(y) < f_i(x),$$

with $(i, j) \in Cp_\varphi$. $I_\varphi(x, y)$ holds since $f_j(y) < f_i(x)$ and $f_j(x) < f_i(y)$, $I_\varphi(y, z)$ holds since $f_j(z) < f_i(y)$ and $f_j(y) < f_i(z)$ and $P_\varphi(x, z)$ holds since $\varphi_i(x, z) = 0$ for all i. Therefore I_φ is not transitive.

2. $\forall (i, j) \in Cp_\varphi, i = j$ and $|Cp_\varphi| > 1.$ Let (i, i) and (j, j) be two different pairs belonging to Cp_φ, with $i < j$. Then using (10), $f_i(y) \geq f_i(x) \wedge f_j(x) \geq f_j(y) \Longrightarrow I_\varphi(x, y)$. For a positive real M large enough (e.g. $M \geq 4$), let x, y, z be three n-point intervals such that

- x: $\forall t \in \{1, \dots, i-1\},\ 1 < f_t(x) < M$; $f_i(x) = 3M+1$; $\forall t \in \{i+1, \dots, j-1\},\ 4M < f_t(x) < 5M$; $f_j(x) = 7M + 2$ and $\forall t \in \{j + 1, \dots, n\},\ 8M < f_t(x) < 9M$;
- y: $\forall t \in \{1, \dots, i-1\},\ f_t(y) < 3M + 3$; $\forall t \in \{i, \dots, j\},\ 3M + 3 < f_t(y) < 7M + 1$; and $\forall t \in \{j + 1, \dots, n\},\ 7M + 1 < f_t(y)$;

- z: $\forall t \in \{1, \ldots, i-1\},\ 2M < f_t(z) < 3M$; $f_i(z) = 3M + 2$; $\forall t \in \{i+1, \ldots, j-1\},\ 6M < f_t(z) < 7M$; $f_j(z) = 7M + 3$ and $\forall t \in \{j+1, \ldots, n\},\ 10M < f_t(z) < 11M$.

$I_\varphi(x, y)$ holds since $f_i(x) = 3M + 1 < 3M + 3 < f_i(y)$ and $f_j(y) < 7M + 1 < 7M + 2 = f_j(x)$; $I_\varphi(y, z)$ holds since $f_j(y) < 7M + 1 < 7M + 3 = f_j(z)$ and $f_i(z) = 3M + 2 < 3M + 3 < f_i(y)$; $P_\varphi(z, x)$ since by construction $\forall i \in \{0, \ldots, n\},\ f_i(x) < f_i(z)$. Therefore I_φ is not transitive. ∎

This result shows that within our framework, the structures being defined by comparing the positions of two different points of the real line have an intransitive indifference relation. Such a result is not surprising since the numerical representation of a large number of preference structures known in the literature as having intransitive indifference uses intervals. This is the case with semiorders, interval orders, split interval orders, etc (see below for definitions).

Propositions 9 and 10 show how weak orders are obtained in our framework.

Definition 8 *A binary relation $P \cup I$ is a weak order if and only if P is transitive, I is reflexive and transitive and $P \cup I$ is complete.*

We have the reflexivity of I_φ and the completeness of $P_\varphi \cup I_\varphi$ by construction.

Corollary 1 *Let P_φ and I_φ be respectively the preference and the indifference relation obtained by applying a simple decision rule as described in definition 6. Let Cp_φ be the component set associated to the decision rule. $P_\varphi \cup I_\varphi$ is a weak order if and only if*

$$\exists i \in \{1, \ldots, n\},\ Cp_\varphi = \{(i, i)\} \tag{14}$$

Such a result allows for the existence of different rules leading to weak orders when n-points intervals are used. The following assertion is easily verified.

Proposition 11 *Let m be the number of different φ when n-point intervals are used such that $P_\varphi \cup I_\varphi$ is a weak order, then*

$$m = n. \tag{15}$$

For instance, with 2-point intervals there exist two ways for obtaining weak orders: $Cp_\varphi = \{1, 1\}$ and $Cp_\varphi = \{2, 2\}$ (for more details see Section 5).

Another class of ordered sets is that of interval orders for which indifference is not transitive. A couple of relations (P, I) (forming a preference structure) has to fulfill the Ferrers property (see [24]) in order to be an interval order.

Definition 9 *A binary relation R has the Ferrers property, and we all it a Ferrers* *relation, if and only if*

$$\forall x, y, z, t \in A, R(x, y) \wedge R(z, t) \Longrightarrow R(x, t) \vee R(z, y) \tag{16}$$

One can also give an alternative characterization of a Ferrers relation using its separation on symmetric and asymmetric relation:

Theorem 1 *Let R be a binary relation and P (respectively I) the asymmetric (resp. the symmetric) part of R, then the two following sentences are equivalent:*

i. *R is a Ferrers relation*
ii. $\forall x, y, z, t \in A,\ P(x, y) \wedge I(y, z) \wedge P(z, t) \Longrightarrow P(x, t)$ *(we denote it by $P.I.P \subset P$).*

The asymmetric part of a Ferrers relation is transitive.

Proposition 12 *Let R be a Ferrers relation and P (respectively I) the asymmetric (resp. the symmetric) part of R, then relation P is transitive.*

Proof.
Since the identity relation is included in I, we have $\forall x, y, z \in A,\ P(x, y) \wedge I(y, y) \wedge P(y, t) \Longrightarrow P(x, z)$ ■

The following result provides a characterization of a Ferrers relation within our framework.

Proposition 13 *Let P_φ and I_φ be binary relations obtained by applying a simple decision rule as described in definition 6 and Cp_φ be the corresponding component set. $P_\varphi \cup I_\varphi$ is guaranteed to be a Ferrers relation on all sets of n-point intervals if and only if*

$$|Cp_\varphi| = 1 \tag{17}$$

Proof.
The proof of this result follows from lemmas 3 and 4 below:

If $|Cp_\varphi| = 1$ then $P_\varphi.I_\varphi.P_\varphi \subset P_\varphi$: see lemma 3

If $P_\varphi.I_\varphi.P_\varphi \subset P_\varphi$ then $|Cp_\varphi| = 1$: see lemma 4.

Lemma 3 *Let P_φ and I_φ be binary relations obtained by applying a simple decision rule as described in definition 6 and Cp_φ be the corresponding component set then*

$$\textit{if } |Cp_\varphi| = 1 \textit{ then } P_\varphi.I_\varphi.P_\varphi \subset P_\varphi. \tag{18}$$

Proof.:
If $|Cp| = \{i, j\}$ then $\forall x, y\ P_\varphi(x, y) \iff f_i(y) < f_j(x)$ and $I_\varphi(x, y) \iff (f_i(y) \geq f_j(x)) \wedge (f_i(x) \geq f_j(y))$.

Let x, y, z, t be four n-point intervals with $P_\varphi(x, y)$, $I_\varphi(y, z)$ and $P_\varphi(z, t)$ then:

$P_\varphi(x, y) \iff f_i(y) < f_j(x)$,

$I_\varphi(y, z) \iff (f_i(y) \geq f_j(z)) \wedge (f_i(z) \geq f_j(y))\,,$

$P_\varphi(z, t) \iff f_i(t) < f_j(z)$.

