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On the continuous extension of a four-valued logic and its interpretation in possibility theory

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Abstract Four-valued logics and possibility theory are two different tools used to deal with uncertainty, tools largely applied in decision theory and preference modelling. Their semantics and their properties are different. Nevertheless there exist strong relationships between them. In this paper, we explore the interpretation of a continuous extension of a four-valued logic as a necessity degree (in possibility theory). It turns out that, in order to take full advantage of the four values, we have to consider "sub-normalised" necessity measures. Under such a hypothesis four-valued logics become the natural logical frame for such an approach.

Keywords: four-valued logic, possibility theory, preference modelling, decision making

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1 Introduction

A common way to formalise a problem is to define it in terms of logical sentences. However, classic logic is not always suitable to formalise real life problem situations and common sense reasoning since it is unable to handle uncertainty as well as situations characterised by incomplete and/or inconsistent information. In decision aiding such situations are regularly encountered and indeed classic logic has often been criticised of being the language used as formalism of decision support models (see Dubois and Prade, 1988, 2001; Roy, 1989; Tsoukiàs and Vincke, 1995; Perny and Roubens, 1998). Under such a perspective we study the possibility to extend a well-known four valued logic (see Belnap, 1976, 1977) in situations where it is possible to make continuous valuations on the presence of truth.

The four values (t, f, k, u) introduced by Belnap have a clear epistemic nature. Given a proposition α, four situations may appear: an agent can receive the information that α holds or that α does not hold or he can get both kind of information or none of them. We capture this idea through the following situations (or truth values, the symbol Δ representing the presence of truth):
- true (t): there is evidence that α is true ($\Delta\alpha$) and there is no evidence that α is false ($\neg\Delta\neg\alpha$)
- false (f): there is no evidence that α is true ($\neg\Delta\alpha$) and there is evidence that α is false ($\Delta\neg\alpha$)
- contradictory (k): there is evidence that α is true ($\Delta\alpha$) and there is evidence that α is false ($\Delta\neg\alpha$)
- unknown (u): there is no evidence that α is true ($\neg\Delta\alpha$) and there is no evidence that α is false ($\neg\Delta\neg\alpha$)

However, the sources of uncertainty are not limited to pure incomplete and/or contradictory situations. The evidences "for" or "against" a certain sentence might not be necessarily of a crisp nature. In this case, we can introduce "positive reasons" and "negative reasons" supporting or not a certain sentence (see Tsoukiàs et al., 2002). By considering a continuous valuation of such reasons, we can introduce a continuous extension of any four-valued logic. This continuous extension may help us to deal with uncertainty due to doubts about the validity of the knowledge; imprecision due to the difficulty in the wording of the situation (insufficient knowledge in numerical values, vagueness of the natural language terms); incompleteness due to the absence of information or the partial knowledge; apparent inconsistency due to contradictory statements. Such situations are all the more relevant in decision aiding and preference modelling.

More precisely in this paper, we consider two variants of Belnap's logic: DDT logic (Tsoukiàs, 2002) which extends Belnap's logic to a first order language and is a Boolean algebra and its continuous extension suggested in Perny and Tsoukiàs (1998).

The aim of this paper is to verify whether it is possible to associate to the DDT logic an uncertainty distribution, possibly of the possibility/necessity type and if so, under which conditions. Section 2 introduces the basic concepts of the four-valued logic and its continuous extension through the concept of positive and negative membership. In Section 3 we try to establish a first relation between four-valued logic and possibility theory. Some related problems are discussed. In Section 4 we suggest the use of "sub-normalised" necessity distributions and we show why four-valued logic can be considered a language for associating such type of uncertainty distributions. In the concluding section further research directions are discussed.

