Multiple Criteria Decision Analysis

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Outline

Example

Consider the following evaluation table concerning four candidates (A,B,C and D) assessed against four criteria H1,H2,H3 and H4.

	H1	H2	H3	H4
A	7	5	9	6
B	8	4	7	8
C	5	8	10	4
D	9	3	5	10

Who is the best?

What is the problem?

- Given a set $A = \{x, y, z, w, \cdots\}$;
- Given (possibly) a set of profiles P;
- Given a set of attributes D;
- Given the assessment of A against D;

Partition the set A in the best possible way.

What are the primitives?

Primitive 1

The primitives are binary relations on A: $\succeq_j \subseteq A \times A$ to be read "at least as good as" or binary relations on A: $\approx_l \subseteq A \times A$ to be read "similar to". (Unsupervised Decision Procedure).

Primitive 2

The primitives are binary relationa between A and P: $\succeq \subseteq A \times P \cup P \times A$ to be read "at least as good as" or binary relations between A and P: $\approx_l \subseteq A \times A$ to be read "similar to". P being the set of external "norms" characterising some classes $C_1 \cdots C_n$. (Supervised Decision Procedure).

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Borda vs. Condorcet

Four candidates and seven examiners with the following preferences.

	a	b	c	d	e	f	g
A	1	2	4	1	2	4	1
B	2	3	1	2	3	1	2
C	3	1	3	3	1	2	3
D	4	4	2	4	4	3	4

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B(x)
15
14
16
25

The Borda count gives B>A>C>D

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If D is not there then A>B>C, instead of B>A>C

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The Condorcet principle gives A>B>C>A !!!!

Arrow's Theorem

Given N rational voters over a set of more than 3 candidates can we found a social choice procedure resulting in a social complete order of the candidates such that it respects the following axioms?

- Universality: the method should be able to deal with any configuration of ordered lists;
- Unanimity: the method should respect a unanimous preference of the voters;
- Independence: the comparison of two candidates should be based only on their respective standings in the ordered lists of the voters.

YES!

There is only one solution: the dictator!!

If we add no-dictatorship among the axioms then there is no solution.

Gibbard-Satterthwaite's Theorem

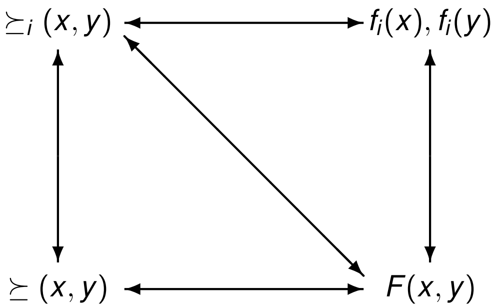
When the number of candidates is larger than two, there exists no aggregation method satisfying simultaneously the properties of universal domain, non-manipulability and non-dictatorship.

Why MCDA is not Social Choice?

Social Choice	MCDA
Total Orders	Any type of order
Equal importance of voters	Variable importance of criteria
As many voters as necessary	Few coherent criteria
No prior information	Existing prior information

The Problem

Suppose we have n preference relations $\succeq_1 \cdots \succeq_n$ on the set A. We are looking for an overall preference relation $\succeq$ on A "representing" the different preferences.



Counting values

$$x \succeq y \;\Leftrightarrow\; \sum_j r_j(x) \geq \sum_j r_j(y)$$

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the primitives: $\succeq_j \subseteq A \times A$
Differences of preferences:
- $(xy)_1 \succcurlyeq (zw)_1$
- $(xy)_1 \succcurlyeq (zw)_2$

How do we learn that?

- Directly through a standard protocol.
- Indirectly:
  - through pairwise comparisons (AHP, MACBETH etc.);
  - through learning from examples (regression, rough sets, decision trees etc.).

Is this sufficient?

NO!

Are preferences independent?
$r \succ w$
$f \succ m$
But *rf* is not better than *wf* ...

Non linear aggregation procedures

What is the output?

- Value functions on each criterion.
- A global value function.
- Rankings, choices, but also ratings if relevant reference points are provided on the value function.

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Counting preferences

$$x \succeq y \Leftrightarrow H_{xy} \geq H_{yx}$$

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the primitives: $\succeq_j \subseteq A \times A$
An ordering relation on $2^{\succeq_j}$

How do we learn that?