These inequalities yield: $f_i(t) < f_j(z) \leq f_i(y) < f_j(x)$, hence we obtain $f_i(t) < f_j(x)$ which is equivalent to $P_\varphi(x, t)$.

As a conclusion we have: $(P_\varphi(x, y) \wedge I_\varphi(y, z) \wedge P_\varphi(z, t)) \implies P_\varphi(x, t)$. This completes the proof. ∎

Lemma 4 *Let P_φ and I_φ be binary relations obtained by applying a simple decision rule as described in definition 6 and Cp_φ be the corresponding component set then*

$$\textit{if}\ |Cp_\varphi| \geq 2\ \textit{then}\ not\,(P_\varphi.I_\varphi.P_\varphi \subset P_\varphi)\,. \tag{19}$$

Proof.:
Let P_φ be a binary relation defined as :

$\forall x, y\ P_\varphi(x, y) \iff \bigwedge_{(i,j)\in Cp_\varphi} f_i(y) < f_j(x)$ where $|Cp_\varphi| \geq 2$.

We analyze two cases: $\exists (i, j) \in Cp_\varphi,\ i < j$ and $\forall (i, j) \in Cp_\varphi,\ i \geq j$.

- If $\exists (i, j) \in Cp_\varphi$, such that $i < j$ then the preference relation P_φ is not transitive (see proposition 9). Using Proposition 12 we conclude that $P_\varphi \cup I_\varphi$ is not Ferrers.

-If $\forall (i, j) \in Cp_\varphi,\ i \geq j$: using (10), we have

$$\forall x, y\ I_\varphi(x, y) \iff \bigvee_{(i,j),(l,m)\in Cp_\varphi} (f_l(y) \geq f_m(x) \wedge f_i(x) \geq f_j(y)).$$

Since $|Cp_\varphi| \geq 2$, $\exists (i, j), (l, m) \in Cp_\varphi$ where $(i, j) \neq (l, m)$, $f_l(x) \geq f_m(y) \wedge f_i(y) \geq f_j(x) \implies I_\varphi(x, y)$.
We suppose here that we have $j \leq i < m \leq l$ (the proof of the case $j < m < i < l$, being similar, is omitted). For a positive real M large enough, let w, x, y, z be four n-point intervals such that

- w: $\forall t \in \{1, \ldots, i\},\ M < f_t(w) < 2M$; $\forall t \in \{i+1, \ldots, n\},\ 5M < f_t(w) < 6M$;
- x: $\forall t \in \{1, \ldots, m-1\},\ 0 < f_t(x) < M$; $\forall t \in \{m, \ldots, n\},\ 4M < f_t(x) < 5M$;
- y: $\forall t \in \{1, \ldots, n\},\ 3M < f_t(y) < 4M$;
- z: $\forall t \in \{1, \ldots, n\},\ 2M < f_t(z) < 3M$.

These four intervals satisfy the following relations:

- $P_\varphi(w, x)$: Indeed $\varphi_t(w, x) = n - m$ for all $t \leq i$ and $\varphi_t(w, x) = 0$ for all $t > i$. Using φ' in lemma 2, yields $\varphi(w, x) \trianglerighteq \varphi' \trianglerighteq \varphi$, hence $P_\varphi(w, x)$.
- $I_\varphi(x, y)$ since $f_m(y) < f_l(x)$ ($3M < f_m(y) < 4M$, $4M < f_l(x) < 5M$) and $f_j(x) < f_i(y)$ ($0 < f_j(x) < M$, $3M < f_i(y) < 4M$).
- $P_\varphi(y, z)$ since $\forall t \in \{1, n\}$, $f_n(z) < f_t(y)$;
- $\neg P_\varphi(w, z)$ since $f_m(z) < f_l(w)$ ($2M < f_m(z) < 3M$, $5M < f_l(w) < 6M$) and $f_j(w) < f_i(z)$ ($M < f_j(w) < 2M$, $2M < f_i(z) < 3M$). ∎

We are able now to characterize an interval order. First, we recall the definition of an interval order.

Definition 10 *A binary relation $P \cup I$ is an interval order if and only if $P \cup I$ is reflexive, complete and Ferrers.*

Corollary 2 *Let P_φ and I_φ be respectively the preference and the indifference relation obtained by applying a simple decision rule as described in definition 6. Let Cp_φ be the component set associated to the decision rule. $P_\varphi \cup I_\varphi$ is guaranteed to be an interval order if and only if*

$$|Cp_\varphi| = 1 \tag{20}$$

As in the case of weak orders, depending on the value n, an interval order can have more than one representation.

Proposition 14 *The number m of relative positions φ yielding a preference structure $P_\varphi \cup I_\varphi$ that is an interval order is*

$$m = \frac{n(n-1)}{2} \tag{21}$$

Proof. If $|Cp_\varphi| = 1$ ($|Cp_\varphi| = \{i, j\}$) then $i \leq j$ (see Proposition 7). Since Cp_φ can be any pair (i, j) with $i < j$, the number m of such Cp_φ is the number of manners of selecting two numbers from a set of n numbers, i.e. $m = \frac{n(n-1)}{2}$ ∎

In the next two sections we analyze simple preference rules that can be applied when 2-point and 3-point intervals are used. Section 5 is devoted to 2-point intervals and Section 6 to 3-point intervals. In each section we analyze in turn all simple preference rules satisfying our axioms, describe the corresponding preference structure and formulate comments. As will be shown, some new preference structures, such as triangle orders, bi-weak orders, etc., will appear in these sections and will receive a characterization in our framework.

5 2-point intervals

In this section we present a complete analysis of 2-point intervals within our framework. With 2-point intervals there are 6 relative positions (see Proposition 3), presented in Figure 8. Figure 9 shows the graph of the cover relation of $\trianglerighteq$ between these six relative positions.