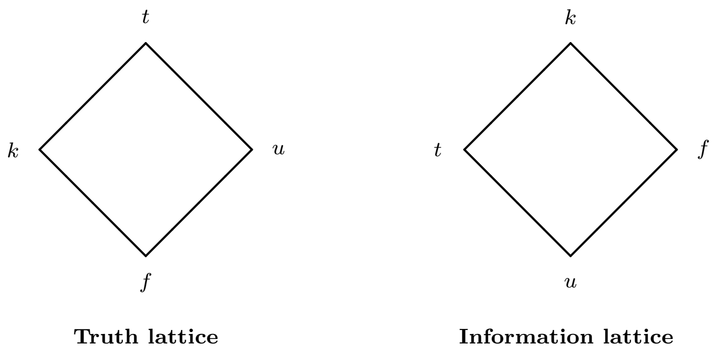

Fig. 2.1. The two lattices suggested by Belnap

α	$\sim\alpha$	$\neg\alpha$	$\not\sim\alpha$
t	f	f	k
k	u	k	t
u	k	u	f
f	t	t	u

Table 2.1. The truth tables of negations and complement

2 Four-valued logic and its continuous extension

2.1 Syntax

Belnap's original proposition (see Belnap, 1976, 1977) aimed to capture situations where hesitation in establishing the truth of a sentence could be associated either to ignorance (poor information) or to contradiction (excess of information). He suggested the use of four truth values forming a bi-lattice (see 2.1). It has been shown that such a bi-lattice is the smallest nontrivial interlaced bi-lattice (see Ginsberg, 1988; Fitting, 1991).

DDT logic (for details see Tsoukiàs, 2002) extended Belnap's logic in a first order language endowed with a weak negation ($\not\sim$). DDT is also a boolean algebra. This logic allows a distinction between the strong negation ($\neg$) and the complementation ($\sim$) as we can remark in table 2.1. It is easy to see that $\sim\alpha \equiv \neg \not\sim \neg \not\sim \alpha$.

In table 2.2, we introduce the truth tables of some basic binary operators: two connectives ($\vee$ ("or"), $\wedge$ ("and")) and implication ($\rightarrow$). One can remark that the implication is defined as follows:

$$\alpha \rightarrow \beta \equiv \neg \not\sim \neg \not\sim \alpha \vee \beta$$

$\wedge$	t	k	u	f
t	t	k	u	f
k	k	k	f	f
u	u	f	u	f
f	f	f	f	f

$\vee$	t	k	u	f
t	t	t	t	t
k	t	k	t	k
u	t	t	u	u
f	t	k	u	f

$\rightarrow$	t	k	u	f
t	t	k	u	f
k	t	t	u	u
u	t	k	t	k
f	t	t	t	t

Table 2.2. The truth tables of conjunction, disjunction and implication

Besides ordinary four valued sentences, in DDT it is possible to formulate two valued sentences such as:
- $\Delta\alpha$ (there is presence of truth in α);
- $\Delta\neg\alpha$ (there is presence of truth in $\neg\alpha$);
- $\mathbf{T}\alpha$ (α is true);
- $\mathbf{K}\alpha$ (α is contradictory);
- $\mathbf{U}\alpha$ (α is unknown);
- $\mathbf{F}\alpha$ (α is false);

through the following formulas:
- $\Delta\alpha \ \equiv \ \not\sim (\alpha \ \wedge \ \not\sim \alpha)$
- $\mathbf{T}\alpha \ \equiv \ \Delta\alpha \ \wedge \ \neg\Delta\neg\alpha$.

2.1.1 Example 1. Why the above is a relevant language in decision aiding problems? Let us take the example of a Parliament which is preparing to vote for a new proposal (α) concerning an ethical issue. Members of the Parliament (MPs) can vote "for" or "against" this proposal or can "not vote". Suppose that the Parliament has the following rule for adopting laws concerning ethics: a "strong" majority has to vote "for" (more than 50%) and less than 1/3 can vote "against" (in order to defend minorities; the reader can see the Nice Treaty establishing the decision rules of the enlarged European Union for more complicated similar examples). This kind of voting can be captured by the four valued logic as in the following: Let us note
$N(\alpha)$: total number of MPs;
$V(\alpha)$: number of MPs voting for α
$V(\neg\alpha)$: number of MPs voting against α
$\Delta\alpha = 1$ iff $\frac{V(\alpha)}{N(\alpha)} \geq 0.51$
$\Delta\neg\alpha = 1$ iff $\frac{V(\neg\alpha)}{N(\alpha)} \geq 0.33$
Four different cases are presented in table 2.3. In the first two cases there is no hesitation since in the first one the proposition is clearly accepted , while in the second is clearly rejected. In the third case, the majority of MPs are for the acceptance of the proposal but at same time the number of MPs against α is remarkable too; the proposition will not be accepted, but is clear that we are facing a conflict, a contradictory case. Finally, in the fourth case, the votes for and against α are insufficient to make a decision which is expressed here with the unknown value. From a decision aiding