- Preferences are “given”.
- Preferences on $2^{\succeq_j}$:
  - directly;
  - coalition games;
  - learning from examples.

Is this sufficient?

NO!

- The relation $\succeq$ is not an ordering relation.
- We need to construct an ordering relation $\succcurlyeq$ "as near as possible" to $\succeq$.
- In order to do so we transform the graph induced by $\succeq$.

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General idea: coalitions

Given a set A and a set of $\succeq_i$ binary relations on A (the criteria) we define:

$$x \succeq y \Leftrightarrow C^+(x,y) \trianglerighteq C^+(y,x) \text{ and } C^-(x,y) \trianglelefteq C^-(y,x)$$

where:
- $C^+(x,y)$: "importance" of the coalition of criteria supporting x wrt to y.
- $C^-(x,y)$: "importance" of the coalition of criteria against x wrt to y.

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Additive Positive Importance

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$$C^{+}(x,y) = \sum_{j \in J^{\pm}} w_j^{+}$$

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Where "positive importance" comes from?

How it works? 2

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$$C^-(x,y) = \max_{j \in J^-} w_j^-$$

where:

w_j^-: "negative importance" of criterion i

$J^- = \{h_j : v_j(x,y)\}$

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$$x \succeq^- y \Leftrightarrow C^-(x,y) \geq \gamma$$

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Where "negative importance" comes from?

Example

The United Nations Security Council

Positive Importance

15 members each having the same positive importance $w_j^+ = \frac{1}{15}$, $\delta = \frac{9}{15}$.

Negative Importance

10 members with 0 negative importance and 5 (the permanent members) with $w_j^- = 1$, $\gamma = 1$.

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Outranking Principle

$$x \succeq y \Leftrightarrow x \succeq^{+} y \text{ and } \neg(x \succeq^{-} y)$$

Thus:

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NB

The relation $\succeq$ is not an ordering relation. Specific algorithms are used in order to move from $\succeq$ to an ordering relation $\succcurlyeq$

What is importance?

Where w_j^+, w_j^- and δ come from?

Further preferential information is necessary, usually under form of multi-attribute comparisons. That will provide information about the decisive coalitions.

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Example

Given a set of criteria and a set of decisive coalitions ($J^{\pm}$) we can solve:

$$\max \delta$$
subject to
$$\sum_{j \in J^{\pm}} w_j \geq \delta$$
$$\sum_j w_j = 1$$

And the final ranking?

- $x \succcurlyeq y \;\Leftrightarrow\; o(x) - i(x) \geq o(y) - i(y)$
- Recursively constructing $\succcurlyeq$:
  - $[x]_1 = \{x \in A : \;\neg\; \exists y \;\; y \succeq x\}$
    $[x]_i = \{x \in A \setminus \cup_{i-1}[x] : \;\neg\; \exists y \;\; y \succeq x\}$
  - $[x]_n = \{x \in A : \;\neg\; \exists y \;\; x \succeq y\}$
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Rating

What if we have preference relations $\succeq_j \subseteq A \times P \cup P \times A$?
The global preference relation remains the same.

- pessimistic rating
  - x is iteratively compared with $p_t \cdots p_1$,
  - as soon as $x \succeq p_h)$ is established, assign x to category c_h.
- optimistic rating
  - x is iteratively compared with $p_1 \cdots p_t$,
  - as soon as is established $p_h \succeq x) \quad \wedge \quad \neg \quad x \succeq p_h)$ then assign x to category c_{h-1}.

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- A global preference relation including incomparabilities.
- An explicit representation of hesitation.
- Robust Rankings, Choices and Ratings.

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Lessons Learned

- We can use social choice inspired procedures for more general decision making processes.
- Care should be taken to model the majority (possibly the minority) principle to be used. The key issue here is the concept of “decisive coalition”.
- We need to “learn” about decisive coalitions, since it is unlike that this information is available. Problem of learning procedures.
- The above information is not always intuitive. However, the intuitive idea of importance contains several cognitive biases.
- A social choice inspired procedure will not deliver automatically an ordering. We need further algorithms (graph theory).

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Resources

- http://www.algodec.org
- http://www.cs.put.poznan.pl/ewgmcda/
- http://www.decision-deck.org
- http://decision-analysis.society.informs.org/
- http://www.mcdmsociety.org/
- http://www.euro-online.org
- http://www.informs.org

Books

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