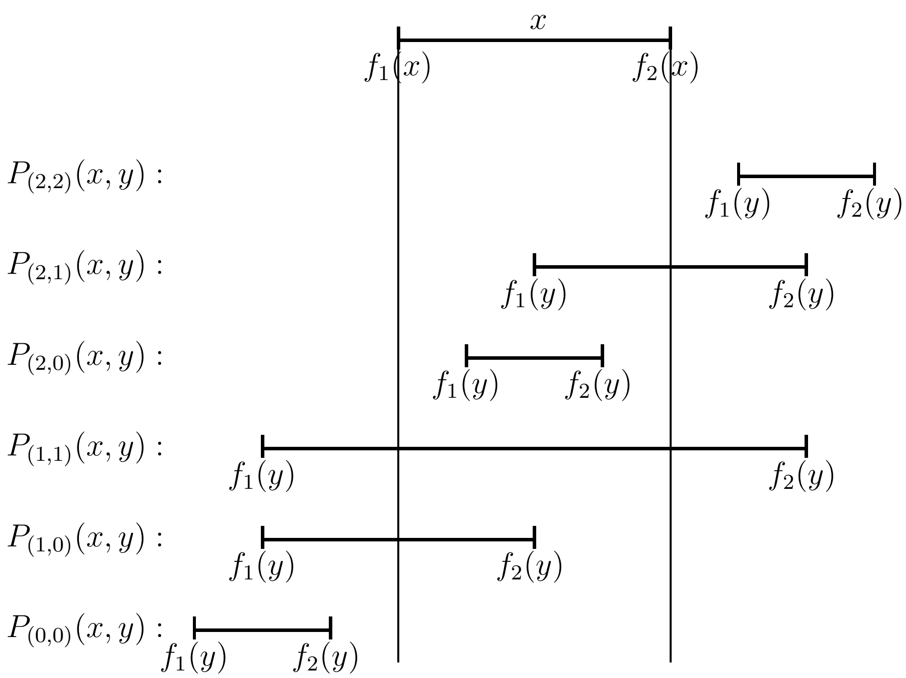


Fig. 8. Relative positions of 2-point intervals

From these six relative positions four P_φ satisfy our axiomatisation (see Proposition 8): $P_{(0,0)}, P_{(1,0)}, P_{(1,1)}$ and $P_{(2,0)}$. These ones correspond to three different well known preference structures: interval orders, weak orders and bi-linear orders.

Weak orders are very commonly used structures. Their caracterisation in term of necessary and sufficient properties of preference and indifference relations is given in Definition 8. Their classical numerical representation makes use of simple numbers: $P \cup I$ on A is a *weak order* if and only if there exists a real-valued function f defined on A such that $\forall x, y \in A,\ xPy \iff f(x) > f(y)$. Their difference from linear orders (total orders) comes from the fact that weak orders may have

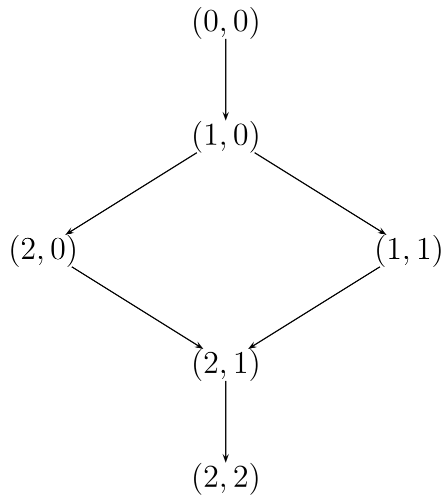


Fig. 9. Graph of the cover relation of $\trianglerighteq$ for 2-point intervals

equivalence classes (two different objects may be considered as indifferent) which is forbidden in the case of linear orders. Note that in our framework where each object of A is represented by an n-point interval, the characterisation of weak orders is as in the following: $\forall \varphi$, ($\forall A, P_\varphi \cup I_\varphi$ on A is a *weak order*) if and only if $\exists i \in \{1, \ldots, n\},\ Cp_\varphi = \{(i,i)\}$ (see Corollary 1). This result shows that when 2-point intervals are used, there are two different comparison rules providing a weak order, the corresponding component sets being $Cp_{(1,1)} = \{1,1\}$ and $Cp_{(2,0)} = \{2,2\}$. The first one consists in comparing the minimum values of objects; the second one the maximum values of objects.

Bi-weak orders are also known structures. They are defined as the intersection of two weak orders and are equivalent to bilinear orders (the interested reader may find more details in [11]). Their classical numerical characterisation is the following : $P \cup I$ on A is a *bi-weak order* if and only if there exist two real-valued functions $f_1$and f_2 defined on A such that

$$\forall x, y \in A,\ xPy \Longleftrightarrow \begin{cases} f_1(x) > f_1(y) \\ f_2(x) > f_2(y) \end{cases}$$

The reader will note that in such a definition the two functions f_1 and f_2 do not necessary represent an interval since they are not ordered (we do not know their relative position). Such an ambiguity can be easily resolved thanks to an old theorem of Dushnik and Miller ([9]). We present in the following the interval characterisation of such structures, an interested reader may find more details in [19].

Theorem 2 *[9] A relation $P \cup I$ on a finite set A is a biweak order if and only if* *there exist two real-valued functions f_1 and f_2 on A such that*

$$\begin{cases} \forall x, y \in A,\ xPy \Longleftrightarrow \begin{cases} f_2(x) > f_2(y), \\ f_1(x) > f_1(y), \end{cases} \\ \forall x,\ f_2(x) \geq f_1(x). \end{cases}$$

This comparison rule is the one represented by $Cp_{(1,0)} = \{(1,1),(2,2)\}$. It means that when 2 point-intervals are used, the object x is preferred to object y if and only if its minimum value is greater than the minimum value of y **and** its maximum value is greater than the maximum value of y. The following result generalises the characterisation of bi-weak orders in the case of n-point intervals (such a result will be useful for the following section).

Proposition 15 *Let P_φ and I_φ be respectively the preference and the indifference relation obtained by applying a simple decision rule as described in definition 6. Let Cp_φ be the component set associated to the decision rule. $P_\varphi \cup I_\varphi$ is a bi-weak order if and only if $\exists i, j \in \{1, \ldots, n\},\ Cp_\varphi = \{(i,i),(j,j)\}$*

Proof: obvious.

Proposition 16 *Let m be the number of different Cp_φ characterising a bi-weak order as in Proposition 15 when n-point intervals are used. Then*

$$m = \binom{n}{2} \tag{22}$$

Proof: obvious.

Hence, when 2-point intervals are used in our framework, the only comparison rule providing necessarily a bi-weak order is $Cp_\varphi = \{(1,1),(2,2)\}$.

Interval orders were introduced in preference modelling in order to have a representation in presence of a threshold: object x is preferred to object y if and only if the evaluation of x is greater than the evaluation of y plus a threshold. The introduction of such thresholds violates transitivity of the indifference relation. The characterisation of interval orders by necessary and sufficient properties is given in Definition 10. We present here their numerical representation: $P \cup I$ on A is an *interval order* if there exists two real-valued functions f_1 and f_2, defined on A such that $\begin{cases} \forall x, y \in A,\ xPy \Longleftrightarrow f_1(x) > f_2(y) \\ \forall x \in A,\ f_2(x) > f_1(x) \end{cases}$

We showed in Section 4 that $\forall \varphi, (\forall A, P_\varphi \cup I_\varphi$ on A is an *interval order*) if and only if $|Cp_\varphi| = 1$ (see Proposition 13). There are three comparison rules satisfying this affirmation: $Cp_{(1,1)} = \{1,1\}, Cp_{(2,0)} = \{2,2\}$ and $Cp_{(0,0)} = \{2,1\}$. The first two ones are weak orders which are special cases of interval orders (interval orders with a threshold equal to 0) and the last one is a proper interval order, $i.e.$ if this comparison rule is used one can always find a set of objects which is not a weak order but an interval order.

Summarising, when 2-point intervals are used, it is possible to define four different comparison rules satisfying our axioms and from these four rules three different preference structures may be obtained which are weak orders, bi-weak orders and interval orders (see Table 2).