Case	$N(\alpha)$	$V(\alpha)$	$V(\neg\alpha)$	$\Delta\alpha$	$\Delta\neg\alpha$	Value
1	100	75	20	1	0	True
2	100	48	40	0	1	False
3	100	60	40	1	1	Contradictory
4	100	41	25	0	0	unknown

Table 2.3. The truth table of example 1

point of view is clear that the recommendation of an analyst towards a decision maker facing any of the above situations will be different. In the third case is necessary to work towards the opposants (perhaps negotiating in order to meet some of their claims), while in the fourth case is necessary to convince the "non voters" (perhaps strengthening the contents of the law).

2.2 Semantics

Consider a universe of discourse A and a predicate S of arity n. Consider an instance of S: $\langle x_1^S, \cdots x_n^S \rangle$. The basic idea in a four-valued logic is to let become independent the membership of such an instance to the models of S from the membership to the models of the negation of S (in the sense of not being necessarily complementary). In other terms we consider that the set of all possible instances A^n contains a subset of elements which are the models of S (denoted S^+) and a subset of elements which are the models of the negation of S (denoted S^-), but that such two subsets do not constitute a partition of A^n. The result is that A^n will be partitioned to four subsets:

$$S^t = S^+ \cap \sim S^- \quad S^k = S^+ \cap S^- \quad S^u = \sim S^+ \cap \sim S^- \quad S^f = \sim S^+ \cap S^-$$

where $\sim S^+$ ($\sim S^-$) is the complement of S^+ (S^-) and S^t, S^k, S^u, S^f, represent the true, contradictory, unknown and false extensions of the predicate S within the universe A^n. Hence $(\neg S)^+, (\neg S)^-, (\sim S)^+ and (\sim S)^-$ are defined as follows:

$$(\neg S)^+ = S^- \quad (\neg S)^- = (S^+) \quad (\sim S)^+ = \sim (S^+) \quad (\sim S)^- = \sim (S^-)$$

Finally, we obtain the following results:

$$S^t \cup S^k = S^+ \qquad S^t = (\neg S)^f = (\sim S)^f$$
$$S^f \cup S^k = S^- \qquad S^k = (\neg S)^k = (\sim S)^u$$
$$S^t \cup S^u = \sim S^- \qquad S^u = (\neg S)^u = (\sim S)^k$$
$$S^f \cup S^u = \sim S^+ \qquad S^f = (\neg S)^t = (\sim S)^t$$
$$S^t \cap S^k = S^t \cap S^u = S^t \cap S^f = S^f \cap S^k = S^f \cap S^u = S^k \cap S^u = \emptyset$$
$$S^t \cup S^k \cup S^u \cup S^f = A^n$$

The above set of calculus is captured in DDT and similar logics (see Thomason and Horty, 1987) through the introduction of positive and negative entailment (satisfaction of a sentence and its negation; for details see Tsoukiàs, 2002).

2.3 Continuous Extension

For the continuous extension of a four valued logic, S^+ and S^- can be considered as fuzzy sets and two membership functions can be introduced: μ_S^+ and μ_S^-. They can be considered for instance as degrees which represent to what extent we believe in $S(a)$ and non $S(a)$ respectively :

$$\mu_S^+ : \quad X \to [0,1] \qquad\qquad \mu_S^- : \quad X \to [0,1]$$

We then have to define the fuzzy subsets S^t, S^k, S^u, S^f. Hence, we have to make explicit the intersection, the union and the complementarity to fuzzy subsets of X. To define these operators, we introduce a De Morgan triple (N, T, V) where N is a strict negation on [0, 1], T a continuous t-norm and V is a continuous co-norm such that $V(x, y) = N(T(N(x), N(y)))$. If we denote $u = \mu_{S^t}(a)$, $v = \mu_{S^k}(a)$, $x = \mu_{S^+}(a)$, $y = \mu_{S^-}(a)$, we have:

$$u = T(x, N(y)) \qquad v = T(x, y) \qquad x = V(u, v) \qquad y = V(N(u), v)$$

As a consequence we should get:

$$\forall x, y \in [0,1], \quad x = V(T(x, N(y)), T(x, y))$$

Unfortunately, for such an equation generally there is no De Morgan triplet satisfying it. Thus, we have to investigate partial solutions. Following Perny et Tsoukiàs (Perny and Tsoukiàs (1998)) we denote:

$$\mu_S^+(\alpha) = B(\alpha) \qquad\qquad \mu_S^-(\alpha) = B(\neg\alpha)$$

Thus the four truth values $t(\alpha)$, $k(\alpha)$, $u(\alpha)$, $f(\alpha)$ can be defined through $B(\alpha)$ and $B(\neg\alpha)$ as follows (using different T-norms):

$$\mu_{S^t}(\alpha) = t(\alpha) = T_1(B(\alpha), N(B(\neg\alpha))) \tag{2.1}$$

$$\mu_{S^k}(\alpha) = k(\alpha) = T_2(B(\alpha), (B(\neg\alpha)) \tag{2.2}$$

$$\mu_{S^u}(\alpha) = u(\alpha) = T_3(N(B(\alpha)), N(B(\neg\alpha))) \tag{2.3}$$

$$\mu_{S^f}(\alpha) = f(\alpha) = T_4(N(B(\alpha)), (B(\neg\alpha)) \tag{2.4}$$

In order to fulfill the definition of fuzzy partition ($t(\alpha)+k(\alpha)+u(\alpha)+f(\alpha) = 1$) we can use the following:

$$N = LN_\phi \qquad T_2 = T_3 = LT_\phi \qquad V = LV_\phi \qquad T_1 = T_4 = \min$$

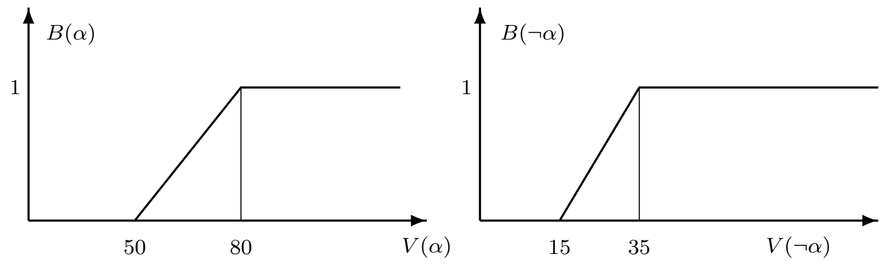

Fig. 2.2. $B(\alpha)$ and $B(\neg\alpha)$ for example 2

Where (LN $_\phi$, LT $_\phi$, LV $_\phi$) is *the Lukasiewicz triple* (see Schweizer and Sclar, 1983). We thus get

$$t(\alpha) = min(B(\alpha), 1 - B(\neg\alpha)) \quad (2.5)$$
$$k(\alpha) = max(B(\alpha) + B(\neg\alpha) - 1, 0) \quad (2.6)$$
$$u(\alpha) = max(1 - B(\alpha) - B(\neg\alpha), 0) \quad (2.7)$$
$$f(\alpha) = min(1 - B(\alpha), B(\neg\alpha)) \quad (2.8)$$

The reader can see further details within the Appendix A (based on Perny and Tsoukiàs (1998)) For another approach using the continuous extension of four-valued logic the reader can see Fortemps and Slowinski (2002). In this case the fuzzy partition property is dropped.

2.3.1 Example 2. We take again the example of the Parliament, but this time we are going to value the positive and negative reasons within the $[0, 1]$ interval. Positive reasons become strictly positive when at least 50% of the MPs vote "for" and become sure (equal to 1) when at least 80% vote "for". Negative reasons become strictly positive when at least 15% vote "against" and become sure (equal to 1) when at least 35% vote "against". The model is shown in figure 2.2.