Preference Structure	$\langle P_\varphi, I_\varphi \rangle$ interval representation
Interval Orders	$Cp_{(0,0)} = \{(1,2)\}$
Weak Orders	$Cp_{(2,0)} = \{(2,2)\}$ $Cp_{(1,1)} = \{(1,1)\}$
Bi-Weak Orders	$Cp_{(1,0)} = \{(1,1),(2,2)\}$

Table 2
Preference structures with 2-point interval representation

6 3-point intervals

In this section we present a complete analysis of 3-point intervals within our framework (a brief presentation of these results can be found in [20]). With 3-point intervals there are 20 relative positions (see Proposition3) which are presented in two separated figures (figures 10, 11). The separation is done in a way that the kth relative position of the figure 11 corresponds to the converse of the kth relative position of the figure 10 (when the two compared 3-point intervals do not have any point in common) and each relative position is stronger than or incomparable with the relative positions which are presented above it. Figure 4 in Section 2 presents the graph of the cover relation of $\trianglerighteq$ between these twenty relative positions.

From these twenty relative positions only fifteen P_φ satisfy our axiomatisation (see Proposition 8): $P_{(0,0,0)}$, $P_{(1,0,0)}$, $P_{(1,1,0)}$, $P_{(2,0,0)}$, $P_{(1,1,1)}$, $P_{(2,1,0)}$, $P_{(2,2,0)}$, $P_{(2,1,1)}$, $P_{(2,2,1)}$, $P_{(2,2,2)}$, $P_{(3,0,0)}$, $P_{(3,1,0)}$, $P_{(3,2,0)}$, $P_{(3,1,1)}$ and $P_{(3,3,0)}$. These ones correspond to seven different preference structures: weak orders, bi-weak orders, three-weak orders, interval orders, split interval orders, triangle orders and structures with intransitive strict preference.

As in the previous section, we will analyse one by one these seven structures: we will introduce first of all their definition and their classical numerical represen-

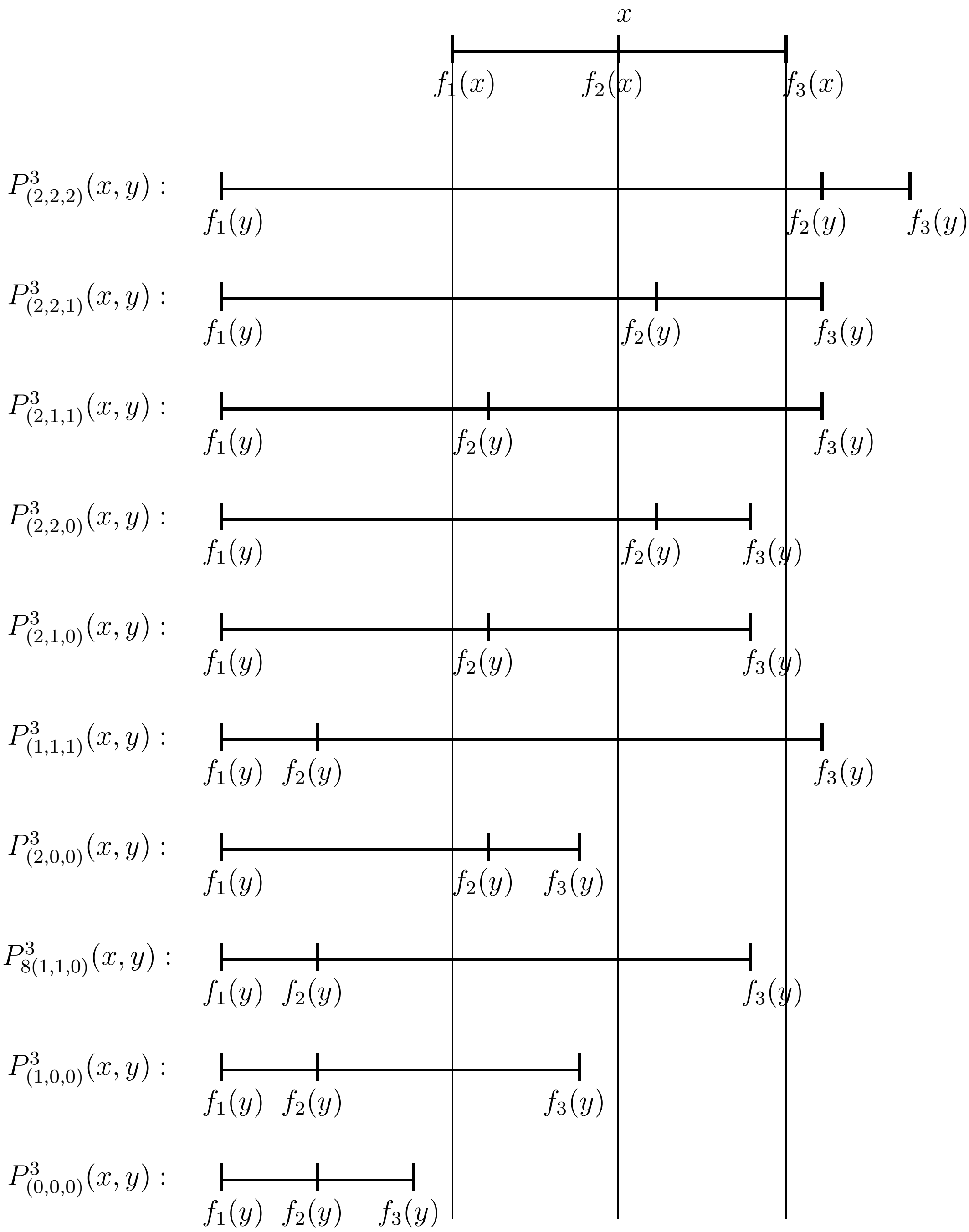


Fig. 10. Relative positions of 3-points intervals: part 1

tation, then show their characterization within our framework and conclude with some remarks.

The definition, the classical numerical representation and the characterisation in our framework of weak orders, bi-weak orders and interval orders are already given in Section 5.