In table 2.4 we show the simulation of a number of votes on a set of issues. How can the decomposition in positive and negative reasons help a decision maker? First of all it is easy to observe that (with that precise decision rule) negative reasons grow faster than positive ones. Cases 1 to 3 show that convincing two non voters to vote "for" will not improve acceptability ($t(a)$), while convincing two opponents to not vote will do. Cases 4 and 5 show how acceptability and opposition will change due to opinion shifts

Case	$V(a)$	$V(\neg a)$	$B(a)$	$B(\neg a)$	$t(a)$	$k(a)$	$u(a)$	$f(a)$
1	75	20	0.83	0.25	0.75	0.08	0	0.17
2	75	18	0.83	0.15	0.83	0	0.02	0.15
3	77	20	0.9	0.25	0.75	0.15	0	0.1
4	82	18	1	0.15	0.85	0.15	0	0
5	78	22	0.93	0.35	0.65	0.28	0	0.07
6	58	26	0.26	0.55	0.26	0	0.19	0.55
7	58	17	0.26	0.1	0.26	0	0.64	0.1
8	58	35	0.26	1	0	0.26	0	0.74
9	68	26	0.6	0.55	0.45	0.15	0	0.4
10	68	17	0.6	0.1	0.6	0	0.3	0.1

Table 2.4. The truth table for example 2

from "for" to "against" when there are no "non voters". Cases 5 to 10 show the appearance of hesitation due to ignorance or conflict. The analysis of the positive and negative reasons helps in showing to a decision maker in what direction he should concentrate his efforts in order to pursue his policy.

3 B(a) as a standard necessity

In this section we first briefly recall some definitions of possibility theory which will be useful for the rest of the paper (the reader can see more details in Dubois and Prade (1988)). Possibility measures are expected to provide an ordinal representation of uncertainty as follows:

Definition 3.1 ***Possibility Measure***
Given a set of events Ω, a possibility measure Π is a function defined on the power set 2^Ω, ($\Pi : 2^\Omega \mapsto [0,1]$ such that:

1. $\Pi(\emptyset) = 0$, $\Pi(\Omega) = 1$
2. $A \subseteq B \in 2^\Omega \rightarrow \Pi(A) \leq \Pi(B)$
3. $\forall A, B \in 2^\Omega$, $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$

The dual of the possibility measure, denoted necessity measure is defined as $N(a) = 1 - \Pi(\neg a)$.

Definition 3.2 ***Necessity measure***
Given a set of events Ω, a necessity measure N is a function defined on the power set 2^Ω, ($N : 2^\Omega \mapsto [0,1]$, such that:

1. $N(\emptyset) = 0$, $N(\Omega) = 1$,
2. $A \subseteq B \in 2^\Omega \rightarrow N(A) \leq N(B)$
3. $\forall A, B \in 2^\Omega$, $N(A \cap B) = \min(N(A), N(B))$

The disjunction of the necessity measure and the conjunction of the possibility measure are not compositional.

$$N(\alpha \vee \beta) \geqslant max(N(\alpha), N(\beta))$$
$$\Pi(\alpha \wedge \beta) \leqslant min(\Pi(\alpha), \Pi(\beta))$$

As a result, we obtain the following properties:

$$max(\Pi(a), \Pi(\neg a)) = 1 \quad (3.1)$$
$$\Pi(a) + \Pi(\neg a) \geq 1 \quad (3.2)$$
$$\Pi(a) \geq N(a) \quad (3.3)$$
$$If \quad N(a) \neq 0, \quad then \quad \Pi(a) = 1 \quad (3.4)$$
$$If \quad \Pi(a) \neq 1, \quad then \quad N(a) = 0 \quad (3.5)$$

By definition we can consider a possibility measure as the upper bound of the uncertainty associated to an event (or a sentence), the one carrying the less specific information. Dually the necessity measure will represent the lower bound: how sure we are about an event (or a sentence). Clearly three extreme situations are possible:
- $N(a) = 1, N(\neg a) = 0$, a is the case;
- $N(a) = 0, N(\neg a) = 1$, $\neg a$ is the case;
- $N(a) = 0, N(\neg a) = 0$, nothing is sure and everything is possible.