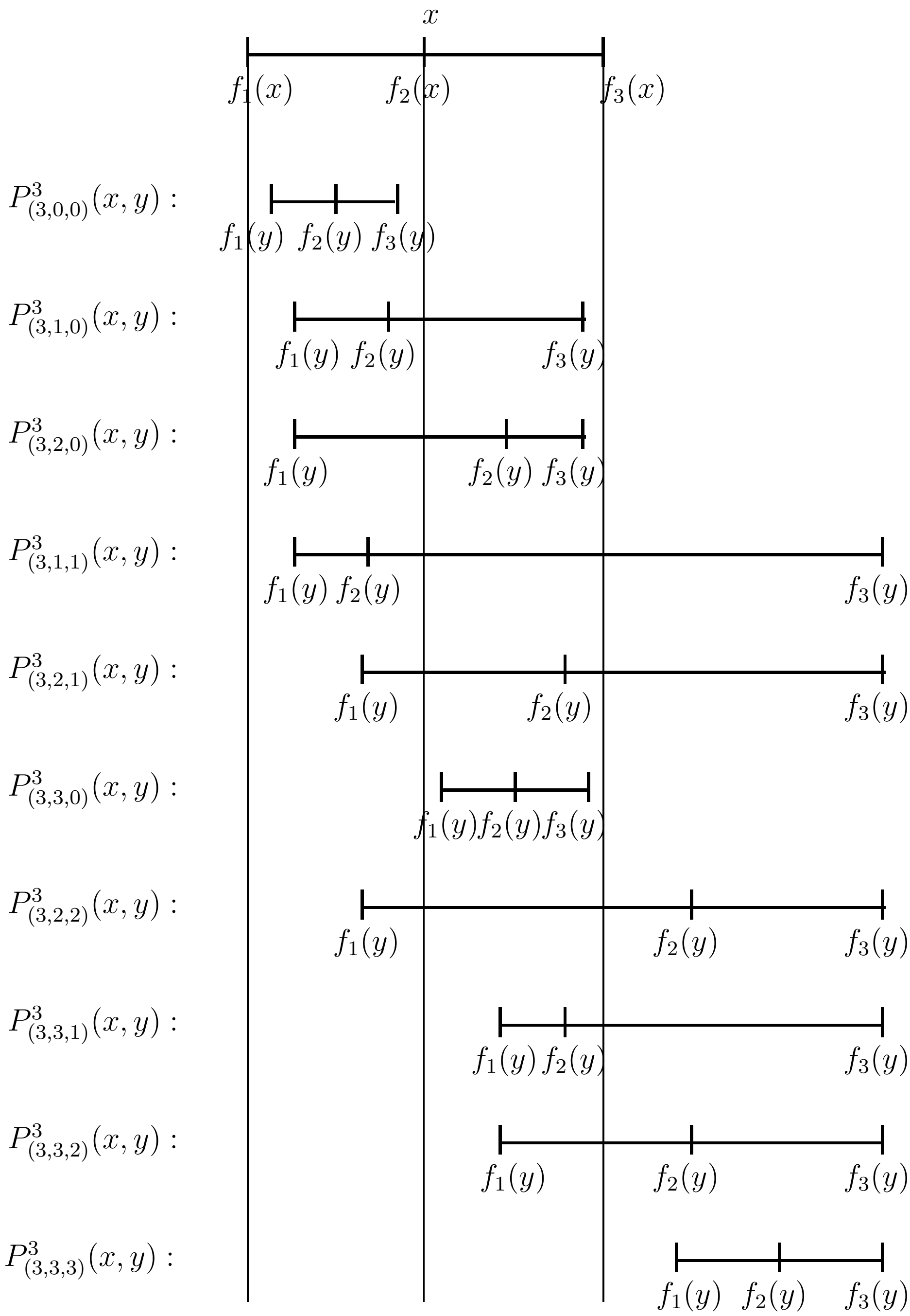


Fig. 11. Relative positions of 3-points intervals: part 2

6.1 Weak, Bi-weak and Interval Orders

When 3-point intervals are used, three different comparison rules provide weak orders, these are given by $Cp_{(3,3,0)} = \{(3,3)\}$, $Cp_{(3,1,1)} = \{(2,2)\}$ and $Cp_{(2,2,2)} = \{(1,1)\}$. They consist respectively in comparing objects with respect to their maximum (resp. their median, their minimum) values.

Bi-weak orders are represented by three comparison rules when 3-point intervals are used: $Cp_{(3,1,0)} = \{(2,2),(3,3)\}$, $Cp_{(2,1,1)} = \{(1,1),(2,2)\}$ and $Cp_{(2,2,0)} = \{(1,1),(3,3)\}$. For instance the first one consists in saying that object x is prefered to object y if and only if the median value of x is greater than the median value of y and the maximum value of x is greater than the maximum value of y.

When objects are presented by three ordered points three comparison rules provide interval orders (except the ones which provide weak orders which are special cases of interval orders): $Cp_{(0,0,0)} = \{(3,1)\}$, $Cp_{(3,0,0)} = \{(3,2)\}$ and $Cp_{(1,1,1)} = \{(2,1)\}$. It is easy to notice that all comparisons of type "object x is prefered to object y if and only if the ith evaluation of x is greater than the jth evaluation of y (j being greater than i) " ($i.e.$, comparing the minimum value of x with the medium or maximum value of y or comparing the medium value of x with the maximum value of y) produce an interval order.

6.2 3-Weak Orders

Three-weak orders generalise bi-weak orders (for more details see [18]). They are defined as the intersection of three weak orders. Their classical numeric representation makes use of three functions as follows: $P \cup I$ on A is a *3-weak order* if there exist three real-valued functions f_1, f_2 and f_3 defined on A such that

$$\left\{ \forall x, y \in A,\ xPy \Longleftrightarrow \begin{cases} f_1(x) > f_1(y), \\ f_2(x) > f_2(y), \\ f_3(x) > f_3(y). \end{cases} \right. \tag{23}$$

As in the case of bi-weak orders, such a representation does not necessary results to an interval since the order between $f_1(x)$, $f_2(x)$ and $f_3(x)$ is not fixed. Naturally, one can find easily an interval representation for such structures (this can be seen as a generalisation of the theorem of Dushnik and Miller [9]):

Proposition 17 *$P \cup I$ on a finite set A is a three-week order if and only if there exist three real-valued functions f_1, f_2 and f_3 on A such that*

$$\begin{cases} \forall x, y \in A,\ xPy \Longleftrightarrow \begin{cases} f_3(x) > f_3(y), \\ f_2(x) > f_2(y), \\ f_1(x) > f_1(y), \end{cases} \\ \forall x,\ f_3(x) \geq f_2(x) \geq f_1(x). \end{cases} \tag{24}$$

Proof.
- ($24 \Longrightarrow 23$): Obvious.
- ($23 \Longrightarrow 24$): Supposing that there exist 3 real-valued functions f_i ($i \in \{1,2,3\}$), defined on A, such that, $\forall x, y \in A,\ xPy \iff \forall i \in \{1,2,3\},\ f_i(x) > f_i(y)$, we will show that one can always find 3 real-valued functions f'_i ($i \in \{1,2,3\}$) defined on A satisfying (24).

We define a constant M such that $M = \max_i \max_{x \in A} |f_i(x)|$ (A is a finite set) and we define $\forall x \in A$, $f'_i(x) = f_i(x) + i \times (2M)$. It is easy to see that $f_i(x) > f_i(y) \iff f'_i(x) > f'_i(y)$.

For the second inequality of the proposition, we have $f'_{i+1}(x) - f'_i(x) = f_{i+1}(x) - f_i(x) + 2|M|$ and $2|M| \geq f_{i+1}(x) - f_i(x)$ by definition. Hence we obtain $\forall x,\ \forall i \in \{1,2\},\ f'_{i+1}(x) \geq f'_i(x)$. ■

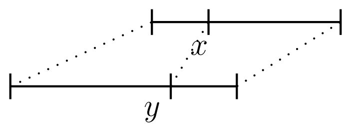
Hence when each object is represented by three ordered points, there is one comparison rule providing a 3-weak order : $Cp_{(2,1,0)} = \{(1,1),(2,2),(3,3)\}$.

The following result generalises the characterisation of 3-weak orders in the case of n-point intervals.

Proposition 18 *Let P_φ and I_φ be respectively the preference and the indifference relation obtained by applying a simple decision rule as described in definition 6. Let Cp_φ be the component set associated to the decision rule. $P_\varphi \cup I_\varphi$ is a three-weak order) if and only if $\exists i, j, k \in \{1, \ldots, n\},\ Cp_\varphi = \{(i,i),(j,j),(k,k)\}$*

Proof: obvious.

Figure 12 illustrates the presentation of a 3-weak order.



xPy
Three-weak order

Fig. 12. d-weak order

Proposition 19 *Let m be the number of different Cp_φ characterizing a 3-weak order as in Proposition 18 when n-point intervals are used, then*

$$m = \binom{n}{3} \tag{25}$$

Proof. Obvious. ■

6.3 Triangle Orders

Triangle orders are defined as the intersection of a weak order and an interval order. Their classical numeric representation is as in the following: $P \cup I$ on a finite set A is a triangle order if and only if there exist 2 real-valued functions f_i ($i \in \{1,2\}$) defined on A and one nonnegative function q on the set A such that

$$\forall x, y \in A,\ xPy \Longleftrightarrow \begin{cases} f_1(x) > f_1(y), \\ f_2(x) > f_2(y) + q(y). \end{cases} \tag{26}$$

Using a similar approach to the case of 3-weak orders, one can propose easily an interval representation for triangle orders.

Proposition 20 *$P \cup I$ on a finite set A is a triangle order if and only if there exist 3 real-valued functions f_i ($i \in \{1,2,3\}$) defined on A, such that*

$$\begin{cases} \forall x, y \in A,\ xPy \Longleftrightarrow \begin{cases} f_1(x) > f_1(y), \\ f_2(x) > f_3(y), \end{cases} \\ \forall x,\ \forall i \in \{1,2\},\ f_{i+1}(x) \geq f_i(x). \end{cases} \tag{27}$$

Proof.
-(27 $\Longrightarrow$ 26): Suppose that there exist 3 real-valued functions f_i ($i \in \{1,2,3\}$) defined on A satisfying the assertion 27. One can always define 2 real-valued functions f'_i ($i \in \{1,2\}$) and one nonnegative function q on the set A such that $\forall x \in A$, $f'_1(x) = f_1(x),\ f'_2(x) = f_2(x)$ and $q(x) = f_3(x) - f_2(x)$. These functions satisfy the assertion 26.
-(26 $\Longrightarrow$ 27): Suppose that there exist 2 real-valued functions f_i ($i \in \{1,2\}$) and one nonnegative function q on the set A satisfying the assertion 26. Let us define three real-valued functions f'_i ($i \in \{1,2,3\}$) defined on A, such that $\forall x$,

- $f'_i(x) = f_i(x) + i|M|,\ \forall i \in \{1,2\}$,

- $f'_3(x) = f_2(x) + 2|M| + q(x)$

where $M = 2 \times \max_i \max_x(f_i(x))$. Hence, $\forall x, y$, ($f_1(x) > f_1(y)$ and $f_2(x) > f_2(y) + q(y)$) is equivalent to ($f'_1(x) > f'_1(y)$ and $f'_2(x) > f'_3(y)$).

The last inequality of 27 is also satisfied since

- $\forall x,\ f'_2(x) - f'_1(x) = f_2(x) - f_1(x) + |M|$ and by definition of M, $\forall x,\ f_2(x) - f_1(x) \leq |M|$;

-$\forall x,\ f'_3(x) - f'_2(x) = q(x)$ and q is a nonnegative function. ■

Such a representation is an interval one since the points are ordered, moreover it is a geometrical one : placing the minimum values of objects on one line (real axis) and the median and the maximum values on another one, each object gets a triangle representation as in Figure 13. When the orientation of these two lines are from left to right a triangle order consists in saying that object x is preferred to object y if and only if its associated triangle is completely on the right of the triangle of y. Figure 13 illustrates such a preference relation.

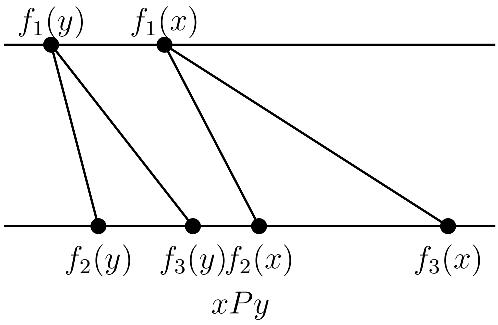


Fig. 13. Triangle Order

Remark that our proposition provides triangles oriented to the left. However, other representations where triangles are oriented to the right can provide identical ordered sets.

Proposition 21 *$P \cup I$ on a finite set A is a triangle order if and only if there exist 3 real-valued functions f_i ($i \in \{1, 2, 3\}$) defined on A, such that*

$$\begin{cases} \forall x, y \in A,\ xPy \Longleftrightarrow \begin{cases} f_3(x) > f_3(y), \\ f_1(x) > f_2(y), \end{cases} \\ \forall x,\ \forall i \in \{1, 2\},\ f_{i+1}(x) \geq f_i(x). \end{cases} \tag{28}$$

Proof. Similar to the proof of Proposition 20. ■

Note that even if the comparison $Cp_\varphi = \{2, 2\}$ provides a weak order and the comparison $Cp_\varphi = \{1, 3\}$ provides an interval order, their intersection gives an interval order (note that interval orders are special cases of triangle orders) which corresponds to the comparison rule $Cp_\varphi = \{1, 3\}$ since $\forall x, y,\ (f_3(y) < f_1(x)) \implies$

$(f_2(y) < f_2(x))$. This special case shows that one can not have $Cp_\varphi = \{(i,i),(j,k)\}$, with $j > i > k$ since the couple (i,i) is redundant with the couple (j,k).

Propositions 20 and 21 show that when 3-point intervals are used two comparison rules provide triangle orders: $Cp_{1,1,0} = \{(2,1),(3,3)\}$ and $Cp_{(2,0,0)} = \{(1,1),(3,2)\}$. Such representations can be easily generalized in the case of n-point intervals:

Proposition 22 *Let P_φ and I_φ be respectively the preference and the indifference relation obtained by applying a simple decision rule as described in definition 6. Let Cp_φ be the component set associated to the decision rule. $P_\varphi \cup I_\varphi$ is a triangle order) if and only if $\exists (i,j,k),\ Cp_\varphi = \{(i,i),(j,k)\},$ where $j > k > i$ or $i > j > k$.*