A first attempt to interpret the continuous valuation of "presence of truth in α" and "presence of truth in $\neg\alpha$" could be to consider them as necessity measures. Coming back to our notation, we consider $B(\alpha)$, as a standard necessity; as a consequence we have:

$$B(\alpha) = N(\alpha) = 1 - \Pi(\neg\alpha)$$
$$B(\neg\alpha) = N(\neg\alpha) = 1 - \Pi(\alpha)$$

Hence, we obtain the following definitions:

$$t(\alpha) = min(N(\alpha), \Pi(\alpha)) \quad (3.6)$$
$$k(\alpha) = max(N(\alpha) - \Pi(\alpha), 0) \quad (3.7)$$
$$u(\alpha) = max(\Pi(\alpha) - N(\alpha), 0) \quad (3.8)$$
$$f(\alpha) = min(\Pi(\neg\alpha), N(\neg\alpha)) \quad (3.9)$$

However, since $\Pi(\alpha) > N(\alpha)$ we can reformulate the equations 3.6-3.9 as follows:

$$t(\alpha) = N(\alpha) \quad (3.10)$$
$$k(\alpha) = 0 \quad (3.11)$$
$$u(\alpha) = \Pi(\alpha) - N(\alpha) \quad (3.12)$$
$$f(\alpha) = N(\neg\alpha) = 1 - \Pi(\alpha) \quad (3.13)$$

We first observe that interpreting $B(\alpha)$ as a standard necessity measure leads to $k(\alpha) = 0$. This is not surprising given the semantics of necessity. Let us study separately two situations, i.e $N(\alpha) = 0$ and $N(\alpha) > 0$:
When N(α)> 0: we have $\Pi(\alpha) = 1$ and we get

$$t(\alpha) = N(\alpha) \tag{3.14}$$
$$k(\alpha) = f(\alpha) = 0 \tag{3.15}$$
$$u(\alpha) = \Pi(\neg\alpha) \tag{3.16}$$

When N(α)= 0, we get:

$$t(\alpha) = k(\alpha) = 0 \tag{3.17}$$
$$u(\alpha) = \Pi(\alpha) \tag{3.18}$$
$$f(\alpha) = N(\neg\alpha) \tag{3.19}$$

Going further on and using the definitions of Appendix A we can establish the functions concerning more complex formula. For instance we get for $t(\alpha \vee \beta)$:

$$\begin{aligned}
t(\alpha \vee \beta) =& min(max(t(\alpha) + u(\alpha), t(\beta) + u(\beta)), \\
& max(t(\alpha) + k(\alpha), t(\beta) + k(\beta))) \\
t(\alpha \vee \beta) =& min(max(N(\alpha) + \Pi(\alpha) - N(\alpha), N(\beta) + \\
& \Pi(\beta) - N(\beta)), max(N(\alpha) + 0, N(\beta) + 0)) \\
t(\alpha \vee \beta) =& max(N(\alpha), N(\beta))
\end{aligned}$$

We obtain here a result which is not expected since a standard necessity measure should be:

$$\begin{aligned}
& t(\alpha \vee \beta) = N(\alpha \vee \beta) \\
& N(\alpha \vee \beta) \geq max(N(\alpha), N(\beta))
\end{aligned}$$

It is interesting to observe that the problem of obtaining two different representations for such formulae becomes more severe when implication is concerned. This is not really a surprise since associating an uncertainty distribution to a boolean algebra will not result to a boolean algebra and therefore it is necessary to do representation choices. We are not going to further discuss this issue in this paper. We only retain from this section that the use of a standard necessity reduces the expressive power of the four valued logic since it requires that one of the four values becomes always zero. For this reason we are going to consider other less standard uncertainty measures.