Proof. Obvious. ■

Proposition 23 *Let m be the number of different Cp_φ characterizing a triangle order as in Proposition 22 when n-point intervals are used, then*

$$m = \frac{n(n^2 - 3n + 2)}{3} \tag{29}$$

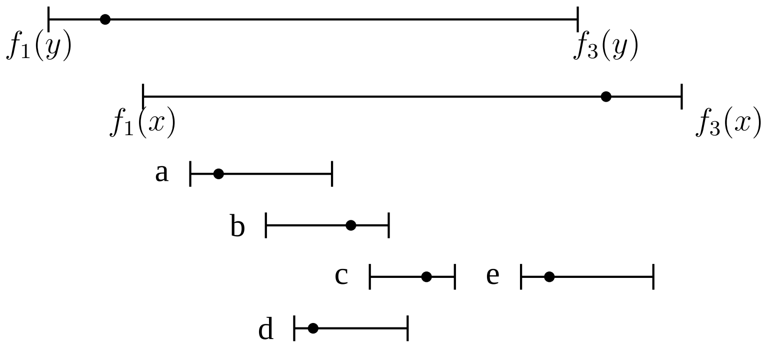
Proof. Recall that a triangle order is an intersection of a weak order and an interval order. Let us fix to i the point establishing the weak order part as in Proposition 22. Then the points related to the interval order part ($(j,k) \in Cp_\varphi$) can be either to the right of this point (there are $n - i$ points to the right of i), in this case we have $\frac{(n-i)(n-i-1)}{2}$ possibilities for j and k (see Proposition 14) or to the left of i (there are $i - 1$ points to the left of i) and in this case we have $\frac{(i-1)(i-2)}{2}$ possibilities for j and k. Summing this value for all i we get $\sum_{i=1}^{n}(\frac{(n-i)(n-i-1)}{2}) + (\frac{(i-1)(i-2)}{2})$. This is equal to $\frac{1}{2}\sum_{i=1}^{n}(n^2 - n + 2) - (2n+2)i + 2i^2$. Using $\sum_{i=1}^{n}(i^2) = \frac{n(n+1)(2n+1)}{6}$, we obtain $\frac{n(n^2-3n+2)}{3}$. ■

6.4 Split Interval Orders

Split interval orders are especially studied in mathematics ([13], [17], [30]) and allow the representation of sophisticated preferences. Their numerical representation is the following: $P \cup I$ is a split interval order if and only if there exist three real-valued functions f_1, f_2 and f_3 defined on A such that

$$\begin{cases} \forall x, y \in A, xPy \Longleftrightarrow \begin{cases} f_1(x) > f_2(y), \\ f_2(x) > f_3(y), \end{cases} \\ \forall x \in A,\ f_3(x) \geq f_2(x) \geq f_1(x) \end{cases} \tag{30}$$

Some instances of the preference and indifference relations of a split interval order are illustrated in figure 14. This example is proposed by Fishburn in his paper [11].



xPy, $aPbPcPe$, $dPcPe$ and I otherwise

Fig. 14. Split Interval Order

Hence when 3-point intervals are used there is one comparison rule satisfying formula 30: $Cp_\varphi = \{(3,2),(2,1)\}$ associated to the preference $P_{(1,0,0)}$. More generally, when n-point intervals are used, we get the following characterisation.

Proposition 24 *Let P_φ and I_φ be respectively the preference and the indifference relation obtained by applying a simple decision rule as described in definition 6. Let Cp_φ be the component set associated to the decision rule. $P_\varphi \cup I_\varphi$ is a split interval order if and only if $\exists(i,j,k), Cp_\varphi = \{(i,j),(j,k)\},$ where $i > j > k$.*

Proof. Obvious. ■

Proposition 25 *Let m be the number of different Cp_φ characterising a triangle order as in Proposition 24 when n-point intervals are used, then*

$$m = \frac{n(n-1)(n-2)}{6} \tag{31}$$

Proof. Once again we fix the point i of Proposition 24. Then there are $\sum_{t=1}^{n-1-i} t$ possibilities for j and k. Summing for all the positions of i we get $\sum_{i=1}^{n-2}\sum_{t=1}^{n-1-i} t$. This is equal to $\sum_{i=1}^{n-2}(i(n-i-1))$ which gives $\frac{(n-1)(n-2)(n-1)}{2} - \frac{(n-2)(n-1)(2(n-2)+1)}{6}$. Hence we obtain $= \frac{n(n-1)(n-2)}{6}$. ■

We have analysed for the moment thirteen comparison rules among the fifteen allowed in our framework; the two remaining ones are $Cp_\varphi = \{(3,3),(2,1)\}$ and $Cp_\varphi = \{(1,1),(2,3)\}$. Such rules provide intransitive preference relations (see Proposition 9). These rules seem to be constructed as the intersection of two rules, the first one providing a weak order ($(3,3) \in Cp_\varphi$ or $(1,1) \in Cp$), and the second one ($(2,1) \in Cp_\varphi$ or $(2,3) \in Cp$) providing the non transitivity of the preference relation. Remark that the second rule can not be used alone within our framework since it violates the asymmetry of the preference relation. Even if preference structures having a non transitive strict preference seem marginal, they are used in some special domains (for instance in biology when cellules are compared or in chemistry when the connection between molecules are analysed). The comparison rule consisting in associating a circle representation to each object and saying that an object is preferred to another one if and only if the circle representing the first object is completely to the right of the circle representing the second one (circles may have different diameters) provides structures with non transitive preference relation ([1], [2]). More generally, when n-point intervals are used, the comparison rules similar to these two ones have the following component set: $Cp_\varphi = \{(i,i)(k,l)\}$ with $i > k > l$ or $i < k < l$. The number of comparison rules having such component set when n-point intervals are used is $\sum_{i=1}^{n}(\frac{(n-i)(n-i-1)}{2}) + (\frac{(i-1)(i-2)}{2})$ which is equivalent to $\frac{n(n^2-3n+2)}{3}$ (the computation of this number is similar to the case of triangle orders, see proof of Proposition 23).

Table 3 summarises the different comparison rules that can be applied when 3-point intervals are used.

Some preference structures are special cases of other ones, for instance weak orders may be seen as interval orders with a threshold equal to 0. Under such a perspective each weak order can be seen as an interval order but not the contrary. Thus, we can consider an inclusion relation between different structures. In Figure 15 each box represents one preference structure, these boxes are partially ordered by inclusion from top to bottom according to the arrows. Such inclusions are either obvious or known from the literature ([5], [11]). However, a complete study of this relation is beyond the scope of this paper and will be left for future work.

7 Conclusion

Handling imprecise, inaccurate and uncertain information is a common problem both in human reasoning and in automatic devices aiming at supporting decision processes and more generally when information is manipulated. One way to take into account such type of information is under form of intervals who are expected to

Preference Structure	$\langle P_\varphi, I_\varphi \rangle$ interval representation
Weak Orders	$Cp_{(3,3,0)} = \{(3,3)\}$ $Cp_{(3,1,1)} = \{(2,2)\}$ $Cp_{(2,2,2)} = \{(1,1)\}$
Bi-weak Orders	$Cp_{(3,1,0)} = \{(2,2),(3,3)\}$ $Cp_{(2,1,1)} = \{(1,1),(2,2)\}$ $Cp_{(2,2,0)} = \{(1,1),(3,3)\}$
Three-Weak Orders	$Cp_{(2,1,0)} = \{(1,1),(2,2),(3,3)\}$
Interval Orders	$Cp_{(0,0,0)} = \{(3,1)\}$ $Cp_{(3,0,0)} = \{(3,2)\}$ $Cp_{(1,1,1)} = \{(2,1)\}$
Split Interval Orders	$Cp_{(1,0,0)} = \{(3,2),(2,1)\}$
Triangle Orders	$Cp_{(1,1,0)} = \{(2,1),(3,3)\}$ $Cp_{(2,0,0)} = \{(1,1),(3,2)\}$
Structures with nontransitive preference	$Cp_{(3,2,0)} = \{(3,3),(1,2)\}$ $Cp_{(2,2,1)} = \{(1,1),(2,3)\}$

Table 3
Preference structures with 3-point interval representation

represent the lower and upper bound of the possible values of a variable, a time or space interval, a gap, an error. Intervals allow also to capture a limited discrimination power such that in order to distinguish two objects we need to use a threshold (when measuring a certain feature).