4 B(a) as a sub-normalised necessity measure

An important feature of four-valued logics is the separation of negation from complementarity. Possibility theory does not make any difference between these two operators since it has been conceived as an uncertainty measure to be associated to classic logic. In this section, we suggest the idea of associating an uncertainty measure to a formalism such as DDT and study the consequences. In order to do that we recall the use in DDT of the "weak negation" $\not\sim$ (to be read as "perhaps"). We remind that such a weak negation is conceived so that the complement of a sentence $\sim\alpha$ can be established as $\neg \not\sim \neg \not\sim \alpha$. We further impose (consistently with the semantics of the DDT language) that an uncertainty distribution associated to a sentence of the language should fulfil the property:

$$B(\alpha) \;=\; B(\not\sim \alpha) \;=\; t(\alpha) + k(\alpha)$$

Denoting the dual measure of B as H ($H(\alpha) = 1 - B(\neg\alpha)$) and recalling that $B(\neg\alpha) = f(\alpha) + k(\alpha)$ as well as the principle of fuzzy partition we get that: $H(\alpha) = t(\alpha) + u(\alpha)$

Further on recall that due to the definitions of section 2 we have:

$$\begin{aligned}
t(\alpha) =& f(\neg\alpha) = f(\neg \not\sim \neg \not\sim \alpha)\\
k(\alpha) =& k(\neg\alpha) = u(\neg \not\sim \neg \not\sim \alpha)\\
u(\alpha) =& u(\neg\alpha) = k(\neg \not\sim \neg \not\sim \alpha)\\
f(\alpha) =& t(\neg\alpha) = t(\neg \not\sim \neg \not\sim \alpha)
\end{aligned}$$

Therefore we have:
$H(\alpha) \;=\; t(\alpha) + u(\alpha) \;=$
$= \; f(\neg \not\sim \neg \not\sim \alpha) + k(\neg \not\sim \neg \not\sim \alpha) \;=$
$= \; t(\not\sim \neg \not\sim \alpha) + k(\not\sim \neg \not\sim \alpha) \;=$
$= \; B(\not\sim \neg \not\sim \alpha) \;=\; B(\neg \sim \alpha)$.

In other terms the dual measure of B is equal to the measure of the negation of the complement. We can summarise the result in table 4.1.

$B(\alpha) =$	$B(\nsim \alpha) =$	$H(\nsim \neg \nsim \alpha) =$	$H(\nsim \neg\alpha)$
$B(\neg\alpha) =$	$B(\nsim \neg\alpha) =$	$H(\neg \nsim \neg \nsim \alpha) =$	$H(\nsim \alpha)$
$B(\neg \nsim \neg \nsim \alpha) =$	$B(\neg \nsim \neg\alpha) =$	$H(\neg\alpha) =$	$H(\neg \nsim \alpha)$
$B(\nsim \neg \nsim \alpha) =$	$B(\neg \nsim \alpha) =$	$H(\alpha) =$	$H(\neg \nsim \neg\alpha)$

Table 4.1. Equivalence between B and H

Remark 4.1. Table 4.1 shows that the introduction of the weak negation reduces the dual measures of the type necessity/possibility to a single one. Indeed we just need to know an uncertainty measure of a sentence and of its negation in order to know all about the uncertainty associated to this sentence.

Remark 4.2. Let us consider the first column of table 4.1. If we consider that only one uncertainty distribution is defined (say B) there is no reason to claim that $B(\not\sim \neg \not\sim \alpha) > B(\alpha)$ (the uncertainty associated to the negation of the complement of a sentence is not necessarily larger than the uncertainty associated to the sentence itself; they should be unrelated). However, since $B(\not\sim \neg \not\sim \alpha) = H(\alpha)$, if the above relation does not hold we are practically relaxing the normalisation principle of uncertainty measures used in possibility theory. What we see is that, while it is difficult to justify such distributions in a pure possibility theory frame, the use of the DDT logic allows to give a logical justification for their existence.

5 Conclusion

In this paper we discuss two distinct tools used to deal with uncertainty: four valued logics and uncertainty distributions. In a first stage, we limited our study in interpreting uncertainty as a standard necessity measure. However, the results appear poor since within such a frame only three values can be considered. A further analysis enabled to show that, in order to fully capture the basic idea behind four valued logics (distinguish between negation and complement of a sentence), is necessary to consider sub-normalised uncertainty measures.