Although the concept of "interval" is naturally associated with an interval of the reals, defined by the two extreme values, there exist situations where more than two values are associated with the same object. For instance consider a variable where we know its lowest possible value, its greatest possible value, but also the one more likely to occur (3 values). Or consider the case where the two extremes of the interval are imprecisely known: we have a lower and an upper bound for the minimum value and a lower and an upper bound for the maximum (4 values). In order to study systematically the problem of how to compare intervals we first generalize the concept itself of interval as a vector of n ordered real numbers, what we call a "n-points interval".

In this paper we propose a general framework about intervals comparison aiming

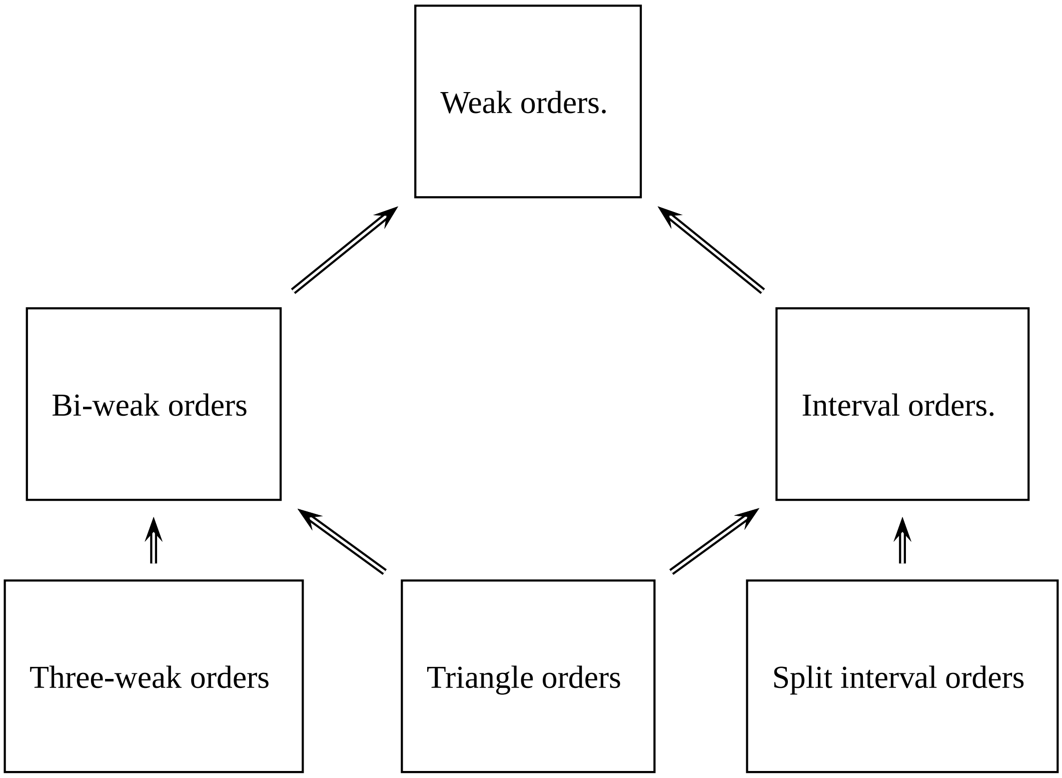


Fig. 15. Inclusions between structures obtained by comparison rules on 2 and 3-point intervals

at producing a classic preference model. The problem has two aspects.
1. On the one hand we want to know all different ways to compare n-points intervals in order to obtain a $\langle P, I\rangle$ preference structure (P being asymmetric, I being symmetric, and both forming a partition of $A \times A$).
2. On the other hand we want know, given a set of preference statements of an agent, to what type of preference structure do these correspond and in case it turns out that intervals have to be used in order to obtain a numerical representation, what type of intervals should be considered?

In the paper we first consider the problem of coding the comparison information in a compact way. It turns out that all the information we need is the "relative position" of two intervals (intuitively showing how "far" is the actual position of the two intervals w.r.t. to complete disjunction: one interval completely on the right of the other). Such a difference can be captured by a binary relation "at least as strong as" providing a partial order among all possible relation positions with complete disjunction as the maximal element. This binary relation defines a complete and distributive lattice on the set of all relative positions. We also show that it is possible to code the information about relative positions in a compact way through the "component set" associated with each relative position (where all redundant information is discarded).

Having defined the tools allowing to conduct a study of intervals comparison we impose the necessary requirements in order to identify within the lattice of relative positions all possible relations establishing $\langle P, I \rangle$ preference structures. These correspond to sub-lattices which have a unique lower bound (the upper bound being always the strongest position: complete disjunction). The particular structure of the lattice is such that the relation P corresponds to the lower bound of the sub-lattice, the inverse relation P^{-1} corresponds to the upper bound of the symmetric complement of the sublattice, I being the rest.

With such definitions it has been possible to conduct an exhaustive study of 2-points and 3-points intervals comparison, summarized in Tables 2 and 3. It turns out that the comparison of 2-points intervals allow to establish 3 different preference structures: 2 types of weak orders, a bi-weak order and an interval order. The use of 3-points intervals allows to establish 7 types of preference structures: 3 types of weak orders, 3 types of bi-weak orders, 3 types of interval orders, a 3-weak order, a split-interval order, a triangle order and 2 types of intransitive preference structures. In the paper we show the equivalences between the usual definitions of such preferences structures, their numerical representation and the properties they characterize them. Such results confirm the descriptive power of our framework which allows to provide a complete characterization for preference structures until today never studied, in common with other structures well known in the literature (for instance we are able to interpret within the same framework triangle orders and weak orders).

The paper opens the way to several research directions. Obviously the major issue is the to generalize the findings for generic n-points intervals identifying the regularities and invariants within our framework. Another research direction consists in associating to the n-points intervals comparison preference structures with more than two relations of the type $\langle P_1 \cdots P_m, I \rangle$ where P_i are asymmetric relation and I is symmetric and they all form a partition of $A \times A$. A more specific research direction concerns the study of 3-points intervals and more precisely the completion of Figure 15. It is worth to note that when using 3-points intervals we start getting structures whose numerical representation needs possibly (triangle orders) or necessary (intransitive structures) more complex geometric figures (such as triangles or circles).

We consider that the general framework we introduced in this paper is sufficiently wide to allow a systematic study of any type of intervals comparison, a major problem in different areas including decision theory, computer science and artificial intelligence and beyond.

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