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Appendix A

Let (x, y), (z, w) be two ordered pairs; the lower and upper bound and the pseudo-complementarity are defined as follows when the min triplet is used:

$$\begin{aligned}\bot_1[(x,y),(z,w)] =&(min\{x,z\}, max\{y,w\}\\ \top_1[(x,y),(z,w)] =&(max\{x,z\}, min\{y,w\}\\ N_1(x,y) =&(y,x)\end{aligned}$$

In the case the Lukasiewicz triple (LN, LT, LV) is adopted we get:

$$\begin{aligned}\bot_2[(x,y),(z,w)] =&(LV\{x,z\}, LT\{y,w\}\\ \top_2[(x,y),(z,w)] =&(LT\{x,z\}, LV\{y,w\}\\ N_2(x,y) =&(LN(x), LN(y))\end{aligned}$$

Hence we get for any ψ transformation:

$$\begin{aligned}\bot_j^M(X,Y) =&\psi^{-1}(\bot_j[\psi(X),\psi(Y)])\\ \top_j^M(X,Y) =&\psi^{-1}(\top_j[\psi(X),\psi(Y)])\\ N_j^M(X) =&\psi^{-1}(N_j[\psi(X)])\end{aligned}$$

We can now define all the operators:

$$\begin{aligned}v(\neg\alpha) =&N_1^M(v(\alpha))\\ v(\alpha\wedge\beta) =&\bot_j^M(v(\alpha),v(\beta))\\ v(\alpha\vee\beta) =&\top_j^M(v(\alpha),v(\beta))\\ v(\alpha\ \rightarrow\ \beta) =&\top_2^M[N_2^M(v(\alpha),v(\beta))]\\ v(\alpha\equiv\beta) =&v((\alpha\ \rightarrow\ \beta)\wedge v(\beta\ \rightarrow\ \alpha))\end{aligned}$$

and we get the following formulae:

$$
\begin{aligned}
t(\alpha \wedge \beta) &= \min\{t(\alpha), t(\beta)\}\\
k(\alpha \wedge \beta) &= \max\{\min\{t(\alpha) + k(\alpha), t(\beta) + k(\beta)\} - \min\{t(\alpha) + u(\alpha), t(\beta) + u(\beta)\}, 0\}\\
u(\alpha \wedge \beta) &= \max\{\min\{t(\alpha) + u(\alpha), t(\beta) + u(\beta)\} - \min\{t(\alpha) + k(\alpha), t(\beta) + k(\beta)\}, 0\}\\
f(\alpha \wedge \beta) &= \min\{\max\{f(\alpha) + u(\alpha), f(\beta) + u(\beta)\}, \max\{f(\alpha) + k(\alpha), f(\beta) + k(\beta)\}
\end{aligned}
$$

$$
\begin{aligned}
t(\alpha \vee \beta) &= \min\{\max\{t(\alpha) + u(\alpha), t(\beta) + u(\beta)\}, \max\{t(\alpha) + k(\alpha), t(\beta) + k(\beta)\}\}\\
k(\alpha \vee \beta) &= \max\{\min\{f(\alpha) + k(\alpha), f(\beta) + k(\beta)\} - \min\{f(\alpha) + u(\alpha), f(\beta) + u(\beta)\}, 0\}\\
u(\alpha \vee \beta) &= \max\{\min\{f(\alpha) + u(\alpha), f(\beta) + u(\beta)\} - \min\{f(\alpha) + k(\alpha), f(\beta) + k(\beta)\}, 0\}\\
f(\alpha \vee \beta) &= \min\{f(\alpha), f(\beta)\}
\end{aligned}
$$

$$
\begin{aligned}
t(\alpha \to \beta) &= \min\{f(\alpha) + u(\alpha) + t(\beta) + k(\beta), f(\alpha) + k(\alpha) + t(\beta) + u(\beta), 1\}\\
k(\alpha \to \beta) &= \max\{\min\{f(a) + u(a) + t(\beta) + k(\beta), 1\} - \min\{f(\alpha) + k(\alpha) + t(\beta) + u(\beta), 1\}, 0\}\\
u(\alpha \to \beta) &= \max\{\min\{f(a) + k(a) + t(\beta) + u(\beta), 1\} - \min\{f(\alpha) + u(\alpha) + t(\beta) + k(\beta), 1\}, 0\}\\
f(\alpha \to \beta) &= \min\{\max\{t(a) + k(a) - t(\beta) - k(\beta), 0\}, \max\{f(\beta) + k(\beta) - f(\alpha) - k(\alpha), 0\}\}
\end{aligned}
$$